Arithmetic Algebraic Geometry
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Finiteness results for motivic cohomology

Shuji Saito (University of Tokyo)

joint work with Uwe Jannsen (University of Regensburg)

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Let X be a regular scheme of finite type over $\mathbb{F}_p$ or $\mathbb{Z}$.

There is the following 'folklore conjecture', generalizing the analogous conjecture of Bass on K-groups:

Conjecture A $H^q_M(X, \mathbb{Z}(r))$ is finitely generated for all q, r.

Here

$$H^q_M(X, \mathbb{Z}(r)) = H^q_{Zar}(X, \mathbb{Z}(r)_M) = CH^r(X, 2r - q)$$

is the *motivic cohomology of* X, defined as the Zariski cohomology of the motivic complex $\mathbb{Z}(r)_M$ or, equivalently, as the *Bloch's higher Chow group*.

A quick review on *Bloch's higher Chow group* :

$z^r(X,q)$ =free abelian group on the set of closed integral subschemes of codimension r of $X \times \Delta^q$ which intersects all faces $\Delta^s \subset \Delta^q$ properly,

where

$$\Delta^q = \mathrm{Spec}(\mathbb{Z}[t_0,\ldots,t_q]/(\sum_{i=0}^{q} t_i - 1))$$

is the algebraic standard q-simplex and faces $\Delta^s \subset \Delta^q$ are given by substituting 0 for some of $t_0,\ldots,t_q$.

One can view Δ^q as an algebraic analogue of the standard q-simplex in topology and $z^r(X,q)$ as an analogue of the free abelian group on the set of continuous maps $\Delta^q \to X$.

We then get *Bloch's cycle complex* :

$$z^r(X,\bullet)\ :\ \cdots z^r(X,q) \xrightarrow{\partial} \cdots z^r(X,1) \xrightarrow{\partial} z^r(X,0),$$

where the boundary map ∂ is the alternating sum of pull-back maps to faces of codimension 1.

Bloch's higher Chow group is defined as :

$$\mathrm{CH}^r(X,q) = H_q(z^r(X,\bullet)).$$

We note

$$\mathrm{CH}^r(X,0) = \mathrm{CH}^r(X),$$

the Chow group of algebraic cycles of codimension r on X modulo rational equivalence.

Remark $z^r(X,q)$ is a free abelian group of infinite rank.

The only known general results on Conjecture A are:

Theorem(Dirichlet, Mordell-Weil) The groups

$$H^2_M(X,\mathbb{Z}(1)) = Pic(X), \quad H^1_M(X,\mathbb{Z}(1)) = H^0(X,\mathcal{O}_X^\times)$$

are finitely generated for X of arbitrary dimension (e.g. for $X = \mathrm{Spec}(\mathcal{O}_k)$ with a number field k, LHS is its ideal class group and RHS is its unit group).

Theorem(Quillen) Conjecture A holds for X of dimension 1.

Theorem (Bloch, Kato-S., Colliot-Thélène-Sansuc-Soulé) If X is regular and projective of dimension d over $\mathbb{F}_p$ or $\mathbb{Z}$,

$$H^{2d}_M(X,\mathbb{Z}(d)) = \mathrm{CH}^d(X) = \mathrm{CH}_0(X)$$

is finitely generated.

Observation: All these results are obtained by considering some *regulator maps* or *cycle maps*.

E. g., the last result comes from *higher class field theory*: The following *reciprocity map* is shown to be injective:

$$\rho_X \ : \ \mathrm{CH}^d(X) = \mathrm{CH}_0(X) \longrightarrow \pi_1^{ab}(X),$$

X : a regular proper scheme of dimension d over $\mathbb{F}_p$ or $\mathbb{Z}$,

$\pi_1^{ab}(X)$: the abelian fundamental group of X.

In case $X = \mathrm{Spec}(\mathcal{O}_k)$ for a number field k, this is the *Hilbert class field theory*.

Finiteness result on $\pi_1^{ab}(X)$ due to Katz-Lang then implies that $CH^d(X)$ is finitely generated.

Second observation: By a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^d(X)/n & \xrightarrow{\rho_X} & \pi_1^{ab}(X)/n \\ \| & & \downarrow \simeq \\ H^{2d}_M(X, \mathbb{Z}/n\mathbb{Z}(d)) & \xrightarrow{\rho_X^{2d,d}} & H^{2d}_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(d)), \end{array}$$

where $d = \dim(X)$,

ρ_X is interpreted as a *cycle map into étale cohomology*. Here

$$H^q_M(X, \mathbb{Z}/n\mathbb{Z}(r)) = \mathrm{CH}^r(X, 2r - q; \mathbb{Z}/n\mathbb{Z})$$

is the *motivic cohomology with finite coefficients* defined by

$$\mathrm{CH}^r(X, q; \mathbb{Z}/n\mathbb{Z}) = H_q(z^r(X, \bullet) \otimes^{\mathbb{L}} \mathbb{Z}/n\mathbb{Z}).$$

We have a short exact sequence:

$$0 \to H^q_M(X, \mathbb{Z}(r))/n \to H^q_M(X, \mathbb{Z}/n\mathbb{Z}(r)) \\ \to H^{q+1}_M(X, \mathbb{Z}(r))[n] \to 0,$$

where $M[n] = \mathrm{Ker}(M \xrightarrow{n} M)$ for a module M.

We are led to the following weak form of Conjecture A :

Conjecture B $H^q_M(X, \mathbb{Z}/n\mathbb{Z}(r))$ is finite for any integer $n > 0$ (for X regular of finite type over $\mathbb{F}_p$ or $\mathbb{Z}$).

Conjecture B relates to Conjecture A like the weak Mordell-Weil theorem relates to the strong one.

Now we discuss the target of the cycle map and our results.

The definition of the suitable étale cohomology needs some care, because we have to consider p-torsion in characteristic p.

There are two cases.

(geometric case) : X is a smooth scheme over $\mathrm{Spec}(\mathbb{F})$ where $\mathbb{F}$ is a finite field,

(arithmetic case) : X is a regular scheme flat of finite type over $\mathrm{Spec}(\mathbb{Z})$.

First consider the geometric case.

Let X be a smooth variety over a finite field $\mathbb{F}$.

Geisser and Levine constructed an étale cycle map:

$$\rho_X^{s,r} \; : \; H^s_M(X, \mathbb{Z}/n\mathbb{Z}(r)) \to H^s_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(r)),$$

where $\mathbb{Z}/n\mathbb{Z}(r)$ on RHS denotes the following complex of étale sheaves on X:

$$\mathbb{Z}/n\mathbb{Z}(r) = \mu_m^{\otimes r} \oplus W_\nu \Omega^r_{X,log}[-r],$$

if $n = mp^\nu$ with $p = \mathrm{ch}(\mathbb{F})$ and m invertible in $\mathbb{F}$. Here

$\mu_m =$ the étale sheaf of m-th roots of unity, and

$W_\nu \Omega^r_{X,log} =$ logarithmic part of de Rham-Witt sheaf $W_\nu \Omega^r_X$.

Theorem (M. Artin, Milne) $H^s_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(r))$ is finite.

Thus the injectivity of

$$\rho_X^{s,r} \; : \; H^s_M(X, \mathbb{Z}/n\mathbb{Z}(r)) \to H^s_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(r)),$$

would imply that $H^s_M(X, \mathbb{Z}/n\mathbb{Z}(r))$ is finite.

Higher class field theory implies $\rho_X^{s,d}$ for $d = \dim(X)$, induces

$$H^s_M(X, \mathbb{Z}/n\mathbb{Z}(d)) \xrightarrow{\simeq} H^s_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(d)) \quad \text{for } s = 2d.$$

We also remark

$$H^s_M(X, \mathbb{Z}/n\mathbb{Z}(d)) = \mathrm{CH}^d(X, 2d - s; \mathbb{Z}/n\mathbb{Z}) = 0 \text{ for } s > 2d.$$

Theorem 1 (Jannsen - S.)

Let X be smooth projective of dimension d over $\mathbb{F}$.

Let $q \geq 0$ be an integer and ℓ be a prime.

Then $\rho_X^{s,d}$ induces an isomorphism

$$H^s_M(X, \mathbb{Z}/\ell^n\mathbb{Z}(d)) \xrightarrow{\simeq} H^s_{\text{ét}}(X, \mathbb{Z}/\ell^n\mathbb{Z}(d)) \quad \text{for } 2d \geq s \geq 2d - q,$$

if $\mathbf{(BK)}_{q+2,\ell}$ holds over X, and one of the following holds :

(1) $q \leq 2$, or
(2) $\mathbf{(RS)}_d$, or
(3) $\mathbf{(RES)}_q$.

We explain and discuss the assumptions:

(BK)$_{q,\ell}$ over X :The cup product for Galois cohomology

$$H^1_{Gal}(L, \mathbb{Z}/\ell\mathbb{Z}(1))^{\otimes q} \to H^q_{Gal}(L, \mathbb{Z}/\ell\mathbb{Z}(q))$$

is surjective for any field L finitely generated over a residue field of X (Milnor-Bloch-Kato conjecture).

The surjectivity of the above map is known to hold if

$q = 2$ (Merkurjev-Suslin) or

$\ell = \mathrm{ch}(L)(= \mathrm{ch}(\mathbb{F}))$ (Bloch-Gabber-Kato) or

$\ell = 2$ (Voevodsky).

It has been announced by Rost and Voevodsky that it holds in general.

(RS)$_d$: For any Y integral and proper of dimension $\leq d$ over $\mathbb{F}$, there exists a proper birational morphism $\pi : Y' \to Y$ such that Y' is smooth over $\mathbb{F}$.

For any U smooth of dimension $\leq d$ over $\mathbb{F}$, there is an open immersion $U \hookrightarrow Y$ such that Y is projective smooth over $\mathbb{F}$ and $Y - U$ is a simple normal crossing divisor on Y.

(RES)$_q$: For

Z, a regular scheme of fintie type over $\mathbb{F}$,

$D \subset Z$, a simple normal crossing divisor on Z,

$Y \subset Z$, an integral closed subscheme of dimension $\leq q$ such that $Y \backslash D$ is regular,

there exists a projective morphism $\pi : Z' \to Z$ such that

(1) Z' is regular and $\pi^{-1}(Z - D) \simeq Z - D$,
(2) $D' = \pi^{-1}(D)$ is a simple normal crossing divisor on Z',
(3) the proper transform Y' of Y in X' is regular and normally crossing to D'.

Theorem 1 in case $q \leq 2$ follows from that under the assumption $\textbf{(RES)}_q$, in view of the following:

Theorem R (Hironaka, Cossart, Cossart-Jannsen-S.)
$\textbf{(RES)}_q$ holds for $q \leq 2$.

The proof is based on the theory of characteristic polyhedra developed in 1960's by Hironaka.

A method of Zariski (refined later by Lipman) shows that any excellent surface can be desingularized by a succession of blowups and normalizations.

In the above theorem only blowups with a regular center normally crossing to boundaries ($=$ inverse images of D) are allowed in the process of desingularization.

Using a seminal result of Soulé, further generalized by Geisser, Kahn, and Jannsen, Theorem 1 implies the following:

Corollary 1 Let the assumption be as in Theorem 1. Assume further that X is *finite-dimensional* in the sense of Kimura and O'Sullivan. Then we have

$$H^s_M(X, \mathbb{Z}(d)) \simeq \bigoplus_{\text{all prime } \ell} H^s_{\text{ét}}(X, \mathbb{Z}_\ell(d)) \qquad \text{for } 2d-1 \geq s \geq 2d-q,$$

if $q \leq 2$, or $\textbf{(RS)}_d$ holds, or $\textbf{(RES)}_q$ holds.

Remark : It is conjectured that any smooth projective vairiety X over a finite field is *finite-dimensional*. The conjecture is true in case there exists a surjective finite map $Y \to X$ where Y is a product of abelian varieties and curves.

Now we consider the arithmetic case.

Here we assume :

X : regular and flat of finite type over $S = \mathrm{Spec}(\mathcal{O}_k)$,

$\mathcal{O}_k$: the ring of integers in k,

k : a number field or its completion at a finite place.

We fix a prime $p > 2$.

We assume that all reduced closed fibers of $X \longrightarrow S$ are SNCD and that the fibers over every point of $\mathrm{Spec}(\mathcal{O}_k)$ of characteristic p is reduced.

Then, by work of Geisser and K. Sato, there is a cycle map

$$\rho_X^{s,r} \;:\; H^s_M(X, \mathbb{Z}/p^n\mathbb{Z}(r)) \to H^s_{\text{ét}}(X, \mathbb{Z}/p^n\mathbb{Z}(r)),$$

where $\mathbb{Z}/p^n\mathbb{Z}(r)$ is an incarnation of the étale motivic complex

$$\mathbb{Z}(r)^{\text{ét}}_M \otimes^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}$$

constructed by Sato.

Theorem (Sato) $H^s_{\text{ét}}(X, \mathbb{Z}/p^n\mathbb{Z}(r))$ is finite.

(The proof depends on results of Bloch-Kato, Hyodo and Tsuji on p-adic vanishing cycles.)

Thus the injectivity of $\rho_X^{r,s}$ would implies $H^s_M(X, \mathbb{Z}/p^n\mathbb{Z}(r))$ is finite.

Theorem 2 (Jannsen - S.) Let the assumption be as above and let $d = \dim(X)$.

(1) Assume k is a local field and that p is different from the residue characteristic. Assuming $\mathbf{(BK)}_{4,p}$ over X,

$$H^s_M(X, \mathbb{Z}/p^n\mathbb{Z}(d)) \xrightarrow{\simeq} H^s_{\text{ét}}(X, \mathbb{Z}/p^n\mathbb{Z}(d)) \text{ for } 2d \geq s \geq 2d-2.$$

(2) Assume k is a number field and that X has good reduction over every point of $\mathrm{Spec}(\mathcal{O}_k)$ of characteristic p. Assuming $\mathbf{(BK)}_{3,p}$ over X,

$$H^s_M(X, \mathbb{Z}/p^n\mathbb{Z}(d)) \xrightarrow{\simeq} H^s_{\text{ét}}(X, \mathbb{Z}/p^n\mathbb{Z}(d)) \text{ for } 2d \geq s \geq 2d-1.$$

Here Theorem 1 (geometic case) has been only partially extended to arithmetic case due to a technical obstacle in p-adic Hodge theory.

We explain the basic idea of the proof of the above results.

First look at the case that $\dim(X) = 1$, i.e.,

X is a proper smooth curve over a finite field $\mathbb{F}$, or

$X = \mathrm{Spec}(\mathcal{O}_k)$ for a number field k.

We have the localization sequence

$$H^1(\eta, \mathbb{Z}/n\mathbb{Z}(1)) \to \bigoplus_{x \in X_0} H^2_x(X, \mathbb{Z}/n\mathbb{Z}(1)) \to H^2(X, \mathbb{Z}/n\mathbb{Z}(1))$$

$$\to H^2(\eta, \mathbb{Z}/n\mathbb{Z}(1)) \to \bigoplus_{x \in X_0} H^3_x(X, \mathbb{Z}/n\mathbb{Z}(1)) \to 0\,,$$

η : the generic point of X,

X_0 : the set of the closed points of X.

Using the Kummer theory and the purity theorem for étale cohomology, it induces an exact sequence

$$0 \to \mathrm{CH}^1(X)/n \xrightarrow{\rho_X^{2,1}} H^2_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(1)) \to \mathrm{Ker}\Big(Br(k) \xrightarrow{\iota} \bigoplus_{x \in X_0} Br(k_x)\Big),$$

k is the residue field of η ($=$ the function field of X),

k_x is the completion of k at x, and

$Br(k)$ (resp. $Br(k_x)$) is the Brauer group of k (resp. k_x).

By a fundamental result in number theory (*Hasse principle for central simple algebras over* k), the last term vanishes, which shows that $\rho_X^{2,1}$ is an isomorphism.

In order to extend the above localization argument to the higher dimensional case, we introduce

$KC_\bullet(X, \mathbb{Z}/n\mathbb{Z})$: (*Kato complex for X of finite type over $\mathbb{Z}$*)

$$\cdots \bigoplus_{x \in X_a} H^{a+1}_{Gal}(\kappa(x), \mathbb{Z}/n\mathbb{Z}(a)) \to \bigoplus_{x \in X_{a-1}} H^{a}_{Gal}(\kappa(x), \mathbb{Z}/n\mathbb{Z}(a-1)) \to \cdots$$

$$\cdots \to \bigoplus_{x \in X_1} H^{2}_{Gal}(\kappa(x), \mathbb{Z}/n\mathbb{Z}(1)) \to \bigoplus_{x \in X_0} H^{1}_{Gal}(\kappa(x), \mathbb{Z}/n\mathbb{Z}),$$

$X_a = \{x \in X \mid \dim \overline{\{x\}} = a\}$, and

the term in degree a is the direct sum of Galois cohomology of residue fields $\kappa(x)$ of $x \in X_a$.

In case X is a proper smooth curve over a finite field $\mathbb{F}$ or $X = \mathrm{Spec}(\mathcal{O}_k)$ for a number field k,

$KC_\bullet(X, \mathbb{Z}/n\mathbb{Z}) =$

$$\bigoplus_{x \in X_1} H^2_{Gal}(\kappa(x), \mathbb{Z}/n\mathbb{Z}(1)) \to \bigoplus_{x \in X_0} H^1_{Gal}(\kappa(x), \mathbb{Z}/n\mathbb{Z}),$$

which is identified with

$$Br(k)[n] \xrightarrow{\iota} \bigoplus_{x \in X_0} Br(k_x)[n],$$

where

$M[n] = \mathrm{Ker}(M \xrightarrow{n} M)$ for a module M.

We define the Kato homology of X by

$$KH_a(X, \mathbb{Z}/n\mathbb{Z}) = H_a(KC_\bullet(X, \mathbb{Z}/n\mathbb{Z})) \quad (a \geq 0).$$

Lemma 1 Let $d = \dim(X)$. Assuming $\mathbf{(BK)}_{q+1,\ell}$ over X, we have a long exact sequence

$$KH_{q+2}(X, \mathbb{Z}/\ell^n) \to$$
$$H_M^{2d-q}(X, \mathbb{Z}/\ell^n(d)) \xrightarrow{\rho_X^{2d-q,d}} H_{\text{ét}}^{2d-q}(X, \mathbb{Z}/\ell^n(d))$$
$$\to KH_{q+1}(X, \mathbb{Z}/\ell^n)$$

In conclusion the bijectivity of $\rho_X^{*,d}$ for X regular proper of dimension d over $\mathbb{F}$ or $\mathbb{Z}$ follows from the following conjecture:

Conjecture K (Kato) Let X be either

a smooth proper variety over $\mathrm{Spec}(\mathbb{F})$, or

a regular proper flat scheme over $\mathrm{Spec}(\mathcal{O}_k)$ for the ring $\mathcal{O}_k$ of integers in a number field k.

In the latter case assume that either n is odd or k is totally imaginary.

Then:

$$KH_a(X, \mathbb{Z}/n\mathbb{Z}) \simeq \begin{cases} \mathbb{Z}/n\mathbb{Z} & a = 0 \\ 0 & a \neq 0\,. \end{cases}$$

The case $\dim(X) = 1$ rephrases the classical result on the Brauer group of a global field.

Kato proved the conjecture in case $\dim(X) = 2$.

In general the following result has been known due to Colliot-Thélène and Suwa (geometric case) and Jannsen-S. (arithmetic case)

Theorem Let the assumption be as above. Then

$$KH_a(X, \mathbb{Q}/\mathbb{Z}) \simeq \begin{cases} \mathbb{Q}/\mathbb{Z} & a = 0 \\ 0 & 0 < a \leq 3 \end{cases}$$

where

$$KH_a(X, \mathbb{Q}/\mathbb{Z}) = \varinjlim_{n} KH_a(X, \mathbb{Z}/n\mathbb{Z}).$$

Theorem 1 follows from the following result.

Theorem 3 (Jannsen-S.):

Let X be projective smooth of dimension d over $\mathbb{F}$.

Let $t \geq 1$ be an integer.

Then we have

$$KH_a(X, \mathbb{Q}/\mathbb{Z}) \simeq \begin{cases} \mathbb{Q}/\mathbb{Z} & a = 0 \\ 0 & 0 < a \leq t \end{cases}$$

if either $t \leq 4$ or $\mathbf{(RS)}_d$, or $\mathbf{(RES)}_{t-2}$ holds.

The same conlusion holds by replacing $\mathbb{Q}/\mathbb{Z}$ by $\mathbb{Z}/\ell^n\mathbb{Z}$ assuming $\mathbf{(BK)}_{t,\ell}$.

We now discuss the proof of Theorem 3.

Define étale homology of separated schemes of finite type $f : X \to \mathrm{Spec}(\mathbb{F})$ by

$$H_a^{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}) := H_{ét}^{-a}(X, Rf^!\mathbb{Z}/n\mathbb{Z}) .$$

Using purity results for étale cohomology, a method of Bloch-Ogus gives rise to the *niveau spectral sequence*

$$E^1_{p,q}(X, \mathbb{Z}/n\mathbb{Z}) = \bigoplus_{x \in X_p} H_{Gal}^{p-q}(\kappa(x), \mathbb{Z}/n\mathbb{Z}(p)) \quad \Rightarrow \quad H^{\text{ét}}_{p+q}(X, \mathbb{Z}/n\mathbb{Z}) .$$

Theorem (Jannsen-S.-Sato): There is an isomophism of complexes

$$E^1_{*,-1}(X, \mathbb{Z}/n\mathbb{Z}) \cong KC_*(X, \mathbb{Z}/n\mathbb{Z})^{(-)},$$

where $C^{(-)}$ means to take the negative differentials in a complex C. Hence

$$KH_a(X, \mathbb{Z}/n\mathbb{Z}) \cong E^2_{a,-1}(X, \mathbb{Z}/n\mathbb{Z}).$$

Theorem 3 on the Kato conjecture now follows by studying the niveau spectral sequence carefully.

The first key observation is that the Kato conjecture implies the following fact : For

X, projective smooth over $\mathbb{F}$,

$Y = Y_1 \cup Y_2 \cup \cdots \cup Y_N \subset X$, SNCD,

$KH_a(U, \mathbb{Z}/n\mathbb{Z})$ of $U = X - Y$ has a combinatoric description as the homology of the complex

$$(\mathbb{Z}/n\mathbb{Z})^{\pi_0(Y^{(d)})} \longrightarrow (\mathbb{Z}/n\mathbb{Z})^{\pi_0(Y^{(d-1)})} \longrightarrow \ldots \longrightarrow (\mathbb{Z}/n\mathbb{Z})^{\pi_0(Y^{(1)})} \longrightarrow (\mathbb{Z}/n\mathbb{Z})^{\pi_0(X)},$$

$\pi_0(Y^{(a)})$ is the set of the connected components of the sum of all a-fold intersections of the irreducible components $Y_1, \ldots, Y_N$ of Y.

Conversely the Kato conjecture for X is deduced from such a combinatoric description of $KH_a(U, \mathbb{Z}/n\mathbb{Z})$ for a suitable choice of $U = X - Y$.

On the other hand,
a weight argument using the *affine Lefschetz theorem* and the *Weil conjecture* proved by Deligne shows :

if one of the divisors $Y_1, \dots, Y_N$ on X is very ample, the étale homology $H_a^{\text{ét}}(U, \mathbb{Q}/\mathbb{Z})$ for $a \le d$ has the same combinatoric description.

(Here we need take the coefficient $\mathbb{Q}/\mathbb{Z}$, not $\mathbb{Z}/n\mathbb{Z}$ since we use a weight arguemnt.)

Since the niveau spectral sequence

$$E^1_{p,q}(U,\mathbb{Z}/n\mathbb{Z}) = \bigoplus_{x\in U_p} H^{p-q}_{Gal}(\kappa(x),\mathbb{Z}/n\mathbb{Z}(p)) \Rightarrow H^{\text{ét}}_{p+q}(U,\mathbb{Z}/n\mathbb{Z})$$

satisfies

$$E^2_{a,-1}(U,\mathbb{Q}/\mathbb{Z}) \simeq KH_a(U,\mathbb{Q}/\mathbb{Z}),$$
$$E^1_{a,q}(U,\mathbb{Q}/\mathbb{Z}) = 0 \quad \text{for } q < -1,$$

the above combinatoric desription of $KH_a(U,\mathbb{Q}/\mathbb{Z})$ is deduced from the vanishing of the higher terms of the niveau spectral sequence for such U :

$$E^\infty_{a,q}(U,\mathbb{Q}/\mathbb{Z}) = 0 \quad \text{for } q > -1.$$

In order to show the vanishing, we pick up any element

$$\alpha \in E^{\infty}_{a,q}(U, \mathbb{Q}/\mathbb{Z})$$

and then take a hypersurface section of high degree

$Z \subset X$ containg the support $Supp(\alpha)$ of α

so that α is killed under the restriction

$$E^{\infty}_{a,q}(U, \mathbb{Q}/\mathbb{Z}) \to E^{\infty}_{a,q}(U \backslash Z, \mathbb{Q}/\mathbb{Z}).$$

The point is that the assumption $\textbf{(RES)}_q$ allows us to make a very careful choice of Z (after desingularizing $Supp(\alpha)$) such that the above restriction map is shown to be injective by the induction on $\dim(U)$. This implies $\alpha = 0$. Q.E.D.