Based on a joint work with Alberto Merici and Kay Rülling

TAME COHOMOLOGY AND ITS APPLICATIONS LECTURES AT INSTITUTE OF SCIENCE TOKYO IN NOV. 2025

SHUJI SAITO

Contents

Introduction

These lectures are based on a joint work with Alberto Merici and Kay Rülling.

Let K be a field and $k \subseteq K$ be the algebraic closure of its prime subfield. A consequence of the result of this paper is the following:

Theorem 0.1. *Let X/K be a smooth and proper variety and let $\varphi \in \mathrm{Aut}(X)$. Then for all $i \in \mathbb{N}$ the number* $\mathrm{dét}(\varphi^*|H^i(X, \mathcal{O}_X))$ *lies in* $k^\times$. *If* $\mathrm{ch}(K) = 0$, *it lies in* $\mathcal{O}_k^\times$, *where* $\mathcal{O}_k$ *is the integral closure of* $\mathbb{Z} \subseteq k$.

The assertion is reduced to the case K is finitely generated over the prime subfield so that k is a finite filed $\mathbb{F}_q$ or a number field. Then $\mathbb{F}_q$ and $\mathcal{O}_k$ are the integral closure of the image of the characteristic map $\mathbb{Z} \to K$, which in turns is the intersection of all the discrete valuation rings whose fraction field is K. Therefore, it is enough to show that for every discrete valuation ring $\mathcal{O}_K \subseteq K$ and every X smooth and proper over K, there exists an $\mathcal{O}_K$-lattice $W \subset H^i(X, \mathcal{O})$, which is preserved by φ^* for any $\varphi \in \mathrm{Aut}(X)$.

In case resolutions of singularities hold, we φ extends to a morphism of regular models $\mathcal{X}' \to \mathcal{X}$, and by results of Chatzistamatiou–Rülling ([?, Theorem 2] for $\mathrm{ch}(K) = p$ and [?, Theorem 1.1.] for $\mathrm{ch}(K) = 0$), we have that $H^i(\mathcal{X}, \mathcal{O}) \to H^i(\mathcal{X}', \mathcal{O})$ is an isomorphism so we can choose W as the image of $H^i(\mathcal{X}, \mathcal{O}) \to H^i(X, \mathcal{O})$, which implies the desired assertion.

Our aim is to provide an unconditional proof of a stronger result on the existence of a canonical integral structure on $H^i(X, \mathcal{O})$. A Key idea is to use rigid analytic geometry to by pass resolution of singularities.

Let K be a complete discrete valuation field with the ring $R = \mathcal{O}_K$ of integers and a prime element π. Let Sm_K be the category of smooth schemes separated of finite type over K and $\mathrm{PrSm}_K \subset \mathrm{Sm}_k$ be the full subcategory of proper K-schemes.

Let Mod_R be the abelian category of R-modules and Mod_R^0 (resp. Mod_R^f) be the Serre subcategory of Mod_R consisting of such M that annihilated by π^e for some $e > 0$ (resp. such M that $M_{\mathrm{tor}} \in \mathrm{Mod}_R^0$ and M/M_{tor} is finite over R). Let $\mathcal{D}(R)$ be the derived (or ∞) category of complexes of R-modules and $\mathcal{D}(R)^f$ be its full subcategory of consisting of complexes whose cohomology groups are in Mod_R^f.

In the following theorem, we consider $\mathbb{Q}$ as a category by the total order.

Theorem 0.2. *There is an object of* $\mathrm{Fun}(\mathbb{Q}^{op} \times \mathrm{PrSm}_K^{op}, \mathcal{D}(R)^f)$*:*

$$r \in \mathbb{Q}^{op} \to \mathcal{F}(r) \in \mathrm{Fun}(\mathrm{PrSm}_K^{op}, \mathcal{D}(R)^f) \tag{0.2.1}$$

equipped with natural equivalences in $\mathrm{Fun}(\mathrm{PrSm}_K^{op}, \mathcal{D}(K))$

$$\varinjlim_{s \to -\infty} \mathcal{F}(s) \simeq R\Gamma(-, \mathcal{O}) \simeq \mathcal{F}(r) \otimes_R K,$$

where the latter equivalence is compatible with transition maps $\mathcal{F}(r) \to \mathcal{F}(r')$ *for* $r > r'$.

If $\mathrm{ch}(K) = 0$[1]*, the above functors extend to an object of* $\mathrm{Fun}(\mathbb{Q}^{op} \times \mathrm{Sm}_K^{op}, \mathcal{D}(R)^f)$*:*

$$r \in \mathbb{Q}^{op} \to \mathcal{F}(r) \in \mathrm{Fun}(\mathrm{Sm}_K^{op}, \mathcal{D}(R)^f) \tag{0.2.2}$$

enjoying the following properties:

(i) For $X \in \mathrm{Sm}_K$*, we have an equivalence*

$$\varinjlim_{s \to -\infty} \mathcal{F}(s)(X) \simeq R\Gamma(\overline{X}, \mathcal{O}) \simeq \mathcal{F}(r)(X) \otimes_R K,$$

for every smooth compactification $\overline{X}$ *of* X *over* K*, where the latter equivalence is compatible with transition maps* $\mathcal{F}(r) \to \mathcal{F}(r')$ *for* $r > r'$*. For a map* $\overline{X}' \to \overline{X}$ *of such compactifcations, the above equivalences are compatible in an obvious sense.*

(ii) ($\mathbf{A}^1$*-invariance)* $\mathcal{F}(r)(X) \simeq \mathcal{F}(r)(X \times_K \mathbf{A}^1_K)$ *for* $X \in \mathrm{Sm}_K$.

(iii) (Birational invariance) $\mathcal{F}(r)(X) \simeq \mathcal{F}(r)(U)$ *for any dense open immersion* $U \hookrightarrow X$ *in* Sm_K.

[1] But the characteristic of the residue field of $\mathcal{O}_K$ may be positive.

(iv) *(Tame descent) For $X \in \mathrm{Sm}_K$ and $Y \to X$ an $\mathcal{O}_K$-tame covering in the sense of* [17], *we have an equivalence*

$$\mathcal{F}(r)(X) \simeq \varprojlim_{\Delta} \Big(\mathcal{F}(r)(Y) \rightrightarrows \mathcal{F}(r)(Y \times_X Y) \to \mathcal{F}(r)(Y \times_X Y \times_X Y) \cdots \Big)$$

and a descent spectral sequence

$$E_1^{pq} = H^q(\mathcal{F}(r)(Y^{\times_X (q+1)})) \Rightarrow H^{p+q}(\mathcal{F}(r)).$$

Here, a morphism of schemes $f : Y \to X$ is an $\mathcal{O}_K$-tame covering if it is an étale covering and for any $x \in X$ and a valuation v on $\kappa(x)$ trivial over $\mathcal{O}_K$, there exists $y \in Y$ lying over x and a valuation w on $\kappa(y)$ extending v such that $\mathcal{O}_w/\mathcal{O}_v$ is tame, i.e. $[\mathrm{Frac}(\mathcal{O}_w^{sh}) : \mathrm{Frac}(\mathcal{O}_v^{sh})]$ is prime to the exponential characteristic of the residue field of $\mathcal{O}_v$, where $(-)^{sh}$ denotes the strict henselization.

To construct such $\mathcal{F}(r)$, we introduce a variant of the tame topology defined by Hübner and Schmidt [17]. For a scheme S, let $\mathbf{Sch}_S$ be the category of schemes separated of finite type over S. For a morphism $U \to \tilde{U}$ of schems, let $\mathrm{Spa}(U, \tilde{U})$ be the set of triples (x, v, ε) such that $x \in U$, v is a valuation on $\kappa(x)$ and $\varepsilon\colon \mathrm{Spec}(\mathcal{O}_v) \to \tilde{U}$ is a map compatible with $\mathrm{Spec}(\kappa(x)) \to X$ (see Definition 5.1).

Definition 0.3. Let K be a complete discrete valuation field with the ring $R = \mathcal{O}_K$ of integers and put $S = \mathrm{Spec}(\mathcal{O}_K)$ and $\eta = \mathrm{Spec}(K)$. Let $\mathbf{Sch}_{(\eta,S)}$ be the category whose objects are pairs $(U, \tilde{U})$ equipped with an open immersion $U \hookrightarrow \tilde{U}$ over S such that $U \to S$ factors through $\eta \hookrightarrow S$. Morphisms $(V, \tilde{V}) \to (U, \tilde{U})$ are pairs of morphisms $f : V \to U$ in $\mathbf{Sch}_K = \mathbf{Sch}_\eta$ and $\tilde{f} : \tilde{V} \to \tilde{U}$ in $\mathbf{Sch}_S$ satisfying the obvious compatibility..

The *tame* topology on $\mathbf{Sch}_{(\eta,S)}$ is generated by a family $\{(f_i, \tilde{f}_i) : (V_i, \tilde{V}_i) \to (U, \tilde{U})\}_i$ of maps such that for every $(x, v, \varepsilon_v) \in \mathrm{Spa}(U, \tilde{U})$, there is $i \in I$ and $(y, w, \varepsilon_w) \in \mathrm{Spa}(V_i, \tilde{V}_i)$ such that $f_i(y) = x$, $w_{|k(x)} = v$, and $\mathcal{O}_w/\mathcal{O}_v$ is tame and the following diagram commutes:

$$\begin{tikzcd} \mathrm{Spec}(\mathcal{O}_w) \arrow[r, "\varepsilon_w"] \arrow[d] & \tilde{V}_i \arrow[d, "\tilde{f}_i"] \\ \mathrm{Spec}(\mathcal{O}_v) \arrow[r, "\varepsilon_v"] & \tilde{U}. \end{tikzcd}$$

The corresponding sites are denoted by $\mathbf{Sch}_{(\eta,S),t}$. For $(X, \tilde{X}) \in \mathbf{Sch}_{(\eta,S)}$, let $(X, \tilde{X})_t$ be the site whose underlying category is the category of objects $(U, \tilde{U})$ over $(X, \tilde{X})$ with $U \to X$ étale, endowed with the induced topology.

For $X \in \mathbf{Sch}_K$, choose a Nagata compactification $X \hookrightarrow \tilde{X}$ of $X \to S$ and define

$$R\Gamma_t(X/\mathcal{O}_K, F) = R\Gamma((X, \tilde{X})_t, F_{|(X,\tilde{X})_t}) \text{ for } F \in \mathbf{Shv}(\mathbf{Sch}_{(\eta,S),t}).$$

We prove that $R\Gamma_t(X/\mathcal{O}_K, F)$ does not depend on the choice of $\tilde{X}$ and extends to a functor (Lemma 6.7)

$$R\Gamma_t(-/\mathcal{O}_K, -) : (\mathbf{Sch}_K)^{op} \times \mathbf{Shv}(\mathbf{Sch}_{(\eta,S),t}) \to \mathcal{D}(\mathbb{Z}).$$

We also prove that the following persheaf on $\mathbf{Sch}_{(\eta,S)}$ belong to $\mathbf{Shv}(\mathbf{Sch}_{(\eta,S),t})$.

Example 0.4. (1) The presheaf $\mathcal{O}$ given by $\mathcal{O}(U, \tilde{U}) = \mathcal{O}(U)$.

(2) The presheaf $\mathcal{O}^t$ given by $\mathcal{O}^t(U, \tilde{U}) = \mathcal{O}(\tilde{U}^{\mathrm{int}})$, where $\tilde{U}^{\mathrm{int}}$ is the integral closure of $\tilde{U}$ in U (Lemma 7.6).

(3) For $r = n/m \in \mathbb{Q}$ with $m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$, the presheaf given by

$$\mathcal{O}^t(r)(U, \tilde{U}) = \{a \in \mathcal{O}(U) |\ a^m \in \pi^n \sqrt{\pi \mathcal{O}(\tilde{U}^{\mathrm{int}})}\}$$

(cf. Lemma ??). Note $\mathcal{O}^t(r) \subset \mathcal{O}^t(r') \subset \mathcal{O}$ for $r \geq r'$ and

$$\varinjlim_{s\to -\infty} \mathcal{O}^t(s) \simeq \mathcal{O} \simeq \mathcal{O}^t(r) \otimes_{\mathcal{O}_K} K.$$

(4) The presheaf $\Omega^{q,t}$ given by

$$\Omega^{q,t}(U,\tilde{U}) := \left\{ a \in \Gamma(U,\Omega^q_U) \,\middle|\, \begin{array}{l} \text{for all } (x,v,\varepsilon) \in \mathrm{Spa}(U,\tilde{U}) \text{ there exists} \\ \text{a finite tame extension } (L,w)/(k(x),v), \\ \text{such that } a_L \in \Omega^q_{\mathcal{O}_w}(\log) \subset \Omega^q_L \end{array} \right\},$$

where $a_L \in \Omega^q_L$ denotes the pullback of $a \in F(U)$ along $\mathrm{Spec}\, L \to \mathrm{Spec}\, k(x) \to U$ and $\Omega^q_{\mathcal{O}_w}(\log)$ is the degree q-part of the graded $\Omega^*_{\mathcal{O}_w}$-subalgebra of Ω^*_L generated by $\Omega^*_{\mathcal{O}_w}$ and $\mathrm{dlog}(L^\times)$ (see Lemma 7.5).

Definition 0.5. $\mathcal{F}(r) := R\Gamma_t(-/\mathcal{O}_K, \mathcal{O}^t(r)) \in \mathrm{Fun}((\mathbf{Sch}_K)^{op}, \mathcal{D}(\mathbb{Z}))$.

We will prove $\mathcal{F}(r)$ satisfies the properties of Theorem 0.2: The properties (i) and (iv) follows immediately from the definition. A key ingredient of the proof of (ii), (iii) and the fact that $\mathcal{F}(r)$ for $X \in \mathrm{Sm}_K$ takes values in $\mathcal{D}(R)^f$ is the following comparison theorem: Fixing $c \in (0,1)$, equip a norm $|-| = c^{v_K(-)}$ on K with v_K the normalized valuation of K.

Theorem 0.6. *(Theorem ??) Let $X \in \mathbf{Sch}_K$ and $X \hookrightarrow \tilde{X} \in \mathcal{N}(X/\mathcal{O}_K)$. Let $\mathfrak{X}^{\mathrm{rig}}$ be the rigid space over K associated to the formal completion $\widehat{\tilde{X}}$ of $\tilde{X}$ along the special fiber. If X is proper over K, there exists a canonical equivalence*

$$\mathcal{F}(r)(X) \simeq R\Gamma(\mathfrak{X}^{\mathrm{rig}}, \mathcal{O}(s)),$$

where $s = c^r$ and $\mathcal{O}(s)$ is a sheaf on $\mathfrak{X}^{\mathrm{rig}}$ given by $\mathcal{O}(s)(U) = \{f \in B \mid |f|_{\sup} < s\}$ for an affinoid subdomain $U = \mathrm{Sp}(B) \subset \mathfrak{X}^{\mathrm{rig}}$, where $|-|_{\sup}$ is the sup norm on B:

$$|f|_{\sup} = \sup_{x\in \mathrm{Sp}(B)} |f(x)|.$$

The same holds if $\mathrm{ch}(K) = 0$ and X is smooth (not necessarily proper) over K.

Theorem 0.6 in case X proper over K is a special case of the following more general statement, Theorem 0.7: For $(X,\tilde{X}) \in \mathbf{Sch}_{(\eta,S)}$ with $X = \tilde{X} \otimes_{\mathcal{O}_K} K$, we will introduce a subcategory $\mathbf{Shv}((X,\tilde{X})_t)_{\mathcal{O}^{int}-coh}$ of $\mathbf{Shv}((X,\tilde{X})_t)$ consisting of *coherent $\mathcal{O}^{\mathrm{int}}$-modules on* $(X,\tilde{X})_t$ and construct a functor (see (??))

$$\widehat{(-)}^{\mathrm{rig}} : \mathbf{Shv}((X,\tilde{X})_t)_{\mathcal{O}^{int}-coh} \to \mathbf{Shv}(\mathfrak{X}^{\mathrm{rig}}) \ : \ F \to \widehat{F}^{\mathrm{rig}}$$

such that $\widehat{\mathcal{O}^t(r)}^{\mathrm{rig}} = \mathcal{O}(s)$ with $s = c^r$. We have the following integral refinement of GAGA (cf. [10, II, 9.4.2]):

Theorem 0.7. *(Theorem ??) If $\tilde{X}$ is proper over $\mathcal{O}_K$, $\widehat{(-)}^{\mathrm{rig}}$ induces an equivalence*

$$R\Gamma((X,\tilde{X})_t, F) \simeq R\Gamma(\mathfrak{X}^{\mathrm{rig}}, \widehat{F}^{\mathrm{rig}}) \text{ for } F \in \mathbf{Shv}((X,\tilde{X})_t)_{\mathcal{O}^{int}-coh}.$$

(iii) of Theorem 0.2 follows immediately from Theorem 0.6 and (ii) follows from an equivalence

$$R\Gamma(\mathfrak{X}^{\mathrm{rig}}, \mathcal{O}(s)) \to R\Gamma(\mathfrak{X}^{\mathrm{rig}} \times \mathbf{P}^{1,\mathrm{rig}}_K, \mathcal{O}(s)),$$

where $\mathbf{P}^{1,\mathrm{rig}}_K$ is the rigid projective line over K, which follows from the following.

Lemma 0.8. *(Lemma ??) For a rigid space V and the projection $q : \mathbf{P}^{n,\mathrm{rig}} \times V \to V$, we have $R^i q_* \mathcal{O}(s) = 0$ for $i > 0$.*

Using the base change theorem in rigid geometry (see [9, Th.2.7.4]), the lemma is deduced from the following theorem due to Bartenwerfer [5, Theorem] and van der Put [26, Thm. 3.15]. Note that even though K is assumed to be a discrete valuation field in Theorem 0.2, we need the following theorems 0.9 and 0.10 for K with a non-archimedean norm which corresponds to a non-discrete valuation of rank one since we need consider stalks of sheaves over analytic points.

Theorem 0.9. *Let K be a non-archimedean field, i.e. a field which is complete with respect to a nontrivial non-archimedean absolute value $|-|: K \to \mathbb{R}_{\geq 0}$. For a generalized polydisk $D \subset \mathbb{B}^d_K = \mathrm{Sp}(K\langle z_1, \ldots, z_d\rangle)$, we have*

$$H^i(D, \mathcal{O}(r)) = 0 \text{ for all } r > 0 \text{ and integers } i > 0.$$

Again, using the base change theorem in rigid geometry, Theorem 0.9 is reduced to the case $d = 1$, which is proved by using the Mittag-Leffler decomposition of analytic functions on the unit disc $\mathbf{B}^1_K$: For $c_1, \ldots, c_m \in K$ and $r_1, \ldots, r_m \in \mathbb{R}_{>0}$, an analytic function f on $\mathbf{B}^1_K - \cup_{1\leq\nu\leq m}\{|z - c_\nu| < r_\nu\}$ is written as

$$f = g + \sum_{1\leq\nu\leq m}\sum_{i=1}^{\infty} \frac{a_{\nu,i}}{(z-c_\nu)^i} \text{ with } g \in K\langle z\rangle,\ a_{\nu,i} \in K.$$

Finally, the fact that $\mathcal{F}(r)$ for $X \in \mathrm{Sm}_K$ takes values in $\mathcal{D}(R)^f$ follows from the following theorem which was first shown by Bartenwerfer in [4], [5, Folgerung 3] (see [21, Th.18 and Cor.18]).

Theorem 0.10. *Let K be as in Theorem 0.9 and X be a smooth affinoid space over K. For $r \in \mathbb{R}_{>0}$, there exists $\pi \in F$ with $|\pi| < 1$ such that $\pi H^i(X, \mathcal{O}(r)) = 0$ for all $i > 0$.*

Now we explain the strategy of the proof of Theorems 0.6 and 0.7. A key ingredient is the following.

Theorem 0.11. *Let $X \hookrightarrow \tilde{X}$ be an open immersion of noetherian schemes. Let F be a sheaf of abelian groups on $(X, \tilde{X})_t$[2] such that the following condition is satisfied:*

(p) for every $(U, \tilde{U}) \in (X, \tilde{X})_t$ and $x \in \tilde{U}$, $F(\mathrm{Spec}(\mathcal{O}_{\tilde{U},x}) \times_{\tilde{U}} U, \mathrm{Spec}(\mathcal{O}_{\tilde{U},x}))$ is a $\mathbb{Z}_{(p_x)}$-module, where p_x is the exponential characteristic of $\kappa(x)$.

Then, we have a canonical equivalence

$$H^i((X, \tilde{X})_t, F) \cong \varinjlim_{(Y,\tilde{Y})\to(X,\tilde{X})} H^i_{\text{ét}}(\tilde{Y}, F_{\tilde{Y}_{\text{ét}}}), \quad i \geq 0,$$

where $F_{\tilde{Y}_{\text{ét}}}$ is the étale sheaf on $\tilde{Y}_{\text{ét}}$ given by $\tilde{V}/\tilde{Y} \mapsto F(Y \times_{\tilde{Y}} \tilde{V}, \tilde{V})$ and the colimit is indexed by the (cofiltered) category $\Lambda_{\tilde{X}}$ of modifications $(Y, \tilde{Y}) \to (X.\tilde{X})$.

For $F \in \mathbf{Shv}((X, \tilde{X})_t)_{\mathcal{O}^{int}-coh}$, Theorem 0.7 follows from a series of equivalences:

$$\begin{aligned} R\Gamma((X,\tilde{X})_t, F) &\overset{(*1)}{\simeq} \varinjlim_{(Y,\tilde{Y})\in\Lambda_{\tilde{X}}} R\Gamma_{\text{ét}}(\tilde{Y}, F_{\tilde{Y}_{\text{ét}}}) \overset{(*2)}{\simeq} \varinjlim_{(Y,\tilde{Y})\in\Lambda_{\tilde{X}}} R\Gamma_{\text{zar}}(\tilde{Y}, F_{\tilde{Y}_{\text{zar}}}) \\ &\overset{(*3)}{\simeq} \varinjlim_{(Y,\tilde{Y})\in\Lambda_{\tilde{X}}} R\Gamma_{\text{zar}}(\hat{\tilde{Y}}, \hat{F}_{\tilde{Y}}) \overset{(*4)}{\simeq} \varinjlim_{\mathfrak{Y}\to\hat{\tilde{X}}} R\Gamma_{\text{zar}}(\hat{\tilde{Y}}, \hat{F}_{\tilde{Y}}) \overset{(*5)}{\simeq} R\Gamma(\mathfrak{X}^{\text{rig}}, \hat{F}^{\text{rig}}) \end{aligned}$$

where $F_{\tilde{Y}_{\text{zar}}}$ is the Zariski sheaf on $\tilde{Y}$ defined by the same way as $F_{\tilde{Y}_{\text{ét}}}$ which is a coherent $\mathcal{O}_{\tilde{Y}}$-module by the definition of $\mathbf{Shv}((X, \tilde{X})_t)_{\mathcal{O}^{int}-coh}$, and $\hat{\tilde{Y}}$ (resp. $\hat{F}_{\tilde{Y}}$) is the formal completion of $\tilde{Y}$ (resp. $F_{\tilde{Y}_{\text{zar}}}$) along the special fiber and $\mathfrak{Y} \to \hat{\tilde{X}}$ range over all admissible

[2]This tame site is defined for any quasi-compact open immersion $X \hookrightarrow \tilde{X}$ of qcqs schemes not only for $(X, \tilde{X}) \in \mathbf{Sch}_{(\eta,S)}$ as in Definition 0.3 (see Definition 6.1).

blowup of $\widehat{\tilde{X}}$. $(*1)$ follows from Theorem 0.11, $(*2)$ from the étale (flat) descent for quasi-coherent sheaves on $\tilde{Y}_{\text{ét}}$ ([35, 03P2]), $(*3)$ from GAGF ([10, Ch.I Th.9.2.1]), $(*4)$ from the fact that by Raynaud-Gruson [35, Tag 081R], any modification $\tilde{Y} \to \tilde{X}$ is refined by an admissible blowup of $\tilde{X}$ which induces an admissible blowup of $\widehat{\tilde{X}}$ while any admissible blowup of $\widehat{\tilde{X}}$ is a base change of an admissible blowup of $\tilde{X}$, and $(*5)$ from natural equivalences of categories (see Theorem 2.32 and [10, Ch.0, 4.4.3])

$$\mathbf{Shv}(\mathfrak{X}^{\mathrm{rig}}) \simeq \mathbf{Shv}(\mathrm{RZ}(\widehat{\tilde{X}})) \simeq \varprojlim_{\mathfrak{Z}\to\widehat{\tilde{X}}} \mathbf{Shv}(\widehat{\tilde{Y}}_{\mathrm{zar}}).$$

.

Theorem 0.11 is proved using the following comparison of the tame cohomology with the Čech cohomology

Theorem 0.12. *Let F be a sheaf of abelian groups on $(X,\tilde{X})_t$, and assume that every finite set of points in $\tilde{X}$ is contained in an affine open. Then the natural map*

$$\check{H}^q((X,\tilde{X})_t, F) \to H^q((X,\tilde{X})_t, F).$$

is an isomorphism for all q, where the left hand side is the Čech cohomology.

The proof of Theorem 0.12 is similar to Artin's proof that the étale cohomology is computed as the Čech cohomology ([1]). For this, we need the following result describing the local rings of the tame topology: We let $\widetilde{(X,\tilde{X})}_\tau$ be the category of pairs $\mathcal{T} = (T,\tilde{T})$ of affine schemes such that that there exists a cofiltered system $\{\mathcal{T}_i = (T_i,\tilde{T}_i)\}_{i\in I}$ of affine objects of $(X,\tilde{X})_\tau$ such that $T = \varprojlim_{i\in I} T_i$ and $\tilde{T} = \varprojlim_{i\in I} \tilde{T}_i$. We say that a pair $\mathcal{T} = \varprojlim_{i\in I} \mathcal{T}_i \in \widetilde{(X,\tilde{X})}_\tau$ is tame local if for every tame covering $\mathcal{V} \xrightarrow{u} \mathcal{U}$ in $(X,\tilde{X})_t$, the morphism of sets

$$\varinjlim_{i\in I} \mathrm{Hom}_{(X,\tilde{X})_\tau}(\mathcal{T}_i,\mathcal{V}) \to \varinjlim_{i\in I} \mathrm{Hom}_{(X,\tilde{X})_\tau}(\mathcal{T}_i,\mathcal{U})$$

is surjective.

Proposition 0.13. *A pair $\mathcal{T} = \varprojlim_{i\in I} \mathcal{T}_i \in \widetilde{(X,\tilde{X})}_\tau$ is tame-local if and only if $\mathcal{T}$ is a coproduct of objects of the form $(\mathrm{Spec}(S), \mathrm{Spec}(\widetilde{S}))$ such that $\widetilde{S}$ is strictly henselian local and S is henselian local with $S = \widetilde{S}[1/f]$ for a non-zero divisor $f \in \widetilde{S}$, and that $\widetilde{S} = S \times_k \mathcal{O}_v$, where k is the residue field of S equipped with a valuation v such that (k,v) is tamely closed and $\mathcal{O}_v$ is its valuation ring.*

Part 1. Reviews on basic theories

1. Topos theory

1.1. Functoriality of presheaves. A functor $u : \mathcal{C} \to \mathcal{D}$ induces

$$u^p : \mathbf{PSh}(\mathcal{D}) \to \mathbf{PSh}(\mathcal{C})$$

given by $u^p F = F \circ u$, in other words $u^p F(V) = F(u(V))$ for $V \in \mathcal{C}$.

Proposition 1.1. *There exists a functor called the left Kan extension of F along u*

$$u_p : \mathbf{PSh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{D})$$

which is a left adjoint to the functor u^p. In other words

$$\mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(F, u^p G) = \mathrm{Hom}_{\mathbf{PSh}(\mathcal{D})}(u_p F, G)$$

holds bifunctorially in $F \in \mathbf{PSh}(\mathcal{C})$ and $G \in \mathbf{PSh}(\mathcal{D})$.

For $V \in \mathcal{D}$, let $I^u(V)$ denote the category whose objects are pairs (U, φ) with $U \in \mathcal{C}$ and $\varphi : V \to u(U)$ and

$$\mathrm{Hom}_{I^u(V)}((U,\varphi),(U',\varphi')) = \{f : U \to U' \text{ in } \mathcal{C} |\ u(f) \circ \varphi = \varphi'\}.$$

We sometimes drop the superscript u from the notation and we simply write $I(V)$. For $F \in \mathbf{PSh}(\mathcal{C})$, we define

$$u_p F(V) = \varinjlim_{(U,\varphi) \in I(V)^{op}} F(U) = \varinjlim_{I(V)^{op}} F_V,$$

where $F_V \in \mathbf{PSh}(I(V), \mathbf{Sets})$ given by

$$F_V : I(V)^{op} \to \mathbf{Sets} \ : \ (U, \varphi) \to F(U).$$

To show that $u_p F \in \mathbf{PSh}(\mathcal{D})$, note that for $g : V' \to V$ in $\mathcal{D}$, we get a functor $g : I(V) \to I(V')$ by setting $g(U, \varphi) = (U, \varphi \circ g)$. It induces a map

$$u_p F(V) = \varinjlim_{(U,\varphi) \in I(V)^{op}} F(U) \to \varinjlim_{(W,\psi) \in I(V')^{op}} F(W) = u_p F(V').$$

A map of $F \to F'$ in $\mathbf{PSh}(\mathcal{C})$ induces for $V \in \mathcal{D}$

$$u_p F(V) = \varinjlim_{(U,\varphi) \in I(V)^{op}} F(U) \to \varinjlim_{(U,\varphi) \in I(V)^{op}} F'(U) = u_p F(V).$$

Thus, we have defined a functor

$$u_p : \mathbf{PSh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{D}).$$

To show that

$$\mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(F, u^p G) = \mathrm{Hom}_{\mathbf{PSh}(\mathcal{D})}(u_p F, G)$$

holds bifunctorially in F and G.

Lemma 1.2. *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume*

(i) $\mathcal{C}$ has a final object e and $u(e)$ is a final object of $\mathcal{D}$,
(ii) $\mathcal{C}$ admits fiber products and u commutes with them.

Then, u_p commutes with fintie limits.

Proof. This follows from the fact that the categories $I^u(V)^{op}$ are filtered by [35, 00X3]. $\square$

1.2. **Sites and sheaves.** .

Definition 1.3. A site is given by a pair $(\mathcal{C}, \tau)$ of a category $\mathcal{C}$ and a Grothendieck pretopology τ which is a function assigning to each object $U \in \mathcal{C}$ a collection $\mathrm{Cov}(U)$ of families of morphisms $\{U_i \to U\}_{i\in I}$, called coverings family of U, satisfying the following axioms:

(i) If $V \to U$ is an isomorphism, we have $\{V \to U\} \in \mathrm{Cov}(U)$.
(ii) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ and $\{V_{ij} \to U_i\}_{jinJ_i} \in \mathrm{Cov}(U_i)$ for each $i \in I$, we have then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(U)$.
(iii) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ and $V \to U$ is a morphism of $\mathcal{C}$, then $U_i \times_U V$ exists for all $i \in I$ and we have $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(V)$.

Example 1.4. For a scheme S, let $\mathbf{Sch}_S$ be the category of schemes of finite presentation over S.

(i) Let $\acute{\mathrm{E}}\mathrm{t}_S$ be the full subcategory of $\mathbf{Sch}_S$ of étale schemes over S. The big étale site $(\mathbf{Sch}_S)_{\acute{e}t}$ is the site whose underlying category is $\mathbf{Sch}_S$ and whose coverings are étale covering[3]. The small étale site $(\mathbf{Sch}_X)_{\acute{e}t}$ is the full subcategory of $(\mathbf{Sch}_S)_{\acute{e}t}$ whose objects are those U/S such that $U \to S$ is étale. A covering of $S_{\acute{e}t}$ is any étale covering $\{U_i \to U\}$ with $U \in S_{\acute{e}t}$.

Definition 1.5. Let $\mathcal{C}$ be a site, and let F be a presheaf of sets on $\mathcal{C}$. We say F is a sheaf if for every $U \in \mathcal{C}$ and every covering $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ the diagram

$$F(U) \to \prod_{i\in I} F(U_i) \underset{pr_1^*}{\overset{pr_0^*}{\rightrightarrows}} \prod_{(i_0,i_1)\in I\times I} F(U_{i_0} \times_U U_{i_1})$$

represents the first arrow as the equalizer of pr_0^* and pr_1^*. We let $\mathbf{Shv}(\mathcal{C}) \subset \mathbf{PSh}(\mathcal{C})$ denote the full subcategory of sheaves (of sets).

Lemma 1.6. *Let $\mathcal{F} : I \to \mathbf{Shv}(\mathcal{C})$ be a diagram. Then $\varprojlim_I \mathcal{F}$ exists and is equal to the limit in $\mathbf{PSh}(\mathcal{C})$.*

Proposition 1.7. *There exists a functor called the sheafification*

$$a : \mathbf{PSh}(\mathcal{C}) \to \mathbf{Shv}(\mathcal{C})$$

which is a left adjoint to the inclusion functor $i : \mathbf{PSh}(\mathcal{C}) \to \mathbf{Shv}(\mathcal{C})$. In other words

$$\mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(F, G) = \mathrm{Hom}_{\mathbf{Shv}(\mathcal{C})}(aF, G)$$

holds bifunctorially in $F \in \mathbf{PSh}(\mathcal{C})$ and $G \in \mathbf{Shv}(\mathcal{C})$. Moreover, a is exact.

Let $F \in \mathbf{PSh}(\mathcal{C})$. For $\mathfrak{U} = \{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$, put

$$H^0(\mathfrak{U}, F) = \mathrm{equalizer}\Big(\prod_{i\in I} F(U_i) \underset{pr_1^*}{\overset{pr_0^*}{\rightrightarrows}} \prod_{(i_0,i_1)\in I\times I} F(U_{i_0} \times_U U_{i_1})\Big)$$

There is a canonical map $F(U) \to H^0(\mathfrak{U}, F)$[4].

For $U \in \mathcal{C}$, let $\mathrm{Cov}(U)$ be the category of all coverings of U in $\mathcal{C}$ whose morphisms are the refinements (see §1.5). Note that $\mathrm{Cov}(U)$ is not empty since $\{id : U \to U\}$ is an object of it. By definition the construction $\mathfrak{U} \mapsto H^0(\mathfrak{U}, F)$ is an object of $\mathbf{PSh}(\mathrm{Cov}(U))$. For $F \in \mathbf{PSh}(\mathcal{C})$, we define

$$F^+(U) = \varinjlim_{\mathfrak{U}\in \mathrm{Cov}(U)^{op}} H^0(\mathfrak{U}, F).$$

Note that $F^+(U) = \check{H}^0(U, F)$ is the zeroth Čech cohomology of F over U (see (1.19.2)).

[3]For $T \in \mathbf{Sch}_S$, an étale covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ in $\mathbf{Sch}_S$ such that each f_i is étale and $T = \cup f_i(T_i)$.

[4]This is the zeroth Čech cohomology of F over U with respect to the covering $\mathfrak{U}$.

Lemma 1.8. *(1) For $F \in \mathbf{PSh}(\mathcal{C})$, F^+ is an object of $\mathbf{PSh}(\mathcal{C})$ equipped with a canonical map $F \to F^+$ in $\mathbf{PSh}(\mathcal{C})$. Moreover, the construction is functorial, i.e. a map $f : F \to G$ in $\mathbf{PSh}(\mathcal{C})$ induces a map $f^+ : F^+ \to G^+$ such that the following diagram commutes in $\mathbf{PSh}(\mathcal{C})$:*

$$\begin{tikzcd} F \arrow[r] \arrow[d, "f"] & F^+ \arrow[d, "f^+"] \\ G \arrow[r] & G^+ \end{tikzcd}$$

(2) The presheaf F^+ is separated.

Proof. [35, 00WB]. □

Proposition 1.9. *For $F \in \mathbf{PSh}(\mathcal{C})$, $(F^+)^+ \in \mathbf{Shv}(\mathcal{C})$ and the induced functor*

$$a = ((-)^+)^+ : \mathbf{PSh}(\mathcal{C}) \to \mathbf{Shv}(\mathcal{C})$$

is a left adjoint to the inclusion functor $\mathbf{PSh}(\mathcal{C}) \to \mathbf{Shv}(\mathcal{C})$. Moreover, a is exact.

Proof. [35, 00WB]. The exactness of a follows from the fact that $\mathrm{Cov}(U)$ is filtered (the point is to show a commutes with finite limits).

□

1.3. **Functoriality of sheaves.**

Definition 1.10. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A functor $u : \mathcal{C} \to \mathcal{D}$ is called continuous if for every $V \in \mathcal{C}$ and every $\{V_i \to V\}_{i\in I} \in \mathrm{Cov}(V)$, we have the following

(i) $\{u(V_i) \to u(V)\}_{i\in I} \in \mathrm{Cov}(u(V))$,
(ii) for any morphism $T \to V$ in $\mathcal{C}$, the morphism $u(T \times_V V_i) \to u(T) \times_{u(V)} u(V_i)$ is an isomorphism.

Example 1.11. For a map $f : T \to S$ of schemes, consider

$$u : \acute{\mathrm{E}}\mathrm{t}_S \to \acute{\mathrm{E}}\mathrm{t}_T \ : \ X \to X \times_S T.$$

Then, u is continuous for the étale topology.

Lemma 1.12. *If $u : \mathcal{C} \to \mathcal{D}$ is continuous, u^p induces*

$$u^s : \mathbf{Shv}(\mathcal{D}) \to \mathbf{Shv}(\mathcal{C}).$$

Proof. Exercise. □

Lemma 1.13. *If $u : \mathcal{C} \to \mathcal{D}$ is continuous, the functor*

$$u_s : \mathbf{Shv}(\mathcal{D}) \to \mathbf{Shv}(\mathcal{C}) \ : \ G \to a(u_p(G))$$

is a left adjoint to u^s.

Proof. Follows directly from Propositions 1.9 and 1.1. □

Definition 1.14. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A morphism of sites $f : \mathcal{D} \to \mathcal{C}$ is given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ such that the functor u_s is exact.

.

Proposition 1.15. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous morphism of sites. Assume*

(i) $\mathcal{C}$ has a final object e and $u(e)$ is a final object of $\mathcal{D}$,
(ii) $\mathcal{C}$ admits fiber products and u commutes with them.

Then, u defines a morphism of sites, i.e. u_s is exact.

Proof. This follows from Lemma 1.2 and the exactness of a from Proposition 1.9 (see [35, 00X6]). □

Definition 1.16. A topos is the category $\mathbf{Shv}(\mathcal{C})$ of sheaves on a site $\mathcal{C}$.

(1) Let $\mathcal{C}$, $\mathcal{D}$ be sites. A morphism of topoi $f : \mathbf{Shv}(\mathcal{D}) \to \mathbf{Shv}(\mathcal{C})$ is given by a adjoint pair of functors
$$f^* : \mathbf{Shv}(\mathcal{C}) \leftrightarrows \mathbf{Shv}(\mathcal{D}) : f_*,$$
namely we have for $G \in \mathbf{Shv}(\mathcal{C})$ and $F \in \mathbf{Shv}(\mathcal{D})$
$$\mathrm{Hom}_{\mathbf{Shv}(\mathcal{D})}(f^*G, F) = \mathrm{Hom}_{\mathbf{Shv}(\mathcal{C})}(G, f_*F)$$
bifunctorially, and the functor f^* commutes with finite limits, i.e., is left exact.

(2) Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be sites. Given morphisms of topoi $f : \mathbf{Shv}(\mathcal{D}) \to \mathbf{Shv}(\mathcal{C})$ and $g : \mathbf{Shv}(\mathcal{E}) \to \mathbf{Shv}(\mathcal{D})$, the composition $f \circ g$ is the morphism of topoi defined by the functors $(f \circ g)_* = f_* \circ g_*$ and $(f \circ g)^* = g^* \circ f^*$.

Lemma 1.17. *Given a morphism of sites $f : D \to C$ corresponding to the functor $u : \mathcal{C} \to \mathcal{D}$, the pair of functors $(f^* = u_s, f_* = u^s)$ is a morphism of topoi.*

Proof. This is obvious from Definition 1.14. □

1.4. **Cohomology.**

Theorem 1.18. *Let $\mathcal{C}$ be a site. Then, the category $\mathbf{Shv}(\mathcal{C}, \mathbf{Ab})$ of abelian sheaves on a site is an abelian category which has enough injectives.*

Proof. [35, 03NU]. □

By the theorem, we can define cohomology as the right-derived functors of the sections functor $F \to F(U)$ for $U \in \mathcal{C}$ and $F \in \mathbf{Shv}(\mathcal{C}, \mathbf{Ab})$ defined as
$$H^i(U, F) := R^i\Gamma(U, F) = H^i(\Gamma(U, I^\bullet)),$$
where $F \to I^\bullet$ is an injective resolution. To do this, we should check that the functor $\Gamma(U, -)$ is left exact. This is true and is part of why the category $\mathbf{Shv}(\mathcal{C}\mathbf{Ab})$ is abelian, see Modules on Sites, Lemma 3.1. For more general discussion of cohomology on sites (including the global sections functor and its right derived functors), see Cohomology on Sites, Section 2. The family of functors $H^i(U, -)$ forms a universal δ-functor $\mathbf{Shv}(\mathcal{C}, \mathbf{Ab}) \to \mathbf{Ab}$.

It sometimes happens that the site $\mathbf{C}$ does not have a final object. In this case, we define the global sections of $F \in \mathbf{PSh}(\mathcal{C}, S_{\text{ét}})$ over $\mathbf{C}$ to be the set
$$\Gamma(\mathcal{C}, F) = \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(e, F),$$
where e is a final object in $\mathbf{PSh}(\mathcal{C}, \mathbf{Sets})$. In this case, given $F \in \mathbf{Shv}(\mathcal{C}, \mathbf{Ab})$, we define the i-th cohomology group of Fon $\mathbf{C}$ as follows
$$H^i(\mathcal{C}, F) = H^i(\Gamma(\mathcal{C}, I^\bullet))).$$
In other words, it is the i-th right derived functor of the global sections functor. The family of functors $H^i(\mathcal{C}, -)$ forms a universal δ-functor $\mathbf{Shv}(\mathcal{C}, \mathbf{Ab}) \to \mathbf{Ab}$.

1.5. **Čech cohomology.** For $U \in \mathcal{C}$ and $\mathfrak{U} = \{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$, write $U_{i_0 \dots i_p} = U_{i_0} \times_U \cdots \times_U U_{i_p}$ for the $(p+1)$-fold fiber product over U of members of $\mathfrak{U}$. Let $F \in \mathbf{PSh}(\mathcal{C}, \mathbf{Ab})$, set
$$\check{C}^p(\mathfrak{U}, F) = \prod_{(i_0 \dots i_p) \in I^{p+1}} F(U_{i_0 \dots i_p}).$$
For $s \in \check{C}^p(\mathfrak{U}, F)$, we denote $s_{i_0 \dots i_p}$ its value in $F(U_{i_0 \dots i_p})$. We define
$$d : \check{C}^p(\mathfrak{U}, F) \to \check{C}^{p+1}(\mathfrak{U}, F)$$
by the formula
$$d(s)_{i_0 \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j (s_{i_0 \dots \hat{i_j} \dots i_{p+1}})_{|U_{i_0 \cdots i_{p+1}}}.$$

It is straightforward to see that $d \circ d = 0$, i.e. $\check{C}(\mathfrak{U}, F)$ is a complex, which we call Čech complex associated to F and $\mathfrak{U}$. Its cohomology groups

$$\check{H}^i(\mathfrak{U}, F) = H^i(\check{C}(\mathfrak{U}, F))$$

are called the Čech cohomology groups associated to F and $\mathfrak{U}$.

Lemma 1.19. *For $U \in \mathcal{C}$ and $\mathfrak{U} = \{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$, there is a transformation of functors:*

$$\mathbf{Shv}(\mathcal{C}, \mathbf{Ab}) \to \mathcal{D}(\mathbb{Z}) \ : \ \check{C}(\mathfrak{U}, -) \to R\Gamma(U, -).$$

Moreover, there is a spectral sequence for $F \in \mathbf{Shv}(\mathcal{C}, \mathbf{Ab})$:

$$E_2^{p,q} = \check{H}^p(\mathfrak{U}, \mathcal{H}^q(F)) \Rightarrow H^{p+q}(U, F), \tag{1.19.1}$$

which is functorial in F, where $\mathcal{H}^q(F) \in \mathbf{PSh}((X, \tilde{X})_t, \mathbf{Ab})$ is given by $\mathcal{U} \to H^q_t(\mathcal{U}, F)$. In particular, if $H^i(U_{i_0} \times_U \cdots \times_U U_{i_p}, F) = 0$ for all $i > 0$, $p \geq 0$ and $i_0, \dots, i_p \in I$, then we have $\check{H}^p(\mathfrak{U}, F) = H^p(U, F)$.

Proof. [35, 03AX, 03AZ, 03F7]. □

For coverings $\mathfrak{U} = \{U_i \to U\}_{i\in I}$ and $\mathfrak{V} = \{V_j \to V\}_{j\in J}$ in $\mathcal{C}$, a morphism $\mathfrak{U} \to \mathfrak{V}$ is given by a morphism $U \to V$ in $\mathcal{C}$, a map of sets $\alpha : I \to J$ and for each $i \in I$ a morphism $U_i \to V_{\alpha(i)}$ such that the diagram

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_{\alpha(i)} \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

is commutative. In the special case $U = V$ and $U \to V$ is the identity, we call $\mathfrak{U}$ a refinement of $\mathfrak{V}$. A remark is that if the above $\mathfrak{V}$ is the empty family, i.e., if $J = \varnothing$, then no family $\mathfrak{U} = \{U_i \to V\}_{i\in I}$ with $I \neq \varnothing$ can refine $\mathfrak{V}$.

For $U \in \mathcal{C}$, let $\mathrm{Cov}(U)$ be the category of all coverings of U in $\mathcal{C}$ whose morphisms are the refinements[5]. Note that $\mathrm{Cov}(U)$ is not empty since $\{id : U \to U\}$ is an object of it. Take $F \in \mathbf{PSh}(\mathcal{C}, \mathbf{Ab})$. By definition the construction $\mathfrak{U} \mapsto \check{C}(\mathfrak{U}, F)$ is a preshesaf on $\mathrm{Cov}(U)$ with values in the category of complexes of abelian groups. We define

$$\check{C}(U, F) := \varinjlim_{\mathfrak{U}\in\mathrm{Cov}(U)^{op}} \check{C}(\mathfrak{U}, F),$$

$$\check{H}^i(U, F) := H^i(\check{C}(\mathfrak{U}, F)) = \varinjlim_{\mathfrak{U}\in\mathrm{Cov}(U)^{op}} \check{H}(\mathfrak{U}, F), \tag{1.19.2}$$

where the last equality holds since $\mathrm{Cov}(U)$ if cofiltered. By Lemma 1.19, we have a transformation of functors:

$$\mathbf{Shv}(\mathcal{C}, \mathbf{Ab}) \to \mathcal{D}(\mathbb{Z}) \ : \ \check{C}(U, -) \to R\Gamma(U, -).$$

(1.19.1) induces a spectral sequence

$$E_2^{p,q} = \check{H}^p(U, \mathcal{H}^q(F)) \Rightarrow H^{p+q}(U, F). \tag{1.19.3}$$

Lemma 1.20. *Let $U \in \mathcal{C}$ and $F \in \mathbf{PSh}(\mathcal{C}, \mathbf{Ab})$.*

1. *(1) $\check{H}^0(U, \mathcal{H}^q(F)) = 0$ for $q > 0$. In particular, for every $\alpha \in H^q(U, F)$, there is $\mathfrak{U} = \{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ such that $\alpha \mapsto 0$ in $H^q(U_i, F)$ for all $i \in I$.*
2. *(2) $\check{H}^i(U, \mathcal{H}^q(F)) = H^i(U, F)$ for $i = 0, 1$ and there is an exact sequence*

$$0 \to \check{H}^2(U, F) \to H^2(U, F) \to \check{H}^1(U, \mathcal{H}^1(F)) \to \check{H}^3(U, F) \to H^3(U, F).$$

[5] By our conventions on sites this is indeed a category, i.e., the collection of objects and morphisms forms a set.

Proof. ([29, Ch.III 2.9 and 2.10]) (2) follow formally from (1) using (1.19.3). To prove (1), we show the following claim. Recall the pair of adjoint functors from Proposition 1.7:

$$a : \mathbf{PSh}(\mathcal{C}) \underset{\leftarrow}{\overset{\rightarrow}{}} \mathbf{Shv}(\mathcal{C}) : i.$$

Claim 1.21. *For $q > 0$, we have $a\mathcal{H}^q(F) = 0$.*

Indeed, take an injective resolution $F \to I^\bullet$ in $\mathbf{Shv}(\mathcal{C}, \mathbf{Ab})$. Then, $\mathcal{H}^q(F)$ is the q-th cohomology presheaf of the complex $i(I^\bullet)$ in $\mathbf{PSh}(\mathcal{C}, \mathbf{Ab})$. Since a is exact and commutes with taking cohomology, $a\mathcal{H}^q(F)$ is the q-th cohomology sheaf of the complex $ai(I^\bullet) = I^\bullet$ in $\mathbf{Shv}(\mathcal{C}, \mathbf{Ab})$ so that it must vanishes.

By Proposition 1.9, we have $a\mathcal{H}^q(F) = (\mathcal{H}^q(F)^+)^+ = 0$. Since $\mathcal{H}^q(F)^+$ is separated by Lemma 1.8, the natural map $\mathcal{H}^q(F)^+ \to (\mathcal{H}^q(F)^+)^+$ is injective. Thus, we get $\mathcal{H}^q(F)^+ = 0$, which implies (1). $\square$

Lemma 1.22. *For $F \in \mathbf{PSh}(\mathcal{C}, \mathbf{Ab})$, the following are equivalent.*

(1) F is flabby, i.e. $H^i(U, F) = 0$ for any $i > 0$ and $U \in \mathcal{C}$.
(2) $\check{H}^i(\mathfrak{U}, F) = 0$ for any $i > 0$, $U \in \mathcal{C}$ and $\mathfrak{U} \in \mathrm{Cov}(U)$.
(3) $\check{H}^i(U, F) = 0$ for any $i > 0$ and $U \in \mathcal{C}$.

Proof. ([29, Ch.III 2.12]) (1)⇒(2). By the assumption, $\mathcal{H}^q(F) = 0$ for $q > 0$ so (1.19.1) implies $\check{H}^i(\mathfrak{U}, F) = H^i(U, F) = 0$.

(2)⇒(3). Pass to the colimit over $\mathfrak{U} \in \mathrm{Cov}(U)$.

(3)⇒(1). Take any $U \in \mathcal{C}$. By the assumption, $\check{H}^q(U, F) = 0$ for any $q > 0$. By Lemma 1.20(2), we get $H^1(U, F) = 0$ which implies $\mathcal{H}^1(F) = 0$. By the long exact sequence in Lemma 1.20(2), we get $H^2(U, F) = 0$ which implies $\mathcal{H}^2(F) = 0$. Assume now $\mathcal{H}^i(F) = 0$ for $i < q$. Since $\check{H}^0(U, \mathcal{H}^q(F)) = 0$ by Lemma 1.20, we get $\check{H}^i(U, \mathcal{H}^j(F)) = 0$ for all $i, j \geq 0$ with $i + j \leq q$. By (1.19.3), it implies $H^q(U, F) = 0$ so that $\mathcal{H}^q(F) = 0$. This complete the proof by induction.

$\square$

2. Classical rigid analytic spaces

Good references for this section are [2]. [3] and [10].

2.1. **Affinoid K-algebras.** Let K be a non-archimedean field, i.e. a field which is complete with respect to a nontrivial non-archimedean absolute value, i.e. a map $|-|: K \to \mathbb{R}_{\geq 0}$ satisfying

(i) $|a| = 0 \Leftrightarrow a = 0$.
(ii) $|ab| = |a||b|$.
(iii) $|a+b| \leq \max\{|a|, |b|\}$.

Note that the map $v : K \to \mathbb{R} \cup \{\infty\}$ given by $v(a) = -\log|a|$ is a valution and there is one-to- one correspondence between non-archimedean absolute values and valuations with value group $\mathbb{R}$ on K, where the inverse is given by $|a| = e^{-v(a)}$. We put

$$\mathcal{O}_K = \{x \in K|\ |x| \leq 1\}$$

and fix $\pi \in K$ with $|\pi| < 1$.

For each $n > 0$, the Tate K-algebra is

$$\begin{aligned} T_n := K\langle T_1, \ldots, T_n\rangle &= \{f = \sum_{\nu \in \mathbb{N}^n} a_\nu T_1^{\nu_1} \cdots T_n^{\nu_n} \mid a_\nu \in K,\ \lim_{|\nu|\to\infty}|a_\nu| = 0\} \\ &= \mathcal{O}_K\{T_1, \ldots, T_n\} \otimes_{\mathcal{O}_K} K, \end{aligned}$$

where $\mathcal{O}_K\{T_1, \ldots, T_n\}$ is the π-adic completion of $\mathcal{O}_K[T_1, \ldots, T_n]$. The Gauss norm[6] $||-|| : T_n \to \mathbb{R}_{\geq 0}$ is given by

$$||f|| = \sup_{\nu \in \mathbb{N}^n} |a_\nu|.$$

Definition 2.1. An affinoid K-algebra is a K-algebra A such that there is a surjective K-algebra homomorphism $\alpha : T_n \to A$. for some $n > 0$. Such a K-affinoid algebra A admits a norm $||-||_\alpha$ given by

$$||\alpha(f)||_\alpha = \inf_{a \in \mathrm{Ker}(\alpha)} ||f - a|| \text{ for } f \in T_n.$$

For another surjective K=algebra homomorphism $\beta : T_m \to A$, there are constants $c, c' > 0$ such that $||-||_\alpha \leq c||-||_\beta \leq c'||-||_\alpha$.

Definition 2.2. For an affinoid K-algebra A, let $\mathrm{Sp}(A)$ be the set of the maximal ideal of A. For $x \in \mathrm{Sp}(A)$, the residue field $K(x)$ of x is a finite extension of K so that it carries a unique extension of $|-|$ on K. For $f \in A$, let $f(x)$ be the image of f in $K(x)$ and $|f(x)|$ be its absolute value under this extension. There is a semi-norm $|-|_{\sup}$ on A on called the supremum norm given by

$$|f|_{\sup} = \sup_{x \in \mathrm{Sp}(A)} |f(x)|.$$

We have the following facts:

(1) $|-|_{\sup}$ is power-multiplicative, i.e. $|f^n|_{\sup} = (|f|_{\sup})^n$ for $f \in A$ and $n > 0$.
(2) For a K-homomorphism $\varphi : A \to B$ of K-affinoid algebras and for $f \in A$, we have $|\varphi(f)|_{\sup} \leq |f|_{\sup}$.
(3) On T_n, the supremum norm coincides with the Gauss norm.
(4) For a surjective K-algebra homomorphism $\alpha : T_n \to A$, we have $|f|_{\sup} \leq ||f||_\alpha$ for all $f \in A$. In particular, $|f|_{\sup} < \infty$.

Theorem 2.3. *(Maximal Principle) For a K-affinoid algebra A and $f \in A$, there exists $x \in \mathrm{Sp}(A)$ such that $|f|_{\sup} = |f(x)|$.*

[6]A map $||-|| : A \to \mathbb{R}_{\geq 0}$ is called a semi-norm if $||0|| = 0$, $||1|| = 1$, $||fg|| \leq ||f||||g||$ and $||f-g|| \leq ||f|| + ||g||$ for $f, g \in A$. It is a norm if $||f|| = 0$ implies $f = 0$. It is non-archimedean if $||f - g|| \leq \max\{||f||, ||g||\}$.

We put

$$A^\circ = \{f \in A|\ |f|_{\sup} \leq 1\} \ \text{ and } \ A^{\circ\circ} = \{f \in A|\ |f|_{\sup} < 1\}.$$

It is easy to see that A° is a subring of A, which is $\mathcal{O}_K$-algebra and $A^{\circ\circ}$ is its ideal. We have the following facts:

(1) A° is π-adically complete and $A = A^\circ \otimes_{\mathcal{O}_K} K$.
(2) A° is the set of power-bounded elements, i.e. those f that $\{||f^n||_\alpha \ (n \in \mathbb{N})\} \subset \mathbb{R}$ is bounded.
(3) $A^{\circ\circ}$ is the set of topologically nilpotent elements, i.e. those f that $\lim\limits_{n\to\infty} ||f^n||_\alpha = 0$.

2.2. **Affinoid K-spaces.** We let AffAlg_K denote the category of affinoid K-algebras and K-algebra homomorphisms. For a morphism $\varphi : A \to B$ in AffAlg_K, we have the induced map $\varphi^* : \mathrm{Sp}(B) \to \mathrm{Sp}(A)$ sending a maximal ideal $\mathfrak{m} \subset B$ to $\varphi^{-1}(\mathfrak{m})$. Thus, we get a functor

$$\mathrm{Sp} : \mathrm{AffAlg}_K \to \mathbf{Sets}\,.$$

In this subsection, we introduce a G-topology in the sense of Definition 2.8 to make $\mathrm{Sp}(A)$ for $A \in \mathrm{AffAlg}_K$ a G-topological space.

Definition 2.4. For $f_1, \cdots, f_r, g \in A$ which generate the unit ideal, let

$$U\big(\frac{f_1, \dots, f_n}{g}\big) = \{x \in \mathrm{Sp}(A)|\ |f_i(x)| \leq |g(x)| \ (i = 1, \dots, r)\}$$

This is called a rational subdomain of $X = \mathrm{Sp}(A)$.

We have the following facts:

Lemma 2.5. *(1) For a rational subdomain $U \subset \mathrm{Sp}(A)$ and a morhpism $\varphi : A \to B$ in AffAlg_K inducing $\varphi^* : \mathrm{Sp}(B) \to \mathrm{Sp}(A)$, $(\varphi^*)^{-1}(U)$ is a rational subdomain of $\mathrm{Sp}(B)$.*
(2) For rational subdomain domains $U, V \subset \mathrm{Sp}(A)$, $U \cap V$ is a rational subdomain.
(3) As a set, $U\big(\frac{f_1,\dots,f_n}{g}\big)$ is identified with $\mathrm{Sp}(A_U)$ with

$$A_U = A\langle\frac{f_1}{g}, \dots, \frac{f_r}{g}\rangle := A\langle w_1, \dots, w_r\rangle/\big(gw_1 - f_1, \dots, gw_r - f_r\big),$$

where $A\langle w_1, \dots, w_r\rangle = A^\circ\{w_1, \dots, w_r\} \otimes_{A^\circ} A$ with $A^\circ\{w_1, \dots, w_r\}$ the π-adic completion of $A^\circ[w_1, \dots, w_r]$.
(4) For rational subdomain domains $U \subset \mathrm{Sp}(A)$ and $V \subset \mathrm{Sp}(A_U)$, V is a rational subdomain of $\mathrm{Sp}(A)$.

Definition 2.6. A subset $U \subset \mathrm{Sp}(A)$ is called an affinoid subdomain if the functor $F_U : \mathrm{AffAlg}_K \to \mathbf{Sets}$ defined by

$$F_U(B) = \{\varphi \in \mathrm{Hom}_{\mathrm{AffAlg}_K}(A, B)|\ \varphi^*(\mathrm{Sp}(B)) \subset U\} \text{ for } B \in \mathrm{AffAlg}_K$$

is representable by $A_U \in \mathrm{AffAlg}_K$: In other words, there is a map $\psi : A \to A_U$ in AffAlg_K such that the image of $\psi^* : \mathrm{Sp}(A_U) \to \mathrm{Sp}(A)$ is contained in U and the following universal property holds: Any morphisms $\varphi : A \to B$ such that the image of $\varphi^* : \mathrm{Sp}(B) \to \mathrm{Sp}(A)$ is contained in U, there is a unique morphism $A_U \to B$ in AffAlg_K which factors $A \to B$.

We have the following facts:

Lemma 2.7. *(1) Under the above notation, ψ^* is injective and $\mathrm{Image}(\psi^*) = U$.*
(2) A rational subdomain is an affinoid subdomain.
(3) For an affinoid subdomain $U \subset \mathrm{Sp}(A)$ and a morhpism $\varphi : A \to B$ in AffAlg_K inducing $\varphi^* : \mathrm{Sp}(B) \to \mathrm{Sp}(A)$, $(\varphi^*)^{-1}(U)$ is an affinoid subdomain of $\mathrm{Sp}(B)$.
(4) If U is an affinoid subdomain of $\mathrm{Sp}(A)$ and V is an affinoid subdomain of U, then V is an affinoid subdomain of $\mathrm{Sp}(A)$.

(5) *(Gerritzen-Grauert) Any affinoid subdomain of* $\mathrm{Sp}(A)$ *is a finite union of rational subdomains.*
(6) *See Theorem 2.27 for a characterization of affinoid subdomains in terms of formal models.*

Definition 2.8. A G-topology τ on a topological space X consists of the following datum:

(i) A category Cat_τ whose objects are open subsets of X and whose morphisms are open immersions. An object of Cat_τ is called an admissible open subset.
(ii) For every $U \in \mathrm{Cat}_\tau$, a family $\mathrm{Cov}_\tau(U)$ of open coverings $\{U_i \to U\}_{i\in I}$. A member of $\mathrm{Cov}_\tau(U)$ is called an admissible covering of U.

It is required to satisfy the following conditions:

(1) If $V \to U$ is an isomorphism in Cat_τ, then $\{V \to U\} \in \mathrm{Cov}_\tau(U)$.
(2) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}_\tau(U)$ and $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}_\tau(U_i)$, then $\{V_{ij} \to U\}_{i\in I.j\in J_i} \in \mathrm{Cov}_\tau(U)$.
(3) $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}_\tau(U)$ and $V \to U$ is a morphism in Cat_τ, then $\{U_i \cap V \to V\}_{i\in I} \in \mathrm{Cov}_\tau(V)$.

A G-topological space is a topological space X with a Grothendieck topology τ. A morphism $(X,\tau) \to (Y,\lambda)$ of G-topological spaces is a continuous morphism $\varphi : X \to Y$ of topological spaces such that for any $U \in \mathrm{Cat}_\lambda$ and $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}_\lambda(U)$, we have $\varphi^{-1}(U) \in \mathrm{Cat}_\tau$ and $\{\varphi^{-1}(U_i) \to \varphi^{-1}(U)\}_{i\in I} \in \mathrm{Cov}_\tau(\varphi^{-1}(U))$.

We let Top^G denote the category of G-topological spaces.

Definition 2.9. A sheaf F on a G-topological space (X,τ) is a presheaf (of sets) on Cat_τ such that for every $U \in \mathrm{Cat}_\tau$ and every $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}_\tau(U)$ the diagram

$$F(U) \to \prod_{i\in I} F(U_i) \overset{pr_0^*}{\underset{pr_1^*}{\rightrightarrows}} \prod_{(i_0,i_1)\in I\times I} F(U_{i_0} \times_U U_{i_1})$$

represents the first arrow as the equalizer of pr_0^* and pr_1^*. We let $\mathbf{Shv}((X,\tau))$ denote the category of sheaves (of sets) on (X,τ).

Definition 2.10. For a K-affinoid algebra A, we equip $X = \mathrm{Sp}(A)$ with a G-topology τ for which the objects of Cat_τ are affinoid subdomains and $\mathrm{Cov}_\tau(U)$ for $U \in \mathrm{Cat}_\tau$ is the family of *finite* coverings of U by affinoid subdomains. We call the G-topological space (X,τ) an *affinoid K-space associated to A* and denote it simply by $\mathrm{Sp}(A)$.

Let $\mathrm{AffSp}_K \subset \mathrm{Top}^G$ denote the full subcategory of affinoid K-spaces and morphismsm of G-topological spaces.

By Lemma 2.5(3), any morphism $\varphi : A \to B$ in AffAlg_K induces a morphism $\varphi^* : \mathrm{Sp}(B) \to \mathrm{Sp}(A)$ in AffAlg_K. Thus, we get a functor

$$(\mathrm{AffAlg}_K)^{op} \to \mathrm{AffSp}_K \;:\; A \to \mathrm{Sp}(A).$$

Theorem 2.11. *(Tate) Let* $\mathcal{O}_X$ *be the presheaf on* (X,τ) *given by* $\mathcal{O}_X(U) = B$ *for an affinoid subdomain* $U = \mathrm{Sp}(B) \subset X$*. Then,* $\mathcal{O}_X$ *is a sheaf on* (X,τ)*.*

Example 2.12. Let $X = \mathrm{Sp}(A)$ be an affinoid K-space. Using Theorem 2.11, one can show that the following presheaves on X is a sheaves.

(1) The presheaf $\mathcal{O}^\circ \subset \mathcal{O}_X$ given by

$$\mathcal{O}^\circ(B) = \{f \in B|\ |f|_{\sup,B} \le 0\} \text{ for an affinoid subdomain } \mathrm{Sp}(B) \subset \mathrm{Sp}(A),$$

(2) For $r \in \mathbb{R}_{>0}$, the presheaf $\mathcal{O}(r) \subset \mathcal{O}_X$ given by

$$\mathcal{O}(r)(B) = \{f \in B|\ |f|_{\sup,B} < r\} \text{ for an affinoid subdomain } \mathrm{Sp}(B) \subset \mathrm{Sp}(A).$$

where $|-|_{\sup,B}$ is the supremum norm on B.

2.3. Rigid analytic K-spaces.

Definition 2.13. A G-ringed K-space is a pair $(X, \mathcal{O}_X)$, where X is a G-topological space and $\mathcal{O}_X$ is a sheaf of K-algebras on it. $(X, \mathcal{O}_X)$ is called a locally G-ringed K-space if, in addition, all stalks $\mathcal{O}_{X,x}$ for $x \in X$ are local rings. A morphism of G-ringed K-spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair (φ, φ^*), where $\varphi : X \to Y$ is a morphism of G-topological spaces, and φ^* is a system of K-homomorphisms $\varphi^*_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(\varphi^{-1}(V))$ with V varying over the admissible open subsets of Y. It is required that the φ^*_V are compatible with restriction map, i.e. for $W \subset V$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}_Y(V) \arrow[r, "\varphi^*_V"] \arrow[d] & \mathcal{O}_X(\varphi^{-1}(V)) \arrow[d] \\ \mathcal{O}_Y(W) \arrow[r, "\varphi^*_W"] & \mathcal{O}_X(\varphi^{-1}(W)) \end{tikzcd}$$

If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally G-ringed K-spaces, a morphism(φ, φ^*) is called a morphism of locally G-ringed K-spaces if the ring homomorphisms

$$\varphi^*_x : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x} \text{ for } x \in X$$

induced from the φ^*_V are local.

If $X = \mathrm{Sp}(A)$ is an affinoid K-space, we can consider the associated locally G-ringed K-space $(X, \mathcal{O}_X)$, where X is the affinoid K-space associated to A from Definition 2.10 and $\mathcal{O}_X$ is the structure sheaf from Theorem 2.11.

Definition 2.14. A rigid (analytic) K-space is a locally G-ringed K-space $(X, \mathcal{O}_X)$ such that X admits an admissible covering $X = \cup_{i \in I} X_i$ such that $(X_i, \mathcal{O}_{X|X_i})$ is an affinoid K-space for all $i \in I$. A morphism of rigid K-spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of locally G-ringed K-spaces. Let Rig_K be the category of rigid K-spaces and morphismsm of locally G-ringed K-spaces. The G-topology on a rigid (analytic) K-space $(X, \mathcal{O}_X)$ is called the *admissible topology*. For an admissible open subset $U \subset X$, the induced locally G-ringed K-space $(U, \mathcal{O}_{X|U})$ is a rigid K-space again, which is called an open subspace of $(X, \mathcal{O}_X)$.

Remark 2.15. It is clear that every morphism of affinoid K-spaces $\varphi : X \to Y$ induces a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ between associated locally G-ringed K-spaces. Thus, we get a functor

$$(\mathrm{AffAlg}_K)^{op} \to \mathrm{Rig}_K \ : \ A \to (X = \mathrm{Sp}(A), \mathcal{O}_X),$$

Remark 2.16. By a formal reason, the sheaves $\mathcal{O}^\circ$ and $\mathcal{O}(r)$ defined on affinoid spaces from Example 2.12 extends to sheaves $\mathcal{O}^\circ$ and $\mathcal{O}(r)$ on rigid K-spaces.

2.4. Formal schemes and Raynaud's theorem.

Definition 2.17. An $\mathcal{O}_K$-algebra A is called of topologically finite type if there is a surjective homomorphism $\varphi : \mathcal{O}_K\{T_1, \ldots, T_n\} \to A$ of $\mathcal{O}_K$-algebras. It is of topologically finite presentation if, furthermore $\mathrm{Ker}(\varphi)$ is finitely generated. It is admissible if furthermore, A does not have π-torsion .

Lemma 2.18. *(1) An $\mathcal{O}_K$-algebra A of topologically finite type is π-adically comoplete and separated.*
(2) An $\mathcal{O}_K$-algebra A of topologically finite type with no π-torsion is of topologically finite presentation.

Proof. [3, §.3 Cor.5 and Cor. 7]. □

Definition 2.19. A formal $\mathcal{O}_K$-scheme $\mathfrak{X}$ is called locally of topologically finite type (resp. locally of topologically finite presentation, resp. admissible) if there is an open affine covering $\mathfrak{X} = \cup_{\imath \in I} \mathfrak{U}_i$ with $\mathfrak{U}_i = \mathrm{Spf}(A_i)$, where A_i is an $\mathcal{O}_K$-algebra of topologically finite type (resp. of topologically finite presentation, resp. an admissible $\mathcal{O}_K$-algebra).

Let $\mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}$ be the category of formal $\mathcal{O}_K$-schemes locally of topologically finite type and $\mathrm{fSch}^{\mathrm{aff,tft}}_{\mathcal{O}_K}$ be its full subcategory of affine formal $\mathcal{O}_K$-schemes . We have an association

$$\text{(2.19.1)} \qquad \mathrm{rig} : \mathrm{fSch}^{\mathrm{aff,tft}}_{\mathcal{O}_K} \to \mathrm{AffSp}_K \;:\; \mathfrak{X} = \mathrm{Spf}(A) \to \mathfrak{X}^{\mathrm{rig}} = \mathrm{Sp}(A \otimes_{\mathcal{O}_K} K).$$

Note that $A \otimes_{\mathcal{O}_K} K$ is an affinoid K-algebra since $\mathcal{O}_K\{T_1, \dots, T_n\} \otimes_{\mathcal{O}_K} K = K\langle T_1, \dots, T_n\rangle$. Since any morphism $\mathrm{Spf}(A) \to \mathrm{Spf}(B)$ in $\mathrm{fSch}^{\mathrm{aff,tft}}_{\mathcal{O}_K}$ is induced by a unique $\mathcal{O}_K$-homomorphism $B \to A$ of $\mathcal{O}_K$-algebras, this is a functor. Moreover, this functor commutes with localizations: For $f \in A$, we have

$$\text{(2.19.2)} \quad \begin{aligned} A\{f^{-1}\} \otimes_{\mathcal{O}_K} K &= \big(A\{T\}/(1 - fT)\big) \otimes_{\mathcal{O}_K} K \\ &= (A \otimes_{\mathcal{O}_K} K)\langle T\rangle/(1 - fT) = (A \otimes_{\mathcal{O}_K} K)\langle f^{-1}\rangle. \end{aligned}$$

From these, we can deduce the following (see [3, §7.3]).

Proposition 2.20. *The functor* (2.19.1) *extends to a functor*

$$\text{(2.20.1)} \qquad \mathrm{rig} : \mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K} \to \mathrm{Rig}_K \;:\; \mathfrak{X} \to \mathfrak{X}^{\mathrm{rig}}.$$

Remark 2.21. If $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{X}^{\mathrm{rig}}$ coincides pointwise with the set of all closed points of $\mathrm{Spec}(A \otimes_{\mathcal{O}_K} K)$, which is the generic fiber of the ordinary scheme $\mathrm{Spec}(A)$ although it is not visible in $\mathrm{Spf}(A)$ on the level of points. By this, $\mathfrak{X}^{\mathrm{rig}}$ is called the generic fiber of $\mathfrak{X}$.

In view of Proposition 2.20, one would like to describe all formal $\mathcal{O}_K$-schemes $\mathfrak{X}$ whose generic fiber $\mathfrak{X}^{\mathrm{rig}}$ coincides with a given rigid K-space X. Such a formal $\mathcal{O}_K$-scheme is called a *formal model of* X. To answer this question, we introduce the following.

Definition 2.22. Let $\mathfrak{X} = \varinjlim_{n\in\mathbb{N}} \underline{\mathrm{Spec}}(\mathcal{O}_{\mathfrak{X}}/(\pi^n)) \in \mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}$ and let $\mathcal{A} \subset \mathcal{O}_{\mathfrak{X}}$ be a coherent open[7] ideal. Then the formal $\mathcal{O}_K$-scheme

$$\mathfrak{X}_{\mathcal{A}} = \varinjlim_{n\in\mathbb{N}} \mathrm{Proj}\,\Big(\bigoplus_{d=0}^{\infty} \mathcal{A}^d \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}/(\pi^n)\Big)$$

together with the canonical projection $\mathfrak{X}_{\mathcal{A}} \to \mathfrak{X}$ is called the formal blowup of $\mathfrak{X}$ in $\mathcal{A}$. Any such blowup is referred to as an *admissible formal blowup of* $\mathfrak{X}$. Note $\mathfrak{X}_{\mathcal{A}} \in \mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}$ by the construction.

Definition 2.23. Let $\mathcal{C}$ be a category and S be a class of morphisms in $\mathcal{C}$. A localization of $\mathcal{C}$ by S is a category $\mathcal{C}_S$ together with a functor $L_S : \mathcal{C} \to \mathcal{C}_S$ such that:

(i) $L_S(s)$ is an isomorphism in $\mathcal{C}_S$ for every $s \in S$.
(ii) If $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $F(s)$ is an isomorphism for every $s \in S$, then F admits a unique factorization as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "L_S"] \arrow[d, "F"'] & \mathcal{C}_S \arrow[dl, "G"] \\ \mathcal{D} & \end{tikzcd}$$

where the commutativity of the diagram, as well as the uniqueness of G are meant up to natural equivalence of functors.

It is known that localizations of categories do always exist.

Proposition 2.24. *For* $\mathfrak{X} \in \mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}$ *and an admissible blowup* $\mathfrak{Y} \to \mathfrak{X}$*, the induced map* $\mathfrak{Y}^{\mathrm{rig}} \to \mathfrak{X}^{\mathrm{rig}}$ *is an isomorphism in* Rig_K*. In particular, the functor* (2.20.1) *factors through the localization* $\mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K} \to (\mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K})_\Sigma$ *by the class* Σ *of admissible blowups.*

Proof. See [3, §8.4, Pr. 2]. $\square$

[7] namely, $\pi^n \in \mathcal{A}$ for some $n > 0$.

Theorem 2.25. *(Raynaud) Let* $\mathrm{Rig}_K^{qcqs} \subset \mathrm{Rig}_K$ *be the full subcategory of quasi-compact quasi separate rigid* K*-spaces. Let* $\mathrm{fSch}^{\mathrm{ad}}_{\mathcal{O}_K} \subset \mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}$ *be the full subcategory of quasi-compact quasi-separate admissible* $\mathcal{O}_K$*-formal schemes and* $(\mathrm{fSch}^{\mathrm{ad}}_{\mathcal{O}_K})_\Sigma$ *be its localization by the class of admissible blowups. Then, the functor* rig *from* (2.20.1) *induces an equivalence of categories*

$$\mathrm{rig} : (\mathrm{fSch}^{\mathrm{ad}}_{\mathcal{O}_K})_\Sigma \simeq \mathrm{Rig}_K^{qcqs}. \tag{2.25.1}$$

Proof. See [3, §8.4, Th.3]. □

Remark 2.26. For $\mathfrak{X} \in \mathrm{fSch}^{\mathrm{ad}}_{\mathcal{O}_K}$, the category $\Sigma_{\mathfrak{X}}$ of admissible blowups $\mathfrak{X}' \to \mathfrak{X}$ admits finite limits so that is cofiltered. This implies that for $\mathfrak{Y} \in \mathrm{fSch}^{\mathrm{ad}}_{\mathcal{O}_K}$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Rig}_K}(\mathfrak{X}^{\mathrm{rig}}, \mathfrak{Y}^{\mathrm{rig}}) = \varinjlim_{\mathfrak{X}' \to \mathfrak{X} \in \Sigma_{\mathfrak{X}}} \mathrm{Hom}_{\mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}}(\mathfrak{X}', \mathfrak{Y}). \tag{2.26.1}$$

Theorem 2.27. *(Geritzen and Grauert) Let* $\mathfrak{X} = \mathrm{Spf}(A) \in \mathrm{fSch}^{\mathrm{aff,tft}}_{\mathcal{O}_K}$ *and* $X = \mathfrak{X}^{\mathrm{rig}} = \mathrm{Sp}(A \otimes_{\mathcal{O}_K} K)$*. A subset* $U \subset X$ *is an affinoid subdomain in the sense of Definition 2.6 if and only if there is* $\mathfrak{Y} \in \Sigma_{\mathfrak{X}}$ *and an affine open* $\mathfrak{U} \hookrightarrow \mathfrak{Y}$ *such that* $U = \mathfrak{U}^{\mathrm{rig}}$*.*

2.5. **Riemann-Zariski spaces.**

Definition 2.28. Let $\mathfrak{X} \in \mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}$ and $\Sigma_{\mathfrak{X}}$ be the category of admissible blowups $\mathfrak{Y} \to \mathfrak{X}$. Let $\mathrm{RZ}(\mathfrak{X}) \subseteq \mathrm{Arr}(\mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K})$ be the category whose objects are morphisms $\mathfrak{U} \to \mathfrak{Y}$ where $\mathfrak{Y} \to \mathfrak{X} \in \Sigma_{\mathfrak{X}}$ and $\mathfrak{U} \to \mathfrak{Y}$ is a Zariski open immersion. We abbreviate $\mathfrak{U} \to \mathfrak{Y}$ to $(\mathfrak{U}/\mathfrak{Y})$. The morphism $(\mathfrak{U}'/\mathfrak{Y}') \to (\mathfrak{U}/\mathfrak{Y})$ in $\mathrm{RZ}(\mathfrak{X})$ are commutative squares in $\mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}$:

$$\begin{tikzcd} \mathfrak{U}' \arrow[r] \arrow[d] & \mathfrak{U} \arrow[d] \\ \mathfrak{Y}' \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

Remark 2.29. $\mathrm{RZ}(\mathfrak{X})$ admits finite limits, and they are calculated termwise. Indeed, the category $\mathrm{Arr}(\mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K})$ of arrows admits finite limits and they are calculated component wise: $\varprojlim(A_i/B_i) = (\varprojlim A_i/\varprojlim B_i)$. If each (A_i/B_i) is in $\mathrm{RZ}(\mathfrak{X})$, then one checks that $\varprojlim(A_i/B_i)$ is again in $\mathrm{RZ}(\mathfrak{X})$.

Definition 2.30. We equip $\mathrm{RZ}(\mathfrak{X})$ with the Grothendieck topology τ generated by:

(1) families of $\{(\mathfrak{U}_i/\mathfrak{Y}) \to (\mathfrak{U}/\mathfrak{Y})\}_{i\in I}$ such that $\{\mathfrak{U}_i \to \mathfrak{U}\}_{i\in I}$ is a Zariski covering,
(2) families of $\{(\mathfrak{Y}' \times_{\mathfrak{Y}} \mathfrak{U}/\mathfrak{Y}') \to (\mathfrak{U}/\mathfrak{Y})\}$ for morphisms $\mathfrak{Y}' \to \mathfrak{Y}$ in $\Sigma_{\mathfrak{X}}$.

The site $(\mathrm{RZ}(\mathfrak{X}), \tau)$ is called the Riemann-Zariski space of $\mathfrak{X}$. We will write $\mathbf{Shv}(\mathrm{RZ}(\mathfrak{X}))$ for the topos associated to the topology generated by coverings of the form (1) and (2).

Remark 2.31. Using that for $\mathfrak{Y}' \to \mathfrak{Y}$ in $\Sigma_{\mathfrak{X}}$, the diagonal $\mathfrak{Y}' \to \mathfrak{Y}' \times_{\mathfrak{Y}} \mathfrak{Y}'$ is a morphism in $\Sigma_{\mathfrak{X}}$, one can show that a presheaf on $\mathrm{RZ}(\mathfrak{X})$ satisfies descent for all families of the form (2) if and only if it sends each $(\mathfrak{Y}' \times_{\mathfrak{Y}} \mathfrak{U}/\mathfrak{Y}') \to (\mathfrak{U}/\mathfrak{Y})$ to an isomorphism. This implies

$$\mathbf{Shv}(\mathrm{RZ}(\mathfrak{X})) \simeq \varprojlim_{\mathfrak{Y} \in \Sigma_{\mathfrak{X}}} \mathbf{Shv}(\mathfrak{Y}_{\mathrm{zar}}) \tag{2.31.1}$$

where the limit is along pushforwards $f_* : \mathbf{Shv}(\mathfrak{Y}'_{\mathrm{zar}}) \to \mathbf{Shv}(\mathfrak{Y}_{\mathrm{zar}})$ for morphisms $f : \mathfrak{Y}' \to \mathfrak{Y}$ in $\Sigma_{\mathfrak{X}}$, namely an object of the RHS of (2.31.1) is given by a system

$$\mathcal{F} = \{F_{\mathfrak{Y}} \in \mathbf{Shv}(\mathfrak{Y}_{\mathrm{zar}})\}_{\mathfrak{Y} \to \mathfrak{X} \in \Sigma_{\mathfrak{X}}} \tag{2.31.2}$$

such that

(♠) $F_{\mathfrak{Y}'}(\mathfrak{U} \times_{\mathfrak{Y}} \mathfrak{Y}') = F_{\mathfrak{Y}}(\mathfrak{U})$ for every $(\mathfrak{U}/\mathfrak{Y}) \in \mathrm{RZ}(\mathfrak{X})$ and $\mathfrak{Y}' \to \mathfrak{Y}$ in $\Sigma_{\mathfrak{X}}$.

If $F_{\mathfrak{Y}}$ are all sheaves of abelian groups, this implies that we have a natural isomorphism

$$\varinjlim_{\mathfrak{Y}\to\mathfrak{X}\in\Sigma_{\mathfrak{X}}} H^i(\mathfrak{Y}, F_{\mathfrak{Y}}) \simeq H^i(\mathrm{RZ}(\mathfrak{X}), \mathcal{F}_{\mathrm{RZ}(\mathfrak{X})}), \tag{2.31.3}$$

where $\mathcal{F}_{\mathrm{RZ}(\mathfrak{X})} = \varprojlim_{\mathfrak{Y}\in\Sigma_{\mathfrak{X}}} F_{\mathfrak{Y}} \in \mathbf{Shv}(\mathrm{RZ}(\mathfrak{X}))$ (see [10, Ch.0, 4.4.1]).

Now, we look at a relation of $\mathbf{Shv}(\mathrm{RZ}(\mathfrak{X}))$ and $\mathbf{Shv}(\mathfrak{X}^{\mathrm{rig}})$ for $\mathfrak{X} \in \mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}$. Using Proposition 2.24, the functor (2.20.1) gives a functor on the categories of open subsets:

$$\mathrm{RZ}(\mathfrak{X}) \to \mathfrak{X}^{\mathrm{rig}} \ : \ (\mathfrak{U}/\mathfrak{Y}) \to \mathfrak{U}^{\mathrm{rig}} \subset \mathfrak{Y}^{\mathrm{rig}} = \mathfrak{X}^{\mathrm{rig}}.$$

By the construction, this is continuous, i.e. maps coverings to coverings so that it defines a morphism of sites

$$\gamma : \mathfrak{X}^{\mathrm{rig}} \to \mathrm{RZ}(\mathfrak{X})$$

which induces a pair of adjoint functors

$$\gamma^* : \mathbf{Shv}(\mathrm{RZ}(\mathfrak{X})) \rightleftarrows \mathbf{Shv}(\mathfrak{X}^{\mathrm{rig}}) : \gamma_*, \tag{2.31.4}$$

where $\gamma_* F(\mathfrak{U}/\mathfrak{Y}) = F(\mathfrak{U}^{\mathrm{rig}})$ for $F \in \mathbf{Shv}(\mathfrak{X}^{\mathrm{rig}})$ and $(\mathfrak{U}/\mathfrak{Y}) \in \mathrm{RZ}(\mathfrak{X})$.

Theorem 2.32. (2.31.4) *induces a natural equivalence of topoi*

$$\mathbf{Shv}(\mathfrak{X}^{\mathrm{rig}}) \simeq \mathbf{Shv}(\mathrm{RZ}(\mathfrak{X})).$$

In particular, for $\mathcal{F}_{\mathrm{RZ}(\mathfrak{X})} = \varprojlim_{\mathfrak{Y}\in\Sigma_{\mathfrak{X}}} F_{\mathfrak{Y}} \in \mathbf{Shv}(\mathrm{RZ}(\mathfrak{X}))$ *from* (2.31.3)*, we have*

$$\varinjlim_{\mathfrak{Y}\to\mathfrak{X}\in\Sigma_{\mathfrak{X}}} H^i(\mathfrak{Y}, F_{\mathfrak{Y}}) \simeq H^i(\mathfrak{X}^{\mathrm{rig}}, \gamma^*\mathcal{F}_{\mathrm{RZ}(\mathfrak{X})}). \tag{2.32.1}$$

Proof. [10, Th.B.2.5]. □

Remark 2.33. By definition, we have

$$\gamma^*\mathcal{F}_{\mathrm{RZ}(\mathfrak{X})}(\mathfrak{U}^{\mathrm{rig}}) = F_{\mathfrak{Y}}(\mathfrak{U}) \text{ for } (\mathfrak{U}/\mathfrak{Y}) \in \mathrm{RZ}(\mathfrak{X}).$$

Since such $\mathfrak{U}^{\mathrm{rig}}$ form a basis of the admissible topology of $\mathfrak{X}^{\mathrm{rig}}$, this determines $\gamma^* F_{\mathrm{RZ}(\mathfrak{X})}$.

Example 2.34. For $\mathfrak{Y} \in \Sigma_{\mathfrak{X}}$ and affine open $\mathfrak{U} \subset \mathfrak{Y}$, define $\mathcal{O}^{\mathrm{int}}_{\mathfrak{Y}}(\mathfrak{U})$ to be the integral closure of $\mathcal{O}_{\mathfrak{Y}}(\mathfrak{U})$ in $\mathcal{O}_{\mathfrak{Y}}(\mathfrak{U}) \otimes_{\mathcal{O}_K} K$. Then, one can check that this assignment extends to a sheaf $\mathcal{O}^{\mathrm{int}}_{\mathfrak{Y}}$ on $\mathfrak{Y}_{\mathrm{zar}}$ and satisfies $f_*\mathcal{O}^{\mathrm{int}}_{\mathfrak{Y}'} = \mathcal{O}^{\mathrm{int}}_{\mathfrak{Y}}$ for $f : \mathfrak{Y}' \to \mathfrak{Y} \in \Sigma_{\mathfrak{X}}$. By Remark 2.31, it gives rise to a sheaf $\mathcal{O}^{\mathrm{int}}_{\mathrm{RZ}(\mathfrak{X})}$ on $\mathrm{RZ}(\mathfrak{X})$. Moreover, we can show $\gamma^*\mathcal{O}^{\mathrm{int}}_{\mathrm{RZ}(\mathfrak{X})} = \mathcal{O}^{\circ}_{\mathfrak{X}^{\mathrm{rig}}}$, where the latter is a sheaf on $\mathfrak{X}^{\mathrm{rig}}$ from Example 2.12 (see also Remark 2.16).

2.6. **Base change theorem.** Let $X = \mathrm{Sp}(A)$ be a K-affinoid space.

Definition 2.35. An analytic point a of is a semi-norm $|-|_a : A \to \mathbb{R}_{\geq 0}$ satisfying:

(1) $|f+g|_a \leq \max\{|f|_a, |g|_a\}$ for $f, g \in A$.
(2) $|fg|_a = |f|_a |g|_a$ for $f, g \in A$.
(3) For $\lambda \in K$, $|\lambda|_a = |\lambda|$, where the latter is the norm on K.
(4) $|-|_a$ is continuous with respect to the norm topology on A.

A filter of the analytic point a consists of the affinoid subdomain $U = \mathrm{Sp}(B) \subset X$ for which $|-|_a$ extends to $B \to \mathbb{R}_{\geq 0}$, i.e. a is also an analytic point of U. For a sheaf F on X, the stalk F at an analytic point a is $F_a = \varinjlim_U F(U)$, where the colimit is indexed by the filter of a.

The set $\mathfrak{m}_a = \{f \in A |\ |f|_a = 0\}$ is a maximal ideal of the stalk $\mathcal{O}_a$ of the structure sheaf $\mathcal{O}_X$ on X and $|-|_a$ induces a norm on $k_a = \mathcal{O}_a/\mathfrak{m}_a$. We let F_a denotes the completion of k_a with respect to this norm.

Definition 2.36. An element $U = \mathrm{Sp}(B)$ of the filter of an analytic point a of X is a wide open neighborhood of a if there are $f_1, \dots, f_n \in B$ which generate B over A such that $|f_i|_a < 1$ for all i. For affinoid subdomains $V \subset U \subset X$, we say that U is a wide neighborhood of V in X, if for any analytic point of V, there is an affinoid wide neighborhood U_a of a in X such that $U_a \subset U$. In this case, we write $V \subset\subset_X U$. This is equivalent to that there are $f_1, \dots, f_n \in \mathcal{O}(U)$ generating $\mathcal{O}(U)$ over $\mathcal{O}(X)$ such that

$$V \subset \{x \in U |\ |f_i(x)| < 1 \text{ for } i = 1, \dots, n\},$$

For a rational subdomain

$$V = \{x \in X |\ |f_i(x)| \le |g(x)|\ (i = 1, \dots, r)\}$$

from Definition 2.4 with $f_1, \cdots, f_r, g \in A$ which generate the unit ideal, the family of the affinoid subdomains

$$V(r) = \{x \in X |\ |f_i(x)| \le r|g(x)|\ (i = 1, \dots, r)\}\ (r > 1, r \in \sqrt{|K^\times|})\}$$

forms a cofinal family of wide open neighborhood of V in X[8].

Definition 2.37. A presheaf F on X is called overconvergent if for any admissible open $V \subset X$, we have

$$F(V) = \varinjlim_{V \subset\subset_X U} F(U).$$

Lemma 2.38. *([9, Lem.2.3.2])*

(1) A constant sheaf is overconvergent. For $r, s \in \mathbb{R}$ with $s < r$, $\mathcal{O}(r, \infty) = \mathcal{O}/\mathcal{O}(r)$ and $\mathcal{O}(s, r) = \mathcal{O}(r)/\mathcal{O}(s)$ are overconvergent.

(2) If $X = \bigcup_{i \in I} U_i$ is a finite affinoid covering, a sheaf F on X is overconvergnet if and only if so is its restriction $F_{|U_i}$ for every $i \in I$.

(3) The category of overconvergnet sheaves (of abelian groups) on X is an exact subcategory of the category of all sheaves.

(4) For a map $f : Y \to X$ of rigid space over K and an overconvergnet sheaf F of abelian groups on Y, $R^i f_* F$ is overconvergent for all i

(5) An overconvergent sheaf F on X is zero if and only if $F_a = 0$ for all analytic points a of X.

Theorem 2.39. *([9, Thm.2.7.4]) Let $f : Y \to X$ be a quasi-compact morphism of rigid analytic spaces over K and F be a sheaf of abelian groups on Y. Then, for any analytic point a of X, there is a canonical isomorphism for all $n \ge 0$*

$$(R^n f_* F)_a \simeq H^n(Y_a, F_{|Y_a}),$$

where Y_a is the fiber of f over a, which is a rigid analytic space over F_a (see [9, §2.7]).

[8] For $\lambda \in K$ with $|\lambda| = r$, $\mathcal{O}(V(r)) = A\langle \frac{f_1}{\lambda g}, \dots, \frac{f_n}{\lambda g} \rangle$ and $V = \{x \in V(r) |\ |\frac{f_i}{\lambda g}(x)| \le \frac{1}{r} \text{ for } i = 1, \dots, n\}$

3. Valuation theory

3.1. **Valuations.** We fix some notations and recall definitions from, e.g., [11, 6.2], [17, 2].

Definition 3.1. A valuation field (K, v) consists of a field K endowed with a surjective group homomorphism $v : K^\times \to \Gamma_v$ onto a totally ordered abelian group Γ_v[9], such that

$$v(x+y) \leq \max\{v(x), v(y)\} \tag{3.1.1}$$

whenever $x+y \neq 0$. We denote by 1 the unit of Γ and the composition law of Γ is denoted by $(x, y) \to xy$. It is easy to check that $\mathcal{O}_v = \{x \in K |\ v(x) \leq 1\}$ is a subring of K and we called it the valuation ring of (K, v).

It is customary to extend v to K, by adding a new element 0 to Γ_v setting $v(0) := 0$. One can then extend the ordering of Γ_v to $\overline{\Gamma}_v := \Gamma_v \cup \{0\}$ by declaring that 0 is the smallest element of $\overline{\Gamma}_v$. By the convention, (3.1.1) holds for every $x \in K$.

We have the following facts (see [11, 6.1,12])

Lemma 3.2. *Let (K, v) be a valuation field with the valuation ring $\mathcal{O}_v$.*

1. *Every finitely generated ideal of $\mathcal{O}_v$ is principal.*
2. *Let L be a field extension of K. Then the integral closure W of $\mathcal{O}_v$ in L is the intersection of all the valuation rings of L containing $\mathcal{O}_v$. In particular, $\mathcal{O}_v$ is integrally closed.*
3. *If L is an algebraic extension of K and W be the integral closure of $\mathcal{O}_v$ in L. Then, for every prime ideal $\mathfrak{p} \subset W$, the localization $W_\mathfrak{p}$ is a valuation ring. Moreover, the assignment $\mathfrak{m} \to W_\mathfrak{m}$ gives a bijection between the set of maximal ideals of W and the set of valuation rings $\mathcal{O}_w$ of L whose associated valuation w extends v.*
4. *Let $\mathcal{O}_v^h$ be the henselization of $\mathcal{O}_v$ with the maximal ideal $\mathfrak{m}_v^h$ and $K^h = \mathrm{Frac}(\mathcal{O}_v^h)$. Then, $\mathcal{O}_v^h$ contains the integral closure W of $\mathcal{O}_v$ in K^h and we have $\mathcal{O}_v^h = W_\mathfrak{q}$, where $\mathfrak{q} := \mathfrak{m}_v^h \cap W$. By (3), this implies that $\mathcal{O}_v^h$ is again a valuation ring. The same argument works also for strict henselizations.*
5. *Any finitely generated torsion-free $\mathcal{O}_v$-module is free and any torsion-free $\mathcal{O}_v$-module is flat. Hence every $\mathcal{O}_v$-module is of Tor-dimension≤ 1.*
6. *A local subring of a field L is a valuation ring of L if and only if it is maximal for the dominance relation on the set of local subrings of L[10].*

Definition 3.3. Let $(K, v : K \to \overline{\Gamma}_v)$ be a valuation field. An extension of valued fields $(E, w : E \to \overline{\Gamma}_w)$ consists of a field extension E/K and a valuation $w : E \to \overline{\Gamma}_w$ together with an embedding $j : \Gamma_v \hookrightarrow \Gamma_w$ such that $w_{|K} = j \circ v$.

Example 3.4. Let $(K, v : K^\times \to \Gamma_v)$ be a valuation field and E/K be a field extension.

1. There always exist valuations on E which extends v ([30, Ch.VI, §1, n.3, Cor.3]).
2. If E/K is algebraic and purely inseparable, then the extension of v to E is unique. ([30, Ch.VI, §8, n.7, Cor.2]).
3. If E is the polynomial ring $K[X]$, we can construct extensions of v on E as follows: Let $\Gamma_v \hookrightarrow \Gamma'$ be an embedding of ordered groups. For every $x_0 in K$ and $\rho \in \Gamma$, we define the Gauss valuation centered at x_0 and with radius ρ:
$$v_{(x_0,\rho)} : K[X] \to \Gamma \cup \{0\},$$
sending $a_0 + a_1(X - x_0) + \cdots + a_n(X - x_0)^n$ to $\max\{v(a_i) \cdot \rho^i |\ i = 0, 1, \ldots, n\}$ ([30, 16, Ch.VI, §10, n.1, Lemma 1]).

[9] written multiplicatively

[10] For local subrings R and S of L, one says that R dominates S if $S \subset R$ and $\mathfrak{m}_S = \mathfrak{m}_R \cap S$, where $\mathfrak{m}_R$ and $\mathfrak{m}_S$ are the maximal ideals of R and S respectively. The relation of dominance defines a partial order structure on the set of local subrings of L.

3.2. **Tame extensions of valuation fields.** Let (K,v) be a valuation field with the valuation ring $\mathcal{O}_v$. Fix an embedding of (K,v) into $(\bar{K},\bar{v})$, where $\bar{K}$ is a separable closure of K and $\bar{v}$ is an extension of v to $\bar{K}$. We denote by (K_v^{sh},v^{sh}) the strict henselization of (K,v) (inside $(\bar{K},\bar{v})$). A finite separable extension $(L,w)/(K,v)$ of valuation fields is called *unramified* (resp. *tame*), if $K_v^{sh}=L_w^{sh}$ (resp. $([L_w^{sh}:K_v^{sh}],p)=1$, where p is the exponential characteristic of the residue field of $\mathcal{O}_v$). The *tame closure* (K^t,v^t) of (K,v) is the union of all finite tame Galois extensions of (K^{sh},v^{sh}). The field K^t is also the fixed field of $\bar{K}$ under the tame ramification group

$$R_{\bar{v}/v}:=\{\sigma\in \mathrm{Gal}(\bar{K}/K)\mid \sigma(\mathcal{O}_{\bar{v}})\subset \mathcal{O}_{\bar{v}} \text{ and } \frac{\sigma(x)}{x}-1\in \mathfrak{m}_{\bar{v}} \text{ for all } x\in \bar{K}^\times\}.$$

We record the following well-known lemma for later reference.

Lemma 3.5. *(1) Let $(L,w)/(K,v)$ be a finite separable extension of valuation fields. Let N/K be a Galois hull of L/K and let $\tilde{w}$ be an extension of w to N. Then $(L,w)/(K,v)$ is tame if and only if $(N,\tilde{w})/(K,v)$ is tame.*

In particular $(L,w)/(K,v)$ is tame if and only if (L,w) is a subextension of $(K^t,v^t)/(K,v)$.

(2) Let $(L,w)/(K,v)$ be a tame extension and let $(K',v')/(K,v)$ be any algebraic extension of valuation fields. Let $L\cdot K'$ be the composition field in an algebraic closure of K and let w' be a valuation extending v'. Then $(L\cdot K',w')/(K',v')$ is tame.

Proof. (1). Note that $N_{\tilde{w}}^{sh}$ is a Galois hull of L_w^{sh}/K_v^{sh}. Therefore we may assume K,L,N are strictly henselian valuation fields of characteristic $p>0$. Thus if $(N,\tilde{w})/(K,v)$ is tame then $[N:K]=[N:L]\cdot[L:K]$ is prime to p and hence $(L,w)/(K,v)$ is tame as well. Now assume $(L,w)/(K,v)$ is tame. Denote by $G_K\supset G_L\supset G_N$ the absolute Galois groups with respect to a fixed separable closure $\bar{K}$ of K, and by P the pro-p-Sylow subgroup of G_K, which is a normal subgroup. The indices satisfy the following equality (of supernatural numbers)

$$[G_K:G_L]\cdot[G_L:P\cap G_L]=[G_K:P]\cdot[P:P\cap G_L].$$

As P is a normal subgroup of G_K, the intersection $P\cap G_L$ is a normal subgroup of G_L and we have an inclusion of profinite groups $G_L/G_L\cap P\hookrightarrow G/P$. Hence $[G:P]$ and $[G_L:P\cap G_L]$ are prime to p. By assumption $[G_K:G_L]=[L:K]$ is prime to p as well. Thus $[P:P\cap G_L]=1$, i.e., $P=P\cap G_L$. The Galois hull of L/K is the composition field (inside $\bar{K}$) of all the $\sigma(L)$, where σ runs through all the embeddings $L\hookrightarrow \bar{K}$. Extending these σ's to K-automorphisms of $\bar{K}$, we find $G_{\sigma(L)}=\sigma G_L\sigma^{-1}$. Hence $G_N=\cap_\sigma \sigma G_L\sigma^{-1}$. As P is a normal subgroup of G_K it follows that P is contained in G_N as well. Thus $[G_K:P]=[G_K:G_N]\cdot[G_N:P]$ is prime to p and hence so is $[N:K]=[G_K:G_N]$.

(2) follows from (1) and the fact that $K'^t=K'\cdot K^t$, see [11, 6.2.18]. □

4. Spectral spaces

Definition 4.1. A topological space is called spectral if it is sober[11], quasi-compact, the intersection of two quasi-compact opens is quasi-compact, and the collection of quasi-compact opens forms a basis for the topology.

Lemma 4.2. *For a topological space X, the following conditions are equivalent.*

(1) X is spectral.
(2) X is a directed inverse limit of finite sober topological spaces.
(3) X is homeomrophic to $\mathrm{Spec}(R)$ *for some commutative ring R.*

Definition 4.3. Let X be a spectral space. The constructible topology on X is the topology which has as a base of opens, the sets U and U^c for a quasi-compact open $U\subset X$.

[11] i.e. every nonempty irreducible closed subset has a unique generic point.

Note that an open U in a spectral space X is retrocompact[12] Hence, the constructible topology can also be characterized as the coarsest topology such that every constructible subset[13] of X is both open and closed. It follows that a subset of X is open (resp. closed) in the constructible topology if and only if it is a union (resp. intersection) of constructible subsets. Since the collection of quasi-compact opens is a basis for the topology on X, we see that the constructible topology is stronger than the given topology on X.

Lemma 4.4. *The constructible topology on a sepctral sapce is Hausdorff, totally disconnected, and quasi-compact.*

Proof. [35, Tag 0901] □

5. Adic spaces

Definition 5.1. For a morphism of schemes $X \to \tilde{X}$, let $\mathrm{Spa}(X, \tilde{X})$ be the set of triples (x, v, ε) such that $x \in X$, v is a valuation on $k(x)$ and $\varepsilon\colon \mathrm{Spec}(\mathcal{O}_v) \to \tilde{X}$ is a map compatible with $\mathrm{Spec}(k(x)) \to X$. Let $Y \to \tilde{Y}$ be a morphism of schemes and $(\varphi, \tilde{\varphi}) : (Y, \tilde{Y}) \to (X, \tilde{X})$ be morphisms such that the following diagram commutative:

(5.1.1)
$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d] & X \arrow[d] \\ \tilde{Y} \arrow[r, "\tilde{\varphi}"] & \tilde{X} \end{tikzcd}$$

Then, we have an induced map $\mathrm{Spa}(Y, \tilde{Y}) \to \mathrm{Spa}(X, \tilde{X})$[14]. We equip $\mathrm{Spa}(X, \tilde{X})$ with a topology as follows: If $X = \mathrm{Spec}(A)$ and $\tilde{X} = \mathrm{Spec}(\tilde{A})$ are affine, the topology is generated by the subset of the form[15]

$$\{(x, v, \varepsilon)|\ v(f_i) \leq v(g) \neq 0\ \forall i = 1, \ldots, m\} \text{ for } f_1, \ldots, f_m, g \in A.$$

In general, we declare that a subset $V \subset \mathrm{Spa}(X, \tilde{X})$ is open if for any commutative diagram (5.1.1) where Y and $\tilde{Y}$ are affine, φ is an open immersion and $\tilde{\varphi}$ is locally of finite type, the inverse image of V in $\mathrm{Spa}(Y.\tilde{Y})$ is open.

Lemma 5.2. *(1) If X and $\tilde{X}$ are quasi-compact and quasi-separated, then* $\mathrm{Spa}(X, \tilde{X})$ *is a spectral space, i.e. homeomorphic to* $\mathrm{Spec}(R)$ *for some commutative ring R. In particular,* $\mathrm{Spa}(X, \tilde{X})$ *is a quasi-compact and quasi-separated topological space.*
(2) Let $(\varphi, \tilde{\varphi})$ be as (5.1.1) and assume that φ is étale and $\tilde{\varphi}$ is locally of finite type. Then, the set of points $(y, w, \varepsilon_w) \in \mathrm{Spa}(Y, \tilde{Y})$ such that the extension $(k(y), w)/(k(\varphi(y)), w_{|k(\varphi(y))})$ is tame is open as well as the set of points $(x, v, \varepsilon_v) \in \mathrm{Spa}(X, \tilde{X})$ such that there exists $(y, w, \varepsilon_w) \in \mathrm{Spa}(Y, \tilde{Y})$ mapping to (x, v, ε_v) such that the extension $(k(y), w)/(k(x), v)$ is tame.

Proof. (1) follows from [17, Lem.4.3] and (2) from [15, Cor.4.4] and [14, Pr.1.7.8]. □

[12] i.e. the inclusion map $U \to X$ is quasi-compact.

[13] i.e. a finite union of subsets of the form $U \cap V^c$ where $U, V \subset X$ are open and retrocompact in X.

[14] sending (y, w, ε') to $(x = \varphi(y), v = w_{|k(x)}, \varepsilon)$ with $\varepsilon : \mathrm{Spec}(\mathcal{O}_v) \to \tilde{X}$ induced by $\mathrm{Spec}(k(x)) \to X \to \tilde{X}$, $\mathrm{Spec}(k(y)) \to Y \to \tilde{X}$ and $\mathrm{Spec}(\mathcal{O}_w) \to \tilde{Y} \to \tilde{X}$ noting $\mathrm{Spec}(\mathcal{O}_v) = \mathrm{Spec}(\mathcal{O}_w) \sqcup_{\mathrm{Spec}(k(y))} \mathrm{Spec}(k(x)$.

[15] This dictates that both $\{(x, v, \varepsilon)|\ v(f) \leq 1\}$ and $\{(x, v, \varepsilon)|\ v(f) \neq 0\}$ be open for $f \in A$.

Part 2. Tame cohomology

6. Tame topos

The site below is very much inspired by the definition of the étale and tame site of a Huber pair (see [14] and [17]). For a scheme S, let $\mathbf{Sch}_S$ be the category of schemes separated of finite type over S.

Definition 6.1. Let $X \to \tilde{X}$ be an open immersion of noetherian schcemes[16].

(1) Let $(X, \tilde{X})_\tau$ be the category of pairs $(U, \tilde{U})$ equipped with an open immersion $U \to \tilde{U}$ in $\mathbf{Sch}_{\tilde{X}}$ such that $U \to \tilde{X}$ factors through an étale morphism $U \to X$. Morphisms $(V, \tilde{V}) \to (U, \tilde{U})$ are pairs of morphisms $f : V \to U$ in $X_{\text{ét}}$ and $\tilde{f} : \tilde{V} \to \tilde{U}$ in $\mathbf{Sch}_{\tilde{X}}$ satisfying the obvious compatibility. This category has fiber products given by
$$(V_1, \tilde{V}_1) \times_{(U,\tilde{U})} (V_2, \tilde{V}_2) = (V_1 \times_U V_2, \tilde{V}_1 \times_{\tilde{U}} \tilde{V}_2),$$
and terminal object $(X, \tilde{X})$.

(2) A morphism $(f, \tilde{f}) : (V, \tilde{V}) \to (U, \tilde{U})$ in $(X, \tilde{X})_\tau$ is a *modification* if f is an isomorphism and $\tilde{f}$ is proper.

(3) A morphism $(f, \tilde{f}) : (V, \tilde{V}) \to (U, \tilde{U})$ in $(X, \tilde{X})_\tau$ is *strict étale* if $\tilde{f}$ is étale, $V = \tilde{V} \times_{\tilde{U}} U$ and $f = \tilde{f} \times_{\tilde{U}} \mathrm{Id}$.

(4) A morphism $(f, \tilde{f}) : (V, \tilde{V}) \to (U, \tilde{U})$ in $(X, \tilde{X})_\tau$ is is *tame over* $(x, v, \varepsilon_v) \in \mathrm{Spa}(U, \tilde{U})$ if there is $(y, w, \varepsilon_w) \in \mathrm{Spa}(V, \tilde{V})$ such that $f(y) = x$, $w_{|k(x)} = v$, and w/v is tamely ramified and the following diagram commutes:
$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_w) & \xrightarrow{\varepsilon_w} & \tilde{V} \\ \downarrow{\scriptstyle (f)_{|y,w\geq 0}} & & \downarrow{\scriptstyle \tilde{f}} \\ \mathrm{Spec}(\mathcal{O}_v) & \xrightarrow{\varepsilon_v} & \tilde{U}. \end{array}$$
It is tame if it is tame over any $(x, v, \varepsilon_v) \in \mathrm{Spa}(U, \tilde{U})$.

On this category, we will consider the following three topologies:

(1) The *strict étale* topology which is generated by strict étale coverings
(2) The *v-étale* topology which is generated by strict étale coverings and modifications.
(3) The *tame* topology generated by tame coverings, where a family $\{(f_i, \tilde{f}_i) : (V_i, \tilde{V}_i) \to (U, \tilde{U})\}_{i \in I}$ in $(X, \tilde{X})_\tau$ is a tame covering if for every $(x, v, \varepsilon_v) \in \mathrm{Spa}(U, \tilde{U})$, there is $i \in I$ such that $(V_i, \tilde{V}_i) \to (U, \tilde{U})$ is tame over (x, v, ε_v).

We let $(X, \tilde{X})_{\text{sét}}$, $(X, \tilde{X})_{\text{vét}}$ and $(X, \tilde{X})_t$ denote the strict étale, the v-étale and the tame site on $(X, \tilde{X})_\tau$ respectively.

$$(X, \tilde{X})_t \xrightarrow{\nu} (X, \tilde{X})_{\text{vét}} \xrightarrow{\mu} (X, \tilde{X})_{\text{sét}} \tag{6.1.1}$$

corresponding to the inclusion functors.

Remark 6.2. Note that by the valuative criterion, if $\tilde{f} : \tilde{U}' \to \tilde{U}$ is separated and universally closed and $U \to \tilde{U}'$ is any map, we have a bijection $\mathrm{Spa}(U, \tilde{U}) \cong \mathrm{Spa}(U, \tilde{U}')$, which is a homeomorphism if $\tilde{f}$ is of finite type (hence proper), see [15, Lemma 2.2]. In particular, every modification in $(X, \tilde{X})_\tau$ is a tame covering.

[16]The construction can be done for qsqs schemes. Here, we only treat noetherian case for simplicity.

Lemma 6.3. *Consider a commutative diagram of rings*

$$\begin{tikzcd}
\widetilde{R} \arrow[r, "\varphi"] \arrow[d, "\tilde{f}"] & R \arrow[d, "f"] \\
\tilde{A} \arrow[r, "\psi"] & A
\end{tikzcd}$$

where f and $\tilde{f}$ are of finite presentation. Let $\{(B_i, \tilde{B}_i)\}_{i\in I}$ be a filtered system of pairs of rings and $(g_i, \tilde{g}_i)_{i\in I} \colon (R, \widetilde{R}) \to \{(B_i, \tilde{B}_i)\}_{i\in I}$ be a system of pairs of maps of rings. Let $B = \varinjlim B_i$ and $\tilde{B} = \varinjlim \tilde{B}_i$ and $(g, \tilde{g}) = \varinjlim (g_i, \tilde{g}_i) : (R, \widetilde{R}) \to (B, \tilde{B})$. Then, we have an isomorphism

$$\varinjlim_{i\in I} \mathrm{Hom}_{(R,\widetilde{R})}((A, \tilde{A}), (B_i, \tilde{B}_i)) \simeq \mathrm{Hom}_{(R,\widetilde{R})}((A, \tilde{A}), (B, \tilde{B})),$$

which means that for all $(h, \tilde{h}) \colon (A, \tilde{A}) \to (B, \tilde{B})$ compatible with $(f, \tilde{f})$ and $(g, \tilde{g})$, there is $i \in I$ and $(h_i, \tilde{h}_i)$ fitting into the following commutative squares of pairs of rings:

$$\begin{tikzcd}
(R, \widetilde{R}) \arrow[r, "(f,\tilde{f})"] \arrow[d, "(g_i,\tilde{g}_i)"'] & (A, \tilde{A}) \arrow[d, "(h,\tilde{h})"] \arrow[dl, "(h_i,\tilde{h}_i)"'] \\
(B_i, \tilde{B}_i) \arrow[r] & (B, \tilde{B}).
\end{tikzcd}$$

Proof. An exercise to use [35, Tag 00QO]. □

Remark 6.4.
(1) As X is quasicompact, the tame topology is finitary: any covering can be refined by a covering of the form $g : (V, \tilde{V}) \to (U, \tilde{U})$ for $(V, \tilde{V})$ and $(U, \tilde{U})$ in $(X, \tilde{X})_t$.
(2) Let $(U, \tilde{U}) \in (X, \tilde{X})_t$ and let $\overline{U}$ be the closure of U in $\tilde{U}$. Then $(U, \overline{U}) \to (U, \tilde{U})$ is a modification, hence it is a v-étale covering and a tame covering.
(3) Note that for any $(U, \tilde{U}) \in (X, \tilde{X})_t$, there exists a *finitely generated* ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\tilde{U}}$ such that the support of $\mathcal{O}_{\tilde{U}}/\mathcal{I}$ is equal to $\tilde{U} \setminus U$. Thus, blowing up $\tilde{U}$ in such an ideal we obtain a modification $(U, \bar{U}) \to (U, \tilde{U})$ such that the complement $\bar{U} \setminus U$ is the support of an effective Cartier divisor.

Lemma 6.5. *Let $(X, \tilde{X}') \to (X, \tilde{X})$ be a modification. Then, for any sheaf $F \in \mathbf{Shv}((X, \tilde{X})_{\text{vét}})$ and $(U, \tilde{U}) \in (X, \tilde{X})_\tau$, we have an isomorphism*

$$F(U, \tilde{U}) \cong F(U, \tilde{U} \times_{\tilde{X}} \tilde{X}').$$

In particular, for $\gamma \in \{\text{vét}, t\}$, the functors

$$\mathbf{Shv}((X, \tilde{X})_\gamma) \to \mathbf{Shv}((X, \tilde{X}')_\gamma)$$

induced by the inclusion $(X, \tilde{X}')_\tau \to (X, \tilde{X})_\tau$ is an equivalence of topoi so that we have equivalences

$$R\Gamma_\gamma((X, \tilde{X}), F) \cong R\Gamma_\gamma((X, \tilde{X}'), F) \text{ for } F \in \mathbf{Shv}((X, \tilde{X})_\gamma, \mathbf{Ab}).$$

Proof. It suffices to prove it for $\gamma = \text{vét}$. Let F be a sheaf of sets on $(X, \tilde{X})_{\text{vét}}$. For $(U, \tilde{U}) \in (X, \tilde{X})_\tau$ with $\tilde{U}' = \tilde{U} \times_{\tilde{X}} \tilde{X}'$, $(U, \tilde{U}') \to (U, \tilde{U})$ and the diagonal map $\delta : (U, \tilde{U}') \to (U, \tilde{U}' \times_{\tilde{U}} \tilde{U}')$ are modifications so coverings in $(X, \tilde{X})_{\text{vét}}$. Hence,

$$\delta^* : F(U, \tilde{U}' \times_{\tilde{U}} \tilde{U}') \to F(U, \tilde{U}')$$

is injective and we find

$$F(U,\tilde U) \cong \mathrm{eq}\left(F(U,\tilde U') \underset{pr_2^*}{\overset{pr_1^*}{\rightrightarrows}} F(U,\tilde U'\times_{\tilde U}\tilde U')\right)$$
$$\cong \mathrm{eq}\left(F(U,\tilde U') \underset{id}{\overset{id}{\rightrightarrows}} F(U,\tilde U')\right) = F(U,\tilde U').$$

Hence, the functor

$$\mathbf{Shv}((X,\tilde X')_{\text{vét}}) \to \mathbf{Shv}((X,\tilde X)_{\text{vét}}):\ G \mapsto \Big((U,\tilde U)\mapsto G(U,\tilde U\times_{\tilde X}\tilde X')\Big)$$

gives a quasi-inverse of the restiction functor $\mathbf{Shv}((X,\tilde X)_{\text{vét}}) \to \mathbf{Shv}((X,\tilde X')_{\text{vét}})$. □

6.1. **Tame cohomology over a base.** Let $\eta \hookrightarrow S$ be an open immersion of noetherian schemes. A basic example is $\eta = \mathrm{Spec}(K)$ and $S = \mathrm{Spec}(\mathcal{O}_K)$ for a complete discrete valuation field K with the ring $\mathcal{O}_K$ of integers, or $\eta = S = \mathrm{Spec}(k)$ for a field k.

Definition 6.6. Let $\mathbf{Sch}_{(\eta,S)}$ be the category whose objects are pairs $(U,\tilde U)$ equipped with an open immersion $U \hookrightarrow \tilde U$ over S such that $U \to S$ factors through $\eta \hookrightarrow S$. Morphisms $(V,\tilde V) \to (U,\tilde U)$ are pairs of morphisms $f: V \to U$ in $\mathbf{Sch}_\eta$ and $\tilde f: \tilde V \to \tilde U$ in $\mathbf{Sch}_S$ satisfying the obvious compatibility.. For $(X,\tilde X)\in \mathbf{Sch}_{(\eta,S)}$, there is a functor $\iota_{(X,\tilde X)}: (X,\tilde X)_\tau \to \mathbf{Sch}_{(\eta,S)}$, which is the identity on objects. We define the tame topology on $\mathbf{Sch}_{(\eta,S)}$ by declaring that the covering families are the images under $\iota_{(X,\tilde X)}$ of the covering families in $(X,\tilde X)_t$ for all $(X,\tilde X)\in\mathbf{Sch}_{(\eta,S)}$. Let $\mathbf{Sch}_{(\eta,S),t}$ denote the corresponding site.

For $F\in \mathbf{Shv}(\mathbf{Sch}_{(\eta,S),t})$ and $X\in\mathbf{Sch}_\eta$, we define

$$R\Gamma_t(X/S,F) = \varprojlim_{(X,\tilde X)} R\Gamma((X,\tilde X)_t, F_{|(X,\tilde X)_t})$$

where the limit is indexed by the category $\mathcal{N}(X/S)$ of all Nagata compactifications $X\hookrightarrow\tilde X$ of $X\to S$. By Lemma 6.5, for every $\tilde X\in\mathcal{N}(X/S)$, the projection induces an equivalence

$$R\Gamma_t(X/S,F) \simeq R\Gamma((X,\tilde X)_t, F_{|(X,\tilde X)_t}). \tag{6.6.1}$$

Lemma 6.7. *The association $X\to R\Gamma_t(X/S,F)$ extends to a functor*

$$R\Gamma_t(-/S,F): \mathbf{Sch}_\eta \to \mathcal{D}(\mathbb{Z}).$$

Proof. For a morphism $f: U\to V$ in $\mathbf{Sch}_\eta$, we can construct a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "f"] & \tilde U \arrow[d, "\tilde f"] \\ V \arrow[r] & \tilde V \end{tikzcd}$$

where $U\hookrightarrow\tilde U$ (resp. $V\hookrightarrow\tilde V$) is a Nagata compactifications of $U\to S$ (resp. $V\to S$). Using (6.6.1), this induces a map

$$f^*: R\Gamma_t(V/S,F) = R\Gamma_t((V,\tilde V),F) \xrightarrow{(f,\tilde f)^*} R\Gamma_t(U/S,F) = R\Gamma_t((U,\tilde U),F).$$

It is standard to check that the construction gives the desired functor.

□

6.2. **Comparison with the tame site of Hübner-Schmidt.** Now we compare our tame site $(X,\tilde{X})_t$ with the tame site $(X/S)_t$ from [17].

Definition 6.8. Let $X \to S$ be a morphism of schemes and let $X_{\text{ét}}$ be the category of étale morphisms $U \to X$. Consider the following Grothendieck topology. A family $\{U_i \to U\}_{i\in I}$ in $X_{\text{ét}}$ is a covering if it is an étale covering and for every $(x,v,\varepsilon_v) \in \operatorname{Spa}(U,S)$, there is $i \in I$ and $(y,w,\varepsilon_w) \in \operatorname{Spa}(V_i,S)$ lying over (x,v,ε_v) such that $(k(y),w)/(k(x),v)$ is tame. We let $(X/S)_t$ denote the site of $X_{\text{ét}}$ with the above topology.

Proposition 6.9. *Let $X \to S$ and $X \hookrightarrow \tilde{X}$ be as before such that there exists a separated and proper map $\tilde{X} \to S$. Then there are adjoint functors*

$$\mathbf{Shv}((X,\tilde{X})_t) \underset{u_*}{\overset{u^*}{\leftrightarrows}} \mathbf{Shv}((X/S)_t),$$

with u^* exact, such that for $F \in \mathbf{Shv}((X,\tilde{X})_t)$ and $U \in (X/S)_t$ affine, $u_*F(U) = F(U,\tilde{U})$ for any choice of a Nagata compactification $U \hookrightarrow \tilde{U}$ of $U \to \tilde{X}$, and for $G \in \mathbf{Shv}((X/S)_t)$, u^*G is the sheafification of the presheaf $(U,\tilde{U}) \mapsto G(U)$. If $(U,\tilde{U}) \in (X,\tilde{X})_t$ with $\tilde{U} \to \tilde{X}$ separated and proper, then $u^*G(U,\tilde{U}) = G(U)$, in particular $G \cong u_*u^*G$.

Proof. See [24]. ∎

Example 6.10. Consider the sheaf $\mathcal{O}^t \in \mathbf{Shv}((X,\tilde{X})_t)$ from Example 0.4. Then,

$$u_*\mathcal{O}^t(U) = \mathcal{O}^t(U,\tilde{U}) = \mathcal{O}(\tilde{U}) = \mathcal{O}(T) \text{ for } U \in X_{\text{ét}},$$

where $U \hookrightarrow \tilde{U}$ is a Nagata compactification of $U \to \tilde{X}$ with $\tilde{U}$ normal and $\tilde{U} \to T \to S$ is the Stein factorization of the proper morphism $\tilde{U} \to S$. Thus, the functor u_* loses information on $\mathcal{O}^t$.

6.3. **Affine objects.**

Definition 6.11. Let $(X,\tilde{X})_{\text{affine},\tau}$ (resp. $(X,\tilde{X})_{\text{int},\tau}$) be the full subcategory of $(X,\tilde{X})_\tau$ whose objects are affine pairs $(U,\tilde{U}) = (\operatorname{Spec}(A),\operatorname{Spec}(\tilde{A}))$ (resp. such that $\tilde{A} \to A$ injective and integrally closed). We make them sites by the restriction of v-étale and tame topologies.

Lemma 6.12. *The inclusions of sites $(X,\tilde{X})_{\text{affine},\text{vét}} \to (X,\tilde{X})_{\text{vét}}$ (resp. $(X,\tilde{X})_{\text{affine},t} \to (X,\tilde{X})_t$) induce equivalence on the topoi. The similar fact holds for $(X,\tilde{X})_{\text{int},\tau}$.*

Proof. We write the proof for vét, the proof for t is analogous. We need to check the properties (1)-(5) of [35, Tag 03A0]: the inclusion is clearly continuous and fully faithful, therefore (2), (3) and (4) are satisfied. In order to check (1), we need to prove that every $(V,\tilde{V}) \in (X,\tilde{X})_\tau$ can be covered by objects in $(X,\tilde{X})_{\text{affine}}$ Consider an affine open cover $\cup_{i\in I}\operatorname{Spec}(B_i)$ of $\tilde{V}$, so that $(V,\tilde{V})$ is covered by $(\operatorname{Spec}(B_i)\cap V,\operatorname{Spec}(V_i))$. Since $\tilde{V}$ is quasi-separated $V \cap \operatorname{Spec}(B_i)$ is quasi-compact. Consider a basic open cover of $\operatorname{Spec}(B_i)\cap V = \cup_{j\in J_i}\operatorname{Spec}(B_i[1/f_{ij}])$, with J_i finite and f_{ij} not nilpotent: let $\mathfrak{I}_i$ be the ideal of $\operatorname{Spec}(B_i)$ generated by f_{ij}: then $V(\mathfrak{I}_i)$ is disjoint from $\operatorname{Spec}(B_i)\cap V$, so that $(\operatorname{Spec}(B_i)\cap V,\operatorname{Spec}(B_i))$ is covered by the modification $(\operatorname{Spec}(B_i)\cap V,\operatorname{Bl}_{V(\mathfrak{I}_i)}(\operatorname{Spec}(B_i)))$. Finally, $\operatorname{Bl}_{V(\mathfrak{I}_i)}(\operatorname{Spec}(B_i))$ is covered by the affine blow-up algebras $\operatorname{Spec}(B_i[\frac{\mathfrak{I}_i}{f_{ij}}])$ whose intersection with $\operatorname{Spec}(B_i)\cap V$ is $\operatorname{Spec}(B_i[1/f_{ij}])$: putting everything together we have a cover $\{(\operatorname{Spec}(B_i[1/f_{ij}]),\operatorname{Spec}(B_i[\frac{\mathfrak{I}_i}{f_{ij}}])) \to (V,\tilde{V})\}_{i\in I,j\in J_i}$. Then (5) also holds in a similar manner. ∎

6.4. **Computation of the v-étale topology.**

Lemma 6.13. *Every composition $\mathcal{V} \xrightarrow{\varphi_1} \mathcal{U} \xrightarrow{\varphi_2} \mathcal{Y}$ in $(X,\tilde{X})_\tau$ with φ_2 strict étale and φ_1 a modificatioon, there is a modification $\psi_2 : \mathcal{T} \to \mathcal{Y}$ such that $\mathcal{U}\times_{\mathcal{Y}}\mathcal{T} \to \mathcal{Y}$ factors through a modification $\mathcal{U}\times_{\mathcal{Y}}\mathcal{T} \to \mathcal{V}$.*

Proof. We may assume $\mathcal{V} \to \mathcal{U} \to \mathcal{Y}$ is of the form $(U, \tilde{V}) \xrightarrow{(id, \tilde{f}_1)} (U, \tilde{U}) \xrightarrow{(f_2, \tilde{f}_2)} (Y, \tilde{Y})$, where $\tilde{f}_2$ is étale and $f_2 = \tilde{f}_2 \times_{\tilde{Y}} Y$ and $\tilde{f}_1$ is proper and the identity on U. The following argument is classical (see [23, Proposition 12.27]): As observed in Remark 6.4, we can assume that Y is dense in $\tilde{Y}$ is dense. By Raynaud-Gruson [35, Tag 081R], there exists $\tilde{g}_1 : \tilde{T} \to \tilde{Y}$ a Y-admissible blow-up such that the strict transform $\tilde{V}'$ of $\tilde{V}$ over $\tilde{T}$ is flat of finite presentation over $\tilde{T}$. Note that the map $\tilde{V}' \to \tilde{T}$ factors as $\tilde{V}' \xrightarrow{\alpha} \tilde{U} \times_{\tilde{Y}} \tilde{T} \xrightarrow{\beta} \tilde{T}$, where β is étale and α is proper inducing an isomorphism over the dense open $U = \tilde{U} \times_{\tilde{Y}} \tilde{T} \times_{\tilde{Y}} Y$. Moreover, α is flat by [20, Lem.4.15] so it is an isomorphism by [20, Lem.4.16]. Hence, we get a morphism $\tilde{U} \times_{\tilde{Y}} \tilde{T} \to \tilde{V}$, which is proper and an isomorphism over U. This completes the proof. □

Lemma 6.14. *For $F \in \mathbf{Shv}((X, \tilde{X})_{\text{sét}})$ and $\mathcal{U} \in (X, \tilde{X})_\tau$, we have*

$$a_{\text{vét}}(F)(\mathcal{U}) = \varinjlim_{\mathcal{V} \to \mathcal{U}} F(\tilde{V}) \tag{6.14.1}$$

where the colimit runs along all modifications of $\mathcal{U}$.

Proof. Let αF be the presheaf on $(X, \tilde{X})_\tau$ defined by the right hand side of (6.14.1). First, we claim that $\alpha F \in \mathbf{Shv}((X, \tilde{X})_{\text{vét}})$: By the definition, αF sends modifications to isomorphisms, so it has descent for those coverings. It remains to prove that αF has descent for every strict étale covering $\{\mathcal{U}_i \to \mathcal{U}\}_{i \in I}$. By a standard reduction, we may assume $I = \{1, 2\}$. Using $F \in \mathbf{Shv}((X, \tilde{X})_{\text{sét}})$, for any modification $\mathcal{V} \to \mathcal{U}$, we have

$$F(\mathcal{V}) = F(\mathcal{V} \times_{\mathcal{U}} \mathcal{U}_1) \times_{F(\mathcal{V} \times_{\mathcal{U}} \mathcal{U}_{12})} F(\mathcal{V} \times_{\mathcal{U}} \mathcal{U}_2),$$

where $\mathcal{U}_{12} = \mathcal{U}_1 \times_{\mathcal{U}} \mathcal{U}_2$. Taking the colimit over $\mathcal{V}$ and using Lemma 6.13 and the fact that filtered colimits commute with fiber products, we get

$$\alpha F(\mathcal{U}) = \alpha F(\mathcal{U}_1) \times_{\alpha F(\mathcal{U}_{12})} \alpha F(\mathcal{U}_2),$$

which proves the claim.

Thus, we get a functor $\alpha : \mathbf{Shv}((X, \tilde{X})_{\text{sét}}) \to \mathbf{Shv}((X, \tilde{X})_{\text{vét}})$. It suffices to show that it is a left adjoint of the inclusion $i : \mathbf{Shv}((X, \tilde{X})_{\text{vét}}) \to \mathbf{Shv}((X, \tilde{X})_{\text{sét}})$. By construction we have a natural transformation $\mathrm{id} \to i\alpha$ and by Lemma 6.5 also a natural isomorphism $\alpha i \xrightarrow{\simeq} \mathrm{id}$. The statement thus follows from [28, IV, §1, Theorem 2(v)]. □

Lemma 6.15. *Let F be a sheaf of abelian groups on $(X, \tilde{X})_{\text{sét}}$. If I is flabby, i.e. $H^i_{\text{sét}}(\mathcal{V}, F) = 0$ for any $i > 0$ and $\mathcal{V} \in (X, \tilde{X})_\tau$, then $a_{\text{vét}} I$ is flabby as a* vét *sheaf.*

Proof. By Lemma 1.22, for any $\mathcal{U} \in (X, \tilde{X})_\tau$ and a strict étale covering $\mathcal{U}' \to \mathcal{U}$, the Čech complex

$$0 \to I(\mathcal{U}) \to I(\mathcal{U}') \to I(\mathcal{U}' \times_{\mathcal{U}} \mathcal{U}') \to \cdots$$

is exact. This implies that for any modification $\mathcal{V} \to \mathcal{U}$, the Čech complex

$$0 \to I(\mathcal{V}) \to I(\mathcal{V} \times_{\mathcal{U}} \mathcal{U}') \to I(\mathcal{V} \times_{\mathcal{U}} \mathcal{U}' \times_{\mathcal{U}} \mathcal{U}') \to \cdots$$

is also exact. Noting that filtered colimits are exact, Lemmas 6.13 and 6.14 implies

$$0 \to a_{\text{vét}} I(\mathcal{U}) \to a_{\text{vét}} I(\mathcal{U}') \to a_{\text{vét}} I(\mathcal{U}' \times_{\mathcal{U}} \mathcal{U}') \to \cdots$$

is exact. Noting $a_{\text{vét}} I(\mathcal{U}) \simeq a_{\text{vét}} I(\mathcal{V})$ for any modification $\mathcal{V} \to \mathcal{U}$, this implies that $a_{\text{vét}} I$ is flabby on $(X, \tilde{X})_{\text{vét}}$ again by Lemma 1.22. □

Definition 6.16. For $\mathcal{U} = (U, \tilde{U}) \in (X, \tilde{X})_{\text{vét}}$, let $\lambda^{\mathcal{U}} : (U, \tilde{U})_{\text{vét}} \to \tilde{U}_{\text{ét}}$ be the morphism of sites defined by the functor

$$\tilde{U}_{\text{ét}} \to (U, \tilde{U})_{\text{vét}} \ : \ \tilde{V} \mapsto (\tilde{V} \times_{\tilde{U}} U, \tilde{V}).$$

It is clear by construction that

$$H^q_{\text{sét}}((U, \tilde{U}), F) = H^q_{\text{ét}}(\tilde{U}, \lambda^{\mathcal{U}}_* F). \tag{6.16.1}$$

Lemma 6.17. *For $F \in \mathbf{Shv}((X,\tilde{X})_{\text{vét}}, \mathbf{Ab})$ and $\mathcal{U} = (U,\tilde{U}) \in (X,\tilde{X})_\tau$, there is a natural isomorphism*

$$H^i_{\text{vét}}(\mathcal{U}, F) = \varinjlim_{\mathcal{V}\to\mathcal{U}} H^i_{\text{ét}}(\tilde{V}, \lambda^{\mathcal{V}}_* F),$$

where the colimit is over the cofiltered category of modifications $\mathcal{V} \to \mathcal{U}$.

Proof. This is very similar to [6, Th. 1.2.2]. Take a flabby resolution $F_{|(X,\tilde{X})_{\text{sét}}} \to I^\bullet$ of the restriction of F on $(X,\tilde{X})_{\text{sét}}$. By Lemma 6.15, this gives a flabby resolution $F \to a_{\text{vét}} I^\bullet$ on $(X,\tilde{X})_{\text{vét}}$. Therefore,

$$\begin{aligned} H^i_{\text{vét}}(\mathcal{U}, F) = H^i(a_{\text{vét}} I^\bullet(\mathcal{U})) &\overset{(*1)}{=} H^i(\varinjlim_{\mathcal{V}\to\mathcal{U}} I^\bullet(\mathcal{V})) \\ &= \varinjlim_{\mathcal{V}\to\mathcal{U}} H^i(I^\bullet(\mathcal{V})) = \varinjlim_{\mathcal{V}\to\mathcal{U}} H^i_{\text{sét}}(\mathcal{V}, F) \overset{(*1)}{=} \varinjlim_{\mathcal{V}\to\mathcal{U}} H^i_{\text{ét}}(\tilde{V}, \lambda^{\mathcal{V}}_* F), \end{aligned}$$

where $(*1)$ follows from Lemma 6.14 and $(*2)$ from (6.16.1). $\square$

7. Construction of tame sheaves

Let $X \hookrightarrow \tilde{X}$ be an open immersion of noetherian schemes. We give a method to extend étale sheaves defined over $X_{\text{ét}}$ to sheaves on $(X, \tilde{X})_t$.

7.1. For $\mathcal{X} = (X, \tilde{X})$, we let $\mathrm{Val}^{\mathrm{s}}_{\mathcal{X}}$ be the category whose objects are triples (L, w, ε), where L is a finite separable field extension of a residue field $k(x)$ of a point $x \in X$, w is a valuation on L, and $\varepsilon\colon \operatorname{Spec}(\mathcal{O}_w) \to \tilde{X}$ is a morphism, which restricts to the map $\operatorname{Spec} L \to \operatorname{Spec} k(x) \to X$, and morphisms $(L, w, \varepsilon) \to (L', w', \varepsilon')$ are given by valued extensions (the compatibility with ε is automatic). Note that for any $(x, v, \varepsilon) \in \operatorname{Spa}(X, \tilde{X})$ and any finite separable extension of valuation fields $(L, w)/(k(x), v)$ uniquely determines an element $(L, w, \varepsilon') \in \mathrm{Val}^{\mathrm{s}}_{\mathcal{X}}$ with ε' equal to the composition $\varepsilon' : \operatorname{Spec} \mathcal{O}_w \to \operatorname{Spec} \mathcal{O}_v \xrightarrow{\varepsilon} \tilde{X}$.

For $(L, w, \varepsilon) \in \mathrm{Val}^{\mathrm{s}}_{\mathcal{X}}$ we set

$$\mathcal{O}^h_{X,L} = \varinjlim_{\operatorname{Spec} L \to U \to X} \mathcal{O}(U) \quad \text{and} \quad \mathcal{O}^h_{\tilde{X},L,w} = \mathcal{O}^h_{X,L} \times_L \mathcal{O}_w, \tag{7.1.1}$$

where the direct limit is over all étale maps $U \to X$ which factor $\operatorname{Spec} L \to X$. Note that $\mathcal{O}^h_{X,L}$ is the unique henselian local ring with residue field L which is finite étale over $\mathcal{O}^h_{X,x}$, corresponding to the field extension $L/k(x)$. In particular, the association $(L, w, \varepsilon) \mapsto \mathcal{O}^h_{X,L}$ defines a functor from $\mathrm{Val}^{\mathrm{s}}_{\mathcal{X}}$ to the category of henselian local rings which are ind-étale over X.

Let F be a sheaf on $X_{\text{ét}}$. We write $F(\mathcal{O}^h_{X,L}) := \varinjlim_{\operatorname{Spec} L \to U \xrightarrow{\text{étale}} X} F(U)$. Let

$$\beta = \{F_w \subset F(\mathcal{O}^h_{X,L})\}_{(L,w,\varepsilon) \in \mathrm{Val}^{\mathrm{s}}_{\mathcal{X}}},$$

be a collection of subsets such that

(β1) for any $(L, w, \varepsilon) \to (L_1, w_1, \varepsilon_1)$ in $\mathrm{Val}^{\mathrm{s}}_{\mathcal{X}}$, the pullback map $F(\mathcal{O}^h_{X,L}) \to F(\mathcal{O}^h_{X,L_1})$ restricts to $F_w \to F_{w_1}$.

For $(U, \tilde{U}) \in (X, \tilde{X})_t$ we define

$$F_\beta(U, \tilde{U}) := \left\{ a \in F(U) \;\middle|\; \begin{array}{l} \text{for all } (x, v, \varepsilon) \in \operatorname{Spa}(U, \tilde{U}) \text{ there exists} \\ \text{a finite tame extension } (L, w)/(k(x), v), \\ \text{such that } a_L \in F_w \end{array} \right\},$$

where a_L denotes the pullback of $a \in F(U)$ along $\operatorname{Spec} \mathcal{O}^h_{X,L} \to U$.

Note that by Lemma 3.5 and (β1) it suffices to consider in the definition of $F_\beta(U, \tilde{U})$ only the finite tame *Galois* extensions $(L, w)/(k(x), v)$.

Proposition 7.2. *The assignment $(U, \tilde{U}) \mapsto F_\beta(U, \tilde{U})$ defines a sheaf on $(X, \tilde{X})_t$.*

Proof. We start by showing that F_β is a presheaf. Let $(u, \tilde{u}) : (U', \tilde{U}') \to (U, \tilde{U})$ be a morphism in $(X, \tilde{X})_t$ and take $a \in F_\beta(U, \tilde{U})$. Let $(y, v, \varepsilon) \in \operatorname{Spa}(U', \tilde{U}')$. Set $x := u(y) \in U$ and denote by $v_x = v_{|k(x)}$ the restriction of v to $k(x)$. Note $\mathcal{O}_{v_x} = \mathcal{O}_v \cap k(x) = \mathcal{O}_v \times_{k(y)} k(x)$. Hence $\varepsilon : \operatorname{Spec} \mathcal{O}_v \to \tilde{U}' \to \tilde{U}$ factors uniquely via a map $\varepsilon_x : \operatorname{Spec} \mathcal{O}_{v_x} \to \tilde{U}$ so that we obtain a point $(x, v_x, \varepsilon_x) \in \operatorname{Spa}(U, \tilde{U})$. By definition there exists a finite tame extension $(L, w)/(k(x), v_x)$ such that $a_L \in F_w$. Denote by L_1 the composition field of L and $k(y)$ in a separable closure of $k(x)$ and choose a valuation v_1 on L_1 extending v. Then the extension $(L_1, v_1)/(k(y), v)$ is tame, by Lemma 3.5(2). Now $u^*(a)_{L_1}$, the pullback of $u^*(a) \in F(U')$ along $F(U') \to F(\mathcal{O}^h_{X,L_1})$, is equal to the image of the pullback of $a_L \in F(\mathcal{O}^h_{X,L})$ under $F(\mathcal{O}^h_{X,L}) \to F(\mathcal{O}^h_{X,L_1})$. As $a_L \in F_w$ we find $u^*(a)_{L_1} \in F_{v_1}$ by (β1) in 7.1. This shows $u^*(a) \in F_\beta(U', \tilde{U}')$. Hence F_β is a presheaf.

As F is an étale sheaf on X, F_β will be a sheaf on $(X, \tilde{X})_t$ if we show the following: Let $\{(U_i, \tilde{U}_i) \to (U, \tilde{U})\}_{i \in I}$ be a tame covering in $(X, \tilde{X})_t$ and let $a \in F(U)$, then

$$a_{|U_i} \in F_\beta(U_i, \tilde{U}_i), \quad \text{for all } i \implies a \in F_\beta(U, \tilde{U}). \tag{7.2.1}$$

Let $(x,v,\varepsilon)\in \mathrm{Spa}(U,\tilde U)$. By definition of tame coverings, we find $i\in I$ and a point $(y,w,\varepsilon')\in \mathrm{Spa}(U_i,\tilde U_i)$ over (x,v,ε) such that $(k(y),w)/(k(x),v)$ is a finite tame extension. As $a_{|U_i}\in F_\beta(U_i,\tilde U_i)$, we find a finite tame extension $(L,w_1)/(k(y),w)$ such that $(a_{|U_i})_L = a_L\in F_{w_1}$. Hence, we get (7.2.1). $\square$

Remark 7.3. By definition, for all F, β as in 7.1 above, we have a pullback diagram

$$\begin{tikzcd} F_\beta(U,\tilde U) \arrow[r] \arrow[d] & \prod \varinjlim F_w \arrow[d] \\ F(U) \arrow[r] & \prod \varinjlim F(\mathcal{O}^h_{U,L}) \end{tikzcd}$$

where:

- the product ranges over all elements (x,v,ε) of $\mathrm{Spa}(U,\tilde U)$
- the colimit ranges over all $(k(x),v)\subseteq (L,w)$ finite tame.

This implies that if $\varphi\colon F\to G$ is a map of sheaves on $X_{\text{ét}}$ and F and G are equipped with β-families $\beta_F:=\{F_w\}$ and $\beta_G:=\{G_w\}$ such that for every $(U,\tilde U)\in (X,\tilde X)_t$ and every $(x,v,\varepsilon)\in \mathrm{Spa}(U,\tilde U)$ there is a cofinal system of tame extensions $S_{(x,v,\varepsilon)}:=\{(k(x),v)\subseteq (L,w)\}$ such that the map $\varphi: F(\mathcal{O}^h_{X,L})\to G(\mathcal{O}^h_{X,L})$ restricts to a map $\varphi_w\colon F_w\to G_w$ for all $(L,w)\in S_{x,v,\varepsilon}$, then φ induces a map $F_{\beta_F}\to G_{\beta_G}$ of sheaves on $(X,\tilde X)_t$, denoted by φ as well.

Remark 7.4. Given a collection β as in (7.1) on a sheaf F, we can define a new family

$$\beta^{div}=\{F^{div}_w\subset F(A_L)\}_{(L,w,\varepsilon)\in \mathrm{Val}^{\mathrm{s}}_{\mathcal{X}}}$$

by setting

$$F^{div}_w := \begin{cases} F_w & \text{if } \operatorname{Spec} L \text{ maps to a generic point of } X \\ & \text{and } w \text{ is a discrete valuation,} \\ F(A_L) & \text{else.}\end{cases}$$

If β satisfies (β1), then so does β^{div}. Here note that if $(L,w,\varepsilon)\to(L',w',\varepsilon')$ is a morphism in $\mathrm{Val}^{\mathrm{s}}_{\mathcal{X}}$ and w is discrete, then w' is discrete as well. We denote by $F^{div}_\beta := F_{\beta^{div}}$ the corresponding tame sheaf. For sections in $F^{div}_\beta(U,\tilde U)$, we only put conditions induced by β along tame extensions of discrete valuations on the generic points of U with center in $\tilde U$, hence $F_\beta\subset F^{div}_\beta$.

Example 7.5. In the following (L,w,ε) is always in $\mathrm{Val}^{\mathrm{s}}_{\mathcal{X}}$. We set $A_L=\mathcal{O}^h_{X,L}$ and $A_{L,w}=\mathcal{O}^h_{\tilde X,L,w}$, see (7.1.1) for notation

(1) Let F be a sheaf on $X_{\text{ét}}$. Set $F_{w,0}=F(A_L)$, if $(L,w,\varepsilon)\in \mathrm{Val}^{\mathrm{s}}_{(X,X)}$, and $F_{w,0}=0$, else. Set $F_{w,\mathrm{triv}}=F(A_L)$, for all (L,w,ε). Then $\beta_0=\{F_{w,0}\}_{(L,w,\varepsilon)}$ and $\beta_{\mathrm{triv}}=\{F_{w,\mathrm{triv}}\}$ are families as in 7.1 and we have

$$\imath_*F_{\beta_0}=j_!F\quad\text{and}\quad \imath_*F_{\beta_{\mathrm{triv}}}=j_*F,$$

where $j:X\to\tilde X$ is the open immersion and $\imath:(X,\tilde X)_t\to \tilde X_{\text{ét}}$ is the morphism of sites induced by the functor $\tilde U\mapsto (\tilde U/\tilde X\times_{\tilde X} X,\tilde U)$.

(2) Given a collection β as in (7.1) on a sheaf F, we can define a new family

$$\beta^{div}=\{F^{div}_w\subset F(A_L)\}_{(L,w,\varepsilon)\in \mathrm{Val}^{\mathrm{s}}_{\mathcal{X}}}$$

by setting

$$F^{div}_w := \begin{cases} F_w & \text{if } \operatorname{Spec} L \text{ maps to a generic point of } X \\ & \text{and } w \text{ is a discrete valuation,} \\ F(A_L) & \text{else.}\end{cases}$$

If β satisfies (β1), then so does β^{div}. Here note that if $(L,w,\varepsilon) \to (L',w',\varepsilon')$ is a morphism in $\mathrm{Val}^{\mathrm{s}}_{\mathcal{X}}$ and w is discrete, then w' is discrete as well. We denote by $F^{div}_{\beta} := F_{\beta^{div}}$ the corresponding tame sheaf. For sections in $F^{div}_{\beta}(U,\tilde{U})$ we only put conditions induced by β along tame extensions of discrete valuations on the generic points of U with center in $\tilde{U}$, hence $F_\beta \subset F^{div}_{\beta}$.

(3) Let F be as above and assume there is a presheaf $\widetilde{F}$ on $\mathbf{Sch}_{\tilde{X}}$ extending F. Then we can define $\beta_{\widetilde{F}} = \{\widetilde{F}(A_{L,w}) \subset F(A_L)\}_{(L,w,\varepsilon)}$. We get a sheaf $F_{\beta_{\widetilde{F}}}$ on $(X,\tilde{X})_t$. Notice that the resulting sheaf $\iota_* F_{\beta_{\widetilde{F}}}$ may be different from $\widetilde{F}_{|\tilde{X}_{\text{ét}}}$, e.g., for $\widetilde{F} = \mathcal{O}$, the structure sheaf on $\mathbf{Sch}_{\tilde{X}}$, the sheaf $\mathcal{O}_{|\tilde{X}_{\text{ét}}}$ is different from $\iota_*\mathcal{O}_{\beta_{\widetilde{F}}} = \iota_*\mathcal{O}^t$, where $\mathcal{O}^t$ is defined in (4) (see Lemma 7.6).

(4) Let $F = \Omega^q$ be the étale sheaf of qth absolute differential forms on $X_{\text{ét}}$. Denote by $\Omega^*_{A_{L,w}}(\log)$ the graded $\Omega^*_{A_{L,w}}$-subalgebra of $\Omega^*_{A_L}$ generated by $\mathrm{dlog}(A_L^\times)$. Note $\Omega^q_{A_{L,w}} \subset \Omega^q_{A_{L,w}}(\log)$. Set

$$\beta_{\log} := \{\Omega^q_{A_{L,w}}(\log) \subset \Omega^q_{A_L}\}_{(L,w,\varepsilon)}.$$

Then $\beta_{\log}$ satisfies (β1) in 7.1 and we get a sheaf

$$\Omega^{q,t} := \Omega^q_{\beta_{\log}} \quad \text{on } (X,\tilde{X})_t. \tag{7.5.1}$$

Note for $q=0$, we write $\mathcal{O}^t = \Omega^{0,t}$. This is a special case of (3), where we take $\widetilde{F}$ to be the structure sheaf $\mathcal{O}$ on $\mathbf{Sch}_{\tilde{X}}$. Note also that the differential of the de Rham complex induces a well-defined differential $d : \Omega^{q,t} \to \Omega^{q+1,t}$ giving rise to a complex of sheaves $\Omega^{\bullet,t}$ on $(X,\tilde{X})_t$. Using Remark (2), we get an inlcusion of sheaves on $(X,\tilde{X})_t$

$$\Omega^{q,t} \subset \Omega^{q,div} := (\Omega^q_{\beta_{\log}})^{div}.$$

Furthermore, given a morphism $\tilde{X} \to S$ we can similarly define the complex $\Omega^{\bullet,t}_{/S} = \Omega^{\bullet}_{/S,\beta_{\log/S}}$, where $\beta_{\log/S} = \{\Omega^q_{A_{L,w}/S}(\log) \subset \Omega^q_{A_L/S}\}_{(L,w,\varepsilon)}$.

(5) Let $W_n\Omega^\bullet$ denote the p-typical de Rham-Witt complex, it is defined as an étale sheaf on all schemes by [13]. Denote by $W_n\Omega^*_{A_{L,w}}(\log)$ the graded $W_n\Omega^*_{A_{L,w}}$-subalgebra of $W_n\Omega^*_{A_L}$ generated by $\mathrm{dlog}[a]$, $a \in {A_L}^\times$, where $[-] : A_L \to W_n(A_L)$ denotes the multiplicative lift. Note $W_n\Omega^q_{A_{L,w}} \subset W_n\Omega^q_{A_{L,w}}(\log)$. Then $\beta_{\log,n} = \{W_n\Omega^q_{A_{L,w}}(\log) \subset W_n\Omega^q_{A_L}\}_{(L,w,\varepsilon)}$ satisfies the assumption from 7.1 and we get a sheaves

$$W_n\Omega^{q,t} := W_n\Omega^q_{\beta_{\log,n}} \subset W_n\Omega^{q,div} := (W_n\Omega^q_{\beta_{\log,n}})^{div} \quad \text{on } (X,\tilde{X})_t.$$

Note that the differential, the restriction map, as well as Frobenius and Verschiebung on the de Rham-Witt complex induce well-defined maps on $W_n\Omega^{q,t}$ and $W_n\Omega^{q,div}$. In particular $W_\bullet\Omega^{\bullet,t}$ and $W_n\Omega^{\bullet,div}$ are Witt complexes in the sense of [13, Definition 4].

Lemma 7.6. *Let $\mathcal{O}^t := \Omega^{0,t}$ and $\mathcal{O}^{div} := \Omega^{0,div}$ in the notation from 7.5(4). For all $(U,\tilde{U}) \in (X,\tilde{X})_t$ we have*

$$\mathcal{O}^t(U,\tilde{U}) = \mathcal{O}(\tilde{U}^{\mathrm{int}}),$$

where $\tilde{U}^{\mathrm{int}}$ denotes the integral closure of $\tilde{U}$ in U. Moreover, if $\tilde{U}$ is a Nagata scheme and U is normal, then we have

$$\mathcal{O}^t(U,\tilde{U}) = \mathcal{O}^{div}(U,\tilde{U}).$$

Proof. By Lemma 6.5 we can assume $\tilde{U} = \tilde{U}^{\mathrm{int}}$. We notice that both sides are contained in $\mathcal{O}(U)$ and that this "$\supset$" inclusion holds by definition of the left hand side. As both sides are sheaves on $\tilde{U}$, it suffices to check the other inclusion for $\tilde{U}$ affine. As an affine open cover $U = \cup_i U_i$ induces an affine open cover $\tilde{U} = \tilde{U}^{\mathrm{int}} = \cup_i \tilde{U}_i^{\mathrm{int}}$ we can assume $(U, \tilde{U}) = (\operatorname{Spec} A, \operatorname{Spec} \tilde{A})$ with $\tilde{A}$ integrally closed in A. Let $x \in A \setminus \tilde{A}$.

Claim 7.7. *There exist a prime ideal $\mathfrak{p}$ in A and a valuation v on $K = \mathrm{Frac}(A/\mathfrak{p})$ such that $\tilde{A} \to A \to K$ factors through $\mathcal{O}_v$ and $v(x) < 0$.*

Admitting the claim, we also have $w(x) < 0$ for all (tame) extensions $(L, w)/(K, v)$ of valuation fields and x cannot lie in $\mathcal{O}^t(U, \tilde{U})$, which completes the proof of the lemma.

We prove the claim. Denote by $C = \tilde{A}[1/x]$ the subring of A_x generated by the image of $\tilde{A}$ and $1/x$. As x is not integral over $\tilde{A}$ it follows from [30, VI, §1, no. 2, Lemma 1], that A_x is not the zero-ring and that there exists a maximal ideal $\mathfrak{m} \subset C$ such that $1/x \in \mathfrak{m}$ and $\mathfrak{m} \cap \tilde{A}$ is a maximal ideal of $\tilde{A}$. Consider the localization $C_\mathfrak{m}$ of C. By the flatness of $C_\mathfrak{m}$ over C, the inclusion $C \hookrightarrow A_x$ induces an injection of rings $C_\mathfrak{m} = C \otimes_C C_\mathfrak{m} \hookrightarrow A_x \otimes_C C_\mathfrak{m}$. As $C_\mathfrak{m}$ is not the zero ring, the ring $A_x \otimes_C C_\mathfrak{m}$ is not zero and hence has a prime ideal; it corresponds to a prime ideal $\mathfrak{p}$ of A, which does not contain x and has empty intersection with $C \setminus \mathfrak{m}$. Set $\mathfrak{p}_0 := C \cap \mathfrak{p} A_x$. By construction we have $\mathfrak{p}_0 \subset \mathfrak{m}$, and in fact this inclusion is strict as else we would have $1/x \in \mathfrak{p} A_x$ since $1/x \in \mathfrak{m}$. Thus, $C/\mathfrak{p}_0$ is not a field. Set $K_0 := \mathrm{Frac}(C/\mathfrak{p}_0) \subset K := \mathrm{Frac}(A_x/\mathfrak{p} A_x) = \mathrm{Frac}(A/\mathfrak{p})$. Let v_0 be a valuation on K_0 such that $C/\mathfrak{p}_0 \subset \mathcal{O}_{v_0} \subset K_0$ and that $\mathfrak{m}_{v_0} \cap C/\mathfrak{p}_0$ is the image of $\mathfrak{m}$ in $C/\mathfrak{p}_0$. Let v be a valuation on K extending v_0. Thus we obtain a commutative diagram

$$\begin{tikzcd}
\tilde{A} \arrow[r] \arrow[d] & C/\mathfrak{p}_0 \arrow[r] \arrow[d] & \mathcal{O}_{v_0} \arrow[r] & \mathcal{O}_v \arrow[d] \\
A \arrow[r] & A_x/\mathfrak{p} A_x \arrow[rr] & & K.
\end{tikzcd}$$

Note $v(x) = v_0(x) < 0$ since $1/x \in \mathfrak{m}$ so that its image in $C/\mathfrak{p}_0$ is in $\mathfrak{m}_{v_0}$. This completes the proof of the claim.

For the last statement, we observe that as $\tilde{U}$ is Nagata the integral closure $\tilde{U}^{int}$ is finite over $\tilde{U}$ and hence is locally noetherian again. As U is normal so is $\tilde{U}^{int}$. We may therefore assume that $\tilde{U}^{int}$ is noetherian, integral and normal. Hence

$$\mathcal{O}^t(U, \tilde{U}) = \mathcal{O}(\tilde{U}^{\mathrm{int}}) = \bigcap_{x \in \tilde{U}^{int,(1)}} \mathcal{O}_{\tilde{U}^{int}, x} = \mathcal{O}^{div}(U, \tilde{U}),$$

where $\tilde{U}^{int,(1)}$ is the set of 1-codimensional points in U^{int} and where the last equality follows from $\mathcal{O}^{div}(U, \tilde{U}) = \mathcal{O}^{div}(U, \tilde{U}^{int})$ and the definition of $\mathcal{O}^{div}$, see Remark (2). ☐

We give some results on $\Omega^{q,t}$ from Example 7.5 without proofs.

Lemma 7.8. *Let T be a normal noetherian scheme, $\tilde{X} \to T$ a smooth morphism, and $X \subset \tilde{X}$ a dense open, such that $\tilde{X} \setminus X$ is the support of a simple normal crossing divisor D over T (i.e., all intersections $D_{i_1} \cap \ldots \cap D_{i_r}$ of the irreducible components of D are smooth over T). Let $j : (X, \tilde{X})_t \to \tilde{X}_{\mathrm{Zar}}$ be the natural morphism of sites induced by the functor $\tilde{X}_{\mathrm{Zar}} \to (X, \tilde{X})_t$ given by $\tilde{U}/\tilde{X} \mapsto (\tilde{U} \times_{\tilde{X}} X, \tilde{U})$. Then*

$$j_* \Omega^{q,t}_{/T} = j_* \Omega^{q,div}_{/T} = \Omega^q_{\tilde{X}/T}(\log D).$$

Lemma 7.9. *Let k be a perfect field of positive characteristic p. Let $\tilde{X}$ be a smooth k-scheme and $X \subset \tilde{X}$ an open subscheme such that the complement $\tilde{X} \setminus X$ is the support of a simple normal crossing divisor D. Let $j : (X, \tilde{X})_t \to \tilde{X}_{\mathrm{Zar}}$ be as above. Then*

$$j_* W_n \Omega^{q,t} = j_* W_n \Omega^{q,div} = W_n \Omega^q_{\tilde{X}}(\log D),$$

where the right hand side denotes q-forms of the logarithmic de Rahm-Witt complex, associated to the smooth log scheme $(\tilde{X}, j_* \mathcal{O}_X^\times \cap \mathcal{O}_{\tilde{X}})$, see [18].

8. Fiber functors

In this section, we characterise fibre functors of the topoi of the sheaves of sets on $(X,\tilde{X})_{\text{vét}}$ and $(X,\tilde{X})_t$. First, we recall the following.

Definition 8.1. Let (C,γ) be a site admitting finite limits. Recall that a *fibre functor* of a topos $\mathbf{Shv}(C,\gamma)$ of sheaves of sets, is a functor $\varphi : \mathbf{Shv}(C,\gamma) \to \mathbf{Sets}$ which preserves colimits and finite limits. Let $\mathrm{Fib}(\mathbf{Shv}(C,\gamma))$ denote the category of fiber functors of $\mathbf{Shv}(C,\gamma)$.

In what follows, let γ denote either the v-étale topology or the tame topology on $(X,\tilde{X})_\tau$. The main result of this section gives a description of $\mathrm{Fib}(\mathbf{Shv}((X,\tilde{X})_\gamma))$. We first introduce some notations (see Proposition 8.5).

Definition 8.2. We let $\widetilde{(X,\tilde{X})_\tau}$ be the category of pairs $\mathcal{T}=(T,\tilde{T})$ of affine schemes such that that there exists a cofiltered projective system $\{\mathcal{T}_i=(T_i,\tilde{T}_i)\}_{i\in I}$ in $(X,\tilde{X})_{\mathrm{affine},\tau}$ such that $T=\varprojlim_{i\in I} T_i$ and $\tilde{T}=\varprojlim_{i\in I}\tilde{T}_i$. Notice that in this case the map $T\to X$ is no longer in general étale (but rather, pro-étale), the map $T\to\tilde{T}$ is no longer in general a quasi-compact open immersion and the map $\tilde{T}\to\tilde{X}$ is no longer locally of finite type. We also consider the full subcategory $\widetilde{(X,\tilde{X})_{\mathrm{int},\tau}}$ of $\widetilde{(X,\tilde{X})_\tau}$ whose objects are cofiltered limits of objects in $(X,\tilde{X})_{\mathrm{int},\tau}$.

Remark 8.3. For $\mathcal{Y}=(\mathrm{Spec}(A),\mathrm{Spec}(\tilde{A}))\in\widetilde{(X,\tilde{X})_{\mathrm{int},t}}$, we observe that:

(1) $\tilde{A}\to A$ is injective and integrally closed, as filtered colimits are exact and the integral closure commutes with filtered colimits.
(2) If $A=A_1\times A_2$ is a product of rings and $\tilde{A}_i$ are the integral closures of $\tilde{A}$ in A_i for $i=1,2$, we have $\mathcal{Y}=\mathcal{Y}_1\sqcup\mathcal{Y}_2$ with $\mathcal{Y}_i=(\mathrm{Spec}(A_i),\mathrm{Spec}(\tilde{A}_i))$.

Definition 8.4. We say that $\mathcal{T}=(T,\tilde{T})\in\widetilde{(X,\tilde{X})_\tau}$ is vét (resp. tamely) local if every vét (resp. tame) covering $\mathcal{V}=(V,\tilde{V})\to\mathcal{U}=(U,\tilde{U})$ in $(X,\tilde{X})_\tau$, the morphism of sets

$$\mathrm{Hom}_{\widetilde{(X,\tilde{X})_\tau}}(\mathcal{T},\mathcal{V})\to\mathrm{Hom}_{\widetilde{(X,\tilde{X})_\tau}}(\mathcal{T},\mathcal{U})$$

is surjective.

By Lemma 6.12, there is an equivalence of categories of fiber functors

$$\mathrm{Fib}(\mathbf{Shv}((X,\tilde{X})_{\mathrm{affine},\gamma}))\simeq\mathrm{Fib}(\mathbf{Shv}((X,\tilde{X})_\gamma)).$$

Hence, by [27, Pro.7.13], there is a bijection between fibre functors of $\mathbf{Shv}((X,\tilde{X})_\gamma)$ and pro-objects

$$\mathcal{P}_\bullet = \text{“}\varprojlim_{\lambda\in\Lambda}\text{”}\mathcal{P}_\lambda \ \text{ with } \mathcal{P}_\lambda=(P,\tilde{P})\in(X,\tilde{X})_{\mathrm{affine},\gamma} \tag{8.4.1}$$

indexed by a cofiltered category Λ, which satisfies the γ-locality condition: For every γ-covering $\mathcal{V}\xrightarrow{u}\mathcal{U}$, the morphism of sets

$$\varinjlim_{\lambda\in\Lambda}\mathrm{Hom}_{(X,\tilde{X})_\tau}(\mathcal{P}_\lambda,\mathcal{V})\to\varinjlim_{\lambda\in\Lambda}\mathrm{Hom}_{(X,\tilde{X})_\tau}(\mathcal{P}_\lambda,\mathcal{U})$$

is surjective. By Lemma 6.3, the latter condition is equivalent to that $\mathcal{T}=\varprojlim_{\lambda\in\Lambda}\mathcal{P}\in\widetilde{(X,\tilde{X})_\tau}$ is γ-local in the sense of Definition 8.4 and the corresponding fiber functor is given by

$$\varphi_{\mathcal{T}}:\mathbf{Shv}((X,\tilde{X})_\gamma)\to\mathbf{Sets}\ ;\ F\to F(\mathcal{T}):=\varinjlim_{\lambda}F(\mathcal{P}_\lambda). \tag{8.4.2}$$

The proof of the following proposition is analogous to [19, Corollary 3.5] and [17, Lemma 10.7]:

Proposition 8.5. *A pair $\mathcal{T} = \varprojlim_{i\in I} \mathcal{T}_i \in \widetilde{(X,\tilde{X})}_\tau$ is v-étale local (resp. tame local) if and only if $\mathcal{T}$ is a coproduct of objects of the form $(\mathrm{Spec}(S), \mathrm{Spec}(\widetilde{S}))$ such that $\widetilde{S}$ is strictly henselian local and S is henselian local with $S = \widetilde{S}[1/f]$ for a non-zero divisor $f \in \widetilde{S}$, and that $\widetilde{S} = S \times_k \mathcal{O}_v$, where k is the residue field of S and $\mathcal{O}_v$ is its valuation ring of k such that (k, v) is strictly henselian (resp. (k, v) is tamely closed). Moreover, in both cases we have $(T, \tilde{T}) \in \widetilde{(X,\tilde{X})}_{\mathrm{int},\tau}$.*

Remark 8.6. By Remark 6.4(1) and Deligne's completeness theorem, [33, Prop.VI.9.0] or [27, Thm.7.44, 7.17], the fibre functor $\varphi_{\mathcal{T}}$ from (8.4.2) for $\mathcal{T}$ satisfying the condition of Proposition 8.5 form a conservative family, i.e. a morphism f in $\mathbf{Shv}((X, \tilde{X}_\gamma)$ is an isomorphism if and only if $\varphi_{\mathcal{T}}(f)$ is an isomorphism of sets for all γ-local $\mathcal{T}$. Equivalently, a morphism $\mathcal{V} \to \mathcal{U}$ in $(X, \tilde{X})_\tau$ is a γ-covering if and only if

$$\mathrm{Hom}_{\widetilde{(X,\tilde{X})}_\tau}(\mathcal{T}, \mathcal{V}) \to \mathrm{Hom}_{\widetilde{(X,\tilde{X})}_\tau}(\mathcal{T}, \mathcal{U})$$

is surjective for all such $\mathcal{T}$, [32, Expo'e IV, Prop.6.5(a)],

Remark 8.7. Let $(\mathrm{Spec}(A), \mathrm{Spec}(A \times_k \mathcal{O}_v))$ be vét local. Let $(\mathrm{Spec}(B), \mathrm{Spec}(\tilde{B})) \to (\mathrm{Spec}(A), \mathrm{Spec}(A \times_k \mathcal{O}_v))$ be a tame covering in the sense of Definition 0.3(4). Since A is henselian local, we can refine it so that $B \to A$ is finite étale asssociated to a finite separable extension of $k \hookrightarrow k'$. By tameness, there exists a valuation w on k' extending v such that $(k', w)/(k, v)$ is tame and $\tilde{B} \to B \to k'$ factors through $\mathcal{O}_w$. This implies that the map $\tilde{B} \to B$ factors through $B \times_{k'} \mathcal{O}_w$, therefore the covering $(\mathrm{Spec}(B), \mathrm{Spec}(\tilde{B}))$ is refined by $(\mathrm{Spec}(B), \mathrm{Spec}(B \times_k \mathcal{O}_w))$. Moreover, by Lemma 3.5(1) we can further refine it so that we have that k'/k is Galois.

8.8. Proof of Proposition 8.5. We need the following technical result.

Lemma 8.9. *Let $(Y, \tilde{Y}) = \varprojlim_{i\in I}(Y_i, \tilde{Y}_i)$ in $\widetilde{(X,\tilde{X})}_\tau$. Let $(f, \tilde{f})\colon (U, \tilde{U}) \to (Y, \tilde{Y})$ in $\widetilde{(X,\tilde{X})}_\tau$ with f an étale covering. Then there exists a cofiltered category J and a system of maps $(f_{ij}, \tilde{f}_{ij})\colon (U_{ij}, \tilde{U}_{ij}) \to (Y_i, \tilde{Y}_i)$ indexed over $I \times J$ such that for all (i, j), $(U_{ij}, \tilde{U}_{ij}) \in (X, \tilde{X})_{\mathrm{int},\tau}$ and f_{ij} is an étale covering, and $\varprojlim_{(i,j)\in I\times J}(U_{ij}, \tilde{U}_{ij}) \to (Y, \tilde{Y})$ refines $(f, \tilde{f})$. Moreover, if $(f, \tilde{f})$ is is a tame covering in the sense of Definition 0.3, then we find such system that $(f_{ij}, \tilde{f}_{ij})$ are tame coverings for all i, j.*

Proof. The first assertion follows from Lemma 6.3 by a standard argument and we omit its proof. We prove the second assertion. Assume that $(f, \tilde{f})$ is a tame covering and prove that $(f_{ij}, \tilde{f}_{ij})$ is a tame covering for a sufficiently large i, j assuming its existence. We proceed as the proof of [17, Theorem 4.6]. Let $Z_i \in \mathrm{Spa}(Y_i, \tilde{Y}_i)$ be the set of triples $(y_i, w_i, \varepsilon_{w_i})$ such that there is no $(x_i, v_i, \varepsilon_{v_i})$ in $\mathrm{Spa}(U_{ij}, \tilde{U}_{ij})$ tame over $(y_i, w_i, \varepsilon_{w_i})$. Since $(f, \tilde{f})$ is tame, we have $\varprojlim_\alpha Z_\alpha = \varnothing$. By Lemma 5.2, Z_i is closed, which implies that it is compact in the constructible topology by Lemma 4.4 since $\mathrm{Spa}(Y_i, \tilde{Y}_i)$ is spectral. Since the inverse limit of nonempty compact spaces is nonempty, we must have $Z_i = \emptyset$ for a sufficiently large i, which completes the proof.

□

Proof of Proposition 8.5: First of all, we observe that $T \to \tilde{T}$ is dense by Remark 6.4, hence T and $\tilde{T}$ have the same number of connected components. We check that $\mathcal{T}$ is local if and only if every connected component is local. Let $\tilde{T} = \coprod_{i\in I} \tilde{T}_i$ and $T = \coprod_{i\in I} T_i$ be the decomposition into the connected components. We claim that $\mathcal{T} = (T, \tilde{T})$ is local if and only $\mathcal{T}_i = (T_i, \tilde{T}_i)$ are local for all i. Indeed, assume that $\mathcal{T}$ is local and take coverings $\mathcal{V}_i \to \mathcal{U}_i$ in $(X, \tilde{X})_\tau$ and maps $\varphi_i : \mathcal{T}_i \to \mathcal{U}_i$ in $\widetilde{(X,\tilde{X})}_\tau$ for $i \in I$. Fixing $i \in I$, it gives rise to a covering $\mathcal{V} := \sqcup_{j\neq i}\mathcal{U}_j \sqcup \mathcal{V}_i \to \mathcal{U} = \sqcup_{i\in I}\mathcal{U}_j$ and a map $\varphi : \mathcal{T} \to \mathcal{U}$ in an obvious way. By the assumption, φ factors through $\mathcal{V}$ and the image of the map $\mathcal{T}_i \to \mathcal{T} \to \mathcal{V}$ lands in $\mathcal{V}_i$ since T_i and $\tilde{T}_i$ are connected. Thus, φ_i factors through $\mathcal{V}_i$ showing that $\mathcal{T}_i$ is local. On the other hand, assume that $\mathcal{T}_i$ are local for all $i \in I$. Let $\mathcal{V} \to \mathcal{U}$ be a covering in $(X, \tilde{X})_\tau$ and $\varphi : \mathcal{T} \to \mathcal{U}$ be a map in $\widetilde{(X, \tilde{X})}_\tau$. For each $i \in I$, the map $\mathcal{T}_i \to \mathcal{T} \to \mathcal{U}$ factors through a map $\psi_i : \mathcal{T}_i \to \mathcal{V}$ since $\mathcal{T}_i$ are local. Then, $\psi = \sqcup_{i\in I}\psi_i : \mathcal{T} \to \mathcal{V}$ gives a lift of φ showing that $\mathcal{T}$ is local. For the rest of the proof, we assume that $(T, \tilde{T})$ is connected.

$\Rightarrow$ Recall that every v-étale covering is also a tame covering. Let $\mathcal{T} = (T, \tilde{T}) = (\mathrm{Spec}(S), \mathrm{Spec}(\widetilde{S}))$ and write

$$\mathcal{T} = \varprojlim_{\alpha\in A} \mathcal{T}_\alpha \quad \text{with } \mathcal{T}_\alpha = (\mathrm{Spec}(S_\alpha), \mathrm{Spec}(\widetilde{S}_\alpha)) \in (X, \tilde{X})_{\mathrm{affine},t}$$

so $S = \varinjlim_\alpha S_\alpha$ and $\widetilde{S} = \varinjlim_\alpha \widetilde{S}_\alpha$. Let $(\widetilde{S})^{\mathrm{int}}$ (resp. $(\widetilde{S}_\alpha)^{\mathrm{int}}$) be the integral closure of $\widetilde{S}$ in S (resp. $\widetilde{S}_\alpha$ in S_α). We have $(\widetilde{S})^{\mathrm{int}} = \varinjlim_\alpha (\widetilde{S}_\alpha)^{\mathrm{int}}$ so we have

$$(\mathrm{Spec}(S), \mathrm{Spec}((\widetilde{S})^{\mathrm{int}})) = \varprojlim_{\alpha\in A} (\mathcal{T}_\alpha)^{\mathrm{int}}$$

with $\mathcal{T}_\alpha^{\mathrm{int}} = (\mathrm{Spec}(S_\alpha), \mathrm{Spec}((\widetilde{S}_\alpha)^{\mathrm{int}})) \in (X, \tilde{X})_{\mathrm{int},t}$. Since $(\mathcal{T}_\alpha)^{\mathrm{int}} \to \mathcal{T}_\alpha$ is a modification and $\mathcal{T}$ is v-étale local, the projection $\mathcal{T} \to \mathcal{T}_\alpha$ factors through $(\mathcal{T}_\alpha)^{\mathrm{int}}$, which implies that $\widetilde{S}_\alpha \to \widetilde{S}$ factors through $(\widetilde{S}_\alpha)^{\mathrm{int}}$. This implies $\widetilde{S} = (\widetilde{S})^{\mathrm{int}}$ and $\mathcal{T} = \varprojlim_{\alpha\in A} (\mathcal{T}_\alpha)^{\mathrm{int}}$ so we may assume $\widetilde{S}_\alpha = (\widetilde{S}_\alpha)^{\mathrm{int}}$ for all $\alpha \in A$.

Noting that the restriction of the v-étale topology on $\tilde{X}$ is finer than the étale topology, $\widetilde{S}$ must be strictly heselian.

We show that $S = \widetilde{S}[1/f]$ for a non-zero divisor $f \in \widetilde{S}$. Fix $\alpha_0 \in A$ and consider a finite collection $f_{\alpha_0,1} \dots f_{\alpha_0,r} \in \widetilde{S}_{\alpha_0}$ such that $S[1/f_{\alpha_0,j}] = \widetilde{S}[1/f_{\alpha_0,j}]$ for all $1 \le j \le r$, giving a standard open covering

$$\sqcup_{j=1\dots r} \mathrm{Spec}(\widetilde{S}_{\alpha_0}[1/f_{\alpha_0,j}]) \to \mathrm{Spec}(S_{\alpha_0}).$$

For $\alpha \in A_{\alpha_0/}$, let $f_{\alpha,j}$ (resp. f_j) be the image of $f_{\alpha_0,j}$ in $\widetilde{S}_\alpha$ (resp. in $\widetilde{S}$). Since the ideal $(f_{\alpha_0,1}, \dots, f_{\alpha_0,r})$ of S_{α_0} is the unit ideal, a fortiori the ideal $\mathfrak{I}_\alpha = (f_{\alpha,1}, \dots, f_{\alpha,r})$ is the unit ideal in S_α, so we have the v-étale covering:

$$\underset{j=1\dots r}{\sqcup} \mathcal{U}_{\alpha,j} \to \mathcal{T}_\alpha \quad \text{with } \mathcal{U}_{\alpha,j} = (\mathrm{Spec}(\widetilde{S}_\alpha[1/f_{\alpha,j}], \mathrm{Spec}(\widetilde{S}_\alpha[\Big[\frac{\mathfrak{I}_\alpha}{f_{\alpha,j}}\Big]])).$$

Since $\mathcal{T}$ is local and connected, the projection $\mathcal{T} \to \mathcal{T}_\alpha$ factors through $\mathcal{U}_{\alpha,j}$ for some j, which implies $S = \widetilde{S}[1/f_j]$, where f_j is the image of $f_{\alpha_0,j}$ in $\widetilde{S}$. We notice that f is not a zero-divisor since the map $\widetilde{S} \to \widetilde{S}[1/f] = S$ is injective by Remark 8.3,(1).

Next, we show that S is a local ring (see [19, Proposition 3.3, "$\Rightarrow$ 2"]) Let x_1 and x_2 closed points in T. Take finite set J_1 and J_2 and $\{g_{j_1}\}_{j_1\in J_1}$ and $\{g_{j_2}\}_{j_2\in J_2}$ in $\widetilde{S}$ such that $\{\mathrm{Spec}(S[1/g_{j_1}])\}_{j_1\in J_1}$ is an open cover of $T - \{x_1\}$ and $\{\mathrm{Spec}(S[1/g_{j_2}])\}_{j_2\in J_2}$ is an open cover of $T - \{x_2\}$. Up to refining the cover, we can assume that g_{j_1} and g_{j_2} are not units in S for all j_1, j_2. If $x_1 \neq x_2$, then the finitely generated ideal $\mathfrak{I} = (g_{j_1}, g_{j_2})_{j_1\in J_1, j_2\in J_2}$ of $\widetilde{S}$ maps to the unit ideal in S. Since $\mathfrak{I}$ is finitely generated, there exists α such that g_{j_1}, g_{j_2} comes from $g_{\alpha,j_1}, g_{\alpha,j_2} \in \widetilde{S}_\alpha$ and that the ideal $\mathfrak{I}_\alpha = (g_{\alpha,j_1}, g_{\alpha,j_2})_{j_1\in J_1, j_2\in J_2}$ of $\widetilde{S}_\alpha$ maps to the unit ideal in S_α. Hence, we get the v-étale covering:

$$\bigsqcup_{j_1\in J_1} \mathcal{U}_{\alpha,j_1} \sqcup \bigsqcup_{j_2\in J_2} \mathcal{U}_{\alpha,j_2} \to \mathcal{T}_\alpha, \quad \text{where}$$

$$\mathcal{U}_{\alpha,j_1} = (\mathrm{Spec}(S_\alpha[1/g_{\alpha,j_1}]), \mathrm{Spec}(\widetilde{S}_\alpha[\frac{\mathfrak{I}_\alpha}{g_{\alpha,j_1}}])), \quad \mathcal{U}_{\alpha,j_2} = (\mathrm{Spec}(S_\alpha[1/g_{\alpha,j_2}]), \mathrm{Spec}(\widetilde{S}_\alpha[\frac{\mathfrak{I}_\alpha}{g_{\alpha,j_2}}])).$$

Since $\mathcal{T}$ is local, the projection $\mathcal{T} \to \mathcal{T}_\alpha$ factors through $\mathcal{U}_{\alpha,j_1}$ for some j_1 or $\mathcal{U}_{\alpha,j_2}$ for some j_2, which implies that there is a splitting $S \to S[1/g_{j_1}] \to S$ for some j_1 or $S \to S[1/g_{j_2}] \to S$ for some j_2. This implies that g_{j_1} or g_{j_2} is a unit in S, which is a contradiction, therefore $x_1 = x_2$, hence S is local.

Let $\mathfrak{p} \subseteq \widetilde{S}$ be the prime ideal such that $\mathfrak{p}S$ is the maximal ideal of S. We show that $\widetilde{S}/\mathfrak{p}$ is a valuation ring (see [19, Proposition 3.3, "⇒ 3"]). Let $a, b \in \widetilde{S} \setminus \mathfrak{p}$. Since $\mathfrak{p}S$ is maximal, a and b are invertible in S. There exists $\alpha \in A$ such that a, b come from $a_\alpha, b_\alpha \in \widetilde{S}_\alpha$. Then, we have the v-étale covering $\mathcal{U}_{\alpha,a} \sqcup \mathcal{U}_{\alpha,b} \to \mathcal{T}_\alpha$ with

$$\mathcal{U}_{\alpha,a} = (\mathrm{Spec}(S_\alpha), \mathrm{Spec}(\widetilde{S}_\alpha\Big[\frac{b_\alpha}{a_\alpha}\Big]), \;\; \mathcal{U}_{\alpha,b} = (\mathrm{Spec}(S_\alpha), \mathrm{Spec}(\widetilde{S}_\alpha\Big[\frac{a_\alpha}{b_\alpha}\Big]).$$

Since $\mathcal{T}$ is local and connected, the projection $\mathcal{T} \to \mathcal{T}_\alpha$ factors through $\mathcal{U}_{\alpha,a}$ or $\mathcal{U}_{\alpha,b}$, so $\widetilde{S}_\alpha \to \widetilde{S}$ factors through either $\widetilde{S}_\alpha\Big[b_\alpha/a_\alpha\Big]$ or $\widetilde{S}_\alpha\Big[a_\alpha/b_\alpha\Big]$. Hence, either $b = ha$ or $a = hb$ for the image $h \in \widetilde{S}$ of b_α/a_α or a_α/b_α, which implies that $\widetilde{S}/\mathfrak{p}$ is a valuation ring.

Next, we show $\widetilde{S} \simeq S \times_{k(\mathfrak{p})} \widetilde{S}/\mathfrak{p}$, where $k(\mathfrak{p}) = S/\mathfrak{p}$ is the fraction field of $\widetilde{S}/\mathfrak{p}$ (see [19, Proposition 3.3, "⇒ 4"]). Since S is local and $\mathfrak{p}S$ is its maximal ideal, we have $S = \widetilde{S}_\mathfrak{p}$. Then, it is enough to check that the map $\mathfrak{p} \to \mathfrak{p}S$ is an isomorphism. The map $\widetilde{S} \to S$ is injective by Remark 8.3,(1), therefore $\mathfrak{p} \to \mathfrak{p}S$ is injective. Recall that $S = \widetilde{S}[1/f]$ for a non-zero divisor $f \in \widetilde{S}$. Let x/f^n in $\mathfrak{p}S$, with $x \in \mathfrak{p}$. There exists $\alpha \in A$ such that x, f come from $x_\alpha, f_\alpha \in \widetilde{S}_\alpha$ such that $f_\alpha \in \widetilde{S}_\alpha$ is a unit in S_α. Then, we have the v-étale covering $\mathcal{U}_{\alpha,f} \sqcup \mathcal{U}_{\alpha,x} \to \mathcal{T}$ with

$$\mathcal{U}_{\alpha,f} = (\mathrm{Spec}(S_\alpha), \mathrm{Spec}(\widetilde{S}_\alpha\Big[\frac{x_\alpha}{f_\alpha^n}\Big])), \;\; \mathcal{U}_{\alpha,x} = (\mathrm{Spec}(S_\alpha[1/x_\alpha]), \mathrm{Spec}(\widetilde{S}_\alpha\Big[\frac{f_\alpha^n}{x_\alpha}\Big])).$$

As before, this implies that $\widetilde{S}_\alpha \to \widetilde{S}$ factors through either $\widetilde{S}[x_\alpha/f_\alpha^n]$ or $\widetilde{S}[f_\alpha^n/x_\alpha]$. In the former case, there is $y \in \widetilde{S}$ such that $yx = f^n$, but this is impossible since $f \notin \mathfrak{p}$. In the latter case, there is $y \in \widetilde{S}$ such that $yf^n = x$, so $x/f^n = y \in \widetilde{S}$, which implies that $\mathfrak{p} \to \mathfrak{p}S$ is surjective.

Next, we show that S is henselian (see [19, Proposition 3.4]). We include the argument of [17, Lemma 10.7] (which is more straightforward). Let $S \to B$ be finite with $\mathrm{Spec}(B)$ connected. Then, B is semilocal, and to show that S is henselian, it is enough to show that $B/\mathfrak{p}B$ is local. Since it is a finite algebra over the field $S/\mathfrak{p}S$, it is enough to check that $\mathrm{Spec}(B/\mathfrak{p}B)$ is connected. Let $\tilde{B}$ be the integral closure of $\widetilde{S}$ in B. Since $\widetilde{S} \to \tilde{B}$ is integral, $\tilde{B}$ is a filtered union of its subrings $\tilde{B}_i$ finite over $\widetilde{S}$. Since $\tilde{B}_i \to \tilde{B} \to B$ are injective, the maps $\mathrm{Spec}(B) \to \mathrm{Spec}(\tilde{B}) \to \mathrm{Spec}(\tilde{B}_i)$ have dense images by [35, Tag 00FL]. Therefore, $\mathrm{Spec}(\tilde{B})$ and $\mathrm{Spec}(\tilde{B}_i)$ are connected since so is $\mathrm{Spec}(B)$. Since $\widetilde{S}$ is henselian, $\tilde{B}_i$ is local henselian for all i. Since the maps $\tilde{B}_i \to \tilde{B}_j$ are finite, they are local maps of henselian local rings. Hence, $\tilde{B}$ is henselian by [35, Tag 04GI], so $\tilde{B}/\mathfrak{p}\tilde{B}$ is henselian.

Claim 8.10. *Every element of $\mathfrak{p}B$ is integral over $\widetilde{S}$.*

Admitting the claim, we have $\mathfrak{p}B \subseteq \tilde{B}$, so $\mathfrak{p}\tilde{B} = \mathfrak{p}B$. Hence, $\tilde{B}/\mathfrak{p}\tilde{B} \to B/\mathfrak{p}B$ is injective, so $\mathrm{Spec}(B/\mathfrak{p}B)$ is connected since so is $\mathrm{Spec}(\tilde{B}/\mathfrak{p}\tilde{B})$ as proved above.

To show the claim, take $y \in B$ and $m \in \mathfrak{p}$. Since B is integral over S, we can write $y^n = \sum_{i=0}^{n-1} a_i y^n$ for $a_i \in S$ so that

$$(my)^n = \sum_{i=0}^{n-1} a_i m^{n-i} (my)^i.$$

Since $n - i \geq 1$ for $i \in [0, n-1]$, we have $a_i m^{n-i} \in \mathfrak{p}S = \mathfrak{p} \subseteq \widetilde{S}$ which proves the claim.

To conclude the proof of the implication $\Rightarrow$, it is enough to further check that (k,v) is strictly henselian in case the v-étale-topology and tamely closed in case the tame topology, where $k = S/\mathfrak{p}S$ and v is the valuation associated to $\widetilde{S}/\mathfrak{p}$. The former case holds since $\widetilde{S}$ is strictly henselian. To show the latter case, take a finite extension k'/k and a valuation v' on k' over v such that v'/v is tame. We want to prove $k = k'$. Since it is separable, there exists $\overline{\varpi} \in \mathcal{O}_{v'}$ such that $k' = k[\overline{\varpi}]$ and $\mathcal{O}_{v'}$ is the integral closure of $\mathcal{O}_v[\overline{\varpi}]$. Let $\overline{p} \in \mathcal{O}_v[T]$ be the monic minimal polynomial of $\overline{\varpi}$ over k. Since S is henselian and $\widetilde{S} = S \times_k \mathcal{O}_v$, there is $p \in \widetilde{S}[T]$ that maps to $\overline{p}$ in $k[T]$ giving a finite étale extension $S \hookrightarrow S' = S[T]/(p)$, with S' henselian local with residue field $k' = S'/\mathfrak{p}S'$. Let $\widetilde{S}' := S' \times_{k'} \mathcal{O}_{v'}$.

Claim 8.11. *$\widetilde{S}'$ is the integral closure of $\widetilde{S}[T]/(p)$ in S'.*

Indeed, let R be the integral closure of $\widetilde{S}[T]/(p)$ in S'. Note that the image of T in S' lies in $\widetilde{S}'$ since its image $\overline{\varpi}$ in $k' = S'/\mathfrak{p}S'$ lies in $\mathcal{O}_{v'}$. Hence, we have that the image of $\widetilde{S}[T]/(p)$ in S' lies in $\widetilde{S}'$, so since $\widetilde{S}'$ is integrally closed in S' we have that $R \subseteq \widetilde{S}'$. It now suffices to show that $\widetilde{S}'$ is integral over R. By Claim 8.10, we have $\mathfrak{p}\widetilde{S}' \subset R$. We have $\widetilde{S}'/\mathfrak{p}\widetilde{S}' = \mathcal{O}_{v'}$ and $\mathcal{O}_{v'}$ is integral over $\mathcal{O}_v[\overline{\varpi}]$. This concludes the proof of the claim.

Recall $\mathcal{T} = (\mathrm{Spec}(S), \mathrm{Spec}(\widetilde{S})) = \varprojlim_\alpha \mathcal{T}_\alpha$ with $\mathcal{T}_\alpha = (\mathrm{Spec}(S_\alpha), \mathrm{Spec}(\widetilde{S}_\alpha))$. We show $\mathcal{T}' := (\mathrm{Spec}(S'), \mathrm{Spec}(\widetilde{S}')) \in \widetilde{(X,\tilde{X})_\tau}$. Since $\widetilde{S} = \varinjlim \widetilde{S}_\alpha$, there exists $\alpha_0 \in A$ and $p_{\alpha_0} \in \widetilde{S}_{\alpha_0}[T]$ mapping to p. Letting p_α be the image of p_{α_0} in $\widetilde{S}_\alpha[T]$, we have $S' = \varinjlim_{\alpha \geq \alpha_0} S'_\alpha$ with $S'_\alpha = S_\alpha[T]/(p_\alpha)$. By construction, $S'_\alpha = S'_{\alpha_0} \otimes_{S_{\alpha_0}} S_\alpha$. By [35, Tag 01SR], $S_\alpha \to S'_\alpha$ is étale for $\alpha \gg \alpha_0$, so $\mathrm{Spec}(\widetilde{S}'_\alpha)$ is étale over X since $\mathrm{Spec}(S_\alpha)$ is étale over X. Let $\widetilde{S}'_\alpha$ be the integral closure of $\widetilde{S}_\alpha[T]/(p_\alpha)$ in S'_α. Since $\mathrm{Spec}(\widetilde{S}_\alpha)$ is ift over $\tilde{X}$, $\mathrm{Spec}(\widetilde{S}'_\alpha)$ is ift over $\tilde{X}$. By Claim 8.11, we have $\varinjlim_\alpha \widetilde{S}'_\alpha = \widetilde{S}'$ noting that taking integral closures commutes with filtered colimits. By construction, $\mathcal{T}'_\alpha := (\mathrm{Spec}(S'_\alpha), \mathrm{Spec}(\widetilde{S}'_\alpha)) \in (X,\tilde{X})_\tau$ and we have $\mathcal{T}' = \varprojlim_\alpha \mathcal{T}'_\alpha$. By Lemma 8.9, $\mathcal{T}'_\alpha \to \mathcal{T}_\alpha$ is a tame covering for $\alpha \gg \alpha_0$. Since $\mathcal{T}$ is tame local, this implies that for $\alpha \gg \alpha_0$, the map $S_\alpha \to S$ factors through S'_α, which implies $k = k'$ as desired.

$\Leftarrow$ Take $\mathcal{T} = (\mathrm{Spec}(S), \mathrm{Spec}(\widetilde{S}))$ with $\widetilde{S} = S \times_k \mathcal{O}_v$ as in Proposition 8.5. We want to show that for any covering $h : \mathcal{V} = (V, \tilde{V}) \to \mathcal{U} = (U, \tilde{U})$ in $(X, \tilde{X})_\gamma$ with $\gamma =$ vét or $\gamma = t$,

$$\mathrm{Hom}_{\widetilde{(X,\tilde{X})_\tau}}(\mathcal{T}, \mathcal{V}) \to \mathrm{Hom}_{\widetilde{(X,\tilde{X})_\tau}}(\mathcal{T}, \mathcal{U}) \tag{8.11.1}$$

is surjective. Clearly, it suffices to consider the generator coverings so we may assume that h is either a modification or strict étale covering in case $\gamma =$ vét and a tame covering in case $\gamma = t$. Write $T = \mathrm{Spec}(S)$, $\tilde{T} = \mathrm{Spec}(\widetilde{S})$ and take $(f, \tilde{f}) : (T, \tilde{T}) \to (U, \tilde{U})$.

If h is a strict étale covering so that $\tilde{V} \to \tilde{U}$ is an étale covering and $V = \tilde{V} \times_{\tilde{U}} U$, $\tilde{f}$ admits a lift $\tilde{g} : \tilde{T} \to \tilde{V}$ since $\widetilde{S}$ is strictly henselian. Moreover, the composite $T \to \tilde{T} \xrightarrow{\tilde{g}} \tilde{V}$ and $f : T \to U$ induce $g : T \to V = \tilde{V} \times_{\tilde{U}} U$ so that $(g, \tilde{g})$ gives a lift of $(f, \tilde{f})$.

Assume that h is a modification. Then, $f : T \to U$ lifts to $g : T \to V$ because $V \to U$ is an isomorphism. By the valuative criterion for properness, the composite $\mathrm{Spec}(k) \hookrightarrow T \xrightarrow{g} V \to \tilde{V}$ extends to a morphism $q : \mathrm{Spec}(\mathcal{O}_v) \to \tilde{V}$. These morphisms factor through some open affine of $\tilde{V}$, so q and $T \xrightarrow{g} V \to \tilde{V}$ glue to give a morphism $\tilde{g} : \tilde{T} = \mathrm{Spec}(\widetilde{S}) \to \tilde{V}$ since $\tilde{T} = \mathrm{Spec}(\mathcal{O}_v \times_k S) = \mathrm{Spec}(\mathcal{O}_v) \sqcup_{\mathrm{Spec}(k)} \mathrm{Spec}(S)$ is the categorical pushout in the category of affine schemes. Thus, we get a lifting $(g, \tilde{g})$ of $(f.\tilde{f})$.

Finally, assume that h is a tame covering and (k, v) is tamely closed. Let $x \in U$ be the image of $\mathrm{Spec}(k) \hookrightarrow T \xrightarrow{f} U$ and v_x be the restriction of v to its residue field $\kappa(x)$. By the assumption, there is $y \in V$ and a valuation v_y on $\kappa(y)$ extending v_x such that $(\kappa(y), v_y)/(\kappa(x).v_x)$ is tame. Since (k, v) is tamely closed, there exists a map of valued fields $(\kappa(y), v_y) \to (k, v)$ which factors $(\kappa(x), v_x) \to (k, v)$. Since $V \to U$ is an étale covering and S is henselian, this implies that $f : T = \mathrm{Spec}(S) \to U$ admits a lift $g : T \to V$. By the same argument as in the case of modifications, g extends to $\tilde{g} : \tilde{T} \to \tilde{V}$ so that we get a lifting $(g, \tilde{g})$ of $(f.\tilde{f})$. This completes the proof.

9. Čech comparison

We fix again an open immersion $X \to \tilde{X}$ of noetherian schemes. The main theorem of this section is the following.

Theorem 9.1. *Let $F \in \mathbf{Shv}((X,\tilde{X})_t, \mathbf{Ab})$ and let $\mathcal{Y} = (Y,\tilde{Y}) \in (X,\tilde{X})_t$ such that $\tilde{Y}$ satisfies the property that every finite set of points is contained in an affine open. Then the natural map*

$$\check{H}^q_t(\mathcal{Y}, F) \to H^q_t(\mathcal{Y}, F)$$

is an isomorphism.

For the proof of the theorem, we need the following result, which is analogous to [1, Theorem 4.1] while its proof is closer to the arguments given in the proof of [17, Proposition 7.14 and Theorem 7.16].

Lemma 9.2. *Let $\mathcal{Y} = (Y,\tilde{Y}) \in (X,\tilde{X})_t$ such that $\tilde{Y}$ satisfies the property that every finite set of points is contained in an affine open. Let $\mathcal{U} = (U,\tilde{U}) \xrightarrow{(\varphi,\tilde{\varphi})} \mathcal{Y}$ be a tame covering. Then, for a tame covering $\mathcal{V} \to \mathcal{U}^{\times_{\mathcal{Y}} n}$, there is a tame covering $\mathcal{U}' \to \mathcal{U}$ such that the composition $\mathcal{U}'^{\times_{\mathcal{Y}} n} \to \mathcal{U}^{\times_{\mathcal{Y}} n}$ factors through $\mathcal{V}$.*

The proof will be given later in §9.3.

We are now ready to prove Theorem 9.1. There is a spectral sequence

$$E_2^{p,q} = \check{H}^p_t(\mathcal{Y}, \mathcal{H}^q(F)) \Rightarrow H^{p+q}_t(\mathcal{Y}, F),$$

where $\mathcal{H}^q(F)$ is the presheaf $\mathcal{U} \to H^q_t(\mathcal{U}, F)$ on $(X,\tilde{X})_t$. It suffices to show $E_2^{p,q} = 0$ for $q > 0$. Every element of $E_2^{p,q}$ is represented by

$$\alpha \in \check{C}^p(\mathcal{U}, \mathcal{H}^q(F)) = \mathcal{H}^q(F)(\mathcal{U}^{\times_{\mathcal{Y}}(p+1)}) = H^q_t(\mathcal{U}^{\times_{\mathcal{Y}}(p+1)}, F)$$

for a tame covering $\mathcal{U} \to \mathcal{Y}$. By Lemma 1.20, we have $\check{H}^0(\mathcal{W}, \mathcal{H}^q(F)) = 0$ for every $\mathcal{W} \in (X,\tilde{X})_t$ so that there is a tame covering $\mathcal{V} \to \mathcal{U}^{\times_{\mathcal{Y}}(p+1)}$ such that $\alpha \mapsto 0$ in $\mathcal{H}^q(F)(\mathcal{V}) = H^q_t(\mathcal{V}, F)$. By Lemma 9.2, there is a tame covering $\mathcal{U}' \to \mathcal{U}$ such that $\mathcal{U}'^{\times_{\mathcal{Y}}(p+1)} \to \mathcal{U}^{\times_{\mathcal{Y}}(p+1)}$ factors through $\mathcal{V}$. Hence, $\alpha \mapsto 0$ in $H^q_t(\mathcal{U}'^{\times_{\mathcal{Y}}(p+1)}, F) = \check{C}^p(\mathcal{U}', \mathcal{H}^q(F))$. This yields the desired vanishing of $E_2^{p,q}$.

9.3. Local objects: We show the existence of local pro-covers, which will be used in the comparison of the tame cohomology with Čech cohomology (see Lemma 9.2).

Lemma 9.4. *For every $\mathcal{Y} \in (X,\tilde{X})_\tau$, there is $\mathcal{W} \in \widetilde{(X,\tilde{X})}_\tau$ v-étale (resp. tamely) local such that $\mathcal{W} \to \mathcal{Y}$ is a cofiltered limit of v-étale (resp. tame) coverings $\mathcal{W}_\lambda \to \mathcal{Y}$ in $(X,\tilde{X})_{\mathrm{affine},\tau}$ with $\mathcal{W}_\lambda \in (X,\tilde{X})_{\mathrm{int},\tau}$.*

Proof. We prove the lemma only for the tame topology. The proof for the v-étale topology is the same. We can suppose $\mathcal{Y} := (\mathrm{Spec}(A), \mathrm{Spec}(\tilde{A})) \in (X,\tilde{X})_{\mathrm{affine,t}}$. We use the same strategy of [7, Lemma 2.2.7] (see [17, Proposition 7.12]). Let I be the set of isomorphism classes of coverings $\mathcal{U} \to \mathcal{Y}$ in $(X,\tilde{X})_{\mathrm{affine,t}}$. For each $i \in I$, pick a representative $(\mathrm{Spec}(B_i), \mathrm{Spec}(\tilde{B}_i)) \to \mathcal{Y}$ and set

$$A_1 := \varinjlim_{J\subset I \text{ finite}} \bigotimes_{j\in J} B_j \qquad \tilde{A}_1 := \varinjlim_{J\subset I \text{ finite}} \bigotimes_{j\in J} \tilde{B}_j, \tag{9.4.1}$$

where the tensor products are over A and $\tilde{A}$ respectively. By construction, we can write

$$\mathcal{Y}_1 := (\mathrm{Spec}(A_1), \mathrm{Spec}(\tilde{A}_1)) = \varprojlim_{\lambda_1\in\Lambda_1} \mathcal{Y}_{\lambda_1} \text{ with } \mathcal{Y}_{\lambda_1} = (\mathrm{Spec}(A_{\lambda_1}), \mathrm{Spec}(\tilde{A}_{\lambda_1}))$$

as a cofiltered limit of coverings $\mathcal{Y}_{\lambda_1} \to \mathcal{Y}$ in $(X,\tilde{X})_{\mathrm{affine,t}}$ such that for every covering $\mathcal{U} \to \mathcal{Y}$ in $(X,\tilde{X})_t$, the map $\mathcal{Y}_1 \to \mathcal{Y}$ factors through $\mathcal{U}$. For each $\lambda_1 \in \Lambda_1$, let I_{λ_1} be the set of isomorphism classes of coverings $\mathcal{U} \to \mathcal{Y}_{\lambda_1}$ in $(X,\tilde{X})_{\mathrm{affine,t}}$ and apply the same construction as (9.4.1) to $(\mathcal{Y}_{\lambda_1}, I_{\lambda_1})$ instead of $(\mathcal{Y}, I)$ to get $(A_{2,\lambda_1}, \tilde{A}_{2,\lambda_1})$ instead of $(A_1, \tilde{A}_1)$ and put $\mathcal{Y}_{2,\lambda_1} = (\mathrm{Spec}(A_{2,\lambda_1}), \mathrm{Spec}(\tilde{A}_{2,\lambda_1}))$. Then, for every covering $\mathcal{U} \to \mathcal{Y}_{\lambda_1}$, the map $\mathcal{Y}_{2,\lambda_1} \to \mathcal{Y}_{\lambda_1}$ factors through $\mathcal{U}$. Put

$$A_2 := \varinjlim_{J \subset \Lambda_1 \text{ finite}} \bigotimes_{\lambda_1 \in J} (A_{2,\lambda_1} \otimes_{A_{\lambda_1}} A_1), \qquad \tilde{A}_2 := \varinjlim_{J \subset \Lambda_1 \text{ finite}} \bigotimes_{\lambda_1 \in J} (\tilde{A}_{2,\lambda_1} \otimes_{\tilde{A}_{\lambda_1}} \tilde{A}_1), \tag{9.4.2}$$

where the tensor products are over A_1 and $\tilde{A}_1$ respectively. Noting $\mathcal{Y}_{2,\lambda_1} \to \mathcal{Y}_{\lambda_1}$ and $\mathcal{Y}_1 \to \mathcal{Y}$ are cofiltered limits of coverings, we can write

$$\mathcal{Y}_2 := (\mathrm{Spec}(A_2), \mathrm{Spec}(\tilde{A}_2)) = \varprojlim_{\lambda_2 \in \Lambda_2} \mathcal{Y}_{\lambda_2}$$

as a cofiltered limit of coverings $\mathcal{Y}_{\lambda_2} \to \mathcal{Y}$ in $(X, \tilde{X})_{\mathrm{affine,t}}$ such that for every $\lambda_1 \in \Lambda_1$ and covering $\mathcal{U} \to \mathcal{Y}_{\lambda_1}$ in $(X, \tilde{X})_t$, the map $\mathcal{Y}_2 \to \mathcal{Y}_1 \to \mathcal{Y}_{\lambda_1}$ factors through $\mathcal{U}$. Iterating the construction, we get a sequence in $\widetilde{(X, \tilde{X})}_\tau$

$$\cdots \to \mathcal{Y}_3 \to \mathcal{Y}_2 \to \mathcal{Y}_1 \to \mathcal{Y} \text{ with } \mathcal{Y}_n = \varprojlim_{\lambda_n \in \Lambda_n} \mathcal{Y}_{\lambda_n}$$

such that for every $\lambda_n \in \Lambda_n$, $\mathcal{Y}_{\lambda_n} \to \mathcal{Y}$ is a covering in $(X, \tilde{X})_{\mathrm{affine,t}}$ and for every covering $\mathcal{U} \to \mathcal{Y}_{\lambda_n}$ in $(X, \tilde{X})_t$, the map $\mathcal{Y}_{n+1} \to \mathcal{Y}_n \to \mathcal{Y}_{\lambda_n}$ factors through $\mathcal{U}$. Set $\mathcal{W} = \varprojlim_n \mathcal{Y}_n \in \widetilde{(X, \tilde{X})}_\tau$. By the construction, $\mathcal{W}$ is a cofiltered limit of coverings of $\mathcal{Y}$. It suffices to show that $\mathcal{W}$ is tame local. Let $\mathcal{V} \to \mathcal{U}$ be a covering in $(X, \tilde{X})_{\mathrm{affine,t}}$ and $\varphi : \mathcal{W} \to \mathcal{U}$ be a map in $\widetilde{(X, \tilde{X})}_\tau$. By Lemma 6.3, φ factors through $\mathcal{Y}_{\lambda_n}$ for some λ_n. Since $\mathcal{Y}_{\lambda_n} \times_{\mathcal{U}} \mathcal{V} \to \mathcal{Y}_{\lambda_n}$ is a covering in $(X, \tilde{X})_t$, the map $\mathcal{Y}_{n+1} \to \mathcal{Y}_n \to \mathcal{Y}_{\lambda_n}$ factors through $\mathcal{Y}_{\lambda_n} \times_{\mathcal{U}} \mathcal{V}$ so that φ lifts to a map $\mathcal{W} \to \mathcal{V}$. This completes the proof. $\square$

Definition 9.5. Let $\mathcal{U} = (U, \tilde{U}) \in (X, \tilde{X})_t$, let $x \in U$ and let $U_x^h = \mathrm{Spec}(\mathcal{O}_{U,x}^h)$ be the henselization at x. An x-local object over $\mathcal{U}$ is $\mathcal{T} := (\mathrm{Spec}(B), \mathrm{Spec}(\tilde{B})) \in \widetilde{(X, \tilde{X})}_t$ with a map $\mathcal{T} \to \mathcal{U}$ such that

(1) B is henselian local with residue field k, $\mathrm{Spec}(B) \to U$ factors through U_x^h and the map $\mathcal{O}_{U,x}^h \to B$ is local and ind-étale;
(2) There is a $\tilde{U}$-admissible valuation v on k such that (k, v) is tamely closed;
(3) $\tilde{B} = B \times_k \mathcal{O}_v$, where $\mathcal{O}_v$ is the valuation ring of (k, v).

Example 9.6. Let $\mathcal{U} = (U, \tilde{U}) \in (X, \tilde{X})_t$, let $x \in U$ and $k(x)$ be its residue field. Let $(x, v, \varepsilon_v) \in \mathrm{Spa}(U, \tilde{U})$ and choose an extension $\overline{v}$ to a separable closure $\overline{k(x)}$ of $k(x)$ and $k(x) \hookrightarrow k(x)_v^t$ be the tame closure of $k(x)$ with respect to the valuation v. Let $\mathcal{O}_{U,x}^h \to \mathcal{O}_{U,x}^t$ be the ind-étale map corresponding to the field extension $k(x) \hookrightarrow k(x)_v^t$ and let $\mathcal{O}_v^t \subseteq k(x)_v^t$ be the valuation ring of the restriction of $\overline{v}$ to $k(x)_v^t$. Then, $\mathcal{U}_{(x,v)} := (\mathrm{Spec}(\mathcal{O}_{U,x}^t), \mathrm{Spec}(\mathcal{O}_{U,x}^t \times_{k(x)_v^t} \mathcal{O}_v^t))$ is an x-local object over $\mathcal{U}$. Moreover, $\mathcal{U}_{(x,v)} \to \mathcal{U}$ is a cofiltered limit of maps $\mathcal{U}_i \to \mathcal{U}$ in $(X, \tilde{X})_t$ which is tame over (x, v, ε_v).

The following Lemma and its proof is very close to [17, Theorem 11.1 and Corollary 11.7].

Lemma 9.7. *Let $\mathcal{Y} = (Y, \tilde{Y}) \in (X, \tilde{X})_t$ such that $\tilde{Y}$ satisfies the property that every finite set of points is contained in an affine open. For $1 \leq i \leq n$, let $x_i \in Y$ and let $\mathcal{P}_i$ be x_i-local objects over $\mathcal{Y}$. Then $\mathcal{T} = \mathcal{P}_1 \times_{\mathcal{Y}} \ldots \times_{\mathcal{Y}} \mathcal{P}_n \in \widetilde{(X, \tilde{X})}_\tau$ is affine and is a disjoint union of x-local objects, where either $x = x_i$ for some i or x is a generization of all x_i, i.e., x_i lie in the closure of x.*

Proof. Let $\mathcal{P}_i = (\mathrm{Spec}(A_i), \mathrm{Spec}(\tilde{A}_i))$ with $\tilde{A}_i = A_i \times_{k(A_i)} V_i$ and let $\mathrm{Spec}(V_i) \to \tilde{Y}$ be the induced map on the valuation rings, and let $\tilde{y}_i$ be the respective images of the closed points. Let $\mathrm{Spec}(\tilde{A}) \subseteq \tilde{Y}$ be an affine open containing x_i and $\tilde{y}_i$ for all i. As $\tilde{y}_i$ is a specialization of x_i, we have natural maps

$$\tilde{A} \to \mathcal{O}_{\tilde{Y},\tilde{y}_i} \to \mathcal{O}_{\tilde{Y},x_i} \to k(x_i) \to k(A_i),$$

and by the definition of $\tilde{y}_i$ its composition factors via $V_i \hookrightarrow k(A_i)$. Therefore all the maps $\mathrm{Spec}(V_i) \to \tilde{Y}$ above factor through $\mathrm{Spec}(\tilde{A})$, therefore $\mathrm{Spec}(\tilde{A}_1) \times_{\tilde{Y}} \ldots \mathrm{Spec}(\tilde{A}_n) = \mathrm{Spec}(\tilde{A}_1 \otimes_{\tilde{A}} \ldots \tilde{A}_n)$. As $\mathrm{Spec}(\tilde{A}) \cap Y$ is quasi-affine we find an open $\mathrm{Spec}(A) \subseteq \mathrm{Spec}(\tilde{A}) \cap Y$ which contains all the x_i. Then all maps $\mathrm{Spec}(A_i) \to Y$ factor through $\mathrm{Spec}(A)$ hence

$$\mathcal{P}_1 \times_{\mathcal{Y}} \ldots \times_{\mathcal{Y}} \mathcal{P}_n = \mathcal{P}_1 \times_{(\mathrm{Spec}(A),\mathrm{Spec}(\tilde{A}))} \cdots \times_{(\mathrm{Spec}(A),\mathrm{Spec}(\tilde{A}))} \mathcal{P}_n.$$

Therefore we are reduced to the case $\mathcal{Y} = (\mathrm{Spec}(A), \mathrm{Spec}(\tilde{A}))$, and now the general case follows from the case where $n = 2$.

Thus it suffices to consider the following situation. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of A. Let $\mathcal{P} = (\mathrm{Spec}(B), \mathrm{Spec}(\tilde{B}))$ and $\mathcal{Q} = (\mathrm{Spec}(C), \mathrm{Spec}(\tilde{C}))$ be $\mathfrak{p}$-local and $\mathfrak{q}$-local objects, respectively, with $\tilde{B} = B \times_{k_B} V_B$ and $\tilde{C} = C \times_{k_C} V_C$ as in Definition 9.5. Denote by $\mathfrak{m}_B \subset B$ and $\mathfrak{m}_C \subset C$ the maximal ideals. By [17, Theorem 6.3 and Theorem 6.4], $B \otimes_A C$ is a product of henselian local A-algebras and the following holds: let D be a factor of $B \otimes_A C$ and denote by $\mathfrak{m}$ its maximal ideal and $L = D/\mathfrak{m}$ its residue field.

(1) If the maps $B \to D$ and $C \to D$ are not local, then L is separably closed;
(2) if $\varphi : B \to D$ is local, then the residue field extension $k_B \to L$ is a separable algebraic extension.

In case (1) the natural map $\tilde{B} \otimes_{\tilde{A}} \tilde{C} \to D$ is surjective. Indeed, we have $\mathfrak{m}_B \subset \tilde{B}$ and $\mathfrak{m}_C \subset \tilde{C}$ and as $B \to D$ and $C \to D$ are not local we have $\mathfrak{m}_B \cdot \mathfrak{m}_C \cdot D = D$. Thus D is integral over $\tilde{B} \otimes_{\tilde{A}} \tilde{C}$ and is strictly henselian by [1, Th.3.4(ii)]. Hence (D, D) is an $\mathfrak{r}$-local object, for $\mathfrak{r} = \mathfrak{m}_D \cap A \subset \mathfrak{p}, \mathfrak{q}$, where we consider the trivial valuation on k_D, and $(\mathrm{Spec}(D), \mathrm{Spec}(D))$ is a component of $\mathcal{P} \times_{\mathcal{Y}} \mathcal{Q}$.

We consider case (2). Denote by $\mathfrak{m}$ its maximal ideal of D and by $L = D/\mathfrak{m}$ its residue field. Let w be the unique valuation on L that extends the valuation v on k_B. Thus (L, w) is a henselian valuation field and its valuation ring $\mathcal{O}_w$ is equal to the integral closure of V_B in L, see, e.g., [30, VI, §8, Proposition 6]. As (k_B, v) is tamely closed so is (L, w). Let $\tilde{D}$ be the integral closure of $\tilde{B} \otimes_{\tilde{A}} \tilde{C}$ in D, and let W be the image of the map $\tilde{D} \to D \to L$, so that we have the following commutative diagram

$$\begin{tikzcd} \tilde{B} \arrow[r] \arrow[d] & \tilde{D} \arrow[r, hook] \arrow[d] & D \arrow[d] \\ V_B \arrow[r] & W \arrow[r, hook] & L. \end{tikzcd} \tag{9.7.1}$$

We claim

$$\mathfrak{m} \subset \tilde{D}. \tag{9.7.2}$$

Assuming (9.7.2) we directly get $\tilde{D} = D \times_L W$. Moreover W contains $\mathcal{O}_w$. Indeed, if $a \in L$ is integral over V_B and $\tilde{a} \in D$ is a lift of a, then we find a monic polynomial $f \in \tilde{B}[X]$ with $f(\tilde{a}) \in \mathfrak{m}$ which by the claim is integral over $\tilde{B} \otimes_{\tilde{A}} \tilde{C}$, hence so is $\tilde{a}$, hence $a \in W$. Thus W is a henselian valuation ring by [17, Lemma 11.4] and its valuation w' is a generization of w, therefore (L, w') is tamely closed. Hence $(\mathrm{Spec}\, D, \mathrm{Spec}\, \tilde{D})$ is a x_1-local object over $\mathcal{Y}$. It remains to prove the claim (9.7.2).

Let $m \in \mathfrak{m}$. As $\varphi : B \to D$ is integral, we find a monic polynomial $f(X) = X^n + a_1 X^{n-1} + \ldots a_n$ in $B[X]$ such that $f^\varphi(m) = 0$, where $f^\varphi = X^n + \varphi(a_1) X^{n-1} + \ldots \varphi(a_n) \in D[X]$. Denote by $\bar{f} \in k_B[X]$ the reduction of f modulo the maximal ideal $\mathfrak{m}_B$. As $0 = f^\varphi(m) \equiv \varphi(a_n) \bmod \mathfrak{m}$, we have $a_n \in \varphi^{-1}(\mathfrak{m}) = \mathfrak{m}_B$. Thus

$$\bar{f} = X^e \cdot \bar{g},$$

for some $e \geq 1$ and $g \in k_B[X]$ monic with $g(0) \neq 0$. As B is henselian, there exist monic polynomials h, $g \in B[X]$ with

$$f = hg, \quad h \equiv X^e \bmod \mathfrak{m}_B, \quad g \equiv \bar{g} \bmod \mathfrak{m}_B.$$

It follows that the constant term of g^φ is a unit in D and hence so is $g^\varphi(m)$. Thus $h^\varphi(m) = 0$ in D. As $h \in X^e + \mathfrak{m}_B[X] \subset \tilde{B}[X]$ we find that m is integral over $\tilde{B}$, hence $m \in \tilde{D}$. This yields claim (9.7.2) and completes the proof of the lemma. $\square$

Prooof Lemma 9.2: By Lemma 9.4, we find a morphism $\mathcal{W} = (W, \tilde{W}) \to \mathcal{U}$ in $\widetilde{(X, \tilde{X})}_t$, which is a limit of tame coverings $\mathcal{W}_\lambda = (W_\lambda, \tilde{W}_) \to \mathcal{U} = (U, \tilde{U})$ in $(X, \tilde{X})_{\mathrm{affine},t}$ with $\mathcal{W}_\lambda \in (X, \tilde{X})_{\mathrm{int},t}$, such that $\mathcal{W}$ is tame local. Note that every connected component $\mathcal{P} = (P, \tilde{P})$ of $\mathcal{W}$ is a x-local object over $\mathcal{Y}$, for some points $x \in Y$. Indeed, by Proposition 8.5, $\mathcal{P}$ satisfies the conditions (2) and (3) of Definition 9.5, so it suffices to check that $\mathcal{P}$ satisfies (1). Denote by x the image in Y of the closed point of P. Then we get a natural local morphism $P \to \mathrm{Spec}(\mathcal{O}^h_{Y,x})$ since P is henselian local. It is ind-étale as P is a component of $\varprojlim_\lambda W_\lambda$ and $W_\lambda \to U$ and $U \to Y$ are étale.

Thus, $\mathcal{W}^n$ (product over $\mathcal{Y}$) is a disjoint union of $\mathcal{P}_1 \times_{\mathcal{Y}} \ldots \times_{\mathcal{Y}} \mathcal{P}_n$, where $\mathcal{P}_j$ are x_j-local objects for some $x_j \in Y$. Hence, by Lemma 9.7 and Proposition 8.5, $\mathcal{W}^n$ is tame local. Thus, for any tame covering $\mathcal{V} \to \mathcal{U}^n$, the map $\mathcal{W}^n \to \mathcal{U}^n$ factors via $\mathcal{V}$. Lemma 6.3 implies that there is λ_0 and a map $\mathcal{W}^n_{\lambda_0} \to \mathcal{V}$ in $(X, \tilde{X})_{\mathrm{affine},t}$ that factors $\mathcal{W}^n \to \mathcal{V}$, hence by choosing $\mathcal{U}' = \mathcal{W}_{\lambda_0}$ we conclude the proof.

10. Computation of tame cohomology

The main result of this section is the following.

Theorem 10.1. *Let $X \hookrightarrow \tilde{X}$ be an open immersion of noetherian schemes. For $\mathcal{V} = (V, \tilde{V}) \in (X, \tilde{X})_t$, let $j^{\mathcal{V}} : (V, \tilde{V})_t \to \tilde{V}_{\text{ét}}$ be the morphisms of sites induced by the functor $\tilde{V}_{\text{ét}} \to (V, \tilde{V})_t$ given by $\tilde{W}/\tilde{V} \mapsto (V \times_{\tilde{V}} \tilde{W}, \tilde{W})$. Let $F \in \mathbf{Shv}((X, \tilde{X})_t, \mathbf{Ab})$ be such that the following condition is satisfied:*

(p) for every $(U, \tilde{U}) \in (X, \tilde{X})_t$ and $x \in \tilde{U}$, $F(\mathrm{Spec}(\mathcal{O}_{\tilde{U},x}) \times_{\tilde{U}} U, \mathrm{Spec}(\mathcal{O}_{\tilde{U},x}))$ is a $\mathbb{Z}_{(p_x)}$-module, where p_x is the exponential characteristic of $\kappa(x)$.

Then, for $\mathcal{U} = (U, \tilde{U})$ in $(X, \tilde{X})_t$, we have canonical isomorphisms

$$H^i_t(\mathcal{U}, F) \cong H^i_{\text{vét}}(\mathcal{U}, \nu_* F) \cong \varinjlim_{\mathcal{V} \to \mathcal{U}} H^i_{\text{ét}}(\tilde{V}, j^{\mathcal{V}}_* F), \quad i \geq 0,$$

where the colimit is over the filtered category of modifications $\mathcal{V} = (V, \tilde{V}) \to \mathcal{U}$.

Remark 10.2. Note that the condition (p) of Theorem 10.1 is satisfied if $F = \Omega^{q,t}$, see 7.5(4), or $F = W_n\Omega^{q,t}$ (see Example 7.5). Also it holds if $\tilde{X}$ is a $\mathbb{Z}_{(p)}$-scheme and F is any tame sheaf of $\mathbb{Z}_{(p)}$-modules.

For the proof, we need the following.

Proposition 10.3. *Let p be a prime and let F be a sheaf of $\mathbb{Z}_{(p)}$-modules on $(X, \tilde{X})_t$. Let $(U, \tilde{U}) \in \widetilde{(X, \tilde{X})}_\tau$ connected and v-étale local. By Proposition 8.5, $\mathcal{U} = (U, \tilde{U}) = (\mathrm{Spec}\, A, \mathrm{Spec}\, \tilde{A})$ where A is a henselian local ring with residue field K and $\tilde{A} = A \times_K \mathcal{O}_v$ for a strictly henselian valuation v . Write $\mathcal{U}$ as the limit of a cofiltered system $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ in $(X, \tilde{X})_\tau$ with $\mathcal{U}_\lambda = (\mathrm{Spec}\, A_\lambda, \mathrm{Spec}\, \tilde{A}_\lambda)$. If the residue characteristic of $\mathcal{O}_v$ is p, then*

$$\varinjlim_{\lambda \in \Lambda} H^i_t(\mathcal{U}_\lambda, F) = 0, \quad \textit{for } i \geq 1.$$

Proof. By Theorem 9.1, we have that

$$\varinjlim_{\lambda \in \Lambda} H^i_t(\mathcal{U}_\lambda, F) = \varinjlim_{\mathcal{V}_\lambda \to \mathcal{U}_\lambda} \varinjlim_{\lambda \in \Lambda} H^{-i} F(\mathcal{V}_\lambda^{\times_{\mathcal{U}_\lambda} \bullet}),$$

where the colimit is indexed over tame covers of $\mathcal{U}_\lambda$. By Lemma 8.9, this is equal to

$$\varinjlim_{\mathcal{V} \to \mathcal{U}} H^{-i} F(\mathcal{V}^{\times_{\mathcal{U}} \bullet}), \tag{10.3.1}$$

where the colimit is indexed over tame covers of $\mathcal{U}$ and F is left Kan extended to $\widetilde{(X, \tilde{X})}_\tau$. By Remark 8.7, we can further suppose that $\mathcal{V}$ are of the form $(V, \tilde{V}) = (\mathrm{Spec}(B), \mathrm{Spec}(\tilde{B}))$ with $A \to B$ is a finite étale map of henselian local rings associated to the residue field extension L/K which is Galois and tame with respect to v and $\tilde{B} = B \times_L \mathcal{O}_w$ with w the valuation on L extending v. Set

$$B_n := B^{\otimes_A n}, \quad V_n := \mathrm{Spec}\, B_n \quad \text{and} \quad \tilde{B}_n := \tilde{B}^{\otimes_{\tilde{A}} n}, \quad \tilde{V}_n := \mathrm{Spec}\, \tilde{B}_n.$$

Let B_n^{int} be the integral closure of $\tilde{B}_n$ in B_n and set $\tilde{V}_n^{int} = \mathrm{Spec}\, \tilde{B}_n^{int}$. As $(V_n, \tilde{V}_n^{int}) \to (V_n, \tilde{V}_n)$ is a modification, see Remark 8.3[17] the desired vanishing follows from the exactness of the complex

$$0 \to F(U, \tilde{U}) \to F(V, \tilde{V}^{\text{int}}) \to F(V_2, \tilde{V}_2^{\text{int}}) \to F(V_3, \tilde{V}_3^{\text{int}}) \to \cdots . \tag{10.3.2}$$

By [30, VI, §8, No. 6, Proposition 6] ,the ring $\mathcal{O}_w$ is also the integral closure of $\mathcal{O}_v$ in L. Hence the Galois group $\mathrm{Gal}(L/K)$ is equal to the decomposition group $\mathrm{Aut}(\mathcal{O}_w/\mathcal{O}_v)$. Moreover, as the category of finite separable field extensions of K is equivalent to the

[17] Precisely speaking, this is not correct since $\tilde{V}_n$ is not noetherian, Indeed, it is a cofiltered limit of modifications.

category of finite local étale A-algebras, we can identify the A-algebra automorphisms of B with $\mathrm{Gal}(L/K)$. Hence $G := \mathrm{Gal}(L/K) = \mathrm{Aut}(B/A)$. As in [29, Example 2.6] the isomorphism $B_2 \to \prod_{\sigma\in G} B$, $b_0 \otimes b_1 \mapsto (\sigma(b_0)b_1)_\sigma$ and induction give the isomorphism for $n \geq 2$

$$\varphi_n : B_n \to \prod_{(\sigma_0,\ldots,\sigma_{n-2})\in G^{n-1}} B$$

with

$$\varphi_n(b_0 \otimes \ldots \otimes b_{n-1})_{(\sigma_0,\ldots,\sigma_{n-2})} = (\sigma_{n-2}\cdots\sigma_0)(b_0)\cdot(\sigma_{n-2}\cdots\sigma_1)(b_1)\cdots\sigma_{n-2}(b_{n-2})\cdot b_{n-1}.$$

As $\tilde{B}$ is integral over $\tilde{A}$, so is $\tilde{B}_n$, hence $\tilde{B}_n^{\mathrm{int}}$ is the integral closure of $\tilde{A}$ in B_n and thus φ_n restricts to an isomorphism

$$\tilde{B}_n^{\mathrm{int}} \to \prod_{(\sigma_0,\ldots,\sigma_{n-2})} \tilde{B}.$$

We thus find isomorphisms

$$(V_n, \tilde{V}_n^{\mathrm{int}}) \cong (V, \tilde{V}) \times G^{n-1}$$

as in [29, III, Example 2.6] and can therefore identify the cohomology of (10.3.2) with Galois cohomology

$$H^{-i}F(\mathcal{V}_\lambda^{\times_{\mathcal{U}_\lambda}\bullet}) = H^i(G, F(V,\tilde{V})).$$

This vanishes as $F(V,\tilde{V})$ is a $\mathbb{Z}_{(p)}$-module and the order of G is invertible in $\mathbb{Z}_{(p)}$ by tameness. This completes the proof. $\square$

Proof of Theorem 10.1: Note that $j^{\mathcal{V}}$ is the composition of the morphism of sites

$$\nu : (V,\tilde{V})_t \to (V,\tilde{V})_{\mathrm{vét}}$$

corresponding to the inclusion functor and the morphism of sites

$$\lambda^{\mathcal{V}} : (V,\tilde{V})_{\mathrm{vét}} \to \tilde{V}_{\mathrm{ét}}$$

defined by the functor $\tilde{V}_{\mathrm{ét}} \to (V,\tilde{V})_{\mathrm{vét}} : \tilde{W}/\tilde{V} \mapsto (\tilde{W}\times_{\tilde{V}} V, \tilde{W})$. By Lemma 6.17, we have

$$H_t^i(\mathcal{U}, F) = H^i_{\mathrm{vét}}(\mathcal{U}, R\nu_*F) = \varinjlim_{\mathcal{V}\to\mathcal{U}} H^i_{\mathrm{ét}}(\tilde{V}, \lambda_*^{\mathcal{V}} R\nu_* F),$$

where the limit is indexed over modifications $\mathcal{V} = (V,\tilde{V}) \to \mathcal{U}$. Hence, it suffices to show $R^i\nu_*F = 0$ for $i \geq 1$. By Remark 8.6, this follows from Proposition 10.3 and the fact that the assumptions of loc.cite. are satisfied by condition (p). This completes the proof.

References

[1] M. Artin, *On the joins of Hensel rings*, Adv. Math. **7** (1971), 282–296.
[2] S. Bosch, U. Güntzer and R. Remmert, *Non-archimedean analysis*
[3] S. Bosch, *Lectureson Formal and Rigid Geometry*, SLN in Math. **2105**,
[4] Bartenwerfer, W. *Die erste "metrische" Kohomologiegruppe glatter affinoider Räume, Nederl. Akad. Wetensch. Proc. Ser. A* 40 (1978), no. 1, 1–14.
[5] Bartenwerfer, W. *Die höheren metrischen Kohomologiegruppen affinoider Räume, Math. Ann.* 241 (1979), no. 1, 11–34.
[6] F. Binda, D. Park, Doosung and P.A. Østvær, *Triangulated categories of logarithmic motives over a field*, Astérisque, **433** (2022).
[7] B. Bhatt and P. Scholze, *The pro-étale topology for schemes*, Astérisque **369** (2019), 99–201.
[8] A. Chatzistamatiou and K. Rülling, *Vanishing of the higher direct images of the structure sheaf*, Compos. Math. **151** (2015), no. 11, 2131–2144.
[9] de Jong, J., van der Put, M. *Etale cohomology of rigid analytic spaces*, *Doc. Math.* 1 (1996), No. 01, 1–56
[10] K. Fujiwara and F. Kato, *Foundations of rigid geometry I*. Monographs in Mathematics, European Mathematical Society Publishing House.
[11] O. Gabber and L. Ramero, *Almost ring theory*, Lecture Notes in Mathematics, **1800**, Springer-Verlag, Berlin.

[12] Gerritzen, L. and van der Put, M. *Schottky groups and Mumford curves*, Lecture Notes in Math. **817**.

[13] L. Hesselholt, *The big de Rham-Witt complex*, Acta Math., **214** (2015) no 1, 135–207.

[14] R. Huber, *Étale Cohomology of Rigid Analytic Varieties and Adic Spaces*, Aspects of Math., **30** (1996), Vieweg+Teubner Verlag Wiesbaden.

[15] K. Hübner, *The Adic Tame Site*, Documenta Math., **26** (2021), 873–945.

[16] K. Hübner, *Logarithmic differentials on discretely ringed adic spaces*, https://arxiv.org/abs/2009.14128.

[17] K. Hübner A. Schmidt, *Tha tame site of a scheme*, Invent. Math., **223** (2020), 397–443.

[18] O. Hyodo and K. Kato, *Semi-stable reduction and cystalline cohomology with logarithmic poles*, Astérisque, **223** (1994).

[19] S. Kelly and H. Miyazaki, *Modulus sheaves with transfers*, https://arxiv.org/pdf/2106.12837.

[20] S. Kelly and S. Saito, *A Pro-cdh toplogy*, arxiv.org/pdf/2401.02699.

[21] M. Kerz, S. Saito and G. Tamme, *Towards a non-archimedean analytic analog of the Bass-Quillen conjecture*, *J. Inst. Math. Jussieu* 19 (2020), no. 6, 1931–1946

[22] J.S. Milne, *Étale cohomology*, Princeton Mathematical Series **33** (1980).

[23] C. Mazza, V. Voevodsky and C. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematical Monographs **2** (2006), American Mathematical Society.

[24] A forthcoming paper by A. Merici, K. Rülling and S. Saito.

[25] M. Raynaud and L. Gruson, *Critère de plâtitude et de projectivité*, Invent. Math., **13** (1971), 1–89.

[26] van der Put, M. *Cohomology on affinoid spaces*, *Compositio Math.* 45 (1982), no. 2, 165–198.

[27] P.T. Johnstone, *Topos theory*, London Mathematical Society Monographs **10** (1977), Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York.

[28] S. MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics **5** (1971), Springer-Verlag, New York-Berlin.

[29] J.S. Milne, *Étale cohomology*, Princeton Mathematical Series **33** (1980), Princeton Univ. Press.

[30] N. Bourbaki, *Algèbre Commutative*, Hermann (1961).

[31] H. Matsumura, *Commutative Algebra*, W.A. Benjamin, INC, New York (1970)

[32] M. Artin and A. Grothendieck and J. L. Verdier, *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas. (SGA 4). Un séminaire dirigé par M. Artin, A. Grothendieck, J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint-Donat. Tome 1: Théorie des topos. Exposés I à IV. 2e éd.*, Lecture Notes in Mathematics **269** (1972), Springer, Cham.

[33] M. Artin and A. Grothendieck and J. L. Verdier, *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas. (SGA 4). Un séminaire dirigé par M. Artin, A. Grothendieck, J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint-Donat. Tome 2: Théorie des topos. Exposés V à VIII. 2e éd.*, Lecture Notes in Mathematics **270** (1972), Springer, Cham.

[34] M. Artin and A. Grothendieck and J. L. Verdier, *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas. (SGA 4). Un séminaire dirigé par M. Artin, A. Grothendieck, J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint-Donat. Tome 2: Théorie des topos. Exposés IX à XIX. 2e éd.*, Lecture Notes in Mathematics **305** (1972), Springer, Cham.

[35] *Stacks Project*, https://stacks.math.columbia.edu (2018).