Cohomological Hasse principle

and applications

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Lecture note available at

http://www.lcv.ne.jp/ smaki/en/index.html

Plan of talk

Part I: Cohomological Hasse principle

Part II: Motivic cohomology

Part III: Special values of zeta function

Part IV: Higher Class field theory

Part V: Resolution of singularities

Cohomological Hasse principle

Hasse principle (Hasse-Minkowski)

$$a_1X_1^2 + \cdots a_nX_n^2 = 0 \quad (a_1, \ldots, a_n \in \mathbb{Q})$$

$\exists$solution in $\mathbb{Q}$

$\Leftrightarrow$ $\exists$solution in $\mathbb{R}$ and $\mathbb{Q}_p$ for $\forall p$

$n = 3$ is the most essential case

Cohomological interpretation

k: field

$$X^2 - aY^2 - bZ^2 = 0 \quad (a, b \in k^\times)$$

$\exists$solution in $k \Leftrightarrow \{a, b\} = 0 \in H^2(k, \mathbb{Z}/2\mathbb{Z})$

$H^*(k, \mathbb{Z}/2\mathbb{Z})$ Galois cohomology of k

Hasse principle ⇔ injectivity of

$$H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}) \to \bigoplus_{p \text{ prime}} H^2(\mathbb{Q}_p, \mathbb{Z}/2\mathbb{Z}) \oplus H^2(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$$

residue **isomorphism**

$$\partial_p : H^2(\mathbb{Q}_p, \mathbb{Z}/2\mathbb{Z}) \simeq H^1(\mathbb{F}_p, \mathbb{Z}/2\mathbb{Z})$$

Hasse principle ⇔ injectivity of

$$H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\partial} \bigoplus_{p \text{ prime}} H^1(\mathbb{F}_p, \mathbb{Z}/2\mathbb{Z}) \oplus H^2(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$$

Generalization

X = Spec($\mathscr{O}_K$) ($[K:\mathbb{Q}] < \infty$), or

X smooth proper curve/$\mathbb{F}_p$, $K = \mathbb{F}_p(X)$

Assume $\nexists$Spec($\mathbb{R}$) $\to X$ for simplicity

$$H^2(K, \mathbb{Z}/n\mathbb{Z}(1)) \xrightarrow{\partial} \bigoplus_{x \in X_{(0)}} H^1(x, \mathbb{Z}/n\mathbb{Z})$$

$X_{(0)}$: set of the closed points of X

$\mathbb{Z}/n\mathbb{Z}(1) = \mu_n$ (if ch(K) $\nmid n$)

Remark $H^2(K, \mathbb{Z}/n\mathbb{Z}(1)) \simeq Br(K)[n]$

H.P. for Br(K) $\Leftrightarrow$ ∂ injective

Cohomological HP in higher dim

X of finite type over $\mathbb{F}_p$ or $\mathbb{Z}$

Kato complex $KC_\bullet(X, \mathbb{Z}/n\mathbb{Z}) =$

$$\cdots \to \bigoplus_{x \in X_{(a)}} H^{a+1}(x, \mathbb{Z}/n\mathbb{Z}(a))$$

$$\to \bigoplus_{x \in X_{(a-1)}} H^{a}(x, \mathbb{Z}/n\mathbb{Z}(a-1)) \to \cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots \to \bigoplus_{x \in X_{(1)}} H^{2}(x, \mathbb{Z}/n\mathbb{Z}(1))$$

$$\to \bigoplus_{x \in X_{(0)}} H^{1}(x, \mathbb{Z}/n\mathbb{Z})$$

$X_{(a)} = \{x \in X \mid \dim \overline{\{x\}} = a\}$

$x \in X_{(a)} \Leftrightarrow \mathrm{trdeg}_{\mathbb{F}_p}(\kappa(x)) = a,$ or
$\mathrm{trdeg}_{\mathbb{Q}}(\kappa(x)) = a - 1$

$\mathbb{Z}/n\mathbb{Z}(a) = \mu_n^{\otimes a}$ if $\mathrm{ch}(\kappa(x)) \not| n$

Kato homology

For $a \in \mathbb{Z}$,

$$\boldsymbol{KH_a(X, \mathbb{Z}/n\mathbb{Z}) = H_a\big(KC_\bullet(X, \mathbb{Z}/n\mathbb{Z})\big)}$$

Conjecture

(Kato's H.P.)

X: smooth proper over $\mathbb{F}_p$, or

X: regular proper flat over $\mathbb{Z}$

Assume $X(\mathbb{R}) = \emptyset$ for simplicity

$\Rightarrow \quad KH_a(X, \mathbb{Z}/n\mathbb{Z}) = 0 \quad$ for $a \neq 0$

Known cases of Kato's H.P.

- $\dim(X) = 1$: Classical

- $\dim(X) = 2$: Kato (1986)

- $KH_a(X, \mathbb{Q}/\mathbb{Z}) = 0$ for $0 < a \leq 3$

  under some mild assumptions

$$KH_a(X, \mathbb{Q}/\mathbb{Z}) = \varinjlim_{n} KH_a(X, \mathbb{Z}/n\mathbb{Z})$$

Colliot-Thélène (1993), Suwa (1995)

Jannsen-S. (2003)

Theorem (Jannsen-S. (2006))

X: smooth projective over $\mathbb{F}_p$

$\Rightarrow \quad KH_a(X, \mathbb{Z}/n\mathbb{Z}) = 0 \quad$ for $a \neq 0$

if $a \leq 4$, or

if admit resolution of singularities

Theorem (Kerz-S. (2009))

X: smooth proper over $\mathbb{F}_p$

$\Rightarrow \quad KH_a(X, \mathbb{Z}/n\mathbb{Z}) = 0 \quad$ for $a \neq 0$

if n is prime to p

Key ingredients in the proof

- Weil conjecture (Deligne's theorem)
- (Refined) alteration (de Jong, Gabber)

Applications

Class number formula

$[K:\mathbb{Q}] < \infty$

$\zeta_K(s)$: Dedekind zeta function of K

$$\zeta_K(0)^* := \lim_{s\to 0} \zeta_K(s)\cdot s^{-\rho_0}$$

$$\rho_0 := \mathrm{ord}_{s=0}\ \zeta_K(s) = \mathrm{rank}(\mathscr{O}_K^\times)$$

$$\zeta_K(0)^* = -\frac{|Cl(K)|}{|(\mathscr{O}_K^\times)_{\mathrm{tors}}|}\cdot R_K$$

$Cl(K)$: ideal class group of K

R_K : Dirichlet's regulator

CNF as arithmetic index theorem?

index (analytic invariant) $=$ characteristic class

(e.g. Euler-Poincaré char.)

Motivic cohomology

X: scheme $\rightsquigarrow H^i_M(X, \mathbb{Z}(r))$ $(i, r \in \mathbb{Z}_{\geq 0})$

For $X = \mathrm{Spec}(\mathcal{O}_K)$ $([K : \mathbb{Q}] < \infty)$,

$$Cl(K) = H^2_M(X, \mathbb{Z}(1))$$

$$\mathcal{O}_K^\times = H^1_M(X, \mathbb{Z}(1))$$

$\Rightarrow$ CNF reads

$$\zeta_K(0)^* = -\frac{|H^2_M(X, \mathbb{Z}(1))_{\mathrm{tors}}|}{|H^1_M(X, \mathbb{Z}(1))_{\mathrm{tors}}|} \cdot R_K$$

Construction of motivic (co)homology

k: field

$DM(k)$: category of *mixed motives*$/k$

equipped with functor

$$M : Sm/k \to DM(k);\ X \to M(X)$$

Sm/k: category of smooth schemes$/k$

Voevodsky (Hanamura, Levine)

Definition $X \in Sm/k$ and $i, r \in \mathbb{Z}_{\geq 0}$

$$H^i_M(X, \mathbb{Z}(r)) = \mathrm{Hom}_{DM(k)}(M(X), \mathbb{Z}(r)[i])$$

$$H_i^M(X, \mathbb{Z}(r)) = \mathrm{Hom}_{DM(k)}(\mathbb{Z}(r)[i], M(X))$$

X finite type over field, Dedekind ring

$\mathrm{CH}^r(X,q)$: Bloch's higher Chow group

$H_i^S(X,\mathbb{Z})$: Suslin homology

defined as homology of

cycle complexes

Algebraic analogue of singular complex

Use algebraic cycles on $X \times \Delta^\bullet$

$\Delta^n = \text{Spec}\,\mathbb{Z}[t_0,\ldots.t_n]/(\sum_{i=0}^n t_i - 1)$

Comparison theorem (Voevodsky)

X smooth over field

$$H_i^M(X,\mathbb{Z}(0)) = H_i^S(X,\mathbb{Z})$$

$$H_M^i(X,\mathbb{Z}(r)) = \mathrm{CH}^r(X,2r-i)$$

Kato's HP and motivic cohomology

Lemma X: regular of finite type over $\mathbb{F}_p$ or $\mathbb{Z}$, $d = \dim(X)$

$\exists$exact sequence:

$$KH_{2d-i+2}(X, \mathbb{Z}/n\mathbb{Z}) \to$$

$$H^i_M(X, \mathbb{Z}/n\mathbb{Z}(d)) \to H^i_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(d))$$

$$\to KH_{2d-i+1}(X, \mathbb{Z}/n\mathbb{Z})$$

$H^i_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(d))$ étale cohomology

Fact $\left|H^i_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(d))\right| < \infty$

Kato's HP provide affirmative result on

Finiteness conjecture of motivic coh

Conjecture

X: regular of finite type over $\mathbb{F}_p$ or $\mathbb{Z}$

$\Rightarrow \quad H^i_M(X, \mathbb{Z}(r))$ finitely generated

Generalization of finiteness result on

$$Cl(K), \quad \mathcal{O}_K^\times, \quad A(K)$$

($[K:\mathbb{Q}] < \infty$, A abelian variety)

(Minkowski, Dirichlet, Mordell-Weil)

Special values via motivic cohomology

X smooth projective over $\mathbb{F}_q$ $(q = p^n)$

$$\zeta(X,s) = \exp\Big(\sum_{n=1}^{\infty} \big|X(\mathbb{F}_{q^n})\big| \frac{q^{-ns}}{n}\Big)$$

For $r \in \mathbb{Z}$,

$$\zeta(X,r)^* := \lim_{s\to r} \zeta(X,s)\cdot(1-q^{r-s})^{\rho_r}$$

$$\rho_r := -\mathrm{ord}_{s=r}\zeta(X,s)$$

Fact (Grothendieck, Deligne, Milne)

- $\zeta(X,r)^* \in \mathbb{Q}^\times$
- $\zeta(X,r)^*$ expressed by using

$$H^i_{\text{ét}}(X,\widehat{\mathbb{Z}}(r)) := \varprojlim_n H^i_{\text{ét}}(X,\mathbb{Z}/n\mathbb{Z}(r))$$

Theorem (Kerz-S.)

X smooth projective over $\mathbb{F}_q$

$d = \dim(X)$

$\Rightarrow H^j_M(X, \mathbb{Z}(d))_{\mathrm{tors}}$ finite (mod p-torsion)

$$\zeta(X,0)^* = \prod_{j=0}^{2d} \left| H^j_M(X, \mathbb{Z}(d))_{\mathrm{tors}} \right|^{(-1)^j}$$

(up to a power of p)

$d \leq 4 \Rightarrow$ True for the p-part

Compared with CNF for $X = \mathrm{Spec}(\mathscr{O}_K)$

$$\zeta(X,0)^* = -\frac{\left| H^2_M(X, \mathbb{Z}(1))_{\mathrm{tors}} \right|}{\left| H^1_M(X, \mathbb{Z}(1))_{\mathrm{tors}} \right|} \cdot R_K$$

Higher class field theory

X regular of finite type over $\mathbb{F}_p$ or $\mathbb{Z}$

$\pi_1^{ab}(X)$ abelian fundamental group of X

$X_{(0)}$ set of closed points of X

$\sigma_x \in \pi_1^{ab}(X)$ Frobenius for $x \in X_{(0)}$

Problem

Determine all relations among σ_x

$$Z_0(X) = \bigoplus_{x \in X_{(0)}} \mathbb{Z} \text{ (group of 0-cycles on } X\text{)}$$

$$\rho_X : Z_0(X) \to \pi_1^{ab}(X)\ ;\ \sum_{x \in X_{(0)}} n_x[x] \to \prod_{x \in X_{(0)}} (\sigma_x)^{n_x}$$

Problem

Determine $\mathrm{Ker}(\rho_X)$

First answer by Kato-Saito (1985)

Theorem (Bloch, Kato-S.)

X regular proper over $\mathbb{F}_p$ or $\mathbb{Z}$

$$\Rightarrow \rho_X \;:\; \mathrm{CH}_0(X) \simeq \pi_1^{ab}(X)$$

$$\mathrm{CH}_0(X) = \mathrm{Coker}\Big(\bigoplus_{y\in X_{(1)}} \kappa(y)^\times \xrightarrow{\delta} \bigoplus_{x\in X_{(0)}} \mathbb{Z}\Big)$$

$X_{(1)}$ set of curves on X

Theorem (Kerz-Schmidt-Wiesend)

X regular of finite type over $\mathbb{Z}$ or $\mathbb{F}_p$

$(n, \mathrm{ch}(K)) = 1$ (K function field of X)

$$\mathrm{Coker}\Big(\bigoplus_{y\in X_{(1)}} \kappa(y)_\Sigma \xrightarrow{\delta} \bigoplus_{x\in X_{(0)}} \mathbb{Z}/n\mathbb{Z}\Big) \simeq \pi_1^{ab}(X)/n$$

$\kappa(y)_\Sigma \subset \kappa(y)^\times$ “congruence subgroup”

($\Sigma =$ “boundary” of X)

Motivic interpretation of CFT

Theorem (Schmidt, Schmidt-Spiess)

Under *tame condition,*

$$\mathrm{Coker}\Big(\bigoplus_{y\in X_{(1)}} \kappa(y)_\Sigma \xrightarrow{\delta} \bigoplus_{x\in X_{(0)}} \mathbb{Z}/n\mathbb{Z}\Big) \simeq H_0^S(X,\mathbb{Z}/n\mathbb{Z})$$

$$\Rightarrow \quad H_0^S(X,\mathbb{Z}/n\mathbb{Z}) \simeq \pi_1^{ab}(X)/n$$

$H_i^S(X,\mathbb{Z}/n\mathbb{Z})$ Suslin homology of X

Consequence of Kato's H.P.

Theorem (Kerz-S.)

X smooth over $\mathbb{F}_p$, $(n,p)=1$

$$\Rightarrow \quad H_i^S(X,\mathbb{Z}/n\mathbb{Z}) \simeq H_{\text{ét}}^{i+1}(X,\mathbb{Z}/n\mathbb{Z})^\vee$$

Remark $H^1_{\text{ét}}(X,\mathbb{Z}/n\mathbb{Z})^\vee \simeq \pi_1^{ab}(X)/n$

$\Rightarrow$ Case $i=0$ is previous theorem

Application to singularities

Quotient signgularities

$G \subset SL_n(\mathbb{C})$ finite group

G act on $X = \mathbb{A}^n_{\mathbb{C}}$

$\pi : Y \to X/G$: "good" resolution s.t.

$E = \pi^{-1}\big((X/G)_{sing}\big)_{red}$ SNCD on Y

Γ_E **configulation complex of E**

$\Gamma_E =$ simplicial complex s.t.

$$\{a\text{-simplices}\} = \pi_0(E^{(a)})$$

$$E^{(a)} = \coprod_{1 \le i_0 < i_1 < \cdots < i_a \le N} E_{i_0} \cap \cdots \cap E_{i_a}$$

$E_1, \ldots, E_N$ irreducible components of E

$\pi_0(-) =$ set of connected components

Case $n = 2$ (Klein singularity)

$0 \in X/G$ isolated singularity

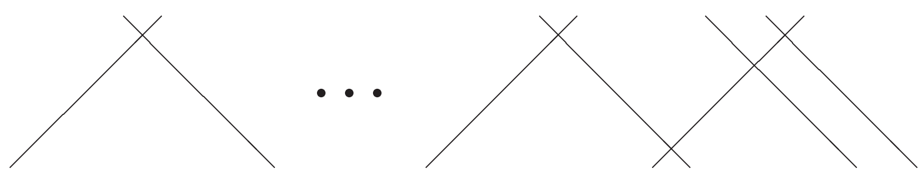
G classified

cyclic, binary dihedral, Platonic solid

Example (binary dihedral)

$G = < \sigma, \tau \mid \sigma^n = 1,\ \tau^2 = -1,\ \tau\sigma\tau = -\sigma^{-1} >$,

$\Rightarrow \quad \mathbb{A}^2_{\mathbb{C}}/G \simeq \text{Spec}\, \mathbb{C}[x, y, z]/(z^2 - yx^2 + 4y^{n+1})$



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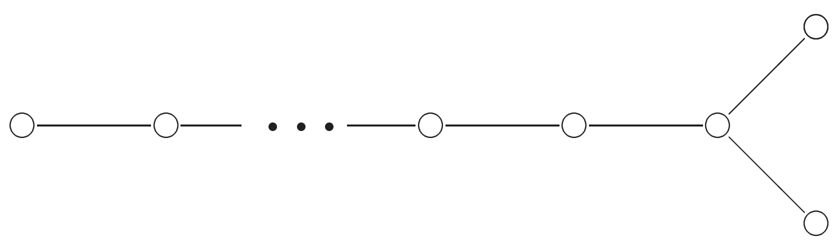

Γ_E

Observation (Mckay)

$\Gamma_E \simeq$ Mckay graph (quiver) of G

determined by irreducible rep'n of G

Case $n > 2$ No general theory exists to compute Γ_E (except toric case)

Question How about $H_a(\Gamma_E, \mathbb{Z})$?

$H_a(\Gamma_E, \mathbb{Z}) =$ homology of complex

$$\cdots \to \mathbb{Z}^{\pi_0(E^{(a)})} \xrightarrow{\partial} \mathbb{Z}^{\pi_0(E^{(a-1)})} \xrightarrow{\partial} \cdots$$
$$\cdots \xrightarrow{\partial} \mathbb{Z}^{\pi_0(E^{(0)})}$$

Theorem (Kerz-S.)

k: field, $p =$ exp. char. of k

X quasi-projective smooth over k

G finite group acting on X

$Z = (X/G)_{sing}$ Assume

(i) Z smooth proper$/k$ (e.g. isolated)

(ii) $\exists \pi : Y \to X/G$ proper birational s.t.

- Y smooth over k,
- $E := \pi^{-1}(Z)_{red}$ SNCD on Y
- $Y - E \simeq X - Z$

(iii) There exists $X \hookrightarrow \overline{X}$

G-equivariant smooth compactification

$\Rightarrow$ $H_a(\Gamma_E, \mathbb{Z}[\frac{1}{p}]) = 0$ for $a \neq 0$

Radicial singularity

k: field of $p = \mathrm{ch}(k) > 0$

For simplicity, assume k perfect

$S = k[x_1, \ldots, x_n] \supset \mathfrak{m} = (x_1, \ldots, x_n)$

$R = (S_{\mathfrak{m}})^{hen}$

$R = k\{x_1, \ldots, x_n\} \subset k[[x_1, \ldots, x_n]]$

For $\alpha \in \mathfrak{m}^2 R$ and $m \in \mathbb{Z}_{>0}$,

$X = \text{Spec } R[y]/(y^{p^m} - \alpha) \to \text{Spec } R$

(finite radicial surjective morphism)

Definition *radicial singularity of dim* n

$(X, 0_X)$ with $0_X = (\mathfrak{m}R, y) \in X$

(most important sing. in positive char)

Theorem (Kerz-S.) Assume

(i) $0_X = (\mathfrak{m}R, y) \in X$ isolated singularity

(ii) $\exists \pi : \widetilde{X} \to X$ proper birational s.t.

$\widetilde{X}$ regular, $\widetilde{X} - \pi^{-1}(0_X) \simeq X - \{0_X\}$,

$E := \pi^{-1}(0_X)_{red}$ SNCD on $\widetilde{X}$

(iii) $\exists$ cartesian diagram:

$$\begin{array}{ccccccc} \widetilde{X} & \xrightarrow{\pi} & X & \to & \text{Spec } R & \ni \mathfrak{m} \\ \downarrow & \Box & \downarrow & \Box & \downarrow \gamma & \\ \widetilde{V} & \to & V & \to & U & \ni 0_U \end{array}$$

$U, \widetilde{V}$ smooth projective over k,

V integral projective over k,

$R \simeq (\mathscr{O}_{U,0_U})^{hen}$ with $0_U = \gamma(\mathfrak{m})$

$$\Rightarrow \quad H_a(\Gamma_E, \mathbb{Z}[\frac{1}{p}]) = 0 \text{ for } a \neq 0$$

Remark Under assumption (i) & (ii), (iii) holds if one admits

(RS)$_n$: $\forall Z$ integral of dim$\leq n$ over k,

$\exists \rho : \widetilde{Z} \to Z$ proper birational s.t.

$\widetilde{Z}$ regular, ρ isomorphism over Z_{reg}

(Z_{reg}: regular locus of Z)

Definition

$(X, 0_X)$ *Medusa singularity of dim* n if

(i) and (ii) hold, and

$H_a(\Gamma_E, \mathbb{Z}[\frac{1}{p}]) \neq 0$ for some $a \neq 0$

Corollary

Medusa singularity of dim n exists

$\Rightarrow$ (RS)$_n$ FALSE

Definition

$(X, 0)$ *quasi Medusa singularity of dim* n **if**

(i) **and** (ii) **hold, and**

$H_a(\Gamma_E, \mathbb{Z}) \neq 0$ **for some** $a \neq 0$

Theorem

quasi Medusa singularity of dim n **exists**

$\Rightarrow$ **either (RS)**$_n$ **or (RS)**$_n^{emb}$ **FALSE**

(RS)$_n^{emb}$**:**

$\forall Z$ **regular of dim**$\leq n$ **over** k**,**

$\forall W \subsetneq Z$ **proper closed subscheme,**

$\exists \pi : \widetilde{Z} \to Z$**: proper birational s.t.**

$\widetilde{Z}$ **regular,** $\widetilde{Z} - \pi^{-1}(W) \simeq Z - W$

$\widetilde{W} = \pi^{-1}(W)_{red} \subset \widetilde{Z}$ **SNCD**

How Kato's HP related to $H_*(\Gamma_E, \mathbb{Z})$

$\mathcal{C} = \{$schemes of separated, f.t. over $k\}$

(1) May assume k finitely generated

(2) Define Kato homology

$$KH_a(X, \mathbb{Z}/n\mathbb{Z}) \text{ for } X \in \mathcal{C}$$

(extend Kato's definition in case $k = \mathbb{F}_p$)

(3) Kato's HP for extended KH_*

Theorem $KH_a(X, \mathbb{Z}/n\mathbb{Z}) = 0$ $(a \neq 0)$

if $X \in \mathcal{C}$ smooth proper and $(n, p) = 1$

(Proof use Weil conjecture by (1))

(4) *Descent property* for KH_*

For $X_\bullet \to X$ "hyperenvelope"

($X_\bullet$ simplicial object in $\mathcal{C}$)

$$E^1_{a,b} = KH_b(X_a, \mathbb{Z}/n\mathbb{Z}) \Rightarrow KH_{a+b}(X, \mathbb{Z}/n\mathbb{Z})$$

(5) HP and Descent for KH_* imply

$$KH_a(E, \mathbb{Z}/n\mathbb{Z}) \simeq H_a(\Gamma_E, \mathbb{Z}/n\mathbb{Z})$$

E SNCD on $Y \in \mathcal{C}$ regular,

E proper over k and $(n,p) = 1$

Theorem on radicial sing follow from

Theorem

$\pi : X' \to X$ finite radicial surjective

$\Rightarrow$ π induces isomorphism

$$KH_a(X', \mathbb{Z}/n\mathbb{Z}) \simeq KH_a(X, \mathbb{Z}/n\mathbb{Z})$$