A procdh topology

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February 20, 2024

Abstract

In this article we propose a definition of a procdh topos. We show that it encodes procdh excision, has bounded homotopy dimension and therefore is hypercomplete and admits a conservative family of fibre functors. We also describe the local rings.

As an application, we show that nonconnective K-theory is the procdh sheafification of connective K-theory, and that the motivic cohomology recently proposed by Elmanto and Morrow is the procdh sheafification of Voevodsky's motivic cohomology.

1 Introduction

It has been known for many decades that blowups of smooth varieties in smooth centres give rise to a long exact sequence in algebraic K-theory (SGA71, Exp.VII), (TT90). As part of his work on the Bloch-Kato conjecture, Voevodsky encoded such blowup exact sequences in a Grothendieck topology—the cdh topology—giving access to the full arsenal of topos theory, (SV00).

For varieties which are not necessarily smooth, the picture has not been so complete. Algebraic K-theory fails to associate long exact sequences to blowups in general, as easy examples show. The failure is in some sense a "quasi-coherent" part of algebraic K-theory. Indeed, quasi-coherent cohomology also fails to have long exact sequences for blowups. On the other hand, if one remembers infinitesimal information around the centre, one *does* get long exact sequences for quasi-coherent cohomology. This is described in Grothendieck's theorem on formal functions, (Gro61, Thm.4.1.5).

The analogous formal blowup sequences for algebraic K-theory have been studied for some 20 years now, (Wei01, §3-4), (KS02), (GH06), (GH11), (Mor18), and feature in Kerz, Strunk, Tamme's celebrated proof of Weibel's conjecture, (KST18a); see (Mor16a) for a historical overview. However, the absence of an associated topology has meant that the topos theoretic techniques so skillfully employed by Voevodsky et al. were not available.

In this article we propose the following definition for a procdh topos.

Definition 1.1 (Definition 2.1)**.** Let S be a qcqs scheme. The *procdh topology* on the category Sch_S of S-schemes of finite presentation is generated by the following coverings.

1. Distinguished Nisnevich coverings: families of the form

$$\{U \xrightarrow{i} X, V \xrightarrow{j} X\}$$

such that i is a quasi-compact open immersion, j is an étale morphism, and j is an isomorphism over $X\backslash U$ equipped with any (equivalently all) closed subscheme structure(s) of finite presentation.

2. Proabstract blowup squares: families of the form

$$\{Z_n \to X\}_{n\in\mathbb{N}} \sqcup \{Y \to X\}$$

where $Y \to X$ is a proper morphism of finite presentation which is an isomorphism outside of a closed subscheme $Z_0 \subseteq X$ of finite presentation, and $Z_n = \underline{\mathrm{Spec}}\,\mathcal{O}_X/\mathcal{I}_Z^n$ is the nth infinitesimal thickening of Z_0.

A sign that this is a "correct" topology is that it captures procdh excision: presheaves with the long exact sequences mentioned above are precisely the procdh sheaves.

Theorem 1.2 (Theorem 6.1, Corollary 7.11)**.** *Let S be a scheme and consider the following conditions on a presheaf of spaces $F \in \mathrm{PSh}(\mathrm{Sch}_S, \mathcal{S})$.*[1]*.*

1. *Excision.* For every distighished Nisnevich square $\{U \to X, V \to X\}$ and every proabstract blowup square $\{Z_n \to X\}_{n\in\mathbb{N}} \sqcup \{Y \to X\}$ in Sch_S, Def.2.1, we have

$$F(X) \xrightarrow{\sim} F(U) \times_{F(U\times_X V)} F(V), \tag{1}$$
$$F(X) \xrightarrow{\sim} F(Y) \times_{\lim_n F(Z_n \times_X Y)} \lim{}_n F(Z_n). \tag{2}$$

2. *Čech descent.* For every procdh covering family $\{Y_\lambda \to X\}_{\lambda\in\Lambda}$ we have

$$F(X) \xrightarrow{\sim} \lim_n F(\mathcal{Y}_n) \tag{3}$$

where we write $F(\mathcal{Y}_n)$ for $\prod_{i\in I^n} F(Y_{i_1} \times_X \cdots \times_X F_{i_n})$. That is, in the terminology of (Lur09), F is a procdh sheaf.

3. *Hyperdescent.* For every $X \in \mathrm{Sch}_S$ and procdh hypercovering $\mathcal{Y}_\bullet \to X$ we have

$$F(X) \xrightarrow{\sim} \lim_n \mathrm{Map}(\mathcal{Y}_n, F). \tag{4}$$

In the terminology of (Lur09), F is a hypercomplete procdh sheaf.

If S is qcqs then (Excision) $\Leftrightarrow$ (Čech descent). If S is qcqs, has finite valuative dimension and Noetherian topological space, then (Čech descent) $\Leftrightarrow$ (Hyperdescent).

Our procdh ∞-topoi have finite homotopy dimension, albeit not quite as optimal as one could hope.

Theorem 1.3 (Theorem 7.9, Example 7.17)**.** *Let S be a qcqs scheme of finite valuative dimension $d \geq 0$ with Noetherian underlying topological space. Then $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S, \mathcal{S})$ has homotopy dimension $\leq 2d$.*

There exists a Noetherian scheme of Krull dimension one with procdh homotopy dimension two.

The appearance of $2d$ instead of d is essentially because the topos $\mathrm{PSh}(\mathbb{N}, \mathcal{S})$ has homotopy dimension one. So every time we perform a $\lim_{\mathbb{N}}$, for example in Eq.(2) above, we potentially increase the homotopy dimension by one. In a future article we consider a way around this using a site involving formal schemes.

Despite the unwanted factor of two, finite homotopy dimension is sufficient to imply that the topos is hypercomplete.

[1] As usual, $\mathcal{S} := N\mathcal{K}an$ is the quasicategory associated to the simplicial category of Kan complexes, and $\mathrm{PSh}(\mathrm{Sch}_S, \mathcal{S})$ is the quasicategory $\mathrm{Fun}(\mathrm{Sch}_S^{\mathrm{op}}, \mathcal{S})$. Before (Lur09), the category $\mathrm{PSh}(\mathrm{Sch}_S, \mathcal{S})$ would have been the category $\mathrm{PSh}(\mathrm{Sch}_S, \mathrm{Set}_\Delta)$ of presheaves of simplicial sets with any model structure for which weak equivalences are objectwise weak equivalences. If one reads the statements using this latter interpretation, all limits should be replaced with homotopy limits as described in (BK72) or (Hir03). Since homotopy limits calculate quasicategorical limits, the difference between the two points of view is superficial.

Corollary 1.4 (Corollary 7.11)**.** *Let S be a qcqs scheme of finite valuative dimension with Noetherian underlying topological space. Then* $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S, \mathcal{S})$ *is hypercomplete.*

Closely related to hypercompleteness is existence of enough points. Recall that a topos, resp. ∞-topos T, is said to have *enough points* when the collection of all geometric morphisms of topoi $\phi^* : T \to \mathrm{Set}$, resp. ∞-topoi $\phi^* : T \to \mathcal{S}$, detect isomorphisms, resp. equivalences.

Theorem 1.5 (Theorem 5.2, Corollary 7.15)**.** *Suppose S is a qcqs scheme with Noetherian topological space of finite Krull dimension. Then both the classical topos* $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S)$*, and the ∞-topos* $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S, \mathcal{S})$ *have enough points.*

We prove the classical version of Theorem 1.5 first and then upgrade it to the ∞-version. Something worth noting is that since the generating covering families of the procdh topology are not finite, the topos is not coherent, so we cannot simply apply Deligne's completeness theorem. We use the following proposition for the upgrade. For sites whose associated topos is coherent, Deligne's theorem combined with Prop.7.14 implies Lurie's version, (Lur18, Thm.A.4.0.5).

Proposition 1.6 (Proposition 7.14)**.** *Let (C, τ) be a small site admitting finite limits such that the topos* $\mathrm{Shv}(C, \mathrm{Set})$ *has enough points in the classical sense. Then the hypercompletion* $\mathrm{Shv}(C, \mathcal{S})^\wedge$ *has enough points as an ∞-topos.*

After all this talk of procdh points, the reader is surely asking for a characterisation of the local rings. Procdh local rings are essentially versions of cdh local rings—henselian valuation rings—with some mild nilpotent thickening. Cf. (Kel24) for some philosophy motivating this.

Proposition 1.7 (Characterisation of procdh local rings)**.** *An affine S-scheme* $\mathrm{Spec}(R) \to S$ *is procdh local, Def.3.2, if and only if it is of the form*

$$R \cong \mathcal{O} \times_K A$$

with A a local ring of Krull dimension zero and $\mathcal{O} \subseteq K$ a henselian valuation ring of $K{=}A/\mathfrak{m}_A$. In other words, $R \subseteq A$ is the set of elements whose residue mod $\mathfrak{m}_A$ lies in $\mathcal{O} \subseteq K = A/\mathfrak{m}_A$.

We also discuss how a general procdh local ring can often be written as a filtered colimit of smaller local rings, see Proposition 3.6 and Theorem 3.10.

In the latter sections, we apply the above theory to show the following topos-theoretic interpretation of the Bass construction.

Theorem 1.8 (Theorem 8.1)**.** *For any Noetherian scheme X with $\dim(X) < \infty$, there exists a natural equivalence*

$$(a_{\mathrm{procdh}}\tau_{\geq 0}K)(X) \simeq K(X).$$

Here $K(X)$ is the non-connective algebraic K-theory of X and $\tau_{\geq 0}K(X)$ the connective K-theory.

The above equivalence is an analogue of the equivalence[2]

$$(a_{cdh}\tau_{\geq 0})K(X) \simeq KH(X) \tag{5}$$

with X still finite dimensional and Noetherian.

[2]Admitting that KH satisfies cdh descent, (Hae04), (Cis13, Thm.3.9), a fast, clean, intuitive way to deduce Eq.(5) is to prove that $K_{\geq 0}(R) \cong K(R) \cong KH(R)$ for valuation rings, (KM21, Thm.1.3), cf.(KM21, Rem.3.4). Equation (5) for finite dimensional Noetherian schemes is also proven in the landmark Kerz-Strunk-Tamme paper (KST18a, Thm.6.3) using derived algebraic geometry.

Using Theorem 8.1 we propose the following construction of a non-$\mathbb{A}^1$-invariant motivic cohomology. For X smooth over a field $\mathbb{F}$, let $Fil^n_{mot}K(X)$ denote the motivic filtration on K-theory whose associated spectral sequence is the Atiyah-Hirzebruch spectral sequence, (FS02), (Lev08),

$$E_2^{p,q} = H^{p-q}_{\mathcal{M}}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X).$$

Here $H^m_{\mathcal{M}}(X, \mathbb{Z}(n)) = R^m\Gamma_{\mathrm{Zar}}(X, \mathbb{Z}(n)^{\mathrm{sm}})$ with $\mathbb{Z}(n)^{\mathrm{sm}} \in \mathrm{PSh}(\mathrm{Sm}_{\mathbb{F}}, D(\mathbb{Z}))$ the graded pieces of $Fil^n_{mot}K$ on the category $\mathrm{Sm}_{\mathbb{F}}$ of smooth $\mathbb{F}$-schemes. Note that $\mathbb{Z}(n)^{\mathrm{sm}}$ is identified with Voevodsky's $\mathbb{A}^1$-invariant motivic complex and Bloch's cycle complex.

Definition 1.9 (Definition 9.3)**.** For integers $n \geq 0$, we define the *procdh local motivic complex*

$$\mathbb{Z}(n)^{\mathrm{procdh}} := a_{\mathrm{procdh}} L^{\mathrm{sm}} \mathbb{Z}(n)^{\mathrm{sm}} \in \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}, D(\mathbb{Z}))),$$

as the procdh sheafification of the left Kan extension $L^{\mathrm{sm}}\mathbb{Z}(n)^{\mathrm{sm}}$ of $\mathbb{Z}(n)^{\mathrm{sm}}$ along $\mathrm{Sm}_{\mathbb{F}} \to \mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}$. Here $\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}$ is the category of qcqs $\mathbb{F}$-schemes.

It follows essentially from the definition, together with a result of Bhatt-Lurie and the fact that the procdh site of a Noetherian scheme has finite cohomological dimension that we get an Atiyah-Hirzebruch spectral sequence.

Theorem 1.10 (Theorem 9.5)**.** *For any Noetherian $\mathbb{F}$-scheme X with $\dim(X) < \infty$, there exists a complete multiplicative decreasing $\mathbb{N}$-indexed filtration $\left\{Fil^n_{\mathrm{procdh}}K(X)\right\}_{n\in\mathbb{N}}$ on $K(X)$ and identifications*

$$\mathrm{gr}^n_{Fil_{\mathrm{procdh}}} K(X) \simeq \mathbb{Z}(n)^{\mathrm{procdh}}(X)[2n].$$

Recently, in (EM23) Elmanto-Morrow have also proposed a non-$\mathbb{A}^1$-invariant motivic cohomology for qcqs $\mathbb{F}$-schemes. We will write $\mathbb{Z}(n)^{\mathrm{EM}}$ for these presheaves. They are constructed by modifying the cdh sheafification $\mathbb{Z}(n)^{\mathrm{cdh}}$ of the left Kan extension of $\mathbb{Z}(n)^{\mathrm{sm}}$ along $\mathrm{Sm}_{\mathbb{F}} \to \mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}$ by using Hodge-completed derived de Rham complexes in case $\mathbb{F} = \mathbb{Q}$ and syntomic complexes in case $\mathbb{F} = \mathbb{F}_p$. The construction is motivated by trace methods in algebraic K-theory using the cyclotomic trace map.

We conclude the article with the following comparison.

Corollary 1.11 (Corollary 9.9)**.** *Let $\mathbb{F} = \mathbb{F}_p$ or $\mathbb{Q}$ and take a Noetherian $\mathbb{F}$-scheme X. Then there are equivalences*

$$\mathbb{Z}(n)^{\mathrm{procdh}}(X) \simeq \mathbb{Z}(n)^{\mathrm{EM}}(X),$$

functorial in X.

1.1 Future work

As mentioned above, a version of the material in this article for formal schemes is in progress. We also have a version for derived schemes.

1.2 Notation and conventions

Throughout we write Sch_S for the category of S-schemes of finite presentation over a scheme S. We write X^{gen} for the set of generic points of a scheme X equipped with the topology induced from the underlying topological space of X.

The first sections deal exclusively with presheaves of sets. When we pass to presheaves of spaces we will write $\mathrm{PSh}(-, \mathcal{S}), \mathrm{PSh}(-, \mathrm{Spt}), \mathrm{PSh}(-, D(\mathbb{Z}))$, etc, and sometimes use $\mathrm{PSh}(-, \mathrm{Set})$ if we want to emphasise that a presheaf takes values in discrete spaces.

1.3 Acknowledgements

We are very grateful to Matthew Morrow for many interesting questions and discussions about the procdh topology and particularly for a very active scientific exchange without which the comparison theorem would not have been possible.

We thank Marc Hoyois for answering the first author's annoying emails about infinity categories.

We also thank Dustin Clausen and Ryomei Iwasa for their encouraging interest in the project.

Much of the research in this article was conducted while the first author was Principal Investigator of JSPS Grant-in-Aid for Early-Career Scientists, Project Number 19K14498 and Bilateral Joint Research Projects (120213206). The second author is Principal Investigator of JSPS Grant-in-Aid for Scientific Research (B), Project Number 20H01791.

2 Definition of a procdh topology

In this section we define a procdh topology.

Definition 2.1. Let S be a scheme. The *procdh topology* on Sch_S is generated by the following coverings.

0. Zariski coverings.
1. Distinguished Nisnevich coverings: families of the form

$$\{U \stackrel{i}{\to} X, V \stackrel{j}{\to} X\}$$

such that i is a quasi-compact open immersion, j is an étale morphism, and j is an isomorphism over $X \backslash U$ equipped with any (equivalently all) closed subscheme structure(s) of finite presentation.

2. Proabstract blowup squares: families of the form

$$\{Z_n \to X\}_{n \in \mathbb{N}} \sqcup \{Y \to X\}$$

where $Y \to X$ is a proper morphism (of finite presentation) which is an isomorphism outside of a closed subscheme $Z_0 \subseteq X$ of finite presentation, and $Z_n = \underline{\mathrm{Spec}}\, \mathcal{O}_X/\mathcal{I}_Z^n$ is the nth infinitesimal thickening of Z_0.

Remark 2.2. More precisely, a family $\mathcal{Y} = \{Y_i \to X\}_{i \in I}$ in Sch_S is a procdh covering if there exists a rooted tree T of finite height and a functor $T \to \mathrm{Sch}_S$, such that the root is sent to X, for each vertex v the family $\{W_c \to W_v\}_{c \in Child(v)}$ is of one of the above forms, and for each leaf l there exists a factorisation $W_l \to Y_{i_l} \to X$ for some $i_l \in I$.

3 Local rings

In this section we characterise procdh local rings, Prop.3.3. We also discuss how a general procdh local ring can often be written as a filtered colimit of smaller local rings, Prop.3.6, Theo.3.10. The latter is used in the proof of the comparison theorem, Corollary 9.9.

Definition 3.1. Recall that a *fibre functor* of a topos $\mathrm{Shv}_\tau(C)$ is a continuous morphism of topoi $\phi^* : \mathrm{Shv}_\tau(C) \rightleftarrows \mathrm{Set} : \phi_*$, or equivalently, a functor $\phi^* : \mathrm{Shv}_\tau(C) \to \mathrm{Set}$ which preserves colimits and finite limits.

For topologies τ on Sch_S for which every scheme is covered by affine ones (e.g., anything finer than the Zariski topology), there is a bijection between fibre functors of $\mathrm{Shv}_\tau(\mathrm{Sch}_S)$ and affine S-schemes $\mathrm{Spec}(R) \to S$ which are *τ-local* in the following sense.

Definition 3.2 (cf.(SGA72a, Thm.III.4.1), (SGA72a, I.8.10.14), (Gro66, Cor.8.13.2))**.** Let τ be a topology on Sch_S such that every scheme is covered by affine ones. An affine S-scheme $\mathrm{Spec}(R) \to S$ is said to be *τ-local* if for every τ-covering $\{Y_i \to X\}_{i\in I}$ the morphism of sets

$$\coprod_{i\in I} \hom(\mathrm{Spec}(R), Y_i) \to \hom(\mathrm{Spec}(R), X) \tag{6}$$

is surjective.

Proposition 3.3 (Characterisation of procdh local rings)**.** *An affine S-scheme* $\mathrm{Spec}(R) \to S$ *is procdh local, Def.3.2, if and only if it is of the form*

$$R \cong \mathcal{O} \times_K A$$

with A a local ring of Krull dimension zero and $\mathcal{O} \subseteq K$ a henselian valuation ring of $K{=}A/\mathfrak{m}_A$. In other words, $R \subseteq A$ is the set of elements whose residue mod $\mathfrak{m}_A$ lies in $\mathcal{O}$.

Proof of Proposition 3.3. ($\Rightarrow$) The following composition of a Zariski covering and the procdh covering

$$\left\{\underline{\mathrm{Spec}}\,\frac{\mathcal{O}_S[x,y]}{\langle x^n, y^n\rangle}\right\}_{n\in\mathbb{N}} \sqcup \left\{\underline{\mathrm{Spec}}\,\mathcal{O}_S[x,\tfrac{y}{x}], \underline{\mathrm{Spec}}\,\mathcal{O}_S[\tfrac{x}{y},y]\right\} \tag{7}$$

shows that procdh local rings satisfy:
($*$) $\forall a, b \in R$; we have $a|b$ or $b|a$ or a and b are both nilpotent.
It follows from this that R_{red} is a valuation ring, and in particular, R has a unique minimal prime ideal $\mathfrak{n}$, which equals the set of nilpotents. All zero divisors are nilpotent by virtue of the procdh covering

$$\left\{\underline{\mathrm{Spec}}\,\frac{\mathcal{O}_S[x,y]}{\langle x^n, xy\rangle}\right\}_{n\in\mathbb{N}} \sqcup \left\{\underline{\mathrm{Spec}}\,\frac{\mathcal{O}_S[x,y]}{\langle y\rangle}\right\} \tag{8}$$

of $\underline{\mathrm{Spec}}\,\frac{\mathcal{O}_S[x,y]}{\langle xy\rangle}$, so $R \to R_\mathfrak{n}$, and therefore $R \to (R/\mathfrak{n}) \times_{k(\mathfrak{n})} R_\mathfrak{n}$ is injective. We claim that the latter is also surjective. It follows from a diagram chase that $\mathfrak{n} \to \mathfrak{n}R_\mathfrak{n}$ is surjective implies $R \to (R/\mathfrak{n}) \times_{k(\mathfrak{n})} R_\mathfrak{n}$ is surjective.

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{n} \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & R/\mathfrak{n} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathfrak{n}R_\mathfrak{n} \arrow[r] & R_\mathfrak{n} \arrow[r] & R_\mathfrak{n}/\mathfrak{n}R_\mathfrak{n} \arrow[r] & 0 \end{tikzcd} \tag{9}$$

So suppose we have $a \in \mathfrak{n}$ and $s \in R \setminus \mathfrak{n}$. We claim there is $b \in \mathfrak{n}$ such that $b/1 = a/s$. Indeed, this follows from $a \in \mathfrak{n}$, $s \in R \setminus \mathfrak{n}$, and ($*$).

So we have shown $R \to (R/\mathfrak{n}) \times_{k(\mathfrak{n})} R_\mathfrak{n}$ is both injective and surjective. The Krull dimension of $R_\mathfrak{n}$ is zero because $\mathfrak{n}$ is a minimal prime, and we have already observed that $R/\mathfrak{n} = R_{\mathrm{red}}$ is a valuation ring, so it suffices to show that $R/\mathfrak{n}$ is henselian. But procdh local rings are Nisnevich local rings, also known as henselian local rings, and quotients of henselian local rings are henselian local rings.

($\Leftarrow$). Suppose $R = \mathcal{O} \times_K A$ as in the statement. We want to show that (6) is an epimorphism for all procdh coverings. Certainly it suffices to consider the generator coverings described in the definition.

We immediately notice that R is henselian: $\mathcal{O} = R_{\mathrm{red}}$ is henselian by assumption and $-\otimes_R (R_{\mathrm{red}})$ induces an equivalence $\mathrm{Et}_R \xrightarrow{\sim} \mathrm{Et}_{R_{\mathrm{red}}}$ of categories of étale algebras, (Sta18, 039R). So the desired lifting condition with respect to Nisnevich coverings is satisfied, cf.(Sta18, 04GG, Item(7)).

Suppose we have a proabstract blowup square $\{Z_n \to X\}_{n\in\mathbb{N}} \sqcup \{Y \to X\}$ and a morphism $\mathrm{Spec}(R) \to X$. If the generic point $\mathrm{Spec}(K)$ of $\mathrm{Spec}(R)$ doesn't land in Z_0, then it lifts through Y because $Y \to X$ is an isomorphism over $X \setminus Z_0$. By the valuative criterion for properness, this lifting extends to a lifting of $\mathrm{Spec}(\mathcal{O})$. Since A is local the morphism $\mathrm{Spec}(K) \to Y$ also extends to $\mathrm{Spec}(A) \to Y$. These three morphisms factor through some open affine of Y, so they glue to give a lifting $\mathrm{Spec}(R) \to Y \to X$ since $\mathrm{Spec}(\mathcal{O} \times_K A) = \mathrm{Spec}(\mathcal{O}) \sqcup_{\mathrm{Spec}(K)} \mathrm{Spec}(A)$ is the categorical pushout in the category of affine schemes.

On the other hand, if $\mathrm{Spec}(K) \to X$ does factor through Z_0, then $\mathrm{Spec}(\mathcal{O}) \to X$ also factors through Z_0. The morphism $\mathrm{Spec}(A) \to X$ doesn't necessarily factor through Z_0 but the (finitely many) generators of $\mathcal{I}_{Z_0}$ are sent inside $\mathfrak{m}_A = \mathrm{Nilpotents}(A)$ so $\mathrm{Spec}(A) \to X$ factors through some Z_n. Then we glue as in the previous case. □

General procdh local rings can be quite large, but they are often filtered colimits of smaller local rings. The rest of this section is devoted to such reductions and some consequences, for example Corollary 5.3. We will use the following non-Nisnevich version of procdh local rings.

Definition 3.4. A *prorh local ring* are rings of the form $\mathcal{O}\times_K A$ with A a local ring of dimension zero, and $\mathcal{O} \subseteq K$ a (not necessarily henselian) valuation ring of $K = A/\mathfrak{m}$.

One way to construct procdh local rings is to first build a prorh local ring, and then take the henselisation.

We write $Q(R)$ for the ring of total fractions. That is, $Q(R) = R[S^{-1}]$ where S is the set of nonzero divisors.

Proposition 3.5. *Suppose that R is a prorh local ring. That is, $R = \mathcal{O} \times_K A$ with $\mathcal{O}$ a (not necessarily henselian) valuation ring, A a dimension zero local ring, and $K = \mathrm{Frac}(\mathcal{O}) \cong A/\mathfrak{m}$. Let $R \to S$ be an étale morphism towards a local ring S.*

Then S is prorh local, and the henselisation R^h is a procdh local ring. Moreover, in this situation we have $\dim S \leq \dim R$, $\mathrm{length}\, Q(R) = \mathrm{length}\, Q(S)$ and $\dim R^h = \dim R$, $\mathrm{length}\, Q(R) = \mathrm{length}\, Q(R^h)$.

Proof. The canonical morphism $S \to (S \otimes_R \mathcal{O}) \times_{(S\otimes_R K)} (S \otimes_R A)$ is an isomorphism because $R \to S$ is flat, (Fer03, Thm.2.2(iv)). Since S is local, $S \otimes_R \mathcal{O} = S \otimes_R (R_{\mathrm{red}})$ is also local, and therefore a valuation ring of Krull dimension $\leq \dim \mathcal{O}$, (Sta18, 0ASJ). It follows that the étale K-algebra $S \otimes_R K$ is actually a finite separable extension of K, and since $(S \otimes_R A)_{\mathrm{red}} = (S \otimes_R (A_{\mathrm{red}}))_{\mathrm{red}} = (S \otimes_R K)_{\mathrm{red}}$, we find that $S \otimes_R A$ has exactly one prime ideal. Since $K \to S\otimes_R K$ is a field extension and $R \to S$ is flat, for any composition series $A \supset I_0 \supset I_1 \supset \dots$, the pullback $S \otimes_R A \supset S \otimes_R I_0 \supset S \otimes_R I_1 \supset \dots$ is a composition series for $S \otimes_R A$. So $\mathrm{length}\, A = \mathrm{length}\, S \otimes_R A$.

In the case that $R \to S$ is a local homomorphism, $R_{\mathrm{red}} \to S_{\mathrm{red}}$ is also local, so it induces a bijection of value groups of these valuation rings, (Sta18, 0ASF), and therefore a bijection of posets of prime ideals. Consequently, $\dim R = \dim R_{\mathrm{red}} = \dim S_{\mathrm{red}} = \dim S$. Now the henselisation R^h is the colimit $\mathrm{colim}\, S_\lambda$ over étale algebras $R \to S_\lambda$ which are local homomorphisms of local rings. We have just seen that all transition morphisms $S_\lambda \to S_\mu$ induce homeomorphisms on Spec, so we deduce that the prime ideals of R^h are in bijection with the prime ideals of R. Therefore, $\dim R^h = \dim R$. As above, since $K \to R^h \otimes_R K$ is a field extension and $R \to R^h$ is flat, pullback preserves composition series so $\mathrm{length}\, A = \mathrm{length}\, R^h \otimes_R A$.

It remains to show that R^h is procdh local. We have $(R^h)_{\mathrm{red}} = (\operatorname{colim} S_\lambda)_{\mathrm{red}} = \operatorname{colim} S_{\lambda,\mathrm{red}}$ is a filtered colimit of valuation rings and therefore a valuation ring. It follows that $\mathfrak{n}^h := \mathrm{Nil}(R^h)$ is a unique minimal prime ideal of R^h, and also that $R^h/\mathfrak{n}^h$ is a *henselian* valuation ring, since quotients of henselian rings are henselian. To conclude it suffices to show that $R^h \to (R^h/\mathfrak{n}^h) \times_{k(\mathfrak{n}^h)} (R^h_{\mathfrak{n}^h})$ is an isomorphism. This follows the fact that filtered colimits commute with fibre products. $\square$

Proposition 3.6. *If S is a Noetherian scheme (e.g., the spectrum of $\mathbb{Z}$ or a field) and $\mathrm{Spec}(R) \to S$ an affine S-scheme such that R is procdh local, then R is a filtered colimit of procdh local $\mathcal{O}_S$-algebras such that each* $\operatorname{length} Q(R_\lambda)$ *is finite. If S has finite valuative dimension, Def. 7.1, then we can also assume each R_λ has finite Krull dimension.*

Proof. We immediately observe that since R is local, we can assume S is affine, say $S = \mathrm{Spec}(B_0)$. Then the idea is that for any factorisation $B_0 \to B \to R$, we can convert B into a procdh local ring in a way which is functorial in B. If we write R as a filtered colimit $R = \operatorname{colim} B_\lambda$ of B_0-algebras B_λ of finite presentation, we can use this fact to functorially convert each B_λ into a procdh local ring R_λ, giving the expression $R = \operatorname{colim} R_\lambda$. To conclude we observe that the R_λ have the properties described in the statement if B_0 is Noetherian, resp. of finite valuative dimension.

Now we carry out this plan. Let $B_0 \to B_\lambda \to R$ be any factorisation, and consider the preimage $\mathfrak{p}_\lambda \subseteq B_\lambda$ of the minimal prime $\mathfrak{n}$ of R. The valuation ring $R/\mathfrak{n}$ of R induces and factors through a valuation ring $\mathcal{O}_\lambda = k(\mathfrak{p}_\lambda) \cap (R/\mathfrak{n})$ of $k(\mathfrak{p}_\lambda)$, and the localisation $R_\mathfrak{n}$ of R factors through the localisation $(B_\lambda)_{\mathfrak{p}_\lambda}$. Form the epi-monic factorisation $(B_\lambda)_{\mathfrak{p}_\lambda} \twoheadrightarrow A_\lambda \hookrightarrow R_\mathfrak{n}$. As $\mathfrak{n}R_\mathfrak{n} = \mathrm{Nil}(R_\mathfrak{n})$, the preimage of this in A_λ also consists entirely of nilpotents. Since this ideal of nilpotents is the maximal ideal of A_λ, it follows that A_λ is a dimension zero local ring. Glueing, we get a prorh local ring $P_\lambda := \mathcal{O}_\lambda \times_{k(\mathfrak{p}_\lambda)} A_\lambda$ and a commutative diagram

$$\begin{tikzcd}
B_\lambda \arrow[rr] \arrow[dd] \arrow[dr] & & \overset{P_\lambda =}{\mathcal{O}_\lambda \times_{k(\mathfrak{p}_\lambda)} A_\lambda} \arrow[rr] \arrow[dd] \arrow[dr] & & R \arrow[dd] \arrow[dr] & \\
& B_\lambda/\mathfrak{p}_\lambda \arrow[rr] \arrow[dd] & & \mathcal{O}_\lambda \arrow[rr] \arrow[dd] & & R/\mathfrak{n} \arrow[dd] \\
(B_\lambda)_{\mathfrak{p}_\lambda} \arrow[rr] \arrow[dr] & & A_\lambda \arrow[rr] \arrow[dr] & & R_\mathfrak{n} \arrow[dr] & \\
& k(\mathfrak{p}_\lambda) \arrow[rr, equal] & & k(\mathfrak{p}_\lambda) \arrow[rr] & & k(\mathfrak{n})
\end{tikzcd}$$

To conclude we take $R_\lambda = P_\lambda^h$ to be the henselisation, which is a procdh local ring by Proposition 3.5.

One checks that the choice of $\mathfrak{p}_\lambda$ =preimage$(\mathfrak{n}) \subseteq B_\lambda$, and the constructions of $k(\mathfrak{p}_\lambda)$, $\mathcal{O}_\lambda = k(\mathfrak{p}_\lambda) \cap (R/\mathfrak{n})$, $(B_\lambda)_{\mathfrak{p}_\lambda}$, A_λ =image$((B_\lambda)_{\mathfrak{p}_\lambda} \to R_\mathfrak{n})$, $P_\lambda = \mathcal{O}_\lambda \times_{k(\mathfrak{p}_\lambda)} A_\lambda$, and $R_\lambda = P_\lambda^h$ are functorial in B_λ. Consider the system of factorisations $B_0 \to B_\lambda \to R$ with $B_0 \to B_\lambda$ of finite presentation. Note, $\operatorname{colim} B_\lambda = R$. We get a factorisation

$$\operatorname{colim} B_\lambda \to \operatorname{colim} P_\lambda \to \operatorname{colim} R_\lambda \to R.$$

The composition is an isomorphism by construction. The maps $A_\lambda \to R_\mathfrak{n}$ and $\mathcal{O}_\lambda \to R/\mathfrak{n}$ are injective by construction, so each $P_\lambda = \mathcal{O}_\lambda \times_{k(\mathfrak{p})} A_\lambda \to R$ is injective, so $\operatorname{colim} P_\lambda \to R$ is injective, and therefore also an isomorphism. Finally, henselisation commutes with filtered colimits of local rings with local transition homomorphisms, (Sta18, 07RP). So $\operatorname{colim} R_\lambda = \operatorname{colim} P_\lambda^h = (\operatorname{colim} P_\lambda)^h = R^h = R$.

If B_0 is Noetherian, then each B_λ is also Noetherian, $(B_\lambda)_{\mathfrak{p}_\lambda}$ is Noetherian, and the quotient $A_\lambda = Q(P_\lambda)$ is Noetherian, hence, of finite length. Included in Proposition 3.5 is that $\operatorname{length} Q(P_\lambda) = \operatorname{length} Q(R_\lambda)$.

If B_0 has finite valuative dimension, then so does each B_λ and $B_\lambda/\mathfrak{p}_\lambda$ so $\dim \mathcal{O}_\lambda = \dim P_\lambda$ is finite. By Proposition 3.5 $\dim P_\lambda = \dim R_\lambda$. $\square$

Sometimes finite length $Q(R)$ is not strong enough, and we would like $\mathrm{Nil}(R)$ to be finitely generated. This is not achievable for general procdh local rings.

Example 3.7. Let R be a procdh local ring such that $\mathfrak{n} = \mathrm{Nil}(R)$ is finitely generated. Then $R = R_\mathfrak{n}$. That is, R is an local ring of Krull dimension zero.

As $\mathfrak{n}$ is finitely generated, there is some n such that $\mathfrak{n}^{n+1} = 0$ and $\mathfrak{n}^n \neq 0$.[3] Since $R = (R/\mathfrak{n}) \times_{k(\mathfrak{n})} (R_\mathfrak{n})$ is defined by a Milnor square, the map $R \to R_\mathfrak{n}$ induces an identification of non-unital rings $\mathfrak{n} \cong \mathfrak{n}R_\mathfrak{n}$, and consequently, an identification $\mathfrak{n}^n \cong (\mathfrak{n}R_\mathfrak{n})^n$. Since $\mathfrak{n}^{n+1}$ (and therefore $(\mathfrak{n}R_\mathfrak{n})^{n+1}$) is zero, $\mathfrak{n}^n$ (resp. $(\mathfrak{n}R_\mathfrak{n})^n$) is an $R/\mathfrak{n}$-module (resp. $k(\mathfrak{n})$-vector space). Moreover, since $\mathfrak{n}$ is a finitely generated R-module, so is $\mathfrak{n}^n$. Changing notation to $\mathcal{O} = R/\mathfrak{n}$ and $M = \mathfrak{n}^n$ to make the consequences of this clearer, we have a finitely generated nonzero module M over a valuation ring $\mathcal{O}$ such that $M \to M \otimes_\mathcal{O} \mathrm{Frac}(\mathcal{O})$ is an isomorphism. This is only possible if $\mathcal{O} = \mathrm{Frac}(\mathcal{O})$.[4] Returning to the previous notation, we have shown that $R/\mathfrak{n} = k(\mathfrak{n})$, and therefore $R = (R/\mathfrak{n}) \times_{k(\mathfrak{n})} (R_\mathfrak{n}) = R_\mathfrak{n}$.

Now we start working towards Theorem 3.10 which says that we can get to finitely generated nilradical if we relax the conditions on the nilpotents slighty. Theorem 3.10 is used in the proof of the comparison theorem, Corollary 9.9.

Lemma 3.8. *Suppose R is a procdh local ring containing a field $\mathbb{F}$ such that $Q(R)_{\mathrm{red}}/\mathbb{F}$ is separably generated. Then the canonical projection $R \to R_{\mathrm{red}}$ admits a section in the category of $\mathbb{F}$-algebras.*

This essentially follows from $Q(R)_{\mathrm{red}}/\mathbb{F}$ being formally étale, and $Q(R) \to Q(R)_{\mathrm{red}}$ being a nilpotent thickening, but we couldn't find a reference with exactly the statement we wanted.

Proof. Since $R = \mathcal{O} \times_K A$ (with $\mathcal{O} = R_{\mathrm{red}}$, $K = Q(R)_{\mathrm{red}}$, $A = Q(R)$), to get a factorisation $\mathcal{O} \dashrightarrow R \to \mathcal{O}$ of the identity (in the category of $\mathbb{F}$-algebras), it is equivalent to find a factorisation $\mathcal{O} \dashrightarrow A \to K$ of the canonical inclusion $\mathcal{O} \subseteq K$ (in the category of $\mathbb{F}$-algebras). Furthermore, since $K = \mathcal{O}[(\mathcal{O} \setminus \{0\})^{-1}]$ and an element of A is a unit if and only if its image in K is nonzero, giving such a factorisation is equivalent to giving a factorisation $K \dashrightarrow A \to K$ of the identity (in the category of $\mathbb{F}$-algebras). That is, it suffices to solve the lifting problem

$$\begin{tikzcd} K & \mathbb{F}_1 \arrow[l] \arrow[dl, dashed] \\ A \arrow[u] & \mathbb{F}_0 \arrow[l] \arrow[u] \end{tikzcd}$$

with $\mathbb{F}_0 = \mathbb{F}$ and $\mathbb{F}_1 = K$. We have assumed $K/\mathbb{F}$ is separably generated, so by assumption there are elements $\{x_\lambda \in K\}_{\lambda \in \Lambda}$ indexed by a well-ordered set Λ such that for every successor $\lambda < \lambda + 1$ in Λ the element $x_{\lambda+1}$ is transcendental or finite separable over the subfield $K(x_\mu :$

[3] We allow $n = 0$, in which case $\mathfrak{n}^0 = R$.

[4] For valuation rings we have torsion free $\Leftrightarrow$ flat, for integral domains we have finitely generated flat $\Leftrightarrow$ finitely generated projective, and for local rings we have projective $\Leftrightarrow$ free, so the isomorphism in question is isomorphic to the canonical inclusion $\mathcal{O}^{\oplus r} \subseteq \mathrm{Frac}(\mathcal{O})^{\oplus r}$ for some $r > 0$.

$\mu \leq \lambda$). So by transfinite induction, we can assume that the field extension $\mathbb{F}_1/\mathbb{F}_0$ is generated by a single element $x \in \mathbb{F}_1$ which is transcendental or finite separable.

Suppose that $\mathbb{F}_1 = \mathbb{F}_0(x)$ with x transcendental. Choosing any element of A in the preimage of x, we get an induced factorisation $\mathbb{F}_0[x] \dashrightarrow A \to K$. Since all non-zero polynomials $f(x) \in \mathbb{F}_0[x]$ are non-zero in $\mathbb{F}_0(x) \subseteq K = A/\mathfrak{m}$, they are sent to units in A, so we get a factorisation $\mathbb{F}_0(x) \dashrightarrow A \to K$.

Now suppose x is finite separable with minimal polynomial $f(X) \in \mathbb{F}_0[X]$, so $\mathbb{F}_1 \cong \mathbb{F}_0[X]/\langle f(X)\rangle$. Choose a lift $a \in A$ of $x \in \mathbb{F}_1 \subseteq K$ and note that $f(a) \in \mathfrak{m}$ since $f(x) = 0$. Since $\mathfrak{m} = \mathrm{Nil}(A)$, we have $f(a)^n = 0$ for some n, so we get an induced commutative diagram

$$\begin{tikzcd}
K & \mathbb{F}_0[X]/\langle f(X)\rangle \arrow[l] \arrow[r, equal] & \mathbb{F}_1 \arrow[dl, dashed] \\
A \arrow[u] & \mathbb{F}_0[X]/\langle f(X)^n\rangle \arrow[l] \arrow[u, "\phi"] & \mathbb{F}_0 \arrow[l] \arrow[u, "\iota"']
\end{tikzcd}$$

with unique diagonal since ι is étale, and $\ker(\phi)$ is a nilpotent ideal, (Sta18, 02HM). ☐

Lemma 3.9. *Suppose $\mathbb{F}$ is any ring and R is any $\mathbb{F}$-algebra such that $R \to R_{\mathrm{red}}$ admits a section in the category of $\mathbb{F}$-algebras. Then we can write R as a filtered colimit $R = \operatorname{colim} R_\lambda$ of sub-$\mathbb{F}$-algebras $R_\lambda \subseteq R$ such that $(R_\lambda)_{\mathrm{red}} \cong R_{\mathrm{red}}$ but $\mathrm{Nil}(R_\lambda)$ is finitely generated for all λ.*

Proof. Given a set of elements $S \subseteq \mathrm{Nil}(R)$, write $R_{\mathrm{red}}[S] \subseteq R$ for the sub-R_{red}-module generated by the monomials $x_{i_1}^{m_1} x_{i_2}^{m_2} \dots x_{i_n}^{m_n}$ for $x_i \in S$. Since elements of $\mathrm{Nil}(R)$ are nilpotent, if S is finite, $\mathrm{Nil}(R_{\mathrm{red}}[S])$ is a finite R_{red}-module. As S_λ ranges over all finite subsets of $\mathrm{Nil}(R)$, we obtain $R = \operatorname{colim} R_\lambda$ with $R_\lambda = R_{\mathrm{red}}[S_\lambda]$ as desired. ☐

Theorem 3.10. *Every procdh local ring over a perfect field $\mathbb{F}$ is a filtered colimit of $\mathbb{F}$-algebras R_λ (not necessarily procdh local rings) such that $\mathrm{Nil}(R_\lambda)$ is finitely generated and $(R_\lambda)_{\mathrm{red}}$ is a finite rank henselian valuation ring.*

Proof. We can assume our initial ring R has finite Krull dimension by Prop.3.6. The map $R \to R_{\mathrm{red}}$ admits a section (as $\mathbb{F}$-algebras) by Lem.3.8, so then the result follows from Lem.3.9. ☐

4 Nisnevich-Riemann-Zariski spaces

In this section we consider a Nisnevich version $\mathrm{RZ}(X_{\mathrm{Nis}})$ of the Riemann-Zariski space associated to a scheme X. These can be considered as small sites, cf.(ILO14, Exposé II). The main result of this section is Corollary 4.13 which says that for each $X \in \mathrm{Sch}_S$, the canonical comparison functor $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S) \to \mathrm{Shv}(\mathrm{RZ}(X_{\mathrm{Nis}}))$ preserves colimits and finite limits, at least if S is a qcqs scheme with Noetherian topological space. This result will be used in Section 5 to show that the topoi have enough points, and in Section 7 to show that the homotopy dimension is finite.

Definition 4.1. By *modification* we will mean a morphism of schemes $Y \to X$ which is proper, of finite presentation, and an isomorphism over a dense qc open $D \subseteq X$. We write $\mathrm{Mod}_X \subseteq \mathrm{Sch}_X$ for the full subcategory of modifications.

Remark 4.2. If X is qcqs and $Y' \to Y$, $Y'' \to Y$ are morphisms in Mod_X then $Y' \times_Y Y''$ is again in Mod_X. In particular, Mod_X admits finite limits, calculated in Sch_X, and is therefore is filtered.

We do not ask modifications to be birational so that finite limits in Mod_X are more nicely behaved. We can of course often refine any object in Mod_X by one which is birational to X, Lem.4.14(1).

Definition 4.3. Let S be a qcqs scheme. For $X \in \mathrm{Sch}_S$ we define

$$\mathrm{RZ}(X_{\mathrm{Nis}}) = \int_{Y \in \mathrm{Mod}_X} Y_{\mathrm{Nis}}.$$

Explicitly, $\mathrm{RZ}(X_{\mathrm{Nis}}) \subseteq \mathrm{Arr}(\mathrm{Sch}_X)$ is the category whose objects are morphisms $U \to Y$ such that $U \in Y_{\mathrm{Nis}}$ and $Y \in \mathrm{Mod}_X$, and morphisms are commutative squares

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

We abbreviate $U \to Y$ to (U/Y).

Remark 4.4. As it is a category of arrows in a category admitting finite limits, $\mathrm{Arr}(\mathrm{Sch}_X)$ admits finite limits and they are calculated component wise: $\lim(A_i/B_i) = (\lim A_i / \lim B_i)$. If each (A_i/B_i) is in $\mathrm{RZ}(X_{\mathrm{Nis}})$, then one checks that $\lim(A_i/B_i)$ is again in $\mathrm{RZ}(X_{\mathrm{Nis}})$.[5] That is $\mathrm{RZ}(X_{\mathrm{Nis}})$ admits finite limits, and they are calculated termwise.

Definition 4.5. The category $\mathrm{RZ}(X_{\mathrm{Nis}})$ is canonically equipped with the Grothendieck topology generated by:

1. families of the form
$$\{(U_i/Y) \to (U/Y)\}_{i \in I} \tag{Nis}$$
such that $\{U_i \to U\}$ is a Nisnevich covering, and
2. families of the form
$$\{(Y' \times_Y U/Y') \to (U/Y)\} \tag{Car}$$
for morphisms $Y' \to Y$ in Mod_X.

We will write $\mathrm{Shv}(\mathrm{RZ}(X_{\mathrm{Nis}}))$ for the topos associated to the topology generated by coverings of the form (Nis) and (Car).

Remark 4.6. Since the diagonal of a modification is again a modification, a presheaf (of sets) satisfies descent for all families (Car) if and only if it sends each $(Y' \times_Y U/Y') \to (U/Y)$ to an isomorphism. Consequently,

$$\mathrm{Shv}_{\mathrm{car}}(\mathrm{RZ}(X_{\mathrm{Nis}})) = \lim_{Y \in \mathrm{Mod}_X} \mathrm{PSh}(Y_{\mathrm{Nis}}),$$

where the limit is along pushforwards $f_* : \mathrm{PSh}(Y'_{\mathrm{Nis}}) \to \mathrm{PSh}(Y_{\mathrm{Nis}})$ for morphisms $f : Y' \to Y$ in Mod_X. This implies

$$\mathrm{Shv}(\mathrm{RZ}(X_{\mathrm{Nis}})) = \lim_{Y \in \mathrm{Mod}_X} \mathrm{Shv}_{\mathrm{Nis}}(Y_{\mathrm{Nis}}). \tag{10}$$

[5] We observed $\lim B_i$ is in Mod_X in Remark 4.2. The fastest way to check that $\lim A_i \to \lim B_i$ is étale, is probably to observe that it is formally étale and of finite presentation, (Sta18, 02HG, 00UR).

Remark 4.7. The same is true for presheaves of spaces (see Section 6 for conventions). Suppose that F satisfies descent for families of the form (Car). If $Y' \to Y$ in Mod_X is a closed immersion then $(Y')^{\times_Y n} = Y'$ so for any (U/Y) in $\mathrm{RZ}(X_{\mathrm{Nis}})$ we have

$$F(U/Y) = \lim_n F((U'/Y')^{\times_{(U/Y)}(n+1)}) = \lim_n F(U'/Y') = F(U'/Y') \tag{11}$$

where $U' = Y' \times_Y U$. It follows that (11) also holds for a general $Y' \to Y$ in Mod_X, since each diagonal $Y' \to (Y')^{\times_Y n}$ is a closed immersion in Mod_X. Conversely, if F sends families of the form (Car) to equivalences, then it clearly satisfies Čech descent for such families.

Proposition 4.8. *Let X be a qcqs scheme and suppose $F \in \mathrm{PSh}(\mathrm{RZ}(X_{\mathrm{Nis}}))$ has descent for the coverings* (Nis)*. Then the sheafification $aF \in \mathrm{Shv}(\mathrm{RZ}(X_{\mathrm{Nis}}))$ satisfies*

$$aF(U/Y) = \mathrm{colim}_{Y' \in (\mathrm{Mod}_X)_{/Y}} F(Y' \times_Y U/Y'). \tag{12}$$

The same is true for presheaves of spaces.

Proof. First we show that the presheaf aF defined via Eq.(12) is a sheaf. By definition, a presheaf on $\mathrm{RZ}(X_{\mathrm{Nis}})$ is a sheaf if and only if it has descent for coverings of the form (Nis) and (Car) in Definition 4.5. The presheaf aF in the statement certainly sends modifications to isomorphisms, resp. equivalences, so it has descent for coverings of the form (Car) by Remark 4.6 and Remark 4.7.

For Nisnevich coverings, we notice that a presheaf F has descent for coverings of the form (Nis) if and only if the restriction to the small Nisnevich site Y_{Nis} for each $Y \in \mathrm{Mod}_X$ has Nisnevich descent if and only if it sends distinguished Nisnevich squares to cartesian squares. If $\{U_0 \to U, U_1 \to U\}$ is a distinguished Nisnevich square, and $(U/Y) \in \mathrm{RZ}(X_{\mathrm{Nis}})$ then for any $Y' \to Y$ in Mod_X we have

$$F(Y' \times_Y U/Y') = F(Y' \times_Y U_0/Y') \times_{F(Y' \times_Y U_{01}/Y')} F(Y' \times_Y U_1/Y')$$

where $U_{01} = U_0 \times_U U_1$ by the assumption that F has descent for (Nis). Taking the colimit over Y' and using the fact that filtered colimits commute with fibre products we find

$$aF(U/Y) = aF(U_0/Y) \times_{aF(U_{01}/Y)} aF(U_1/Y).$$

So aF is a sheaf. To conclude that a is the sheafification functor, it suffices to show that if F is already a sheaf, then $F \to aF$ is an equivalence. But this is clear, since sheaves send modifications to isomorphisms, Rem.4.6, resp. equivalences, Rem.4.7. □

Definition 4.9. Let S be a qcqs scheme. For $X \in \mathrm{Sch}_S$, we consider the canonical projection functor

$$\rho_X : \mathrm{RZ}(X_{\mathrm{Nis}}) \to \mathrm{Sch}_S; \quad (U/Y) \mapsto U$$

and the functor induced by composition

$$\mathrm{PSh}(\mathrm{Sch}_S) \to \mathrm{PSh}(\mathrm{RZ}(X_{\mathrm{Nis}})); \qquad F \mapsto F \circ \rho_X.$$

By composing this with the sheafification functor $\mathrm{PSh}(\mathrm{RZ}(X_{\mathrm{Nis}})) \to \mathrm{Shv}(\mathrm{RZ}(X_{\mathrm{Nis}}))$, we get

$$\rho_X^* : \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S) \to \mathrm{Shv}(\mathrm{RZ}(X_{\mathrm{Nis}})). \tag{13}$$

Remark 4.10. Using Proposition 4.8 we have the following concrete description.

$$(\rho_X^* F)(U/Y) = \mathrm{colim}_{Y' \in \mathrm{Mod}_X} F(Y' \times_X U).$$

Recall that a morphism of sites $\phi : (C,\tau) \to (D,\upsilon)$ is *cocontinuous* if for every $U \in C$ and covering family $\mathcal{U} = \{U_i \to \phi U\}_{i\in I}$ there is a covering family $\{V_i \to U_i\}$ such that $\{\phi V_i \to \phi U\}_{i\in I}$ refines $\mathcal{U}$, (SGA72a, Def.III.2.1), (Sta18, 00XJ).

Proposition 4.11. *Let X be a qcqs scheme whose underlying topological space is Noetherian. Then the morphism ρ is cocontinuous.*

Proof. For $(U/Y) \in \mathrm{RZ}(X_{\mathrm{Nis}})$ and a procdh covering $\{V_i \to U\}_{i\in I}$ in Sch_S, we want to find a covering $\{(U_j/Y_j) \to (U/Y)\}_{j\in J}$ in $\mathrm{RZ}(X_{\mathrm{Nis}})$, a function $J \to I$; $j \mapsto i_j$, and commutative triangles

$$\begin{tikzcd} U_j \arrow[r, dashed] \arrow[dr] & V_{i_j} \arrow[d] \\ & U. \end{tikzcd}$$

Since procdh coverings are refined by finite length compositions of generator procdh coverings, using induction on the height of the tree in Remark 2.2, it suffices to prove the claim for distinguished Nisnevich coverings and proabstract blowup squares.

For Nisnevich coverings the statement is obvious since for any $(U/Y) \in \mathrm{RZ}(X_{\mathrm{Nis}})$, a Nisnevich covering $\{U_i \to U\}_{i\in I}$ of U gives rise to a Nisnevich covering $\{(U_i/Y) \to (U/Y)\}_{i\in I}$ of (U/Y).

Consider $(U/Y) \in \mathrm{RZ}(X_{\mathrm{Nis}})$ and a proabstract blowup square $\mathcal{U} = \{Z_n \to U\}_{n\in\mathbb{N}} \sqcup \{W \to U\}$. Our task is to find a morphism $Y' \to Y$ in Mod_X and a Nisnevich covering $\{V_j \to U' := Y' \times_Y U\}_{j\in J}$ such that for each j, we have a commutative diagram on the left for some n or on the right.

$$\begin{tikzcd} V_j \arrow[r] \arrow[d] & Z_n \arrow[d] \\ U' \arrow[r] \arrow[d] & U \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd} \qquad\qquad \begin{tikzcd} V_j \arrow[r] \arrow[d] & W \arrow[d] \\ U' \arrow[r] \arrow[d] & U \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd} \tag{14}$$

Since U has finitely many generic points, by Lemma 4.17, we can assume that U is locally irreducible. As such it suffices to treat the following two cases.

Case 1: $U^{\mathrm{gen}} \subseteq Z_0$. In this case, $(Z_0)_{\mathrm{red}} = U_{\mathrm{red}}$. This means the (finitely many) generators of $\mathcal{I}_{Z_0}$ are nilpotent, so $Z_n = U$ for some n. Hence, $Y' = Y$ and the trivial covering $\{U \to U\}$ give a square on the left of (14).

Case 2: $U^{\mathrm{gen}} \cap Z_0 = \varnothing$. We will build a square as on the right of (14) with $V_j = U'$. By the assumption $U^{\mathrm{gen}} \cap Z_0 = \varnothing$, the morphism $W \to U$ is an isomorphism over the generic points of U. These all lie over generic points of Y (because $U \to Y$ is étale) so $W \to U \to Y$ is generically flat. More precisely, letting $T \subset Y$ be the closure of the image of Z_0 in Y, the morphism $W \to Y$ is flat over $Y \backslash T$ and T is nowhere dense in Y by the assumption $U^{\mathrm{gen}} \cap Z_0 = \varnothing$. Write $T = \varprojlim_\lambda T_\lambda$ for a cofiltered system of closed immersions $T_\lambda \to T$ of finite presentation. Since Y^{gen} is finite by the assumption, there exists λ such that T_λ is nowhere dense.

By Raynaud-Gruson (RG71, Th.5.2.2), (Sta18, 081R), there is a blowup $Y' \to Y$ (projective, but not necessarily of finite presentation) with a center contained in T_λ such that the strict transform $W' \to Y'$ of $W \to Y$ is flat. Since $U' := Y' \times_Y U \to Y'$ is étale, this implies that $W' \to U'$ is flat, Lemma 4.15. So now we have a flat proper morphism which is generically an isomorphism. This implies it is globally an isomorphism, Lemma 4.16. So we obtain a factorisation $U' \cong W' \to W \to U$.

If $Y' \to Y$ is not of finite presentation, then we can at least write it as a filtered limit $Y' = \lim Y'_\lambda$ such that each $Y'_\lambda \to Y$ is in Mod_Y and the transition morphisms are affine, Lemma 4.14. Setting $U'_\lambda = Y'_\lambda \times_Y U$ we also have $U' = \lim U'_\lambda$ and we can descend $U' \to W$ to some U'_λ, (Gro66, Prop.8.13.1). Noting that $Y'_\lambda \in \mathrm{Mod}_Y$ implies $Y'_\lambda \in \mathrm{Mod}_X$, we have found a square on the right of (14). □

Counterexample 4.12. Let k be a field, I a set, and $k[x^I]$ the polynomial ring in I-many variables. Define $R^I := k[x^I]/\langle x_i x_j : i \neq j \rangle$ and $\mathbb{X}^I = \operatorname{Spec} R^I$. So $\mathbb{X}^I$ is I-many copies of the affine line joined at the origin and $(\mathbb{X}^I)^{\mathrm{gen}}$ is homeomorphic to I with the discrete topology. Given a subset $J \subseteq I$ there is an associated closed immersion $\mathbb{X}^J \subseteq \mathbb{X}^I$ defined by $x_i \mapsto 0$ for $i \notin J$ which is finite presentation if and only if $I \setminus J$ is finite.

For any two cofinite $J, J' \subseteq I$ with $J \cup J' = I$, the set $\{\mathbb{X}^J \to \mathbb{X}^I, \mathbb{X}^{J'} \to \mathbb{X}^I\}$ induces a proabstract blowup square. This cannot be refined by a modification (of finite presentation), so the functor $\mathrm{RZ}(\mathbb{X}^I_{\mathrm{Nis}}) \to \mathrm{Sch}_{\mathbb{X}^I}$ is *not* cocontinuous.

Indeed, by Chevalley's Theorem (Sta18, 054K), the image of any summand Y_0 of a modification $Y_0 \amalg Y_1 \to \mathbb{X}^I$ is a closed subscheme of finite presentation. Since modifications are isomorphisms over a dense open, this implies that for any modification $Y \to \mathbb{X}^I$ the scheme Y is connected. Since it is an isomorphism over the generic points of $\mathbb{X}$, it cannot factor through any $\mathbb{X}^J \to \mathbb{X}^I$ unless $J = I$.

Corollary 4.13. *Let S be a qcqs scheme whose underlying topological space is Noetherian and $X \in \mathrm{Sch}_S$. The canonical functor ρ_X^* from* (13) *preserves colimits and finite limits, (SGA72a, III.2.3), (Sta18, 00XL).*

Here are some lemmas that were used above.

Lemma 4.14. *Suppose X is a qcqs scheme.*

1. *If the underlying topological space of X is Noetherian, then for every $Y \in \mathrm{Mod}_X$ there is a closed immersion $Y' \to Y$ in Mod_X such that $(Y')^{\mathrm{gen}} = X^{\mathrm{gen}}$.*
2. *If $Y \to X$ is a proper morphism, not necessarily of finite presentation, which is an isomorphism over a dense qc open $D \subseteq X$, then Y can be written as a cofiltered limit $Y = \lim Y_\lambda$ with $Y_\lambda \in \mathrm{Mod}_X$ and whose transition morphisms are closed immersions $Y_\lambda \to Y_\mu$ which are isomorphisms over D.*

Proof.

1. Let $D \subseteq X$ be a dense qc open over which $Y \to X$ is an isomorphism. Let $j : D \to Y$ be the induced open immersion. Write $\mathcal{I} = \ker(\mathcal{O}_Y \to j_*\mathcal{O}_D)$ as a filtered colimit $\mathcal{I} = \operatorname{colim} \mathcal{I}_\lambda$ of ideals $\mathcal{I}_\lambda$ of finite presentation, (DG71, Cor.6.9.15). Since $\underline{\mathrm{Spec}}(\mathcal{O}_Y/\mathcal{I}) \to Y$ is an isomorphism over the open $D \subseteq Y$, there is some λ for which $Y'_\lambda := \underline{\mathrm{Spec}}(\mathcal{O}_Y/\mathcal{I}_\lambda) \to Y$ is an isomorphism over $D \subseteq Y$, (Gro66, Thm.8.10.5(i)). Since X has Noetherian topological space, so does Y, Lem.4.18, so up to changing λ, we can assume $Y'_\lambda \cap (Y^{\mathrm{gen}} \setminus X^{\mathrm{gen}}) = \varnothing$, i.e., $(Y'_\lambda)^{\mathrm{gen}} = X^{\mathrm{gen}}$.
2. We can write $Y \to X$ as $Y = \lim Y_\lambda$ with each $Y_\lambda \to X$ proper and of finite presentation and all transition morphisms closed immersions by (Sta18, 09ZR, 09ZQ). The isomorphism $D \times_X Y \cong D$ induces closed immersions $D \to D \times_X Y_\lambda$ which are of finite presentation by (Sta18, 00F4(4)), and therefore defined by coherent sheaves of ideals, (Sta18, 01TV). Since the $D \times_X Y_\lambda$ are all quasi-compact, there is a λ for which $D = D \times_X Y_\mu$ for all $\mu \geq \lambda$.

□

Lemma 4.15. *Suppose that $W \to U$ is any morphism of schemes, $U \to Y$ is étale and $W \to Y$ is flat. Then $W \to U$ is also flat.*

Proof. If $U \to Y$ is not already assumed to be separated, we can reduce to this case by replacing U with an open affine covering. Then the diagonal $\delta : U \to U \times_Y U$ is open (resp. closed) because $U \to Y$ is unramified (resp. separated). Considering the cartesian squares

$$\begin{tikzcd} W \arrow[r, "\delta'"] \arrow[d] & U \times_Y W \arrow[r, "pr_2"] \arrow[d] & W \arrow[d] \\ U \arrow[r, "\delta"] & U \times_Y U \arrow[r, "pr_2"] & U \end{tikzcd}$$

one sees that $\delta' : W \to U \times_Y W$ is also both open and closed. So we have $U \times_Y W = \delta'(W) \sqcup W'$ for some W'. Since $W \to U$ factors as the composition $W \to \delta'(W) \sqcup W' = U \times_Y W \to U$ of the inclusion of a summand, and the pullback $U \times_Y W \to U$ of the flat morphism $W \to Y$, it is flat. $\square$

Lemma 4.16. *Suppose that $f : W \to U$ is a flat, proper morphism of schemes, and $D \subseteq U$ is a schematically dense open such that $D \times_U W \to D$ is an isomorphism. Then $W \to U$ is an isomorphism.*

Proof. First note that f is faithfully flat: Since it is proper, it is closed, so it suffices to show all generic points η of U are in the image. The inclusion of the localisation $U_\eta \to U$ is flat so $U_\eta \times_U D \to U_\eta$ is schematically dense. But U_η has a single point, so $U_\eta \times_U D$ is non-empty, i.e., $\eta \in D$, so $\eta \in f(W)$.

Since $W \to U$ is faithfully flat, to show it is an isomorphism it suffices to show that $pr_1 : W \times_U W \to W$ is an isomorphism, or equivalently, that the diagonal $\delta : W \to W \times_U W$ is an isomorphism. The two morphisms $W \times_U W \stackrel{pr_1}{\to} W \to U$ are flat, so $D \times_U W \to W$ and $D \times_U W \times_U W \to W \times_U W$ are schematically dense. But $W \to U$ is an isomorphism over D so $D \cong D \times_U W \cong D \times_U W \times_U W$. So we have a factorisation $D \stackrel{j}{\to} W \stackrel{\delta}{\to} W \times_U W$ with both j and δj schematically dense. It follows that δ is schematically dense. Since it is also a closed immersion (f is proper, so separated) this implies δ is an isomorphism. $\square$

Lemma 4.17. *Suppose Y is qcqs with Noetherian underlying space, let $U \to Y$ be an étale morphism, and suppose $U^{\mathrm{gen}} = \eta_0 \sqcup \eta_1$ is a decomposition of the space of generic points of U into clopens. Then there exists a* cartesian *square*

$$\begin{tikzcd} U_0' \sqcup U_1' \arrow[r] \arrow[d] & U \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

such that

1. *$Y' \to Y$ is a proper morphism of finite presentation which is an isomorphism over a dense qc open of Y, and*
2. *there are identifications $\eta_0 = (U_0')^{\mathrm{gen}}$ and $\eta_1 = (U_1')^{\mathrm{gen}}$.*

If Y is Noetherian, then we can take $Y' \to Y$ to be a blowup in a centre which is nowhere dense in Y.

Proof. Choose any closed subschemes $U_0 \subseteq U$ and $U_1 \subseteq U$ of finite presentation whose generic schemes are η_0 and η_1 respectively (here we are using that U^{gen} is finite).[6] Since $U_0 \sqcup U_1 \to$

[6] For example, let $\mathcal{I}_Z = \ker(\mathcal{O}_U \to \mathcal{O}_{U_Z^{\mathrm{gen}}})$. Write $\mathcal{I}_Z$ as a filtered union of ideals $\mathcal{I}_{Z,\lambda}$ of finite presentation. Use the fact that U^{gen} is a disjoint union of finitely many local schemes of dimension zero to deduce that there is some $\mathcal{I}_\lambda$ with $(\underline{\mathrm{Spec}}\, \mathcal{O}_U/\mathcal{I}_\lambda) \cap U_W^{\mathrm{gen}} = \varnothing$.

$U \to Y$ is generically étale, there is some dense open $D \subseteq Y$ over which it is étale, (Sta18, 07RP), and in particular, flat and of finite presentation. So we can apply Raynaud-Gruson platification, (Sta18, 081R), to find a blowup $Y' \to Y$ (not necessarily of finite presentation) which is an isomorphism over $D \subseteq Y$, and for which the strict transform $U'_0 \sqcup U'_1 \to Y'$ of $U_0 \sqcup U_1 \to Y$ is flat. By Lemma 4.15 this implies $U'_0 \sqcup U'_1 \to Y' \times_Y U$ is also flat. It is proper and an isomorphism over a dense open by construction, so it is in fact an isomorphism, Lem.4.16. Now the second condition is satisfied, since $U'_0 \to U$ factors through $U'_0 \to U_0$ and this latter is an isomorphism generically by construction.

The morphism $Y' \to Y$ might not be of finite presentation, but it is a cofiltered limit of some $Y'_\lambda \to Y$ which are proper, of finite presentation, and isomorphisms over dense qc opens $D_\lambda \subseteq Y$, Lem.4.14. There is some λ for which we already have a decomposition $U'_\lambda = U'_{\lambda,0} \sqcup U'_{\lambda,1}$ where of course we have set $U'_\lambda = Y_\lambda \times_Y U$ and $U'_{\lambda,i} = Y'_\lambda \times_Y U_i$. Replacing Y' with this Y'_λ, both conditions are now satisfied. □

Lemma 4.18. *Suppose that X is a scheme with Noetherian topological space of finite dimension and $Y \to X$ is a morphism of finite type. Then Y also has Noetherian topological space of finite dimension.*

Proof. We want to show that every descending chain

$$Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \dots$$

of closed subspaces of Y stabilises. Since X has Noetherian topological space, it admits a finite open affine covering $\{\mathrm{Spec}(A_i)\}_{i=1}^n$. It suffices to show that our chain stabilises in each $\mathrm{Spec}(A_i) \times_X Y$, that is, we can assume X is affine.

Similarly, by definition, since $Y \to X$ is finite type it is quasi-compact, so we can also assume Y is affine, say $Y = \mathrm{Spec}(B)$. In particular, this means there is a closed immersion $Y \to \mathbb{A}^n_X$ for some n. So we can assume that $Y = \mathbb{A}^n_X$. By induction we can assume $n = 1$.

Since X has Noetherian topological space, it has finitely many irreducible components. So we can assume that X is integral.

Now we use induction on the dimension of X. If X is dimension -1 it is empty, and we are done (the dimension zero case is also easy).

In general, suppose that

$$\mathbb{A}^1_X \supseteq Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \dots$$

is a decreasing chain of closed subschemes. I claim that it suffices to show that chains of finite presentation closed subschemes stabilise. Indeed, if there exists a strictly decreasing chain of closed subsets, then we can manufacture a strictly decreasing chain of closed subsets of finite presentation: For each i choose a prime $\mathfrak{p}_i \in Z_i \setminus Z_{i+1}$ and an element $f_{i+1} \in I_{i+1} \setminus \mathfrak{p}_i$ where I_{i+1} is the ideal corresponding to Z_{i+1}. Then setting W_i to be the closed corresponding to $\langle f_1, \dots, f_i \rangle$, the chain $W_0 \supseteq W_1 \supseteq W_2 \supseteq \dots$ contains $Z_0 \supseteq Z_1 \supseteq \dots$ as a subchain and also has $\mathfrak{p}_i \in W_i \setminus W_{i+1}$, so it is strictly decreasing.

So we can assume $Z_n = \langle f_1, \dots, f_n \rangle$ for some $f_i \in A[x]$ where $A = \Gamma(X, \mathcal{O}_X)$. Letting K be the quotient field of A, the ring $K[x]$ is Noetherian, so the corresponding chain of ideals (not just closed subsets) in $K[x]$ stabilises. Moreover, $K[x]$ is a principal ideal domain, so there is some $a \in A \setminus \{0\}$ and $g(x) \in A[x]$ such that $\langle \frac{1}{a} g \rangle = \langle f_1, \dots, f_n \rangle$ for all $n \gg 0$. That is, we can write each f_i as $f_i = \frac{1}{a} g \frac{1}{b_i} h_i$ for some $h_i \in A[x]$ and $b_i \in A \setminus \{0\}$, and $1 = \sum \frac{1}{b_i} h_i \frac{1}{c_i} k_i$ for some $k_i \in A[x]$ and $c_i \in A \setminus \{0\}$ (for simplicity, we choose $k_j = 0$ $c_j = 1$ for $j \gg 0$ so that the k_j and c_j are independent of n). If the a, b_i, c_i are all 1, then $\langle g \rangle = \langle f_1, \dots, f_n \rangle$ holds in $A[x]$, not just $K[x]$. That is, our chain stabilises. Indeed, $f_i = g h_i$ and $g = g(\sum h_i k_i) = \sum f_i k_i$.

We now show how to reduce to the case all a, b_i, c_i are 1. Set $a_i = a\prod b_i \prod c_i$. Since $\mathrm{Spec}(A)$ has Noetherian topological space, the chain of opens $\mathrm{Spec}(A[\frac{1}{a_0}]) \subseteq \mathrm{Spec}(A[\frac{1}{a_0a_1}]) \subseteq \mathrm{Spec}(A[\frac{1}{a_0a_1a_2}]) \subseteq \dots$ stabilises. That is, there is some $d \in A$ such that $A[d^{-1}] = A[\frac{1}{a_0a_1\dots a_n}]$ for all $n \gg 0$. Since d is nonzero and A is integral, $\mathrm{Spec}(A/\langle d\rangle)$ has dimension strictly smaller than $\mathrm{Spec}(A)$, so our chain of Z_i's restricted to $\mathbb{A}^1_{\mathrm{Spec}(A/\langle d\rangle)}$ stabilises by the induction hypothesis, and it suffices to show that it stabilises when restricted to $\mathbb{A}^1_{\mathrm{Spec}(A[d^{-1}])}$. By construction, a, b_i, c_i are all invertible in $A[d^{-1}]$. So replacing A with $A[d^{-1}]$, we can assume they are all 1 by replacing g, h_i, k_i with $\frac{1}{a}g$, $\frac{1}{b_i}h_i$ and $\frac{1}{c_i}k_i$. $\square$

5 Conservativity of the fibre functors

Recall that a topos is said to have *enough points* when a morphism f is an isomorphism if and only if $\phi(f)$ is an isomorphism for all fibre functors ϕ, Def.3.1, (SGA72a, Exposé IV, Déf.6.4.1). Equivalently, (SGA72a, Exposé IV, Prop.6.5(a)), a topos of the form $\mathrm{Shv}_\tau(C)$ has enough points when a family $\{Y_i \to X\}_{i\in I}$ in C is a covering family if and only if $\sqcup_{i\in I}\phi(Y_i) \to \phi(X)$ is surjective for all fibre functors ϕ.[7]

Deligne's completeness theorem says that if C is an essentially small category with fibre products, and every τ-covering is refinable by a finite one, then $\mathrm{Shv}_\tau(C)$ has enough points, (SGA72b, Prop.VI.9.0) or (Joh77, Thm.7.44, 7.17).

Example 5.1. If X is a qcqs scheme then $\mathrm{Shv}(\mathrm{RZ}(X_{\mathrm{Nis}}))$ has enough points.

Theorem 5.2. *Suppose S is a qcqs scheme with Noetherian topological space of finite Krull dimension. Then the topos $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S)$ has enough points.*

Proof. We want to show that if $\mathcal{Y} = \{Y_i \to Y\}_{i\in I}$ is a family of morphisms in Sch_S such that the morphism of sets $\sqcup_i\phi(Y_i) \to \phi(Y)$ is surjective for every fibre functor ϕ, then $\mathcal{Y}$ is a procdh-covering. We work by induction on the Krull dimension of Y, the base case being $Y = \varnothing$ with $\dim Y = -1$, where it is clearly true. The inductive hypothesis is:

> (Hyp.) If $\dim X < \dim Y$ and $\sqcup\phi X_i \to \phi X$ is surjective for every fibre functor ϕ, then $\{X_i \to X\}_{i\in I}$ is a procdh covering.

The functor $\rho_Y^* : \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S) \to \mathrm{Shv}\,\mathrm{RZ}(Y_{\mathrm{Nis}})$ from (13) preserves colimits and finite limits, Corollary 4.13. So by composition, every fibre functor ϕ of $\mathrm{Shv}\,\mathrm{RZ}(Y_{\mathrm{Nis}})$ induces a fibre functor

$$\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S) \xrightarrow{\rho_Y^*} \mathrm{RZ}(Y_{\mathrm{Nis}}) \xrightarrow{\phi} \mathrm{Set}.$$

By assumption, our family $\mathcal{Y}$ is sent to a surjection of sets under each such fibre functor $\phi \circ \rho_Y^*$. Since $\mathrm{RZ}(Y_{\mathrm{Nis}})$ has enough points, Exam.5.1, it follows that $\rho_Y^*\mathcal{Y}$ is a surjective family of sheaves in $\mathrm{Shv}(\mathrm{RZ}(Y_{\mathrm{Nis}}))$. This means that, locally, we can lift the section id_Y of $(\rho_Y^*Y)((Y/Y)) = \hom_{\mathrm{Sch}_S}(\rho((Y/Y)), Y) = \hom_{\mathrm{Sch}_S}(Y, Y)$. Explicitly, this means that there exists a covering $\{(U_j/Y') \to (Y/Y)\}_{j\in J}$ such that the family $\{U_j \to Y' \to Y\}_{j\in J}$ refines $\mathcal{Y}$. Since $Y' \to Y$ is a modification, there is a nowhere dense closed subscheme of finite presentation $Z_0 \to Y$ outside of which $Y' \to Y$ is an isomorphism. Since Y has finite Krull dimension and $Z_0 \to Y$ is nowhere dense, $\dim Z_0 < \dim Y$. Since fibre functors commute with finite limits, for each fibre functor ϕ, the morphism of sets $\sqcup\phi(Z_n \times_Y Y_i) = \sqcup\phi(Z_n) \times_{\phi(Y)} \phi(Y_i) \to \phi(Z_n)$ is the pullback of the

[7] Here, we have used the same symbol for an object X of C and the sheafification of the presheaf $\hom(-, X)$ it represents.

surjective morphism of sets $\sqcup\phi(Y_i) \to \phi(Y)$, and therefore is surjective. So by the induction hypothesis, for $n \in \mathbb{N}$ the families $\{Z_n \times_Y Y_i \to Z_n\}_{i\in I}$ are procdh coverings. Then the family

$$\{Z_n \times_Y Y_i \to Z_n \to Y\}_{n\in\mathbb{N}, i\in I} \sqcup \{U_j \to Y' \to Y\}_{j\in J} \tag{15}$$

is the composition of the procdh covering $\{Z_n \to Y\}_{n\in\mathbb{N}} \sqcup \{Y' \to Y\}$, the Nisnevich covering $\{U_j \to Y'\}_{j\in J}$ and the procdh coverings $\{Z_n \times_Y Y_i \to Z_n\}_{j\in J_n}$, and each morphism in Eq.(15) factors through some $Y_i \to Y$. □

Corollary 5.3. *Suppose S is a Noetherian scheme of finite Krull dimension. Then procdh local S-rings R with* $\operatorname{length} Q(R)$ *and* $\dim R$ *finite induce a conservative family of fibre functors of* $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S)$.

Proof. Apply Proposition 3.6. □

6 Excision

In this section we discuss the equivalence of procdh excision and Čech descent.

Sheaves of spaces. Since (Lur09), the default has become to define a *sheaf of spaces* as a presheaf of spaces satisfying Čech descent.[8]

Theorem 6.1. *Let S be a scheme and consider the following conditions on a presheaf of spaces $F \in \mathrm{PSh}(\mathrm{Sch}_S, \mathcal{S})$.*

1. *Excision.* For every distighished Nisnevich square $\{U \to X, V \to X\}$ and every proabstract blowup square $\{Z_n \to X\}_{n\in\mathbb{N}} \sqcup \{Y \to X\}$ in Sch_S, Def.2.1, we have

$$F(X) \xrightarrow{\sim} F(U) \times_{F(U\times_X V)} F(V), \tag{16}$$

$$F(X) \xrightarrow{\sim} F(Y) \times_{\lim F(Z_n \times_X Y)} \lim F(Z_n). \tag{17}$$

2. *Čech descent.* For every procdh covering family $\{Y_\lambda \to X\}_{\lambda\in\Lambda}$ we have

$$F(X) \xrightarrow{\sim} \lim_n F(\mathcal{Y}_n) \tag{18}$$

   where we write $F(\mathcal{Y}_n)$ for $\prod_{i\in I^n} F(Y_{i_1} \times_X \cdots \times_X F_{i_n})$. That is, in the terminology of (Lur09), F is a procdh sheaf.

If S is qcqs then (Excision) ⇔ (Čech descent).

Proof. The proofs from (AHW17, Thm.3.2.5) work verbatim. See (KS24, §6) for complete proofs in our setting. □

7 Homotopy dimension

7.1 Valuative dimension

Definition 7.1. Recall that the *valuative dimension* of a scheme is the supremum of the ranks of all valuation rings of residue fields of generic points of X centred on X.

$$\dim_v(X) = \sup\left\{\dim R \;\middle|\; \begin{array}{c} \exists\; x \in X^{\mathrm{gen}}; R \text{ is a valuation ring of } k(x) \\ \exists\; \mathrm{Spec}(R) \to X \text{ compatible with } R \subseteq k(x) \end{array}\right\}.$$

[8] As opposed to the hyperdescent used in work of Artin, Brown, Deligne, Friedlander, Gersten, Jardine, Joyal, Mazur, Morel, Thomason, Verdier, Voevodsky, and many others in work dating back at least to the 70's.

The valuative dimension is of interest to us because it controls the size of $\mathrm{RZ}(X)$. Indeed, the R appearing in the definition of valuative dimension are precisely the reductions of the local rings of the locally ringed topological space $\lim_{Y \in \mathrm{Mod}_X} Y$.

The following basic properties of valuative dimension follow directly from well-known facts about valuation rings.

Lemma 7.2. *Let X be a scheme.*

1. *If $Y \subseteq X$ is a nowhere dense subscheme and $\dim_v(Y)$ is finite, then*

$$\dim_v Y \lneq \dim_v X.$$

2. *If $f : Y \to X$ is a proper morphism inducing a bijection of sets $Y^{\mathrm{gen}} = X^{\mathrm{gen}}$ and isomorphisms of residue fields $k(y) \cong k(f(y))$ for all $y \in Y^{\mathrm{gen}}$ then*

$$\dim_v Y = \dim_v X.$$

3. *If X has finite valuative dimension then so does every X-scheme of finite type.*

Proof. The statements already appear in (EHIK21, Prop.2.3.2). For more reasonable proofs see (KS24, Lem.7.2). □

7.2 Homotopy dimension

Recall that for $n \geq -2$ one says that a space $K \in \mathcal{S}$ is *n-truncated* if $\mathrm{Map}(D^{n+2}, K) \xrightarrow{\sim} \mathrm{Map}(S^{n+1}, K)$, where D^{n+2} is the $(n+2)$-disc, S^{n+1} is its boundary, the $(n+1)$-sphere, so $S^0 = * \sqcup *$ and $S^{-1} := \varnothing$, (Lur09, Lem.5.5.6.17). The notion of n-truncatedness is extended to a general ∞-category T, such as $T = \mathrm{PSh}(C, \mathcal{S})$, by declaring an object $F \in T$ to be *n-truncated* if the space $\mathrm{Map}(G, F)$ is n-truncated for all objects G of T. If T is presentable[9] then the inclusion $T_{\leq n} \subseteq T$ of the full subcategory $T_{\leq n}$ of n-truncated objects is ω-accessible and the inclusion admits a left adjoint, (Lur09, Prop.5.5.6.18),

$$(-)_{\leq n} : T \to T_{\leq n}.$$

Suppose T, T' are ∞-categories admitting finite limits and $\rho : T' \to T$ is a functor. If ρ preserves finite limits, then it preserves n-truncated objects, cf.(Lur09, Prop.5.5.6.16), and we obtain the commutative square on the right hand side of the following diagram.

$$\begin{tikzcd} T \arrow[r, "\lambda"] \arrow[d, "(-)_{\leq n}"'] & T' \arrow[d, "(-)_{\leq n}"] \\ T_{\leq n} \arrow[r, dashed, "\lambda_{\leq n}"] & T'_{\leq n} \end{tikzcd} \qquad \begin{tikzcd} T & T' \arrow[l, "\rho"'] \\ T_{\leq n} \arrow[u, "inc."] & T'_{\leq n} \arrow[u, "inc."'] \arrow[l, dashed, "\rho_{\leq n}"'] \end{tikzcd} \tag{19}$$

If ρ and the inclusions $T_{\leq n} \subseteq T$, $T'_{\leq n} \subseteq T'$ all admit left adjoints $\lambda, (-)_{\leq n}, (-)_{\leq n}$ respectively, then an adjunction argument shows that the functor $\lambda_{\leq n} := (-)_{\leq n} \circ \lambda \circ inc.$ produces the commutative square on the left hand side of Eq.(19) above. In fact, since the *inc.* are fully faithful, $\lambda_{\leq n}$ is a left adjoint to $\rho_{\leq n}$. Note that *inc.* and $(-)_{\leq n}$ preserve final objects. So if λ also preserves the final object, we have

$$\lambda_{\leq n}(*) = (-)_{\leq n} \circ \lambda \circ inc.(*) = *.$$

Therefore, for $F \in T$ we have $F_{\leq n} \cong *$ implies $\lambda(F)_{\leq n} \cong *$.

[9] This means that T admits all small colimits and is of the form $T = \mathrm{Ind}(T')$ for some small category T'.

Example 7.3.

1. If $\Phi : \mathrm{Shv}_\tau(C, \mathcal{S}) \to \mathcal{S}$ is any fibre functor[10] and $F \in \mathrm{Shv}_\tau(C, \mathcal{S})$, we have $F_{\leq n} \cong * \Rightarrow \Phi(F)_{\leq n} \cong *$.
2. If $X \in C$ is any object and $F \in \mathrm{Shv}(C, \mathcal{S})$, we have $F_{\leq n} \cong * \Rightarrow (F|_X)_{\leq n} \cong *$, where $(-)|_X : \mathrm{Shv}(C, \mathcal{S}) \to \mathrm{Shv}(C_{/X}, \mathcal{S})$ is the restriction functor with $C_{/X}$ equipped with the induced topology: coverings in $C_{/X}$ are precisely those families which are sent to coverings in C; the projection $C_{/X} \to C$ is a continuous and cocontinuous morphism of sites.
3. If X is a qcqs scheme with Noetherian topological space and $F \in \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S)$ we have $F_{\leq n} \cong * \Rightarrow (\rho_X^* F)_{\leq n} \cong *$ in $\mathrm{Shv}(\mathrm{RZ}(X_{\mathrm{Nis}}), \mathcal{S})$, cf. Corollary 4.13.

As a right adjoint, global sections $\mathrm{Map}(*, -)$ does not preserve connectivity in general. Homotopy dimension describes how badly this fails.

Definition 7.4 ((Lur09, Prop.6.5.1.12, Def.7.2.1.1))**.** One says the ∞-topos $\mathrm{Shv}_\tau(C, \mathcal{S})$ has *homotopy dimension* $\leq d$ if for every sheaf $F \in \mathrm{Shv}_\tau(C, \mathcal{S})$ we have

$$F_{\leq d-1} \cong * \text{ implies } \mathrm{Map}(*, F)_{\leq -1} \cong *.$$

Note that the latter condition is equivalent to $\mathrm{Map}(*, F)_{\leq -1} \neq \varnothing$.

Remark 7.5. More generally, if $\mathrm{Shv}_\tau(C, \mathcal{S})$ has homotopy dimension $\leq d$ then we have

$$F_{\leq d+n} \cong * \text{ implies } \mathrm{Map}(*, F)_{\leq n} \cong *$$

for all $n \geq -1$, (Lur09, Def.7.2.1.6, Lem.7.2.1.7).

Remark 7.6. Even more generally, one says that $\mathrm{Shv}_\tau(C, \mathcal{S})$ is *locally of finite homotopy dimension* if for every $X \in C$ there is some $d_X < \infty$ such that $F_{\leq d_X + n} \cong *$ implies $\mathrm{Map}(X, F)_{\leq n} \cong *$ for all $n \geq -1$.

By the canonical adjunction $\mathrm{Shv}_\tau(C, \mathcal{S}) \rightleftarrows \mathrm{Shv}_\tau(C_{/X}, \mathcal{S})$ (cf. Example 7.3(2)), this is equivalent to asking that each $\mathrm{Shv}_\tau(C_{/X}, \mathcal{S})$ has finite homotopy dimension.

Example 7.7. Consider the category $\mathbb{N} = \{0 \to 1 \to 2 \to \dots\}$. An object of $\mathrm{PSh}(\mathbb{N}, \mathcal{S})$ is a diagram $\cdots \to K(2) \to K(1) \to K(0)$ and the global sections functor is $\{K(n)\}_{n \in \mathbb{N}} \mapsto \lim_{n \in \mathbb{N}} K(n)$. If $(K(n))_{\leq 0} \cong *$, that is, $\pi_0 K(n) \cong *$ for all n, then it follows from the short exact sequences of pointed sets, (BK72, §7.4),

$$* \to {\lim_{n \in \mathbb{N}}}^1 \pi_1 K(n) \to \pi_0 \lim_{n \in \mathbb{N}} K(n) \to \lim_{n \in \mathbb{N}} \pi_0 K(n) \to * \tag{20}$$

that $(\lim K(n))_{\leq -1} \cong *$. So the topos $\mathrm{PSh}(\mathbb{N}, \mathcal{S})$ has homotopy dimension ≤ 1. The sequence of non-empty discrete spaces $K(n) = \mathbb{N}_{\geq n}$ shows that the homotopy dimension is $\not\leq 0$ since $K(n)_{\leq -1} = *$ for all n but $\hom(*, \{K(n)\}_{n \in \mathbb{N}})_{\leq -1} = \varnothing$.

Example 7.8 ((CM21, Cor.3.11, Thm.3.18))**.** If C_λ is a filtered system of finitary[11] excisive[12] sites with colimit C, then Clausen and Mathew show that $\mathrm{Shv}(C, \mathcal{S})$ has homotopy dimension $\leq d$ if all $\mathrm{Shv}(C_\lambda, \mathcal{S})$ do. Using this they show that for any qcqs algebraic space whose underlying topological space has Krull dimension $\leq d$, the ∞-topos $\mathrm{Shv}(X_{\mathrm{Nis}}, \mathcal{S})$ has homotopy dimension $\leq d$.

It also follows from this that if X is a qcqs scheme of valuative dimension d then $\mathrm{RZ}(X_{\mathrm{Nis}})$ has homotopy dimension $\leq d$.

[10] As in the case of sets, Φ is a fibre functor if it preserves all colimits and finite limits, cf.(Lur09, Rem.6.3.1.2, Cor.5.5.2.9, Thm.6.1.0.6).

[11] A site is finitary if it has finite limits and every covering family is refineable by a finite one.

[12] A site is excisive if for all $U \in C$, the functor $F \mapsto \mathrm{Map}(U, F)$ commutes with filtered colimits.

Theorem 7.9. *Let S be a qcqs scheme of finite valuative dimension $d \geq 0$ with Noetherian underlying topological space. Then* $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S, \mathcal{S})$ *has homotopy dimension* $\leq 2d$.

Proof. The proof is by induction on the valuative dimension of S. Suppose $F \in \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S, \mathcal{S})$ has $F_{\leq 2d-1} = *$. We want to show that $F(S)_{\leq -1} \cong *$.

Since S has Noetherian topological space,

$$\rho^* = \rho_S^* : \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S, \mathcal{S}) \to \mathrm{Shv}(\mathrm{RZ}(S_{\mathrm{Nis}}), \mathcal{S})$$

is a left adjoint of a morphism of ∞-topoi, Cor.4.13, so $(\rho^* F)_{\leq 2d-1} \cong *$, Exam.7.3. Since the homotopy dimension of $\mathrm{Shv}(\mathrm{RZ}(S_{\mathrm{Nis}}), \mathcal{S})$ is $\leq d$, Exam.7.8, we have $(\rho^* F)(S)_{\leq -1} \cong *$, that is, the space $(\rho^* F)(S)$ is non-empty. Concretely, one can calculate $(\rho^* F)(S)$ as $\mathrm{colim}_{Y \in \mathrm{Mod}_S} F(Y)$, Remark 4.10, so we can find a modification $Y \to S$ such that $F(Y)$ is non-empty.

If $d = 0$, we have $Y = S$ and we are done with this step.

If $d > 0$, up to refining Y we can assume that $Y^{\mathrm{gen}} = S^{\mathrm{gen}}$, Lem.4.14. In particular, there exists a nowhere dense non-empty closed subscheme of finite presentation $Z_0 \subseteq S$ such that $Y \to S$ is an isomorphism over $S \backslash Z_0$ and $E_0 := Z_0 \times_S Y$ is also a nowhere dense closed subscheme of finite presentation. Note we now have $0 \leq \dim_v Z_0 \leq d-1$ and similar for E_0, Lem.7.2. We continue to have $(F|_{\mathrm{Sch}_{Z_n}})_{\leq 2d-1} \cong *$ and $(F|_{\mathrm{Sch}_{E_n}})_{\leq 2d-1} \cong *$, Exam.7.3. By the induction hypothesis $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_{Z_n}, \mathcal{S})$ and $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_{E_n}, \mathcal{S})$ all have homotopy dimension $\leq 2d-2$, so $F(Z_n)_{\leq 1}, F(E_n)_{\leq 1} \cong *$ for all n, Rem.7.5. We have seen that $\mathrm{PSh}(\mathbb{N}, \mathcal{S})$ has homotopy dimension ≤ 1, Exam.7.7 so $(\lim_{n \in \mathbb{N}} F(Z_n))_{\leq 0} \cong *$ and $(\lim_{n \in \mathbb{N}} F(E_n))_{\leq 0} \cong *$. Combining this with $F(Y)_{\leq -1} \cong *$, cartesianness of the square, Thm.6.1,

$$\begin{tikzcd} F(S) \arrow[r] \arrow[d] & F(Y) \arrow[d] \\ \lim_{n \in \mathbb{N}} F(Z_n) \arrow[r] & \lim_{n \in \mathbb{N}} F(E_n) \end{tikzcd}$$

implies that $F(S)_{\leq -1} \cong *$. Indeed, both $\lim_{n \in \mathbb{N}} F(Z_n)$ and $\lim_{n \in \mathbb{N}} F(E_n)$ are non-empty connected and $F(Y)$ is non-empty, so the pullback is also non-empty. □

Corollary 7.10. *Let S be a qcqs scheme of finite valuative dimension $d \geq 0$ with Noetherian underlying topological space. For any sheaf of abelian groups* $F \in \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S, \mathrm{Ab})$ *we have*

$$H^n_{\mathrm{procdh}}(S, F) = 0; \qquad n > 2d.$$

Proof. This is (Lur09, Cor.7.2.2.30). Cf. also (Lur09, Def.7.2.2.14, Rem.7.2.2.17). □

Corollary 7.11. *Let S be a qcqs scheme of finite valuative dimension with Noetherian underlying topological space. Then* $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S, \mathcal{S})$ *is Postnikov complete,*[13] *and hypercomplete.*[14]

Proof. Lemma 7.2(3) and Lemma 4.18 say that each $X \in \mathrm{Sch}_S$ also have finite valuative dimension with Noetherian underlying topological space so by Theorem 7.9 the ∞-topos $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S)$ is locally of finite homotopy dimension, hence, Postnikov comlete, (Lur09, Prop.7.2.1.10), and hypercomplete, (Lur09, Cor.7.2.1.12). □

[13] Postnikov complete means that for every object F we have $F = \lim_n \tau_{\leq n} F$.

[14] Hypercomplete means that for every hypercovering $U_* \to X$ and object F we have $\mathrm{Map}(X, F) = \lim_{n \in \Delta} \mathrm{Map}(U_n, F)$, (Lur09, pg.667).

7.3 Points of ∞-topoi

As in the 1-topos case, one says that an ∞-topos $\mathrm{Shv}(C, \mathcal{S})$ has *enough points* when a morphism f is an equivalence if and only $\phi^* f$ is an equivalence for every geometric morphism of ∞-topoi $\phi^* : \mathrm{Shv}(C, \mathcal{S}) \rightleftarrows \mathcal{S} : \phi_*$, (Lur09, Rem.6.5.4.7). Unlike the 1-topos case, in general, detecting equivalences is not equivalent to detecting coverings unless $\mathrm{Shv}(C, \mathcal{S})$ is hypercomplete, (Lur18, Prop.A.4.2.1).[15]

The purpose of this subsection is to prove Proposition 7.14 which says that if the 1-topos of a site has enough points and the ∞-topos is hypercomplete, then the ∞-topos also has enough points.

Remark 7.12. This should be compared to Lurie's result (Lur18, Thm.A.4.0.5), which says that any ∞-topos which is locally coherent and hypercomplete has enough points. We cannot apply this to the procdh topos because it is not locally coherent.

Conversely, combining Deligne's 1-categorical completeness theorem with our Proposition 7.14 recovers (Lur18, Thm.A.4.0.5).

We apply Proposition 7.14 to the procdh topos in Corollary 7.15.

Lemma 7.13. *Let (C, τ) be a site admitting finite limits. Every fibre functor ϕ^*_{Set} of the 1-topos induces a fibre functor $\phi^*_{\mathcal{S}}$ of the ∞-topos fitting into a commutative square Eq.*(21)*. The same is true with ϕ^*_{Set} and $\phi^*_{\mathcal{S}}$ reversed.*

$$\begin{tikzcd}
\mathrm{Shv}(C,\mathcal{S}) \arrow[r, "(-)_{\leq 0}"] \arrow[d, "\phi^*_{\mathcal{S}}"'] & \mathrm{Shv}(C,\mathrm{Set}) \arrow[d, "\phi^*_{\mathrm{Set}}"] \\
\mathcal{S} \arrow[r, "(-)_{\leq 0}"] & \mathrm{Set}
\end{tikzcd} \tag{21}$$

Proof. Since C admits finite limits, in both the 1-category and ∞-category setting there is a canonical correspondence between fibre functors $\phi^*_?$ of $\mathrm{PSh}(C, ?)$ and pro-objects P of C, given by $(P : \Lambda \to C) \mapsto (F \mapsto \mathrm{colim}_{\lambda \in \Lambda} F(P_\lambda))$ and $\phi^* \mapsto (P : \int_C F \to C)$, (Lur09, Cor.5.3.5.4). So, for $\phi^*_{\mathrm{Set}} : \mathrm{Shv}(C, \mathrm{Set}) \to \mathrm{Set}$ as in the lemma, we automatically get the extended diagram on the left.

$$\begin{tikzcd}
\mathrm{PSh}(C,\mathcal{S}) \arrow[r, "(-)_{\leq 0}"] \arrow[d] \arrow[dd, bend right=60, "\phi^*_{\mathrm{PSh}}"'] & \mathrm{PSh}(C,\mathrm{Set}) \arrow[d] \\
\mathrm{Shv}(C,\mathcal{S}) \arrow[r, "(-)_{\leq 0}"] & \mathrm{Shv}(C,\mathrm{Set}) \arrow[d, "\phi^*_{\mathrm{Set}}"] \\
\mathcal{S} \arrow[r, "(-)_{\leq 0}"] & \mathrm{Set}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{PSh}(C,\mathcal{S}) \arrow[r, "(-)_{\leq 0}"] \arrow[d] & \mathrm{PSh}(C,\mathrm{Set}) \arrow[d] \arrow[dd, bend left=60, "\phi^*_{\mathrm{PSh}}"] \\
\mathrm{Shv}(C,\mathcal{S}) \arrow[r, "(-)_{\leq 0}"] \arrow[d, "\phi^*_{\mathcal{S}}"'] & \mathrm{Shv}(C,\mathrm{Set}) \\
\mathcal{S} \arrow[r, "(-)_{\leq 0}"] & \mathrm{Set}
\end{tikzcd} \tag{22}$$

Since $\mathrm{Shv}(C, \mathcal{S})$ is the topological localisation of $\mathrm{PSh}(C, \mathcal{S})$ generated by covering sieves, (Lur09, Def.5.5.4.5, Def.6.2.1.4) and ϕ^*_{PSh} preserves colimits, to show that ϕ^*_{PSh} factors through $\mathrm{Shv}(C, \mathcal{S})$, it suffices to show it sends covering sieves to equivalences. Any sieve[16] in $\mathrm{PSh}(C, \mathcal{S})$ is sent by ϕ^*_{PSh} to an inclusion of connected components $A \to A \sqcup B$ in $\mathcal{S}$. Such an inclusion is an equivalence if and only if it induces an isomorphism on connected components, i.e., if it is an isomorphism after $\tau_{\leq 0}$. So the result follows from the fact that covering sieves are sent to isomorphisms in Set by the outside square in Eq.(22).

[15] This is unsurprising as the definition of sieve in (Lur09) basically encodes the notion of Čech descent.

[16] Or more generally, any monomorphism, i.e., any morphism $F \to G$ such that $F \to F \times_G F$ is an equivalence.

Conversely, by the same argument, any $\phi^*_{\mathcal{S}}$ gives rise to a ϕ^*_{PSh} as in the diagram on the right. Again, this factors through $\mathrm{Shv}(C, \mathrm{Set})$ if and only if it sends covering sieves to isomorphisms, but this is guaranteed by the fact that $\mathrm{PSh}(C, \mathcal{S}) \to \mathrm{Shv}(C, \mathcal{S})$ sends covering sieves to equivalences. $\square$

Proposition 7.14. *Let (C, τ) be a small site admitting finite limits such that the 1-topos $\mathrm{Shv}(C, \mathrm{Set})$ has enough points as a 1-topos. Then the hypercompletion $\mathrm{Shv}(C, \mathcal{S})^\wedge$ has enough points as an ∞-topos.*

Proof. We have the following facts.

1. $\mathrm{PSh}(C, \mathcal{S})[S^{-1}] \cong (L_S\mathrm{PSh}(C, \mathrm{Set}_\Delta))^\circ$ for any set of morphisms S in $\mathrm{PSh}(C, \mathrm{Set}_\Delta)$. That is, the localisation (in the sense of (Lur09, Def.5.2.7.2)) of the ∞-category $\mathrm{PSh}(C, \mathcal{S})$ at the image of S is the ∞-category associated to the Bousfield localisation of the simplicial model category $\mathrm{PSh}(C, \mathrm{Set}_\Delta)$ (with either the injective or projective model structures).
2. $\mathrm{Shv}(C, \mathcal{S})$ is the localisation of $\mathrm{PSh}(C, \mathcal{S})$ at the class of Čech hypercoverings.
3. $\mathrm{Shv}(C, \mathcal{S})^\wedge$ is the localisation of $\mathrm{PSh}(C, \mathcal{S})$ at the class HR of all hypercoverings, (Lur09, Thm.6.5.3.12).
4. the weak equivalences in $L_{HR}\mathrm{PSh}(C, \mathrm{Set}_\Delta)$ are the morphisms such that each of the induced maps
$$\pi_0 E \to \pi_0 B$$
$$\pi_n(E|_{C_{/X}}, b) \to \pi_n(B|_{C_{/X}}, b), \qquad X \in C, b \in B(X)$$
are isomorphisms of sheaves, (Jar87, Prop.2.8), (DHI04).

Putting these facts together, the result holds almost by definition. $\square$

Corollary 7.15. *Let S be a qcqs scheme of finite valuative dimension $d \geq 0$ with Noetherian underlying topological space. Then $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S, \mathcal{S})$ has enough points as an ∞-topos.*

Proof. The 1-topos $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S, \mathrm{Set})$ has enough points, Thm.5.2, and the ∞-topos $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S, \mathcal{S})$ is hypercomplete, Cor.7.11, so it has enough points by Proposition 7.14. $\square$

7.4 Counterexample

We give an example that suggests the bound on the homotopy dimension in Theorem 7.9 is strict.

Proposition 7.16. *Let S be scheme with finitely many generic points and such that for all $s \in S$ the henselisation $\mathcal{O}^h_{S,s}$ is procdh local (e.g., a smooth curve over a field, or a procdh local scheme). Suppose $F \in \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sch}_S, *)$ is a Nisnevich sheaf of sets / abelian groups / chain complexes of abelian groups / spaces. Then $F(S) = F_{\mathrm{procdh}}(S)$.*

Proof. Since F and F_{procdh} are both Nisnevich sheaves, and the (small) Nisnevich ∞-topos is hypercomplete, it suffices to show that for $s \in S$ we have $\phi^*_{\mathrm{Nis},s}F = \phi^*_{\mathrm{Nis},s}F_{\mathrm{procdh}}$ where $\phi^*_{\mathrm{Nis},s}G$ means $\mathrm{colim}_{s \to V \to S}\, G(V)$ and the colimit is over all factorisations with $V \to S$ étale of finite presentation. By assumption $\mathcal{O}^h_{S,s}$ is procdh local. Note that $\{s \to V \to S : V \in S_{\mathrm{Nis}}\}$ is a pro-object of Sch_S and $\mathrm{Spec}(\mathcal{O}^h_{S,s}) = \lim_{\substack{s \to V \to S \\ V \in S_{\mathrm{Nis}}}} V$ so the functor $\phi^*_{\mathrm{Nis},s}$ is the fibre functor of $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S)$ associated to the procdh local ring $\mathcal{O}^h_{S,s}$. Hence, $\phi^*_{\mathrm{Nis},s}F = \phi^*_{\mathrm{Nis},s}F_{\mathrm{procdh}}$. $\square$

Example 7.17. The idea is to find a scheme S of valuative dimension 1, a proabstract blowup square

$$\begin{tikzcd} \text{“}\operatorname*{colim}_{n\in\mathbb{N}}\text{”} E_n \arrow[r] \arrow[d] & X \arrow[d] \\ \text{“}\operatorname*{colim}_{n\in\mathbb{N}}\text{”} Z_n \arrow[r] & S \end{tikzcd} \tag{23}$$

and a sheaf of abelian groups F such that

$$H^i_{\mathrm{procdh}}(X, F) = 0 \text{ for all } i \tag{24}$$

$$H^i_{\mathrm{procdh}}(Z_n, F) = 0 \text{ for all } i \tag{25}$$

$$H^i_{\mathrm{procdh}}(E_n, F) = 0 \quad i > 0. \tag{26}$$

Then the corresponding long exact sequence is

$$\cdots \to 0 \to H^i_{\mathrm{procdh}}(S, F) \to R^{i-1} \lim_{n\in\mathbb{N}} F(E_n) \to 0 \to \dots .$$

So if we can also arrange that the $\lim^1_{n\in\mathbb{N}}$ is non-zero, we have non-zero $H^2_{\mathrm{procdh}}(S, F)$ giving a counterexample to the statement that $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S, \mathcal{S})$ has homotopy dimension ≤ 1.

Our example is two affine lines joined at the origin, $S = \mathbb{A}^1 \sqcup_{\{0\}} \mathbb{A}^1$, with normalisation $X = \mathbb{A}^1_k \sqcup \mathbb{A}^1_k$ and associated proabstract blowup square

$$\begin{tikzcd} \text{“}\operatorname*{colim}_{n\in\mathbb{N}}\text{”} E_n \arrow[r] \arrow[d] & X \arrow[d] \\ \text{“}\operatorname*{colim}_{n\in\mathbb{N}}\text{”} Z_n \arrow[r] & S \end{tikzcd} \tag{27}$$

where Z_n is the nth thickening of the origin and $E_n = Z_n \times_S X$. Note that E_n decomposes into two components corresponding to the components of X, namely, $E_n = \operatorname{Spec} k[x]/x^n \sqcup \operatorname{Spec} k[y]/y^n =: E_{n,x} \sqcup E_{n,y}$.

To a sequence of abelian groups $\dots A_2 \to A_1 \to A_0$ we associated the presheaf of abelian groups on Sch_S,

$$F : T \mapsto \begin{cases} A_m, & m = \min\{i \mid \exists\, T \to E_{i,x} \to S\} \\ 0, & \forall i,\ \nexists\, T \to E_{i,x} \to S. \end{cases}$$

For $\tau = \mathrm{Nis}, \mathrm{procdh}$, let F_τ be the τ-sheafifiaction of F considered as a presheaf of chain complexes. So $H^i(F_\tau(S)) = H^i_\tau(S, F)$, etc. The only $V \in X_{\mathrm{Nis}}$ admitting an S-morphism $V \to E_{i,x}$ is the empty scheme, so $F_{\mathrm{Nis}}(X) = 0$ and by Proposition 7.16 we have $F_{\mathrm{procdh}}(X) = F_{\mathrm{Nis}}(X)$, so (24) holds. We have (25) for the same reason. The condition (26) holds because E_n is a disjoint union of procdh local schemes. More precisely we have

$$\begin{aligned} F_{\mathrm{procdh}}(E_n) &= F_{\mathrm{procdh}}(E_{n,x}) \times F_{\mathrm{procdh}}(E_{n,y}) \\ &= F(E_{n,x}) \times F(E_{n,y}) = A_n \times \{0\}. \end{aligned}$$

where the 0 is because there are no factorisations $E_{n,y} \to E_{i,x}$ for $n > 1$. So choosing any sequence $(A_n)_{n\in\mathbb{N}}$ with $\lim^1_{n\in\mathbb{N}} A_n \neq 0$ produces an F with $H^2_{\mathrm{procdh}}(S, F) \neq 0$.

Remark 7.18. It seems possible to push the above technique further to at least show that the cohomological dimension of certain surfaces is 4. For example, Gabber proposed trying the surface $S \times S$, where S is the scheme from Example 7.17.

8 Application to K-theory

In this section, we work with the category $\mathrm{Sch}^{\mathrm{qcqs}}$ of qcqs schemes and the full subcategory $\mathrm{Sch}^{\mathrm{noe}}$ of noetherian schemes. For $\mathcal{C} = \mathrm{Spt}, D(\mathbb{Z}), D(\mathbb{Q}), D(\mathbb{F}_p)$, we write $\mathrm{PSh}(\mathrm{Sch}^{\mathrm{qcqs}}, \mathcal{C})$ for the ∞-category of presheaves on $\mathrm{Sch}^{\mathrm{qcqs}}$ with values in $\mathcal{C}$ and write $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}^{\mathrm{qcqs}}, \mathcal{C})$ for the full subcategory of the procdh sheaves with the sheafification functor[17]

$$a_{\mathrm{procdh}} : \mathrm{PSh}(\mathrm{Sch}^{\mathrm{qcqs}}, \mathcal{C}) \to \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}^{\mathrm{qcqs}}, \mathcal{C}).$$

8.1 Algebraic K-theory

By (KST18b), non-connective K-theory satisfies procdh excision for Noetherian schemes. Consequently, by Theorem 6.1 it defines a sheaf $K \in \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}^{\mathrm{noe}}, \mathrm{Spt})$. We have the following topos-theoretic interpretation of the Bass construction.

Theorem 8.1. *For $X \in \mathrm{Sch}^{\mathrm{noe}}$ with $\dim(X) < \infty$, there exists a natural equivalence*

$$(a_{\mathrm{procdh}}\tau_{\geq 0}K)(X) \simeq K(X).$$

Proof. For X as above, we have the descent spectral sequences

$$E_2^{p,q} = H^p_{\mathrm{procdh}}(X, \widetilde{K}_{-q}) \Rightarrow K_{-p-q}(X),$$

$$E_2^{p,q} = H^p_{\mathrm{procdh}}(X, \tau_{\geq 0}\widetilde{K}_{-q}) \Rightarrow \pi_{-p-q}(a_{\mathrm{procdh}}\tau_{\geq 0}K(X)),$$

where $\widetilde{K}_i$ is the procdh sheafification of the presheaf $K_i = \pi_i K$ of abelian groups on Sch_X, and $\tau_{\geq 0}\widetilde{K}_i = \widetilde{K}_i$ for $i \geq 0$ and $\tau_{\geq 0}\widetilde{K}_i = 0$ for $i < 0$. By Corollary 7.10, the spectral sequences are bounded. So, it suffices to show that $\widetilde{K}_i = 0$ for $i < 0$. By Theorem 5.2 the procdh 1-topos has enough points, so it is enough to show that $\phi^* K_i = 0$ for all $i < 0$ and fibre functors $\phi^* : \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_X) \to \mathrm{Set}$. Since K-theory commutes with filtered colimits of rings, we have $\phi^* K_i = K_i(R)$ where $R = \operatorname{colim} R_\lambda$ for $(\mathrm{Spec}(R_\lambda) \to X)_{\lambda \in \Lambda}$ the proobject corresponding to ϕ^*. Let $\mathfrak{N} \subset R$ be the ideal of nilpotent elements. Then, for $i < 0$, we get $K_i(R) = K_i(R/\mathfrak{N}) = 0$, where the first equality follow from the nil-invariance of the negative K-theory and the last equality follows from (KM21, Th.1.3) since $R/\mathfrak{N}$ is a valuation ring. This completes the proof. □

Remark 8.2. The proof of Theorem 8.1 shows the following. Note that we use Corollary 5.3 to reduce to the smaller class of those procdh local rings R with $\operatorname{length} Q(R)$ and $\dim R$ finite.

Proposition 8.3. *Take $\mathbb{F} = \mathbb{F}_p$ or $\mathbb{Q}$, set $\mathcal{C} = \mathrm{Spt}$ or $D(\mathbb{Z})$, and let $\mathcal{E} \in \mathrm{PSh}(\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}, \mathcal{C})$ be a presheaf. Consider the following conditions.*

(Desc) For each Noetherian $\mathbb{F}$-scheme X the restriction $\mathcal{E}|_{\mathrm{Sch}_X}$ is a procdh sheaf.

(Fin) $\mathcal{E}$ is finitary, in the sense that it preserves filtered colimits of $\mathbb{F}$-algebras.

(BB)$_{\geq N}$ For each procdh local ring R with $\operatorname{length} Q(R)$ and $\dim R$ finite, the procdh stalk $\mathcal{E}(R)$ is homologically bounded below N. That is, $\pi_i \mathcal{E}(R) = 0$ for $i < N$.

If $\mathcal{E}$ satisfies (Desc), (Fin), and (BB)$_N$ then for every Noetherian $\mathbb{F}$-scheme X there is a natural equivalence

$$a_{\mathrm{procdh}}(F_{\geq N})(X) \simeq F(X). \tag{28}$$

We will produce a number of presheaves that satisfy the conditions of Proposition 8.3 using the following definition.

[17] For existence of sheafification for certain large sites (including ours) see (BS13, Rem.4.1.2).

Definition 8.4. Let $\mathbb{F}$ and $\mathcal{C}$ be as in Proposition 8.3. For $\mathcal{E} \in \mathrm{PSh}(\mathrm{Sch}_{\mathbb{F}}^{\mathrm{qcqs}}, \mathcal{C})$, define

$$\mathrm{Nil}\,\mathcal{E} := \mathrm{fib}(\mathcal{E} \to a_{\mathrm{cdh}}\mathcal{E}) \in \mathrm{PSh}(\mathrm{Sch}_{\mathbb{F}}^{\mathrm{qcqs}}, \mathcal{C}), \tag{29}$$

where $a_{\mathrm{cdh}} : \mathrm{PSh}(\mathrm{Sch}_{\mathbb{F}}^{\mathrm{qcqs}}, \mathcal{C}) \to \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}_{\mathbb{F}}^{\mathrm{qcqs}}, \mathcal{C})$ is the cdh sheafification functor.

8.2 Negative cyclic homology

Recall that for $k \to R$ a morphism of commutative rings, one defines $\mathrm{HC}^-(R/k) = HH(R/k)^{hS^1}$ as the homotopy fixed points of the Hochsdhild homology and for qcqs k-schemes using Zariski descent. In (Ant19, Thm.1.1), Antieau defines a functorial complete decreasing multiplicative $\mathbb{Z}$-indexed filtration,[18]

$$\left\{ Fil^n_{\mathrm{HKR}}\,\mathrm{HC}^-(X/k) \right\}_{n\in\mathbb{Z}} \text{ on } \mathrm{HC}^-(X/k) \tag{30}$$

and natural equivalences

$$\mathrm{gr}^n_{Fil_{\mathrm{HKR}}}\,\mathrm{HC}^-(X/k) \simeq \widehat{L\Omega}^{\geq n}_{X/k}[2n]. \tag{31}$$

Here, $\widehat{L\Omega}_{X/\mathbb{Q}}$ is the Hodge-completed derived de Rham complex equipped with the Hodge filtration $\{\widehat{L\Omega}^{\geq n}_{X/k}\}_{n\in\mathbb{N}}$.[19] The graded pieces of this filtration are computed by

$$gr^n \widehat{L\Omega}_{X/k} \simeq \wedge^n L_{X/k}[-n] \tag{32}$$

where $L_{X/k} \in \mathrm{PSh}(\mathrm{Sch}_k^{\mathrm{qcqs}}, D(k)))$ is the cotangent complex, (Bha12, Construction 4.1).

By (EM23, Lem.4.5), the cofibre $\mathrm{cofib}(Fil^0_{\mathrm{HKR}}\,\mathrm{HC}^-(-/\mathbb{Q}) \to \mathrm{HC}^-(-/\mathbb{Q}))$ is a cdh sheaf on $\mathrm{Sch}_{\mathbb{Q}}^{\mathrm{qcqs}}$, so the canonical morphism $\mathrm{Nil}\,Fil^0_{\mathrm{HKR}}\,\mathrm{HC}^-(-/\mathbb{Q}) \xrightarrow{\sim} \mathrm{Nil}\,\mathrm{HC}^-(-/\mathbb{Q})$ is an equivalence of presheaves. Therefore, applying Nil to (30) produces a complete and exhaustive $\mathbb{N}$-indexed filtration

$$\left\{ Fil^n_{\mathrm{HKR}}\,\mathrm{Nil}\,\mathrm{HC}^-(-/\mathbb{Q}) \right\}_{n\in\mathbb{N}} \text{ on } \mathrm{Nil}\,\mathrm{HC}^-(-/\mathbb{Q}) \tag{33}$$

with identifications

$$\mathrm{gr}^n_{Fil_{\mathrm{HKR}}}\,\mathrm{Nil}\,\mathrm{HC}^-(-/\mathbb{Q}) \simeq \mathrm{Nil}\,\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}}[2n]. \tag{34}$$

Lemma 8.5. *The presheaves* $\mathrm{Nil}\,\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}} \in \mathrm{PSh}(\mathrm{Sch}_{\mathbb{Q}}^{\mathrm{qcqs}}, D(\mathbb{Q}))$ *satisfy (Desc), (Fin), and (BB)*$_{\geq -n}$.

Proof.

(Desc) By (Mor16b, Thm.2.10), the $\wedge^n L_{-/\mathbb{Q}} \in \mathrm{PSh}(\mathrm{Sch}_{\mathbb{Q}}^{\mathrm{qcqs}}, D(\mathbb{Q}))$ are procdh sheaves on $\mathrm{Sch}_{\mathbb{Q}}^{\mathrm{noe}}$. Hence, the $L\Omega^{<n}_{-/\mathbb{Q}}$, and their limit $\widehat{L\Omega}_{-/\mathbb{Q}} = \lim_n L\Omega^{<n}_{-/\mathbb{Q}}$ are also procdh sheaves. From this we deduce that the fibres $\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}} = \mathrm{fib}(\widehat{L\Omega}_{-/\mathbb{Q}} \to L\Omega^{<n}_{-/\mathbb{Q}})$ are procdh sheaves. Since all cdh sheaves are procdh sheaves, the fibres $\mathrm{Nil}\,\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}} = \mathrm{fib}(\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}} \to a_{\mathrm{cdh}}\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}})$ are procdh sheaves.

[18] It is not necessarily exhaustive in general, but is exhaustive if X/k is $L_{X/k}$ has Tor-amplitude contained in $[0,1]$.

[19] That is, $\widehat{L\Omega}_{X/k} = \varprojlim_n L\Omega^{<n}_{X/k}$ where for affines X, the complex $L\Omega^{<n}_{X/k}$ is the totalisation of the simplicial chain complex $[r] \mapsto \sigma^{<n}\Omega^*_{P_r/A}$ for $P_\bullet \to B$ a polynomial A-algebra resolution of B and where $\sigma^{<n}$ refers to the stupid truncation in cohomological degrees $< n$.

(Fin) Above we saw that $\widehat{L\Omega}_{-/\mathbb{Q}}$ is a procdh sheaf. In fact, it is a cdh sheaf, (EM23, Lem.4.5), so $\operatorname{Nil}\widehat{L\Omega}_{-/\mathbb{Q}} = 0$ leading to an equivalence

$$\operatorname{Nil}\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}} \cong \operatorname{Nil} L\Omega^{<n}_{-/\mathbb{Q}}[-1]. \tag{35}$$

So it suffices to show finitarity of $\operatorname{Nil} L\Omega^{<n}_{-/\mathbb{Q}}$. The presheaves $\wedge^i L_{-/\mathbb{Q}}$ are finitary and $L\Omega^{<n}_{-/\mathbb{Q}}$ admits a finite filtration whose graded quotients are shifts of $\wedge^i L_{-/\mathbb{Q}}$, so it follows that the $L\Omega^{<n}_{-/\mathbb{Q}}$ are finitary. The cdh sheafification functor preserves finitary sheaves so the $a_{\mathrm{cdh}} L\Omega^{<n}_{-/\mathbb{Q}}$, and therefore $\operatorname{Nil} L\Omega^{<n}_{-/\mathbb{Q}}$ are finitary.

$(\mathrm{BB})_{\geq -n}$ By Eq.(35) it suffices to show that $L\Omega^{<n}_{R/\mathbb{Q}}$ is supported in cohomological degree $\leq n-1$ for all R as in Proposition 8.3. We have

$$(a_{\mathrm{cdh}} L\Omega^{<n}_{-/\mathbb{Q}})(R) = (a_{\mathrm{cdh}} L\Omega^{<n}_{-/\mathbb{Q}})(R/\mathfrak{N}) = L\Omega^{<n}_{(R/\mathfrak{N})/\mathbb{Q}}, \tag{36}$$

where the first (resp. second) equality holds since any finitary cdh sheaf is nil-invariant (resp. a valuation ring is a point of the cdh topos). Since $L\Omega^{<n}_{A/\mathbb{Q}}$ for a local $\mathbb{Q}$-algebra A is supported in degrees $\leq n-1$, we are reduced to showing the surjectivity of the map

$$H^{n-1}(L\Omega^{<n}_{R/\mathbb{Q}}) \to H^{n-1}(L\Omega^{<n}_{(R/\mathfrak{N})/\mathbb{Q}}).$$

This holds since the map is identified with $\Omega^{n-1}_{R/\mathbb{Q}} \to \Omega^{n-1}_{(R/\mathfrak{N})/\mathbb{Q}}$.

□

Proposition 8.6. *Let X be a Noetherian $\mathbb{Q}$-scheme of finite Krull dimension. Then* $\operatorname{Nil}\mathrm{HC}^-(-/\mathbb{Q}) \in \operatorname{PSh}(\mathrm{Sch}_X, D(\mathbb{Q}))$ *satisfy (Desc), (Fin), and $(\mathrm{BB})_{\geq 0}$. Therefore*

$$a_{\mathrm{procdh}}(\operatorname{Nil}\mathrm{HC}^-(-/\mathbb{Q}))_{\geq 0}(X) \simeq \operatorname{Nil}\mathrm{HC}^-(X/\mathbb{Q}).$$

Proof. The property (Desc) follows from the corresponding property for the graded pieces $\operatorname{Nil}\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}}$ of the filtration (33). For (Fin) and $(\mathrm{BB})_{\geq 0}$ consider the spectral sequence induced by (33)

$$E_2^{i,j} = H^{i-j}(\operatorname{Nil}\widehat{L\Omega}^{\geq -j}_{X/\mathbb{Q}}) \Rightarrow H^{i+j}\operatorname{Nil}\mathrm{HC}^-(X/\mathbb{Q}).$$

If $X \in \mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{Q}}$ has finite valuative dimension d with Noetherian underlying topological spaces, Corollary 7.10 and $(\mathrm{BB})_{\geq j}$ for $\operatorname{Nil}\widehat{L\Omega}^{\geq -j}_{-/\mathbb{Q}}$ from Lemma 8.5 imply $E_2^{i,j} = 0$ for $i-j > 2d-j$, which implies that the spectral sequence is bounded. Hence, (Fin) and $(\mathrm{BB})_{\geq 0}$ for $\mathrm{HC}^-(-/\mathbb{Q})$ follows from (Fin) and $(\mathrm{BB})_{\geq j}$ for $\operatorname{Nil}\widehat{L\Omega}^{\geq -j}_{-/\mathbb{Q}}$, Lemma 8.5. □

8.3 Integral topological cyclic homology

For $X \in \mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}_p}$ write $\mathrm{TC}(X)$ for the integral topological cyclic homology of X. By (BMS19), for any qcqs $\mathbb{F}_p$-scheme X, there exists a functorial complete decreasing $\mathbb{N}$-indexed filtration[20]

$$\left\{ Fil^n_{\mathrm{BMS}}\,\mathrm{TC}(X) \right\}_{n\in\mathbb{N}} \text{ on } \mathrm{TC}(X) \tag{37}$$

with associated graded quotients

$$\mathrm{gr}^n_{Fil_{\mathrm{BMS}}}\,\mathrm{TC}(X) \simeq \mathbb{Z}(n)^{\mathrm{syn}}(X)[2n]$$

for a natural object $\mathbb{Z}(n)^{\mathrm{syn}} \in \operatorname{PSh}(\mathrm{Sch}_{\mathbb{F}_p}, D(\mathbb{Z}_p))$ called the syntomic complex.

[20] (BMS19) treats quasi-syntomic rings and it is extended to all p-complete rings in (AMMN22).

Lemma 8.7. *The presheaf* $\mathrm{Nil}\,\mathbb{Z}(n)^{\mathrm{syn}} \in \mathrm{PSh}(\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}_p}, \mathrm{Spt})$ *satisfies (Desc), (Fin) and (BB)*$_{\geq -n}$.

Proof.

(Desc) This property for $\mathrm{Nil}\,\mathbb{Z}(n)^{\mathrm{syn}}$ follows from the fact that $\mathbb{Z}(n)^{\mathrm{syn}}$ is a procdh sheaf on $\mathrm{Sch}^{\mathrm{noe}}_{\mathbb{F}_p}$ by (EM23, Th.8.6) and Theorem 6.1.

(Fin) The argument is taken from the proof of (EM23, Th.4.24(4)). First, we claim $\mathrm{Nil}\,\mathbb{Z}(n)^{\mathrm{syn}}[\frac{1}{p}] = 0$. Using the fact that $Fil^{\bullet}_{\mathrm{BMS}}\,\mathrm{TC}$ from (37) naturally splits after inverting p, the claim is reduced to $\mathrm{Nil}\,\mathrm{TC}[\frac{1}{p}] = 0$. To prove this, we use the following pullback square in $\mathrm{PSh}(\mathrm{Sch}^{\mathrm{qcqs}}, \mathrm{Spt})$

$$\begin{tikzcd} K \arrow[r, "\mathrm{tr}"] \arrow[d] & \mathrm{TC} \arrow[d] \\ KH \arrow[r, "\mathrm{tr}^{cdh}"] & a_{\mathrm{cdh}}\,\mathrm{TC} \end{tikzcd}, \tag{38}$$

where KH is the homotopy K-theory and tr is the cyclotomic trace ((BHM93), (DGM13)), and tr^{cdh} is induced by tr via the equivalence $KH \simeq a_{\mathrm{cdh}}K$, (KST18b, Th.6.3), (Cis13), (KM21).

The pullback square follows from the latter equivalence and the fact that the fiber of tr is a cdh sheaf by (LT19, Th. A.3). By (38), $\mathrm{Nil}\,\mathrm{TC}[\frac{1}{p}] = 0$ follows from $\mathrm{fib}(K \to KH)[\frac{1}{p}] = 0$ by (TT90, Th. 9.6). Thus, it suffices to show that $\mathbb{Z}(n)^{\mathrm{syn}}(-)/p$ is finitary noting that the cdh sheafification of a finitary presheaf is finitary. By (EM23, Lem.4.16), $\mathbb{Z}(n)^{\mathrm{syn}}(-)/p$ admits a finite increasing filtration whose graded pieces are some shifts of $\wedge^i L\Omega_{-/\mathbb{F}_p}$ with $i \leq n$, so the desired assertion follows from the finitarity of $\wedge^i L\Omega_{-/\mathbb{F}_p}$.

(BB)$_{\geq -n}$ By the finitarity and Theorem 3.10, it suffices to show that $\mathrm{Nil}\,\mathbb{Z}(n)^{\mathrm{syn}}(R) \in D(\mathbb{Z}_p))^{\leq n}$ for any henselian local ring R such that the ideal $\mathfrak{N} \subset R$ of nilpotent elements is finitely generated and $R/\mathfrak{N}$ is a valuation ring. Similarly to (36), we have

$$a_{\mathrm{cdh}}\mathbb{Z}(n)^{\mathrm{syn}}(R) = a_{\mathrm{cdh}}\mathbb{Z}(n)^{\mathrm{syn}}(R/\mathfrak{N}) = \mathbb{Z}(n)^{\mathrm{syn}}(R/\mathfrak{N}).$$

Note that $\mathbb{Z}(n)^{\mathrm{syn}}$ and $a_{\mathrm{cdh}}\mathbb{Z}(n)^{\mathrm{syn}}$ are not finitary in general so the first equality requires the assumption that $\mathfrak{N}$ is finitely generated. Hence, (BB)$_{\geq -n}$ follows from (AMMN22, Th.5.2) noting $(R, \mathfrak{N})$ is a henselian pair. This completes the proof of the claim.

□

Proposition 8.8. *The presheaf* $\mathrm{Nil}\,\mathrm{TC} \in \mathrm{PSh}(\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}_p}, \mathrm{Spt})$ *satisfies (Desc), (Fin), and (BB)*$_{\geq 0}$. *Consequently, for any Noetherian* $\mathbb{F}_p$*-scheme* X *with* $\dim(X) < \infty$ *we have*

$$a_{\mathrm{procdh}}(\mathrm{Nil}\,\mathrm{TC})_{\geq 0}(X) \simeq \mathrm{Nil}\,\mathrm{TC}(X).$$

Proof. The desired properties follow from Lemma 8.7 by the same argument deducing Proposition 8.6 from Lemma 8.5: The property (Desc) follows from (Desc) for $\mathrm{Nil}\,\mathbb{Z}(n)^{\mathrm{syn}}$ by the filtration (37). The properties (Fin) and (BB)$_{\geq 0}$ for $\mathrm{Nil}\,\mathrm{TC}$ follow from (Fin) and (BB)$_{\geq j}$ for $\mathrm{Nil}\,\mathbb{Z}_p(-j)^{syn}$ by using the spectral sequence

$$E_2^{i,j} = H^{i-j}(\mathrm{Nil}\,\mathbb{Z}_p(-j)^{syn}(X)) \Rightarrow \pi_{-i-j}\,\mathrm{Nil}\,\mathrm{TC}(X),$$

arising from the filtration (37). Note that the spectral sequence is bounded since Corollary 7.10 and (BB)$_{\geq j}$ for $\mathrm{Nil}\,\mathbb{Z}_p(-j)^{syn}$ from Lemma 8.7 imply $E_2^{i,j} = 0$ for $i - j > 2d - j$. □

9 Procdh motivic complex

In this section, we address the following conjecture.

Conjecture 9.1 (Beilinson, (Beĭ87), cf. also (EM23, Introduction))**.** For any reasonable scheme X, there is a natural spectral sequence:

$$E_2^{p,q} = H_{\mathcal{M}}^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X) \tag{39}$$

where $K_*(X)$ is the non-connective algebraic K-theory of X, and $H^i_{\mathcal{M}}(X, \mathbb{Z}(n))$ is the *motivic cohomology* of X yet to be defined.

When X is smooth over a field k, an answer was given by the following theorem.

Theorem 9.2 ((FS02) and (Lev08))**.** *Let* Sm_k *denote the category of smooth schemes separated of finite type over a field k. There exists a complete multiplicative decreasing* $\mathbb{N}$*-indexed filtration* $\left\{Fil^n_{\mathrm{mot}}K(X)\right\}_{n\in\mathbb{N}}$ *on* $K(X)$ *and identifications of spectra*

$$\mathrm{gr}^n_{Fil_{\mathrm{mot}}} K(X) \simeq \mathbb{Z}(n)^{\mathrm{sm}}(X)[2n] \tag{40}$$

functorial in $X \in \mathrm{Sm}_k$*.*

Above, $\mathbb{Z}(n)^{\mathrm{sm}}(X)[2n]$ is a chain complex of abelian groups regarded as a spectrum via the Eilenberg-Maclane functor $D(\mathbb{Z}) \to \mathrm{Spt}$. As a complex of abelian groups, it can be defined as

$$\mathbb{Z}(n)^{\mathrm{sm}}(X) = \underline{C}_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}))(X)[-n] \quad \text{for } X \in \mathrm{Sm}_k, \tag{41}$$

where $\underline{C}_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}))[-q]$ is Voevodsky's $\mathbb{A}^1$-invariant motivic complex defined in (SV00). This is strictly functorial in Sm_k in the sense that it defines a functor between the 1-category Sm_k and the 1-category of chain complexes of abelian groups. Scheme-wise this is shown to be quasi-isomorphic to Bloch's cycle complex, (Blo86), in (Voe02, Cor.2). Of course, another approach is to just take Eq.(40) as the definition for $\mathbb{Z}(n)^{\mathrm{sm}}(X)[2n]$.

In what follows, we write $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$ and continue to let $\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}$ be the category of qcqs schemes over $\mathbb{F}$ and $\mathrm{Sch}^{\mathrm{noe}}_{\mathbb{F}}$ be its full subcategory of noetherian schemes. Recently, Elmanto-Morrow (EM23) extended Theorem 9.2 to all $X \in \mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}$ by using instead of $\mathbb{Z}(n)^{\mathrm{sm}}$ a new motivic complex

$$\mathbb{Z}(n)^{\mathrm{EM}} \in \mathrm{PSh}(\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}, D(\mathbb{Z}))).$$

They construct $\mathbb{Z}(n)^{\mathrm{EM}}$ by modifying the cdh sheafification $\mathbb{Z}(n)^{\mathrm{cdh}}$ of the left Kan extension of $\mathbb{Z}(n)^{\mathrm{sm}}$ along $\mathrm{Sm}_{\mathbb{F}} \to \mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}$ by using Hodge-completed derived de Rham complexes in case $\mathbb{F} = \mathbb{Q}$ and syntomic complexes in case $\mathbb{F} = \mathbb{F}_p$. The construction is motivated by trace methods in algebraic K-theory using the cyclotomic trace map tr from (38). The purpose of this section is to give a different approach to Conjecture 9.1 by using our procdh topology.

Definition 9.3. For integers $n \geq 0$, we define the *procdh-local motivic complex*

$$\mathbb{Z}(n)^{\mathrm{procdh}} := a_{\mathrm{procdh}} L^{\mathrm{sm}}\mathbb{Z}(n)^{\mathrm{sm}} \in \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}, D\mathbb{Z})),$$

as the procdh sheafification of the left Kan extension $L^{\mathrm{sm}}\mathbb{Z}(n)^{\mathrm{sm}}$ of $\mathbb{Z}(n)^{\mathrm{sm}}$ along $\mathrm{Sm}_{\mathbb{F}} \to \mathrm{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}$.

Remark 9.4. Note that there have been various constructions of motivic cohomology on smooth schemes over Dedekind domains, (Gei04), (Lev01), (Spi18), (CD19), etc., and one could apply the same construction, i.e., procdh sheafification of left Kan extension, to any of these.

Theorem 9.5. *For $X \in \mathrm{Sch}_{\mathbb{F}}^{\mathrm{noe}}$ with $\dim(X) < \infty$, there exists a complete multiplicative*[21] *decreasing $\mathbb{N}$-indexed filtration $\left\{ Fil^n_{\mathrm{procdh}} K(X) \right\}_{n \in \mathbb{N}}$ on $K(X)$ and identifications*

$$\mathrm{gr}^n_{Fil_{\mathrm{procdh}}} K(X) \simeq \mathbb{Z}(n)^{\mathrm{procdh}}(X)[2n].$$

Proof. By Bhatt-Lurie (see (EHK+20, Ex. A.0.6)), there is a natural equivalence

$$K_{\geq 0} \simeq L^{\mathrm{sm}}(K_{|\mathrm{Sm}_{\mathbb{F}}}),$$

where the right hand side is the left Kan extension of $K_{|\mathrm{Sm}_{\mathbb{F}}}$ along $\mathrm{Sm}_{\mathbb{F}} \to \mathrm{Sch}_{\mathbb{F}}^{\mathrm{qcqs}}$. Using Theorem 8.1, we obtain a filtration on K by procdh sheafifying the left Kan extension along $\mathrm{Sm}_{\mathbb{F}} \to \mathrm{Sch}_{\mathbb{F}}$ of $Fil^{\bullet}_{\mathrm{mot}} K_{|\mathrm{Sm}_{\mathbb{F}}}$ from Theorem 9.2. The identification of the graded pieces is clear. Completeness follows from the boundedness described in Eq.(44) below. ☐

Remark 9.6. By definition, Eq.(41), we have $\mathcal{H}^i_{\mathrm{Zar}}(\mathbb{Z}(n)^{\mathrm{sm}}) = 0$ for $i > n$, where $\mathcal{H}^i_{\mathrm{Zar}}(\mathbb{Z}(n)^{\mathrm{sm}})$ is the Zariski cohomology sheaf of $\mathbb{Z}(n)^{\mathrm{sm}}$. It implies that for any local k-algebra A, we have

$$H^i((L^{\mathrm{sm}}\mathbb{Z}(n)^{\mathrm{sm}})(A)) = 0 \quad \text{for } i > n. \tag{42}$$

Thus, Corollary 7.10 implies that for $X \in \mathrm{Sch}_{\mathbb{F}}^{\mathrm{qcqs}}$ of finite valuative dimension d with Noetherian underlying topological space, we have

$$H^i(\mathbb{Z}(n)^{\mathrm{procdh}}(X)) = 0 \quad \text{for } i > 2d + n. \tag{43}$$

In particular, when X is noetherian, we have for each $n \in \mathbb{N}$

$$\pi_{-i} Fil^n_{\mathrm{procdh}} K(X) = 0; \qquad i > 2d - n \tag{44}$$

so the induced spectral sequence (39) with

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) := H^i(\mathbb{Z}(n)^{\mathrm{procdh}}(X))$$

is bounded.

Now, it is a natural question if two constructions $\mathbb{Z}(n)^{\mathrm{EM}}$ and $\mathbb{Z}(n)^{\mathrm{procdh}}$ coincide.

Theorem 9.7. *Assume given $\mathcal{Z}(n) \in \mathrm{PSh}(\mathrm{Sch}_{\mathbb{F}}^{\mathrm{qcqs}}, D(\mathbb{Z})))$ and consider the following conditions.*

(a) There is a natural comparison map

$$\psi : \mathbb{Z}(n)^{\mathrm{sm}} \to \mathcal{Z}(n)_{|\mathrm{Sm}_{\mathbb{F}}}$$

in $\mathrm{PSh}(\mathrm{Sm}_{\mathbb{F}}, D(\mathbb{Z})))$, whose induced map $\phi : L^{\mathrm{sm}}\mathbb{Z}(n)^{\mathrm{sm}} \to \mathcal{Z}(n)$ has the properties that $\phi(R)$ is an equivalence for all procdh local rings R with $\mathrm{length}\, Q(R)$ and $\dim R$ finite.

(b) $\mathcal{Z}(n)$ is finitary. That is,

$$\mathcal{Z}(n)(\lim_{\lambda} P_\lambda) = \operatorname*{colim}_{\lambda} \mathcal{Z}(n)(P_\lambda)$$

for any cofiltered system $(P_\lambda)_{\lambda \in \Lambda}$ in $\mathrm{Sch}_{\mathbb{F}}^{\mathrm{qcqs}}$ with affine transition morphisms.

[21] For compatibility of the monoidal structures with left Kan extension see (EM23, §2.3). Sheafification is also compatible since it is defined using filtered colimits, see (Lur09, Prop.6.2.2.7) which works for a very general class of coefficient categories.

(c) $\mathcal{Z}(n)$ is a procdh sheaf on Noetherian schemes. That is,

$$\mathcal{Z}(n) \in \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_{\mathbb{F}}^{\mathrm{noe}}, D(\mathbb{Z}))).$$

If (a), (b), and (c) are satisfied then, ϕ induces an equivalence $\mathbb{Z}(n)^{\mathrm{procdh}}(X) \simeq \mathcal{Z}(n)(X)$ for any $X \in \mathrm{Sch}_{\mathbb{F}}^{\mathrm{noe}}$.

Proof. For procdh sheaves of chain complexes $\mathcal{E} \in \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_{\mathbb{F}}^{\mathrm{noe}}, D(\mathbb{Z}))$, we have a descent spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{procdh}}(X, \mathcal{H}^q_{\mathrm{procdh}}\mathcal{E}) \Rightarrow H^{p+q}\mathcal{E}(X),$$

which converges by finiteness of cohomological dimension, Cor.7.10. Here $\mathcal{H}^q_{\mathrm{procdh}}\mathcal{E}$ is the procdh sheafification of the presheaf of abelian groups $X \mapsto \pi_{-q}\mathcal{E}(X)$. So since $\mathcal{Z}(n)$ and $\mathbb{Z}(n)^{\mathrm{procdh}}$ are procdh sheaves, Assumption (c), it suffices to show that the morphism

$$\mathcal{H}^q_{\mathrm{procdh}}\mathbb{Z}(n)^{\mathrm{procdh}} \to \mathcal{H}^q_{\mathrm{procdh}}\mathcal{Z}(n)$$

of procdh cohomology sheaves of abelian groups is an isomorphism on Noetherian schemes. Since the 1-topos $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_X)$ has enough points, Thm.5.2, and cohomology commutes with filtered colimits, and both $\mathcal{Z}(n)$ and $L^{\mathrm{sm}}\mathbb{Z}(n)^{\mathrm{sm}}$ are finitary, Assumption (b), it suffices to show that for all procdh local rings $\mathrm{Spec}(R) \to X$ with R as in Assumption (a), the comparison

$$H^q L^{\mathrm{sm}}\mathbb{Z}(n)^{\mathrm{sm}}(R) \to H^q\mathcal{Z}(n)(R)$$

is an isomorphism for all $q \in \mathbb{Z}$. This is precisely Assumption (a). $\square$

Theorem 9.8 (Elmanto-Morrow)**.** *The presheaf $\mathbb{Z}(n)^{\mathrm{EM}}$ satisfies the conditions (a), (b), (c) of Theorem 9.7.*

Proof. Condition (b), i.e., finitary-ness, is (EM23, Th.4.10(5), Th.4.24(4)) and Condition (c), i.e., procdh descent, is (EM23, Th.8.2).

We prove (a). Existence of the morphism $\mathbb{Z}(n)^{\mathrm{sm}} \to \mathcal{Z}(n)_{|\mathrm{Sm}_{\mathbb{F}}}$ is contained in (EM23, Th.1.1(9)), although we don't need the full force of their theorem since we are only asking for existence, not the stronger fact that the morphism is an equivalence. Now we want to show that the induced map $L^{\mathrm{sm}}\mathbb{Z}(n)^{\mathrm{sm}} \to \mathcal{Z}(n)$ is an equivalence on procdh local rings R with $\mathrm{length}\, Q(R)$ and $\dim R$ finite.

By (EM23, Th.7.7), it suffices to prove $\mathbb{Z}(n)^{\mathrm{EM}}(R)$ is supported in cohomological degrees $\leq n$ for any procdh local ring R. By (EM23, Th.4.10(2), Th.4.24.(2)), there are fiber sequences (cf. Eq.(29))

$$\mathrm{Nil}\,\widehat{L\Omega}^{\geq n}_{R/\mathbb{Q}} \to \mathbb{Z}(n)^{\mathrm{EM}}(R) \to \mathbb{Z}(n)^{\mathrm{cdh}}(R) \quad \text{if } \mathbb{F} = \mathbb{Q},$$

$$\mathrm{Nil}\,\mathbb{Z}(n)^{\mathrm{syn}}(R) \to \mathbb{Z}(n)^{\mathrm{EM}}(R) \to \mathbb{Z}(n)^{\mathrm{cdh}}(R) \quad \text{if } \mathbb{F} = \mathbb{F}_p,$$

where $\mathbb{Z}(n)^{\mathrm{cdh}} = a_{\mathrm{cdh}}L^{\mathrm{sm}}\mathbb{Z}(n)^{\mathrm{sm}}$. Note that $\mathrm{Nil}\,\widehat{L\Omega}^{\geq n}_{R/\mathbb{Q}}$ and $\mathrm{Nil}\,\mathbb{Z}(n)^{\mathrm{syn}}(R)$ are supported in degrees $\leq n$ by Lemma 8.5 and Lemma 8.7. Noting that $\mathbb{Z}(n)^{\mathrm{sm}}$ is finitary, the same argument as Lemma 8.7 gives

$$\mathbb{Z}(n)^{\mathrm{cdh}}(R) = \mathbb{Z}(n)^{\mathrm{cdh}}(R/\mathfrak{N}) = (L^{\mathrm{sm}}\mathbb{Z}(n)^{\mathrm{sm}})(R/\mathfrak{N}).$$

So, $\mathbb{Z}(n)^{\mathrm{cdh}}(R)$ is supported in degrees $\leq n$ by Eq.(42), which proves the desired assertion. $\square$

Corollary 9.9. *There is an equivalence $\mathbb{Z}(n)^{\mathrm{procdh}} \simeq \mathbb{Z}(n)^{\mathrm{EM}}$ in $\mathrm{PSh}(\mathrm{Sch}_{\mathbb{F}}^{\mathrm{noe}}, D(\mathbb{Z})))$.*

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