Based on a joint work with Alberto Merici and Kay Rülling

TAME COHOMOLOGY AND ITS APPLICATIONS LECTURES AT INSTITUTE OF SCIENCE TOKYO IN NOV. 2025

SHUJI SAITO

Contents

Part 1. Reviews on basic theories

1. Topos theory

1.1. **Functoriality of presheaves.** A functor $u : \mathcal{C} \to \mathcal{D}$ induces

$$u^p : \mathbf{PSh}(\mathcal{D}) \to \mathbf{PSh}(\mathcal{C})$$

given by $u^pF = F \circ u$, in other words $u^pF(V) = F(u(V))$ for $V \in \mathcal{C}$.

Proposition 1.1. *There exists a functor called the left Kan extension of F along u*

$$u_p : \mathbf{PSh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{D})$$

which is a left adjoint to the functor u^p. In other words

$$\mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(F, u^pG) = \mathrm{Hom}_{\mathbf{PSh}(\mathcal{D})}(u_pF, G)$$

holds bifunctorially in $F \in \mathbf{PSh}(\mathcal{C})$ and $G \in \mathbf{PSh}(\mathcal{D})$.

For $V \in \mathcal{D}$, let $I^u(V)$ denote the category whose objects are pairs (U, φ) with $U \in \mathcal{C}$ and $\varphi : V \to u(U)$ and

$$\mathrm{Hom}_{I^u(V)}((U, \varphi), (U', \varphi')) = \{f : U \to U' \text{ in } \mathcal{C} |\ u(f) \circ \varphi = \varphi'\}.$$

We sometimes drop the superscript u from the notation and we simply write $I(V)$. For $F \in \mathbf{PSh}(\mathcal{C})$, we define

$$u_pF(V) = \varinjlim_{(U,\varphi)\in I(V)^{op}} F(U) = \varinjlim_{I(V)^{op}} F_V,$$

where $F_V \in \mathbf{PSh}(I(V), \mathbf{Sets})$ given by

$$F_V : I(V)^{op} \to \mathbf{Sets} \ : \ (U,\varphi) \to F(U).$$

To show that $u_pF \in \mathbf{PSh}(\mathcal{D})$, note that for $g : V' \to V$ in $\mathcal{D}$, we get a functor $g : I(V) \to I(V')$ by setting $g(U,\varphi) = (U, \varphi \circ g)$. It induces a map

$$u_pF(V) = \varinjlim_{(U,\varphi)\in I(V)^{op}} F(U) \to \varinjlim_{(W,\psi)\in I(V')^{op}} F(W) = u_pF(V').$$

A map of $F \to F'$ in $\mathbf{PSh}(\mathcal{C})$ induces for $V \in \mathcal{D}$

$$u_pF(V) = \varinjlim_{(U,\varphi)\in I(V)^{op}} F(U) \to \varinjlim_{(U,\varphi)\in I(V)^{op}} F'(U) = u_pF(V).$$

Thus, we have defined a functor

$$u_p : \mathbf{PSh}(\mathcal{C}) \to \mathbf{PSh}(\mathcal{D}).$$

To show that

$$\mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(F, u^pG) = \mathrm{Hom}_{\mathbf{PSh}(\mathcal{D})}(u_pF, G)$$

holds bifunctorially in F and G.

Lemma 1.2. *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor. Assume*

(i) $\mathcal{C}$ has a final object e and $u(e)$ is a final object of $\mathcal{D}$,
(ii) $\mathcal{C}$ admits fiber products and u commutes with them.

Then, u_p commutes with fintie limits.

Proof. This follows from the fact that the categories $I^u(V)^{op}$ are filtered by [35, 00X3]. $\square$

1.2. Sites and sheaves. .

Definition 1.3. A site is given by a pair $(\mathcal{C}, \tau)$ of a category $\mathcal{C}$ and a Grothendieck pretopology τ which is a function assigning to each object $U \in \mathcal{C}$ a collection $\mathrm{Cov}(U)$ of families of morphisms $\{U_i \to U\}_{i\in I}$, called coverings family of U, satisfying the following axioms:

(i) If $V \to U$ is an isomorphism, we have $\{V \to U\} \in \mathrm{Cov}(U)$.
(ii) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ and $\{V_{ij} \to U_i\}_{jinJ_i} \in \mathrm{Cov}(U_i)$ for each $i \in I$, we have then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(U)$.
(iii) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ and $V \to U$ is a morphism of $\mathcal{C}$, then $U_i \times_U V$ exists for all $i \in I$ and we have $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(V)$.

Example 1.4. For a scheme S, let $\mathbf{Sch}_S$ be the category of schemes of finite presentation over S.

(i) Let $\acute{\mathrm{E}}\mathrm{t}_S$ be the full subcategory of $\mathbf{Sch}_S$ of étale schemes over S. The big étale site $(\mathbf{Sch}_S)_{\acute{e}t}$ is the site whose underlying category is $\mathbf{Sch}_S$ and whose coverings are étale covering[1]. The small étale site $(\mathbf{Sch}_X)_{\acute{e}t}$ is the full subcategory of $(\mathbf{Sch}_S)_{\acute{e}t}$ whose objects are those U/S such that $U \to S$ is étale. A covering of $S_{\acute{e}t}$ is any étale covering $\{U_i \to U\}$ with $U \in S_{\acute{e}t}$.

[1]For $T \in \mathbf{Sch}_S$, an étale covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i\in I}$ in $\mathbf{Sch}_S$ such that each f_i is étale and $T = \cup f_i(T_i)$.

Definition 1.5. Let $\mathcal{C}$ be a site, and let F be a presheaf of sets on $\mathcal{C}$. We say F is a sheaf if for every $U \in \mathcal{C}$ and every covering $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ the diagram

$$F(U) \to \prod_{i\in I} F(U_i) \underset{pr_1^*}{\overset{pr_0^*}{\rightrightarrows}} \prod_{(i_0,i_1)\in I\times I} F(U_{i_0} \times_U U_{i_1})$$

represents the first arrow as the equalizer of pr_0^* and pr_1^*. We let $\mathbf{Shv}(\mathcal{C}) \subset \mathbf{PSh}(\mathcal{C})$ denote the full subcategory of sheaves (of sets).

Lemma 1.6. *Let $\mathcal{F} : I \to \mathbf{Shv}(\mathcal{C})$ be a diagram. Then $\varprojlim_I \mathcal{F}$ exists and is equal to the limit in $\mathbf{PSh}(\mathcal{C})$.*

Proposition 1.7. *There exists a functor called the sheafification*

$$a : \mathbf{PSh}(\mathcal{C}) \to \mathbf{Shv}(\mathcal{C})$$

which is a left adjoint to the inclusion functor $i : \mathbf{PSh}(\mathcal{C}) \to \mathbf{Shv}(\mathcal{C})$. In other words

$$\mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(F, G) = \mathrm{Hom}_{\mathbf{Shv}(\mathcal{C})}(aF, G)$$

holds bifunctorially in $F \in \mathbf{PSh}(\mathcal{C})$ and $G \in \mathbf{Shv}(\mathcal{C})$. Moreover, a is exact.

Let $F \in \mathbf{PSh}(\mathcal{C})$. For $\mathfrak{U} = \{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$, put

$$H^0(\mathfrak{U}, F) = \mathrm{equalizer}\Big(\prod_{i\in I} F(U_i) \underset{pr_1^*}{\overset{pr_0^*}{\rightrightarrows}} \prod_{(i_0,i_1)\in I\times I} F(U_{i_0} \times_U U_{i_1})\Big)$$

There is a canonical map $F(U) \to H^0(\mathfrak{U}, F)$[2].

For $U \in \mathcal{C}$, let $\mathrm{Cov}(U)$ be the category of all coverings of U in $\mathcal{C}$ whose morphisms are the refinements (see §1.5). Note that $\mathrm{Cov}(U)$ is not empty since $\{id : U \to U\}$ is an object of it. By definition the construction $\mathfrak{U} \mapsto H^0(\mathfrak{U}, F)$ is an object of $\mathbf{PSh}(\mathrm{Cov}(U))$. For $F \in \mathbf{PSh}(\mathcal{C})$, we define

$$F^+(U) = \varinjlim_{\mathfrak{U}\in \mathrm{Cov}(U)^{op}} H^0(\mathfrak{U}, F).$$

Note that $F^+(U) = \check{H}^0(U, F)$ is the zeroth Čech cohomology of F over U (see (1.19.2)).

Lemma 1.8. *(1) For $F \in \mathbf{PSh}(\mathcal{C})$, F^+ is an object of $\mathbf{PSh}(\mathcal{C})$ equipped with a canonical map $F \to F^+$ in $\mathbf{PSh}(\mathcal{C})$. Moreover, the construction is functorial, i.e. a map $f : F \to G$ in $\mathbf{PSh}(\mathcal{C})$ induces a map $f^+ : F^+ \to G^+$ such that the following diagram commutes in $\mathbf{PSh}(\mathcal{C})$:*

$$\begin{tikzcd} F \arrow[r] \arrow[d, "f"] & F^+ \arrow[d, "f^+"] \\ G \arrow[r] & G^+ \end{tikzcd}$$

(2) The presheaf F^+ is separated.

Proof. [35, 00WB]. □

Proposition 1.9. *For $F \in \mathbf{PSh}(\mathcal{C})$, $(F^+)^+ \in \mathbf{Shv}(\mathcal{C})$ and the induced functor*

$$a = ((-)^+)^+ : \mathbf{PSh}(\mathcal{C}) \to \mathbf{Shv}(\mathcal{C})$$

is a left adjoint to the inclusion functor $\mathbf{PSh}(\mathcal{C}) \to \mathbf{Shv}(\mathcal{C})$. Moreover, a is exact.

Proof. [35, 00WB]. The exactness of a follows from the fact that $\mathrm{Cov}(U)$ is filtered (the point is to show a commutes with finite limits).

□

[2] This is the zeroth Čech cohomology of F over U with respect to the covering $\mathfrak{U}$.

1.3. **Functoriality of sheaves.**

Definition 1.10. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A functor $u : \mathcal{C} \to \mathcal{D}$ is called continuous if for every $V \in \mathcal{C}$ and every $\{V_i \to V\}_{i\in I} \in \mathrm{Cov}(V)$, we have the following

(i) $\{u(V_i) \to u(V)\}_{i\in I} \in \mathrm{Cov}(u(V))$,
(ii) for any morphism $T \to V$ in $\mathcal{C}$, the morphism $u(T \times_V V_i) \to u(T) \times_{u(V)} u(V_i)$ is an isomorphism.

Example 1.11. For a map $f : T \to S$ of schemes, consider

$$u : \text{Ét}_S \to \text{Ét}_T \ : \ X \to X \times_S T.$$

Then, u is continuous for the étale topology.

Lemma 1.12. *If $u : \mathcal{C} \to \mathcal{D}$ is continuous, u^p induces*

$$u^s : \mathbf{Shv}(\mathcal{D}) \to \mathbf{Shv}(\mathcal{C}).$$

Proof. Exercise. □

Lemma 1.13. *If $u : \mathcal{C} \to \mathcal{D}$ is continuous, the functor*

$$u_s : \mathbf{Shv}(\mathcal{D}) \to \mathbf{Shv}(\mathcal{C}) \ : \ G \to a(u_p(G))$$

is a left adjoint to u^s.

Proof. Follows directly from Propositions 1.9 and 1.1. □

Definition 1.14. Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A morphism of sites $f : \mathcal{D} \to \mathcal{C}$ is given by a continuous functor $u : \mathcal{C} \to \mathcal{D}$ such that the functor u_s is exact.

.

Proposition 1.15. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous morphism of sites. Assume*

(i) $\mathcal{C}$ has a final object e and $u(e)$ is a final object of $\mathcal{D}$,
(ii) $\mathcal{C}$ admits fiber products and u commutes with them.

Then, u defines a morphism of sites, i.e. u_s is exact.

Proof. This follows from Lemma 1.2 and the exactness of a from Proposition 1.9 (see [35, 00X6]). □

Definition 1.16. A topos is the category $\mathbf{Shv}(\mathcal{C})$ of sheaves on a site $\mathcal{C}$.

(1) Let $\mathcal{C}$, $\mathcal{D}$ be sites. A morphism of topoi $f : \mathbf{Shv}(\mathcal{D}) \to \mathbf{Shv}(\mathcal{C})$ is given by a adjoint pair of functors

$$f^* : \mathbf{Shv}(\mathcal{C}) \leftrightarrows \mathbf{Shv}(\mathcal{D}) : f_*,$$

namely we have for $G \in \mathbf{Shv}(\mathcal{C})$ and $F \in \mathbf{Shv}(\mathcal{D})$

$$\mathrm{Hom}_{\mathbf{Shv}(\mathcal{D})}(f^*G, F) = \mathrm{Hom}_{\mathbf{Shv}(\mathcal{C})}(G, f_*F)$$

bifunctorially, and the functor f^* commutes with finite limits, i.e., is left exact.

(2) Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be sites. Given morphisms of topoi $f : \mathbf{Shv}(\mathcal{D}) \to \mathbf{Shv}(\mathcal{C})$ and $g : \mathbf{Shv}(\mathcal{E}) \to \mathbf{Shv}(\mathcal{D})$, the composition $f \circ g$ is the morphism of topoi defined by the functors $(f \circ g)_* = f_* \circ g_*$ and $(f \circ g)^* = g^* \circ f^*$.

Lemma 1.17. *Given a morphism of sites $f : D \to C$ corresponding to the functor $u : \mathcal{C} \to \mathcal{D}$, the pair of functors $(f^* = u_s, f_* = u^s)$ is a morphism of topoi.*

Proof. This is obvious from Definition 1.14. □

1.4. **Cohomology.**

Theorem 1.18. *Let $\mathcal{C}$ be a site. Then, the category* $\mathbf{Shv}(\mathcal{C},\mathbf{Ab})$ *of abelian sheaves on a site is an abelian category which has enough injectives.*

Proof. [35, 03NU]. □

By the theorem, we can define cohomology as the right-derived functors of the sections functor $F \to F(U)$ for $U \in \mathcal{C}$ and $F \in \mathbf{Shv}(\mathcal{C},\mathbf{Ab})$ defined as

$$H^i(U,F) := R^i\Gamma(U,F) = H^i(\Gamma(U,I^\bullet)),$$

where $F \to I^\bullet$ is an injective resolution. To do this, we should check that the functor $\Gamma(U,-)$ is left exact. This is true and is part of why the category $\mathbf{Shv}(\mathcal{C}\mathbf{Ab})$ is abelian, see Modules on Sites, Lemma 3.1. For more general discussion of cohomology on sites (including the global sections functor and its right derived functors), see Cohomology on Sites, Section 2. The family of functors $H^i(U,-)$ forms a universal δ-functor $\mathbf{Shv}(\mathcal{C},\mathbf{Ab}) \to \mathbf{Ab}$.

It sometimes happens that the site $\mathbf{C}$ does not have a final object. In this case, we define the global sections of $F \in \mathbf{PSh}(\mathcal{C}, S_{\text{ét}})$ over $\mathbf{C}$ to be the set

$$\Gamma(\mathcal{C},F) = \mathrm{Hom}_{\mathbf{PSh}(\mathcal{C})}(e,F),$$

where e is a final object in $\mathbf{PSh}(\mathcal{C},\mathbf{Sets})$. In this case, given $F \in \mathbf{Shv}(\mathcal{C},\mathbf{Ab})$, we define the i-th cohomology group of Fon $\mathbf{C}$ as follows

$$H^i(\mathcal{C},F) = H^i(\Gamma(\mathcal{C},I^\bullet))).$$

In other words, it is the i-th right derived functor of the global sections functor. The family of functors $H^i(\mathcal{C},-)$ forms a universal δ-functor $\mathbf{Shv}(\mathcal{C},\mathbf{Ab}) \to \mathbf{Ab}$.

1.5. **Čech cohomology.**

For $U \in \mathcal{C}$ and $\mathfrak{U} = \{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$, write $U_{i_0\dots i_p} = U_{i_0} \times_U \cdots \times_U U_{i_p}$ for the $(p+1)$-fold fiber product over U of members of $\mathfrak{U}$. Let $F \in \mathbf{PSh}(\mathcal{C},\mathbf{Ab})$, set

$$\check{C}^p(\mathfrak{U},F) = \prod_{(i_0\dots i_p)\in I^{p+1}} F(U_{i_0\dots i_p}).$$

For $s \in \check{C}^p(\mathfrak{U},F)$, we denote $s_{i_0\dots i_p}$ its value in $F(U_{i_0\dots i_p})$. We define

$$d : \check{C}^p(\mathfrak{U},F) \to \check{C}^{p+1}(\mathfrak{U},F)$$

by the formula

$$d(s)_{i_0\dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j (s_{i_0\dots\hat{i_j}\dots i_{p+1}})_{|U_{i_0\cdots i_{p+1}}}.$$

It is straightforward to see that $d\circ d = 0$, i.e. $\check{C}(\mathfrak{U},F)$ is a complex, which we call Čech complex associated to F and $\mathfrak{U}$. Its cohomology groups

$$\check{H}^i(\mathfrak{U},F) = H^i(\check{C}(\mathfrak{U},F))$$

are called the Čech cohomology groups associated to F and $\mathfrak{U}$.

Lemma 1.19. *For $U \in \mathcal{C}$ and $\mathfrak{U} = \{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$, there is a transformation of functors:*

$$\mathbf{Shv}(\mathcal{C},\mathbf{Ab}) \to \mathcal{D}(\mathbb{Z}) \ : \ \check{C}(\mathfrak{U},-) \to R\Gamma(U,-).$$

Moreover, there is a spectral sequence for $F \in \mathbf{Shv}(\mathcal{C},\mathbf{Ab})$:

$$E_2^{p,q} = \check{H}^p(\mathfrak{U},\mathcal{H}^q(F)) \Rightarrow H^{p+q}(U,F), \tag{1.19.1}$$

which is functorial in F, where $\mathcal{H}^q(F) \in \mathbf{PSh}((X,\tilde{X})_t,\mathbf{Ab})$ is given by $\mathcal{U} \to H^q_t(\mathcal{U},F)$. In particular, if $H^i(U_{i_0} \times_U \cdots \times_U U_{i_p}, F) = 0$ for all $i > 0$, $p \geq 0$ and $i_0,\dots,i_p \in I$, then we have $\check{H}^p(\mathfrak{U},F) = H^p(U,F)$.

Proof. [35, 03AX, 03AZ, 03F7]. □

For coverings $\mathfrak{U} = \{U_i \to U\}_{i\in I}$ and $\mathfrak{V} = \{V_j \to V\}_{j\in J}$ in $\mathcal{C}$, a morphism $\mathfrak{U} \to \mathfrak{V}$ is given by a morphism $U \to V$ in $\mathcal{C}$, a map of sets $\alpha : I \to J$ and for each $i \in I$ a morphism $U_i \to V_{\alpha(i)}$ such that the diagram

$$\begin{tikzcd} U_i \arrow[r] \arrow[d] & V_{\alpha(i)} \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

is commutative. In the special case $U = V$ and $U \to V$ is the identity, we call $\mathfrak{U}$ a refinement of $\mathfrak{V}$. A remark is that if the above $\mathfrak{V}$ is the empty family, i.e., if $J = \varnothing$, then no family $\mathfrak{U} = \{U_i \to V\}_{i\in I}$ with $I \neq \varnothing$ can refine $\mathfrak{V}$.

For $U \in \mathcal{C}$, let $\mathrm{Cov}(U)$ be the category of all coverings of U in $\mathcal{C}$ whose morphisms are the refinements[3]. Note that $\mathrm{Cov}(U)$ is not empty since $\{id : U \to U\}$ is an object of it. Take $F \in \mathbf{PSh}(\mathcal{C}, \mathbf{Ab})$. By definition the construction $\mathfrak{U} \mapsto \check{C}(\mathfrak{U}, F)$ is a preshesaf on $\mathrm{Cov}(U)$ with values in the category of complexes of abelian groups. We define

$$\check{C}(U, F) := \varinjlim_{\mathfrak{U}\in \mathrm{Cov}(U)^{op}} \check{C}(\mathfrak{U}, F),$$

$$\check{H}^i(U, F) := H^i(\check{C}(\mathfrak{U}, F)) = \varinjlim_{\mathfrak{U}\in \mathrm{Cov}(U)^{op}} \check{H}(\mathfrak{U}, F), \tag{1.19.2}$$

where the last equality holds since $\mathrm{Cov}(U)$ if cofiltered. By Lemma 1.19, we have a transformation of functors:

$$\mathbf{Shv}(\mathcal{C}, \mathbf{Ab}) \to \mathcal{D}(\mathbb{Z}) \ : \ \check{C}(U, -) \to R\Gamma(U, -).$$

(1.19.1) induces a spectral sequence

$$E_2^{p,q} = \check{H}^p(U, \mathcal{H}^q(F)) \Rightarrow H^{p+q}(U, F). \tag{1.19.3}$$

Lemma 1.20. *Let $U \in \mathcal{C}$ and $F \in \mathbf{PSh}(\mathcal{C}, \mathbf{Ab})$.*

(1) $\check{H}^0(U, \mathcal{H}^q(F)) = 0$ for $q > 0$. In particular, for every $\alpha \in H^q(U, F)$, there is $\mathfrak{U} = \{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ such that $\alpha \mapsto 0$ in $H^q(U_i, F)$ for all $i \in I$.

(2) $\check{H}^i(U, \mathcal{H}^q(F)) = H^i(U, F)$ for $i = 0, 1$ and there is an exact sequence

$$0 \to \check{H}^2(U, F) \to H^2(U, F) \to \check{H}^1(U, \mathcal{H}^1(F)) \to \check{H}^3(U, F) \to H^3(U, F).$$

Proof. ([29, Ch.III 2.9 and 2.10]) (2) follow formally from (1) using (1.19.3). To prove (1), we show the following claim. Recall the pair of adjoint functors from Proposition 1.7:

$$a : \mathbf{PSh}(\mathcal{C}) \underset{\leftarrow}{\overset{\rightarrow}{}} \mathbf{Shv}(\mathcal{C}) : i.$$

Claim 1.21. *For $q > 0$, we have $a\mathcal{H}^q(F) = 0$.*

Indeed, take an injective resolution $F \to I^\bullet$ in $\mathbf{Shv}(\mathcal{C}, \mathbf{Ab})$. Then, $\mathcal{H}^q(F)$ is the q-th cohomology presheaf of the complex $i(I^\bullet)$ in $\mathbf{PSh}(\mathcal{C}, \mathbf{Ab})$. Since a is exact and commutes with taking cohomology, $a\mathcal{H}^q(F)$ is the q-th cohomology sheaf of the complex $ai(I^\bullet) = I^\bullet$ in $\mathbf{Shv}(\mathcal{C}, \mathbf{Ab})$ so that it must vanishes.

By Proposition 1.9, we have $a\mathcal{H}^q(F) = (\mathcal{H}^q(F)^+)^+ = 0$. Since $\mathcal{H}^q(F)^+$ is separated by Lemma 1.8, the natural map $\mathcal{H}^q(F)^+ \to (\mathcal{H}^q(F)^+)^+$ is injective. Thus, we get $\mathcal{H}^q(F)^+ = 0$, which implies (1). ☐

Lemma 1.22. *For $F \in \mathbf{PSh}(\mathcal{C}, \mathbf{Ab})$, the following are equivalent.*

(1) F is flabby, i.e. $H^i(U, F) = 0$ for any $i > 0$ and $U \in \mathcal{C}$.

(2) $\check{H}^i(\mathfrak{U}, F) = 0$ for any $i > 0$, $U \in \mathcal{C}$ and $\mathfrak{U} \in \mathrm{Cov}(U)$.

[3]By our conventions on sites this is indeed a category, i.e., the collection of objects and morphisms forms a set.

(3) $\check{H}^i(U,F)=0$ *for any* $i>0$ *and* $U\in\mathcal{C}$.

Proof. ([29, Ch.III 2.12]) (1)⇒(2). By the assumption, $\mathcal{H}^q(F)=0$ for $q>0$ so (1.19.1) implies $\check{H}^i(\mathfrak{U},F)=H^i(U,F)=0$.

(2)⇒(3). Pass to the colimit over $\mathfrak{U}\in\mathrm{Cov}(U)$.

(3)⇒(1). Take any $U\in\mathcal{C}$. By the assumption, $\check{H}^q(U,F)=0$ for any $q>0$. By Lemma 1.20(2), we get $H^1(U,F)=0$ which implies $\mathcal{H}^1(F)=0$. By the long exact sequence in Lemma 1.20(2), we get $H^2(U,F)=0$ which implies $\mathcal{H}^2(F)=0$. Assume now $\mathcal{H}^i(F)=0$ for $i<q$. Since $\check{H}^0(U,\mathcal{H}^q(F))=0$ by Lemma 1.20, we get $\check{H}^i(U,\mathcal{H}^j(F))=0$ for all $i,j\geq 0$ with $i+j\leq q$. By (1.19.3), it implies $H^q(U,F)=0$ so that $\mathcal{H}^q(F)=0$. This complete the proof by induction.

□

2. Classical rigid analytic spaces

Good references for this section are [2] and [3].

2.1. **Affinoid K-algebras.** Let K be a non-archimedean field, i.e. a field which is complete with respect to a nontrivial non-archimedean absolute value, i.e. a map $|-|:K\to\mathbb{R}_{\geq 0}$ satisfying

(i) $|a|=0\Leftrightarrow a=0$.
(ii) $|ab|=|a||b|$.
(iii) $|a+b|\leq\max\{|a|,|b|\}$.

Note that the map $v:K\to\mathbb{R}\cup\{\infty\}$ given by $v(a)=-\log|a|$ is a valution and there is one-to- one correspondence between non-archimedean absolute values and valuations with value group $\mathbb{R}$ on K, where the inverse is given by $|a|=e^{-v(a)}$. We put

$$\mathcal{O}_K=\{x\in K|\ |x|\leq 1\}$$

and fix $\pi\in K$ with $|\pi|<1$.

For each $n>0$, the Tate K-algebra is

$$\begin{aligned}T_n:=K\langle T_1,\dots,T_n\rangle&=\{f=\sum_{\nu\in\mathbb{N}^n}a_\nu T_1^{\nu_1}\cdots T_n^{\nu_n}\mid a_\nu\in K,\ \lim_{|\nu|\to\infty}|a_\nu|=0\}\\&=\mathcal{O}_K\{T_1,\dots,T_n\}\otimes_{\mathcal{O}_K}K,\end{aligned}$$

where $\mathcal{O}_K\{T_1,\dots,T_n\}$ is the π-adic completion of $\mathcal{O}_K[T_1,\dots,T_n]$. The Gauss norm[4] $||-||:T_n\to\mathbb{R}_{\geq 0}$ is given by

$$||f||=\sup_{\nu\in\mathbb{N}^n}|a_\nu|.$$

Definition 2.1. An affinoid K-algebra is a K-algebra A such that there is a surjective K-algebra homomorphism $\alpha:T_n\to A$. for some $n>0$. Such a K-affinoid algebra A admits a norm $||-||_\alpha$ given by

$$||\alpha(f)||_\alpha=\inf_{a\in\mathrm{Ker}(\alpha)}||f-a||\text{ for }f\in T_n.$$

For another surjective K=algebra homomorphism $\beta:T_m\to A$, there are constants $c,c'>0$ such that $||-||_\alpha\leq c||-||_\beta\leq c'||-||_\alpha$.

Definition 2.2. For an affinoid K-algebra A, let $\mathrm{Sp}(A)$ be the set of the maximal ideal of A. For $x\in\mathrm{Sp}(A)$, the residue field $K(x)$ of x is a finite extension of K so that it carries a unique extension of $|-|$ on K. For $f\in A$, let $f(x)$ be the image of f in $K(x)$ and $|f(x)|$

[4] A map $||-||:A\to\mathbb{R}_{\geq 0}$ is called a semi-norm if $||0||=0$, $||1||=1$, $||fg||\leq||f||||g||$ and $||f-g||\leq||f||+||g||$ for $f,g\in A$. It is a norm if $||f||=0$ implies $f=0$. It is non-archimedian if $||f-g||\leq\max\{||f||,||g||\}$.

be its absolute value under this extension. There is a semi-norm $|-|_{\sup}$ on A on called the supremum norm given by

$$|f|_{\sup} = \sup_{x \in \mathrm{Sp}(A)} |f(x)|.$$

We have the following facts:

(1) $|-|_{\sup}$ is power-multiplicative, i.e. $|f^n|_{\sup} = (|f|_{\sup})^n$ for $f \in A$ and $n > 0$.
(2) For a K-homomorphism $\varphi : A \to B$ of K-affinoid algebras and for $f \in A$, we have $|\varphi(f)|_{\sup} \leq |f|_{\sup}$.
(3) On T_n, the supremum norm coincides with the Gauss norm.
(4) For a surjective K-algebra homomorphism $\alpha : T_n \to A$, we have $|f|_{\sup} \leq ||f||_\alpha$ for all $f \in A$. In particular, $|f|_{\sup} < \infty$.

Theorem 2.3. *(Maximal Principle) For a K-affinoid algebra A and $f \in A$, there exists $x \in \mathrm{Sp}(A)$ such that $|f|_{\sup} = |f(x)|$.*

We put

$$A^\circ = \{f \in A| \ |f|_{\sup} \leq 1\} \ \text{ and } \ A^{\circ\circ} = \{f \in A| \ |f|_{\sup} < 1\}.$$

It is easy to see that A° is a subring of A, which is $\mathcal{O}_K$-algebra and $A^{\circ\circ}$ is its ideal. We have the following facts:

(1) A° is π-adically complete and $A = A^\circ \otimes_{\mathcal{O}_K} K$.
(2) A° is the set of power-bounded elements, i.e. those f that $\{||f^n||_\alpha \ (n \in \mathbb{N})\} \subset \mathbb{R}$ is bounded.
(3) $A^{\circ\circ}$ is the set of topologically nilpotent elements, i.e. those f that $\lim\limits_{n\to\infty} ||f^n||_\alpha = 0$.

2.2. **Affinoid K-spaces.** We let AffAlg_K denote the category of affinoid K-algebras and K-algebra homomorphisms. For a morphism $\varphi : A \to B$ in AffAlg_K, we have the induced map $\varphi^* : \mathrm{Sp}(B) \to \mathrm{Sp}(A)$ sending a maximal ideal $\mathfrak{m} \subset B$ to $\varphi^{-1}(\mathfrak{m})$. Thus, we get a functor

$$\mathrm{Sp} : \mathrm{AffAlg}_K \to \mathbf{Sets}\,.$$

In this subsection, we introduce a G-topology in the sense of Definition 2.8 to make $\mathrm{Sp}(A)$ for $A \in \mathrm{AffAlg}_K$ a G-topological space.

Definition 2.4. For $f_1, \cdots, f_r, g \in A$ which generate the unit ideal, let

$$U\Big(\frac{f_1, \ldots, f_n}{g}\Big) = \{x \in \mathrm{Sp}(A)| \ |f_i(x)| \leq |g(x)| \ (i = 1, \ldots, r)\}$$

This is called a rational subdomain of $X = \mathrm{Sp}(A)$.

We have the following facts:

Lemma 2.5. *(1) For a rational subdomain $U \subset \mathrm{Sp}(A)$ and a morhpism $\varphi : A \to B$ in AffAlg_K inducing $\varphi^* : \mathrm{Sp}(B) \to \mathrm{Sp}(A)$, $(\varphi^*)^{-1}(U)$ is a rational subdomain of $\mathrm{Sp}(B)$.*
(2) For rational subdomain domains $U, V \subset \mathrm{Sp}(A)$, $U \cap V$ is a rational subdomain.
(3) As a set, $U\big(\frac{f_1,\ldots,f_n}{g}\big)$ is identified with $\mathrm{Sp}(A_U)$ with

$$A_U = A\langle\frac{f_1}{g}, \ldots, \frac{f_r}{g}\rangle := A\langle w_1, \ldots, w_r\rangle/\big(gw_1 - f_1, \ldots, gw_r - f_r\big),$$

where $A\langle w_1, \ldots, w_r\rangle = A^\circ\{w_1, \ldots, w_r\} \otimes_{A^\circ} A$ with $A^\circ\{w_1, \ldots, w_r\}$ the π-adic completion of $A^\circ[w_1, \ldots, w_r]$.
(4) For rational subdomain domains $U \subset \mathrm{Sp}(A)$ and $V \subset \mathrm{Sp}(A_U)$, V is a rational subdomain of $\mathrm{Sp}(A)$.

Definition 2.6. A subset $U \subset \mathrm{Sp}(A)$ is called an affinoid subdomain if the functor $F_U : \mathrm{AffAlg}_K \to \mathbf{Sets}$ defined by

$$F_U(B) = \{\varphi \in \mathrm{Hom}_{\mathrm{AffAlg}_K}(A, B) |\ \varphi^*(\mathrm{Sp}(B)) \subset U\} \text{ for } B \in \mathrm{AffAlg}_K$$

is representable by $A_U \in \mathrm{AffAlg}_K$: In other words, there is a map $\psi : A \to A_U$ in AffAlg_K such that the image of $\psi^* : \mathrm{Sp}(A_U) \to \mathrm{Sp}(A)$ is contained in U and the following universal property holds: Any morphisms $\varphi : A \to B$ such that the image of $\varphi^* : \mathrm{Sp}(B) \to \mathrm{Sp}(A)$ is contained in U, there is a unique morphism $A_U \to B$ in AffAlg_K which factors $A \to B$.

We have the following facts:

Lemma 2.7. *(1) Under the above notation, ψ^* is injective and* $\mathrm{Image}(\psi^*) = U$.
(2) A rational subdomain is an affinoid subdomain.
(3) For an affinoid subdomain $U \subset \mathrm{Sp}(A)$ and a morhpism $\varphi : A \to B$ in AffAlg_K inducing $\varphi^* : \mathrm{Sp}(B) \to \mathrm{Sp}(A)$, $(\varphi^*)^{-1}(U)$ is an affinoid subdomain of $\mathrm{Sp}(B)$.
(4) If U is an affinoid subdomain of $\mathrm{Sp}(A)$ and V is an affinoid subdomain of U, then V is an affinoid subdomain of $\mathrm{Sp}(A)$.
(5) (Gerritzen-Grauert) Any affinoid subdomain of $\mathrm{Sp}(A)$ is a finite union of rational subdomains.
(6) See Theorem 2.27 for a characterization of affinoid subdomains in terms of formal models.

Definition 2.8. A G-topology τ on a topological space X consists of the following datum:
(i) A category Cat_τ whose objects are open subsets of X and whose morphisms are open immersions. An object of Cat_τ is called an admissible open subset.
(ii) For every $U \in \mathrm{Cat}_\tau$, a family $\mathrm{Cov}_\tau(U)$ of open coverings $\{U_i \to U\}_{i\in I}$. A member of $\mathrm{Cov}_\tau(U)$ is called an admissible covering of U.

It is required to satisfy the following conditions:
(1) If $V \to U$ is an isomorphism in Cat_τ, then $\{V \to U\} \in \mathrm{Cov}_\tau(U)$.
(2) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}_\tau(U)$ and $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}_\tau(U_i)$, then $\{V_{ij} \to U\}_{i\in I. j\in J_i} \in \mathrm{Cov}_\tau(U)$.
(3) $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}_\tau(U)$ and $V \to U$ is a morphism in Cat_τ, then $\{U_i \cap V \to V\}_{i\in I} \in \mathrm{Cov}_\tau(V)$.

A G-topological space is a topological space X with a Grothendieck topology τ. A morphism $(X, \tau) \to (Y, \lambda)$ of G-topological spaces is a continuous morphism $\varphi : X \to Y$ of topological spaces such that for any $U \in \mathrm{Cat}_\lambda$ and $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}_\lambda(U)$, we have $\varphi^{-1}(U) \in \mathrm{Cat}_\tau$ and $\{\varphi^{-1}(U_i) \to \varphi^{-1}(U)\}_{i\in I} \in \mathrm{Cov}_\tau(\varphi^{-1}(U))$.

We let Top^G denote the category of G-topological spaces.

Definition 2.9. A sheaf F on a G-topological space (X, τ) is a presheaf (of sets) on Cat_τ such that for every $U \in \mathrm{Cat}_\tau$ and every $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}_\tau(U)$ the diagram

$$F(U) \to \prod_{i\in I} F(U_i) \underset{pr_1^*}{\overset{pr_0^*}{\rightrightarrows}} \prod_{(i_0,i_1)\in I\times I} F(U_{i_0} \times_U U_{i_1})$$

represents the first arrow as the equalizer of pr_0^* and pr_1^*. We let $\mathbf{Shv}((X, \tau))$ denote the category of sheaves (of sets) on (X, τ).

Definition 2.10. For a K-affinoid algebra A, we equip $X = \mathrm{Sp}(A)$ with a G-topology τ for which the objects of Cat_τ are affinoid subdomains and $\mathrm{Cov}_\tau(U)$ for $U \in \mathrm{Cat}_\tau$ is the family of *finite* coverings of U by affinoid subdomains. We call the G-topological space (X, τ) an *affinoid K-space associated to A* and denote it simply by $\mathrm{Sp}(A)$.

Let $\mathrm{AffSp}_K \subset \mathrm{Top}^G$ denote the full subcategory of affinoid K-spaces and morphismsm of G-topological spaces.

By Lemma 2.5(3), any morphism $\varphi : A \to B$ in AffAlg_K induces a morphism $\varphi^* : \mathrm{Sp}(B) \to \mathrm{Sp}(A)$ in AffAlg_K. Thus, we get a functor

$$(\mathrm{AffAlg}_K)^{op} \to \mathrm{AffSp}_K \ : \ A \to \mathrm{Sp}(A).$$

Theorem 2.11. *(Tate) Let $\mathcal{O}_X$ be the presheaf on (X, τ) given by $\mathcal{O}_X(U) = B$ for an affinoid subdomain $U = \mathrm{Sp}(B) \subset X$. Then, $\mathcal{O}_X$ is a sheaf on (X, τ).*

Example 2.12. Let $X = \mathrm{Sp}(A)$ be an affinoid K-space. Using Theorem 2.11, one can show that the following presheaves on X is a sheaves.

(1) The presheaf $\mathcal{O}^\circ \subset \mathcal{O}_X$ given by

$$\mathcal{O}^\circ(B) = \{f \in B| \ |f|_{\sup,B} \leq 0\} \text{ for an affinoid subdomain } \mathrm{Sp}(B) \subset \mathrm{Sp}(A),$$

(2) For $r \in \mathbb{R}_{>0}$, the presheaf $\mathcal{O}(r) \subset \mathcal{O}_X$ given by

$$\mathcal{O}(r)(B) = \{f \in B| \ |f|_{\sup,B} < r\} \text{ for an affinoid subdomain } \mathrm{Sp}(B) \subset \mathrm{Sp}(A).$$

where $| - |_{\sup,B}$ is the supremum norm on B.

2.3. **Rigid analytic K-spaces.**

Definition 2.13. A G-ringed K-space is a pair $(X, \mathcal{O}_X)$, where X is a G-topological space and $\mathcal{O}_X$ is a sheaf of K-algebras on it. $(X, \mathcal{O}_X)$ is called a locally G-ringed K-space if, in addition, all stalks $\mathcal{O}_{X,x}$ for $x \in X$ are local rings. A morphism of G-ringed K-spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair (φ, φ^*), where $\varphi : X \to Y$ is a morphism of G-topological spaces, and φ^* is a system of K-homomorphisms $\varphi_V^* : \mathcal{O}_Y(V) \to \mathcal{O}_X(\varphi^{-1}(V))$ with V varying over the admissible open subsets of Y. It is required that the φ_V^* are compatible with restriction map, i.e. for $W \subset V$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[r, "\varphi_V^*"] \arrow[d] & \mathcal{O}_X(\varphi^{-1}(V)) \arrow[d] \\
\mathcal{O}_Y(W) \arrow[r, "\varphi_W^*"] & \mathcal{O}_X(\varphi^{-1}(W))
\end{tikzcd}$$

If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally G-ringed K-spaces, a morphism(φ, φ^*) is called a morphism of locally G-ringed K-spaces if the ring homomorphisms

$$\varphi_x^* : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x} \text{ for } x \in X$$

induced from the φ_V^* are local.

If $X = \mathrm{Sp}(A)$ is an affinoid K-space, we can consider the associated locally G-ringed K-space $(X, \mathcal{O}_X)$, where X is the affinoid K-space associated to A from Definition 2.10 and $\mathcal{O}_X$ is the structure sheaf from Theorem 2.11.

Definition 2.14. A rigid (analytic) K-space is a locally G-ringed K-space $(X, \mathcal{O}_X)$ such that X admits an admissible covering $X = \cup_{i \in I} X_i$ such that $(X_i, \mathcal{O}_{X|X_i})$ is an affinoid K-space for all $i \in I$. A morphism of rigid K-spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of locally G-ringed K-spaces. Let Rig_K be the category of rigid K-spaces and morphismsm of locally G-ringed K-spaces. The G-topology on a rigid (analytic) K-space $(X, \mathcal{O}_X)$ is called the *admissible topology*. For an admissible open subset $U \subset X$, the induced locally G-ringed K-space $(U, \mathcal{O}_{X|U})$ is a rigid K-space again, which is called an open subspace of $(X, \mathcal{O}_X)$.

Remark 2.15. It is clear that every morphism of affinoid K-spaces $\varphi : X \to Y$ induces a morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ between associated locally G-ringed K-spaces. Thus, we get a functor

$$(\mathrm{AffAlg}_K)^{op} \to \mathrm{Rig}_K \ : \ A \to (X = \mathrm{Sp}(A), \mathcal{O}_X),$$

Remark 2.16. By a formal reason, the sheaves $\mathcal{O}^\circ$ and $\mathcal{O}(r)$ defined on affinoid spaces from Example 2.12 extends to sheaves $\mathcal{O}^\circ$ and $\mathcal{O}(r)$ on rigid K-spaces.

2.4. **Formal schemes and Raynaud's theorem.**

Definition 2.17. An $\mathcal{O}_K$-algebra A is called of topologically finite type if there is a surjective homomorphism $\varphi : \mathcal{O}_K\{T_1, \ldots, T_n\} \to A$ of $\mathcal{O}_K$-algebras. It is of topologically finite presentation if, furthermore $\mathrm{Ker}(\varphi)$ is finitely generated. It is admissible if furthermore, A does not have π-torsion .

Lemma 2.18. *(1) An $\mathcal{O}_K$-algebra A of topologically finite type is π-adically comoplete and separated.*
(2) An $\mathcal{O}_K$-algebra A of topologically finite type with no π-torsion is of topologically finite presentation.

Proof. [3, §.3 Cor.5 and Cor. 7]. □

Definition 2.19. A formal $\mathcal{O}_K$-scheme $\mathfrak{X}$ is called locally of topologically finite type (resp. locally of topologically finite presentation, resp. admissible) if there is an open affine covering $\mathfrak{X} = \cup_{1\in I}\mathfrak{U}_i$ with $\mathfrak{U}_i = \mathrm{Spf}(A_i)$, where A_i is an $\mathcal{O}_K$-algebra of topologically finite type (resp. of topologically finite presentation, resp. an admissible $\mathcal{O}_K$-algebra).

Let $\mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}$ be the category of formal $\mathcal{O}_K$-schemes locally of topologically finite type and $\mathrm{fSch}^{\mathrm{aff,tft}}_{\mathcal{O}_K}$ be its full subcategory of affine formal $\mathcal{O}_K$-schemes . We have an association

$$\mathrm{rig} : \mathrm{fSch}^{\mathrm{aff,tft}}_{\mathcal{O}_K} \to \mathrm{AffSp}_K \ : \ \mathfrak{X} = \mathrm{Spf}(A) \to \mathfrak{X}^{\mathrm{rig}} = \mathrm{Sp}(A \otimes_{\mathcal{O}_K} K). \tag{2.19.1}$$

Note that $A\otimes_{\mathcal{O}_K} K$ is an affinoid K-algebra since $\mathcal{O}_K\{T_1, \ldots, T_n\}\otimes_{\mathcal{O}_K} K = K\langle T_1, \ldots, T_n\rangle$. Since any morphism $\mathrm{Spf}(A) \to \mathrm{Spf}(B)$ in $\mathrm{fSch}^{\mathrm{aff,tft}}_{\mathcal{O}_K}$ is induced by a unique $\mathcal{O}_K$-homomorphism $B \to A$ of $\mathcal{O}_K$-algebras, this is a functor. Moreover, this functor commutes with localizations: For $f \in A$, we have

$$\begin{aligned} A\{f^{-1}\} \otimes_{\mathcal{O}_K} K &= \big(A\{T\}/(1 - fT)\big) \otimes_{\mathcal{O}_K} K \\ &= (A \otimes_{\mathcal{O}_K} K)\langle T\rangle/(1 - fT) = (A \otimes_{\mathcal{O}_K} K)\langle f^{-1}\rangle. \end{aligned} \tag{2.19.2}$$

From these, we can deduce the following (see [3, §7.3]).

Proposition 2.20. *The functor* (2.19.1) *extends to a functor*

$$\mathrm{rig} : \mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K} \to \mathrm{Rig}_K \ : \ \mathfrak{X} \to \mathfrak{X}^{\mathrm{rig}}. \tag{2.20.1}$$

Remark 2.21. If $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{X}^{\mathrm{rig}}$ coincides pointwise with the set of all closed points of $\mathrm{Spec}(A \otimes_{\mathcal{O}_K} K)$, which is the generic fiber of the ordinary scheme $\mathrm{Spec}(A)$ although it is not visible in $\mathrm{Spf}(A)$ on the level of points. By this, $\mathfrak{X}^{\mathrm{rig}}$ is called the generic fiber of $\mathfrak{X}$.

In view of Proposition 2.20, one would like to describe all formal $\mathcal{O}_K$-schemes $\mathfrak{X}$ whose generic fiber $\mathfrak{X}^{\mathrm{rig}}$ coincides with a given rigid K-space X. Such a formal $\mathcal{O}_K$-scheme is called a *formal model of* X. To answer this question, we introduce the following.

Definition 2.22. Let $\mathfrak{X} = \varinjlim_{n\in\mathbb{N}} \underline{\mathrm{Spec}}(\mathcal{O}_{\mathfrak{X}}/(\pi^n)) \in \mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}$ and let $\mathcal{A} \subset \mathcal{O}_{\mathfrak{X}}$ be a coherent open[5] ideal. Then the formal $\mathcal{O}_K$-scheme

$$\mathfrak{X}_{\mathcal{A}} = \varinjlim_{n\in\mathbb{N}} \mathrm{Proj}\,\big(\bigoplus_{d=0}^{\infty}\mathcal{A}^d \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}/(\pi^n)\big)\big)$$

together with the canonical projection $\mathfrak{X}_{\mathcal{A}} \to \mathfrak{X}$ is called the formal blowup of $\mathfrak{X}$ in $\mathcal{A}$. Any such blowup is referred to as an *admissible formal blowup of* $\mathfrak{X}$. Note $\mathfrak{X}_{\mathcal{A}} \in \mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}$ by the construction.

Definition 2.23. Let $\mathcal{C}$ be a category and S be a class of morphisms in $\mathcal{C}$. A localization of $\mathcal{C}$ by S is a category $\mathcal{C}_S$ together with a functor $L_S : \mathcal{C} \to \mathcal{C}_S$ such that:

[5] namely, $\pi^n \in \mathcal{A}$ for some $n > 0$.

(i) $L_S(s)$ is an isomorphism in $\mathcal{C}_S$ for every $s \in S$.
(ii) If $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $F(s)$ is an isomorphism for every $s \in S$, then F admits a unique factorization as follows:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "L_S"] \arrow[d, "F"] & \mathcal{C}_S \arrow[dl, "G"] \\ \mathcal{D} & \end{tikzcd}$$

where the commutativity of the diagram, as well as the uniqueness of G are meant up to natural equivalence of functors.

It is known that localizations of categories do always exist.

Proposition 2.24. *For $\mathfrak{X} \in \mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}$ and an admissible blowup $\mathfrak{Y} \to \mathfrak{X}$, the induced map $\mathfrak{Y}^{\mathrm{rig}} \to \mathfrak{X}^{\mathrm{rig}}$ is an isomorphism in Rig_K. In particular, the functor (2.20.1) factors through the localization $\mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K} \to (\mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K})_\Sigma$ by the class Σ of admissible blowups.*

Proof. See [3, §8.4, Pr. 2]. □

Theorem 2.25. *(Raynaud) Let $\mathrm{Rig}^{qcqs}_K \subset \mathrm{Rig}_K$ be the full subcategory of quasi-compact quasi separate rigid K-spaces. Let $\mathrm{fSch}^{\mathrm{ad}}_{\mathcal{O}_K} \subset \mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}$ be the full subcategory of quasi-compact quasi-separate admissible $\mathcal{O}_K$-formal schemes and $(\mathrm{fSch}^{\mathrm{ad}}_{\mathcal{O}_K})_\Sigma$ be its localization by the class of admissible blowups. Then, the functor* rig *from* (2.20.1) *induces an equivalence of categories*

$$\mathrm{rig} : (\mathrm{fSch}^{\mathrm{ad}}_{\mathcal{O}_K})_\Sigma \simeq \mathrm{Rig}^{qcqs}_K. \tag{2.25.1}$$

Proof. See [3, §8.4, Th.3]. □

Remark 2.26. For $\mathfrak{X} \in \mathrm{fSch}^{\mathrm{ad}}_{\mathcal{O}_K}$, the category $\Sigma_{\mathfrak{X}}$ of admissible blowups $\mathfrak{X}' \to \mathfrak{X}$ admits finite limits so that is cofiltered. This implies that for $\mathfrak{Y} \in \mathrm{fSch}^{\mathrm{ad}}_{\mathcal{O}_K}$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Rig}_K}(\mathfrak{X}^{\mathrm{rig}}, \mathfrak{Y}^{\mathrm{rig}}) = \varinjlim_{\mathfrak{X}' \to \mathfrak{X} \in \Sigma_{\mathfrak{X}}} \mathrm{Hom}_{\mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}}(\mathfrak{X}', \mathfrak{Y}). \tag{2.26.1}$$

Theorem 2.27. *(Geritzen and Grauert) Let $\mathfrak{X} = \mathrm{Spf}(A) \in \mathrm{fSch}^{\mathrm{aff,tft}}_{\mathcal{O}_K}$ and $X = \mathfrak{X}^{\mathrm{rig}} = \mathrm{Sp}(A \otimes_{\mathcal{O}_K} K)$. A subset $U \subset X$ is an affinoid subdomain in the sense of Definition 2.6 if and only if there is $\mathfrak{Y} \in \Sigma_{\mathfrak{X}}$ and an affine open $\mathfrak{U} \hookrightarrow \mathfrak{Y}$ such that $U = \mathfrak{U}^{\mathrm{rig}}$.*

2.5. **Riemann-Zariski spaces.**

Definition 2.28. Let $\mathfrak{X} \in \mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}$ and $\Sigma_{\mathfrak{X}}$ be the category of admissible blowups $\mathfrak{Y} \to \mathfrak{X}$. Let $\mathrm{RZ}(\mathfrak{X}) \subseteq \mathrm{Arr}(\mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K})$ be the category whose objects are morphisms $\mathfrak{U} \to \mathfrak{Y}$ where $\mathfrak{Y} \to \mathfrak{X} \in \Sigma_{\mathfrak{X}}$ and $\mathfrak{U} \to \mathfrak{Y}$ is a Zariski open immersion. We abbreviate $\mathfrak{U} \to \mathfrak{Y}$ to $(\mathfrak{U}/\mathfrak{Y})$. The morphism $(\mathfrak{U}'/\mathfrak{Y}') \to (\mathfrak{U}/\mathfrak{Y})$ in $\mathrm{RZ}(\mathfrak{X})$ are commutative squares in $\mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}$:

$$\begin{tikzcd} \mathfrak{U}' \arrow[r] \arrow[d] & \mathfrak{U} \arrow[d] \\ \mathfrak{Y}' \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

Remark 2.29. $\mathrm{RZ}(\mathfrak{X})$ admits finite limits, and they are calculated termwise. Indeed, the category $\mathrm{Arr}(\mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K})$ of arrows admits finite limits and they are calculated component wise: $\varprojlim(A_i/B_i) = (\varprojlim A_i/\varprojlim B_i)$. If each (A_i/B_i) is in $\mathrm{RZ}(\mathfrak{X})$, then one checks that $\varprojlim(A_i/B_i)$ is again in $\mathrm{RZ}(\mathfrak{X})$.

Definition 2.30. We equip $\mathrm{RZ}(\mathfrak{X})$ with the Grothendieck topology τ generated by:

(1) families of $\{(\mathfrak{U}_i/\mathfrak{Y}) \to (\mathfrak{U}/\mathfrak{Y})\}_{i\in I}$ such that $\{\mathfrak{U}_i \to \mathfrak{U}\}_{i\in I}$ is a Zariski covering,

(2) families of $\{(\mathfrak{Y}'\times_{\mathfrak{Y}}\mathfrak{U}/\mathfrak{Y}')\to(\mathfrak{U}/\mathfrak{Y})\}$ for morphisms $\mathfrak{Y}'\to\mathfrak{Y}$ in $\Sigma_{\mathfrak{X}}$.

The site $(\mathrm{RZ}(\mathfrak{X}),\tau)$ is called the Riemann-Zariski space of $\mathfrak{X}$. We will write $\mathbf{Shv}(\mathrm{RZ}(\mathfrak{X}))$ for the topos associated to the topology generated by coverings of the form (1) and (2).

Remark 2.31. Using that for $\mathfrak{Y}'\to\mathfrak{Y}$ in $\Sigma_{\mathfrak{X}}$, the diagonal $\mathfrak{Y}'\to\mathfrak{Y}'\times_{\mathfrak{Y}}\mathfrak{Y}'$ is a morphism in $\Sigma_{\mathfrak{X}}$, one can show that a presheaf on $\mathrm{RZ}(\mathfrak{X})$ satisfies descent for all families of the form (2) if and only if it sends each $(\mathfrak{Y}'\times_{\mathfrak{Y}}\mathfrak{U}/\mathfrak{Y}')\to(\mathfrak{U}/\mathfrak{Y})$ to an isomorphism. This implies

$$\mathbf{Shv}(\mathrm{RZ}(\mathfrak{X}))\simeq\varprojlim_{\mathfrak{Y}\in\Sigma_{\mathfrak{X}}}\mathbf{Shv}(\mathfrak{Y}_{\mathrm{zar}}) \tag{2.31.1}$$

where the limit is along pushforwards $f_*:\mathbf{Shv}(\mathfrak{Y}'_{\mathrm{zar}})\to\mathbf{Shv}(\mathfrak{Y}_{\mathrm{zar}})$ for morphisms $f:\mathfrak{Y}'\to\mathfrak{Y}$ in $\Sigma_{\mathfrak{X}}$, namely an object of the RHS of (2.31.1) is given by a system

$$\mathcal{F}=\{F_{\mathfrak{Y}}\in\mathbf{Shv}(\mathfrak{Y}_{\mathrm{zar}})\}_{\mathfrak{Y}\to\mathfrak{X}\in\Sigma_{\mathfrak{X}}} \tag{2.31.2}$$

such that

(♠) $F_{\mathfrak{Y}'}(\mathfrak{U}\times_{\mathfrak{Y}}\mathfrak{Y}')=F_{\mathfrak{Y}}(\mathfrak{U})$ for every $(\mathfrak{U}/\mathfrak{Y})\in\mathrm{RZ}(\mathfrak{X})$ and $\mathfrak{Y}'\to\mathfrak{Y}$ in $\Sigma_{\mathfrak{X}}$.

If $F_{\mathfrak{Y}}$ are all sheaves of abelian groups, this implies that we have a natural isomorphism

$$\varinjlim_{\mathfrak{Y}\to\mathfrak{X}\in\Sigma_{\mathfrak{X}}}H^i(\mathfrak{Y},F_{\mathfrak{Y}})\simeq H^i(\mathrm{RZ}(\mathfrak{X}),\mathcal{F}_{\mathrm{RZ}(\mathfrak{X})}), \tag{2.31.3}$$

where $\mathcal{F}_{\mathrm{RZ}(\mathfrak{X})}=\varprojlim_{\mathfrak{Y}\in\Sigma_{\mathfrak{X}}}F_{\mathfrak{Y}}\in\mathbf{Shv}(\mathrm{RZ}(\mathfrak{X}))$ (see [10, Ch.0, 4.4.1]).

Now, we look at a relation of $\mathbf{Shv}(\mathrm{RZ}(\mathfrak{X}))$ and $\mathbf{Shv}(\mathfrak{X}^{\mathrm{rig}})$ for $\mathfrak{X}\in\mathrm{fSch}^{\mathrm{tft}}_{\mathcal{O}_K}$. Using Proposition 2.24, the functor (2.20.1) gives a functor on the categories of open subsets:

$$\mathrm{RZ}(\mathfrak{X})\to\mathfrak{X}^{\mathrm{rig}}\ :\ (\mathfrak{U}/\mathfrak{Y})\to\mathfrak{U}^{\mathrm{rig}}\subset\mathfrak{Y}^{\mathrm{rig}}=\mathfrak{X}^{\mathrm{rig}}.$$

By the construction, this is continuous, i.e. maps coverings to coverings so that it defines a morphism of sites

$$\gamma:\mathfrak{X}^{\mathrm{rig}}\to\mathrm{RZ}(\mathfrak{X})$$

which induces a pair of adjoint functors

$$\gamma^*:\mathbf{Shv}(\mathrm{RZ}(\mathfrak{X}))\underset{\longleftarrow}{\overset{\longrightarrow}{}}\mathbf{Shv}(\mathfrak{X}^{\mathrm{rig}}):\gamma_*, \tag{2.31.4}$$

where $\gamma_*F(\mathfrak{U}/\mathfrak{Y})=F(\mathfrak{U}^{\mathrm{rig}})$ for $F\in\mathbf{Shv}(\mathfrak{X}^{\mathrm{rig}})$ and $(\mathfrak{U}/\mathfrak{Y})\in\mathrm{RZ}(\mathfrak{X})$.

Theorem 2.32. (2.31.4) *induces a natural equivalence of topoi*

$$\mathbf{Shv}(\mathfrak{X}^{\mathrm{rig}})\simeq\mathbf{Shv}(\mathrm{RZ}(\mathfrak{X})).$$

In particular, for $\mathcal{F}_{\mathrm{RZ}(\mathfrak{X})}=\varprojlim_{\mathfrak{Y}\in\Sigma_{\mathfrak{X}}}F_{\mathfrak{Y}}\in\mathbf{Shv}(\mathrm{RZ}(\mathfrak{X}))$ *from* (2.31.3), *we have*

$$\varinjlim_{\mathfrak{Y}\to\mathfrak{X}\in\Sigma_{\mathfrak{X}}}H^i(\mathfrak{Y},F_{\mathfrak{Y}})\simeq H^i(\mathfrak{X}^{\mathrm{rig}},\gamma^*\mathcal{F}_{\mathrm{RZ}(\mathfrak{X})}). \tag{2.32.1}$$

Proof. [10, Th.B.2.5]. □

Remark 2.33. By definition, we have

$$\gamma^*\mathcal{F}_{\mathrm{RZ}(\mathfrak{X})}(\mathfrak{U}^{\mathrm{rig}})=F_{\mathfrak{Y}}(\mathfrak{U})\ \text{for}\ (\mathfrak{U}/\mathfrak{Y})\in\mathrm{RZ}(\mathfrak{X}).$$

Since such $\mathfrak{U}^{\mathrm{rig}}$ form a basis of the admissible topology of $\mathfrak{X}^{\mathrm{rig}}$, this determines $\gamma^*F_{\mathrm{RZ}(\mathfrak{X})}$.

Example 2.34. For $\mathfrak{Y}\in\Sigma_{\mathfrak{X}}$ and affine open $\mathfrak{U}\subset\mathfrak{Y}$, define $\mathcal{O}^{\mathrm{int}}_{\mathfrak{Y}}(\mathfrak{U})$ to be the integral closure of $\mathcal{O}_{\mathfrak{Y}}(\mathfrak{U})$ in $\mathcal{O}_{\mathfrak{Y}}(\mathfrak{U})\otimes_{\mathcal{O}_K}K$. Then, one can check that this assignment extends to a sheaf $\mathcal{O}^{\mathrm{int}}_{\mathfrak{Y}}$ on $\mathfrak{Y}_{\mathrm{zar}}$ and satisfies $f_*\mathcal{O}^{\mathrm{int}}_{\mathfrak{Y}'}=\mathcal{O}^{\mathrm{int}}_{\mathfrak{Y}}$ for $f:\mathfrak{Y}'\to\mathfrak{Y}\in\Sigma_{\mathfrak{X}}$. By Remark 2.31, it gives rise to a sheaf $\mathcal{O}^{\mathrm{int}}_{\mathrm{RZ}(\mathfrak{X})}$ on $\mathrm{RZ}(\mathfrak{X})$. Moreover, we can show $\gamma^*\mathcal{O}^{\mathrm{int}}_{\mathrm{RZ}(\mathfrak{X})}=\mathcal{O}^{\circ}_{\mathfrak{X}^{\mathrm{rig}}}$, where the latter is a sheaf on $\mathfrak{X}^{\mathrm{rig}}$ from Example 2.12 (see also Remark 2.16).

3. Valuation theory

3.1. **Valuations.** We fix some notations and recall definitions from, e.g., [11, 6.2], [17, 2].

Definition 3.1. A valuation field (K, v) consists of a field K endowed with a surjective group homomorphism $v : K^\times \to \Gamma_v$ onto a totally ordered abelian group Γ_v[6], such that

$$v(x+y) \leq \max\{v(x), v(y)\} \tag{3.1.1}$$

whenever $x+y \neq 0$. We denote by 1 the unit of Γ and the composition law of Γ is denoted by $(x, y) \to xy$. It is easy to check that $\mathcal{O}_v = \{x \in K |\ v(x) \leq 1\}$ is a subring of K and we called it the valuation ring of (K, v).

It is customary to extend v to K, by adding a new element 0 to Γ_v setting $v(0) := 0$. One can then extend the ordering of Γ_v to $\overline{\Gamma}_v := \Gamma_v \cup \{0\}$ by declaring that 0 is the smallest element of $\overline{\Gamma}_v$. By the convention, (3.1.1) holds for every $x \in K$.

We have the following facts (see [11, 6.1,12])

Lemma 3.2. *Let (K, v) be a valuation field with the valuation ring $\mathcal{O}_v$.*

(1) Every finitely generated ideal of $\mathcal{O}_v$ is principal.
(2) Let L be a field extension of K. Then the integral closure W of $\mathcal{O}_v$ in L is the intersection of all the valuation rings of L containing $\mathcal{O}_v$. In particular, $\mathcal{O}_v$ is integrally closed.
(3) If L is an algebraic extension of K and W be the integral closure of $\mathcal{O}_v$ in L. Then, for every prime ideal $\mathfrak{p} \subset W$, the localization $W_\mathfrak{p}$ is a valuation ring. Moreover, the assignment $\mathfrak{m} \to W_\mathfrak{m}$ gives a bijection between the set of maximal ideals of W and the set of valuation rings $\mathcal{O}_w$ of L whose associated valuation w extends v.
(4) Let $\mathcal{O}_v^h$ be the henselization of $\mathcal{O}_v$ with the maximal ideal $\mathfrak{m}_v^h$ and $K^h = \mathrm{Frac}(\mathcal{O}_v^h)$. Then, $\mathcal{O}_v^h$ contains the integral closure W of $\mathcal{O}_v$ in K^h and we have $\mathcal{O}_v^h = W_\mathfrak{q}$, where $\mathfrak{q} := \mathfrak{m}_v^h \cap W$. By (3), this implies that $\mathcal{O}_v^h$ is again a valuation ring. The same argument works also for strict henselizations.
(5) Any finitely generated torsion-free $\mathcal{O}_v$-module is free and any torsion-free $\mathcal{O}_v$-module is flat. Hence every $\mathcal{O}_v$-module is of Tor-dimension≤ 1.
(6) A local subring of a field L is a valuation ring of L if and only if it is maximal for the dominance relation on the set of local subrings of L[7].

Definition 3.3. Let $(K, v : K \to \overline{\Gamma}_v)$ be a valuation field. An extension of valued fields $(E, w : E \to \overline{\Gamma}_w)$ consists of a field extension E/K and a valuation $w : E \to \overline{\Gamma}_w$ together with an embedding $j : \Gamma_v \hookrightarrow \Gamma_w$ such that $w_{|K} = j \circ v$.

Example 3.4. Let $(K, v : K^\times \to \Gamma_v)$ be a valuation field and E/K be a field extension.

(1) There always exist valuations on E which extends v ([30, Ch.VI, §1, n.3, Cor.3]).
(2) If E/K is algebraic and purely inseparable, then the extension of v to E is unique. ([30, Ch.VI, §8, n.7, Cor.2]).
(3) If E is the polynomial ring $K[X]$, we can construct extensions of v on E as follows: Let $\Gamma_v \hookrightarrow \Gamma'$ be an embedding of ordered groups. For every $x_0 inK$ and $\rho \in \Gamma$, we define the Gauss valuation centered at x_0 and with radius ρ:

$$v_{(x_0,\rho)} : K[X] \to \Gamma \cup \{0\},$$

sending $a_0 + a_1(X - x_0) + \cdots + a_n(X - x_0)^n$ to $\max\{v(a_i) \cdot \rho^i |\ i = 0, 1, \ldots, n\}$ ([30, 16, Ch.VI, §10, n.1, Lemma 1]).

[6] written multiplicatively

[7] For local subrings R and S of L, one says that R dominates S if $S \subset R$ and $\mathfrak{m}_S = \mathfrak{m}_R \cap S$, where $\mathfrak{m}_R$ and $\mathfrak{m}_S$ are the maximal ideals of R and S respectively. The relation of dominance defines a partial order structure on the set of local subrings of L.

3.2. **Tame extensions of valuation fields.** Let (K, v) be a valuation field with the valuation ring $\mathcal{O}_v$. Fix an embedding of (K, v) into $(\bar{K}, \bar{v})$, where $\bar{K}$ is a separable closure of K and $\bar{v}$ is an extension of v to $\bar{K}$. We denote by (K_v^{sh}, v^{sh}) the strict henselization of (K, v) (inside $(\bar{K}, \bar{v})$). A finite separable extension $(L, w)/(K, v)$ of valuation fields is called *unramified* (resp. *tame*), if $K_v^{sh} = L_w^{sh}$ (resp. $([L_w^{sh} : K_v^{sh}], p) = 1$, where p is the exponential characteristic of the residue field of $\mathcal{O}_v$). The *tame closure* (K^t, v^t) of (K, v) is the union of all finite tame Galois extensions of (K^{sh}, v^{sh}). The field K^t is also the fixed field of $\bar{K}$ under the tame ramification group

$$R_{\bar{v}/v} := \{\sigma \in \mathrm{Gal}(\bar{K}/K) \mid \sigma(\mathcal{O}_{\bar{v}}) \subset \mathcal{O}_{\bar{v}} \text{ and } \frac{\sigma(x)}{x} - 1 \in \mathfrak{m}_{\bar{v}} \text{ for all } x \in \bar{K}^\times\}.$$

We record the following well-known lemma for later reference.

Lemma 3.5. *(1) Let $(L, w)/(K, v)$ be a finite separable extension of valuation fields. Let N/K be a Galois hull of L/K and let $\tilde{w}$ be an extension of w to N. Then $(L, w)/(K, v)$ is tame if and only if $(N, \tilde{w})/(K, v)$ is tame.*

In particular $(L, w)/(K, v)$ is tame if and only if (L, w) is a subextension of $(K^t, v^t)/(K, v)$.

(2) Let $(L, w)/(K, v)$ be a tame extension and let $(K', v')/(K, v)$ be any algebraic extension of valuation fields. Let $L \cdot K'$ be the composition field in an algebraic closure of K and let w' be a valuation extending v'. Then $(L \cdot K', w')/(K', v')$ is tame.

Proof. (1). Note that $N_{\tilde{w}}^{sh}$ is a Galois hull of L_w^{sh}/K_v^{sh}. Therefore we may assume K, L, N are strictly henselian valuation fields of characteristic $p > 0$. Thus if $(N, \tilde{w})/(K, v)$ is tame then $[N : K] = [N : L] \cdot [L : K]$ is prime to p and hence $(L, w)/(K, v)$ is tame as well. Now assume $(L, w)/(K, v)$ is tame. Denote by $G_K \supset G_L \supset G_N$ the absolute Galois groups with respect to a fixed separable closure $\bar{K}$ of K, and by P the pro-p-Sylow subgroup of G_K, which is a normal subgroup. The indices satisfy the following equality (of supernatural numbers)

$$[G_K : G_L] \cdot [G_L : P \cap G_L] = [G_K : P] \cdot [P : P \cap G_L].$$

As P is a normal subgroup of G_K, the intersection $P \cap G_L$ is a normal subgroup of G_L and we have an inclusion of profinite groups $G_L/G_L \cap P \hookrightarrow G/P$. Hence $[G : P]$ and $[G_L : P \cap G_L]$ are prime to p. By assumption $[G_K : G_L] = [L : K]$ is prime to p as well. Thus $[P : P \cap G_L] = 1$, i.e., $P = P \cap G_L$. The Galois hull of L/K is the composition field (inside $\bar{K}$) of all the $\sigma(L)$, where σ runs through all the embeddings $L \hookrightarrow \bar{K}$. Extending these σ's to K-automorphisms of $\bar{K}$, we find $G_{\sigma(L)} = \sigma G_L \sigma^{-1}$. Hence $G_N = \cap_\sigma \sigma G_L \sigma^{-1}$. As P is a normal subgroup of G_K it follows that P is contained in G_N as well. Thus $[G_K : P] = [G_K : G_N] \cdot [G_N : P]$ is prime to p and hence so is $[N : K] = [G_K : G_N]$.

(2) follows from (1) and the fact that $K'^t = K' \cdot K^t$, see [11, 6.2.18]. □

4. Spectral spaces

Definition 4.1. A topological space is called spectral if it is sober[8], quasi-compact, the intersection of two quasi-compact opens is quasi-compact, and the collection of quasi-compact opens forms a basis for the topology.

Lemma 4.2. *For a topological space X, the following conditions are equivalent.*

(1) X is spectral.
(2) X is a directed inverse limit of finite sober topological spaces.
(3) X is homeomrophic to $\mathrm{Spec}(R)$ *for some commutative ring R.*

Definition 4.3. Let X be a spectral space. The constructible topology on X is the topology which has as a base of opens, the sets U and U^c for a quasi-compact open $U \subset X$.

[8] i.e. every nonempty irreducible closed subset has a unique generic point.

Note that an open U in a spectral space X is retrocompact[9] Hence, the constructible topology can also be characterized as the coarsest topology such that every constructible subset[10] of X is both open and closed. It follows that a subset of X is open (resp. closed) in the constructible topology if and only if it is a union (resp. intersection) of constructible subsets. Since the collection of quasi-compact opens is a basis for the topology on X, we see that the constructible topology is stronger than the given topology on X.

Lemma 4.4. *The constructible topology on a sepctral sapce is Hausdorff, totally disconnected, and quasi-compact.*

Proof. [35, Tag 0901] □

5. Adic spaces

Definition 5.1. For a morphism of schemes $X \to \tilde{X}$, let $\mathrm{Spa}(X, \tilde{X})$ be the set of triples (x, v, ε) such that $x \in X$, v is a valuation on $k(x)$ and $\varepsilon\colon \mathrm{Spec}(\mathcal{O}_v) \to \tilde{X}$ is a map compatible with $\mathrm{Spec}(k(x)) \to X$. Let $Y \to \tilde{Y}$ be a morphism of schemes and $(\varphi, \tilde{\varphi}) : (Y, \tilde{Y}) \to (X, \tilde{X})$ be morphisms such that the following diagram commutative:

(5.1.1)
$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d] & X \arrow[d] \\ \tilde{Y} \arrow[r, "\tilde{\varphi}"] & \tilde{X} \end{tikzcd}$$

Then, we have an induced map $\mathrm{Spa}(Y, \tilde{Y}) \to \mathrm{Spa}(X, \tilde{X})$[11]. We equip $\mathrm{Spa}(X, \tilde{X})$ with a topology as follows: If $X = \mathrm{Spec}(A)$ and $\tilde{X} = \mathrm{Spec}(\tilde{A})$ are affine, the topology is generated by the subset of the form[12]

$$\{(x, v, \varepsilon)|\ v(f_i) \leq v(g) \neq 0\ \forall i = 1, \dots, m\} \text{ for } f_1, \dots, f_m, g \in A.$$

In general, we declare that a subset $V \subset \mathrm{Spa}(X, \tilde{X})$ is open if for any commutative diagram (5.1.1) where Y and $\tilde{Y}$ are affine, φ is an open immersion and $\tilde{\varphi}$ is locally of finite type, the inverse image of V in $\mathrm{Spa}(Y.\tilde{Y})$ is open.

Lemma 5.2. *(1) If X and $\tilde{X}$ are quasi-compact and quasi-separated, then* $\mathrm{Spa}(X, \tilde{X})$ *is a spectral space, i.e. homeomorphic to* $\mathrm{Spec}(R)$ *for some commutative ring R. In particular,* $\mathrm{Spa}(X, \tilde{X})$ *is a quasi-compact and quasi-separated topological space.*

(2) Let $(\varphi, \tilde{\varphi})$ be as (5.1.1) and assume that φ is étale and $\tilde{\varphi}$ is locally of finite type. Then, the set of points $(y, w, \varepsilon_w) \in$ $\mathrm{Spa}(Y, \tilde{Y})$ *such that the extension $(k(y), w)/(k(\varphi(y)), w_{|k(\varphi(y))})$ is tame is open as well as the set of points $(x, v, \varepsilon_v) \in$* $\mathrm{Spa}(X, \tilde{X})$ *such that there exists $(y, w, \varepsilon_w) \in$* $\mathrm{Spa}(Y, \tilde{Y})$ *mapping to (x, v, ε_v) such that the extension $(k(y), w)/(k(x), v)$ is tame.*

Proof. (1) follows from [17, Lem.4.3] and (2) from [15, Cor.4.4] and [14, Pr.1.7.8]. □

References

[1] M. Artin, *On the joins of Hensel rings*, Adv. Math. **7** (1971), 282–296.
[2] S. Bosch, U. Güntzer and R. Remmert, *Non-archimedean analysis*
[3] S. Bosch, *Lectureson Formal and Rigid Geometry*, SLN in Math. **2105**,
[4] Bartenwerfer, W. *Die erste "metrische" Kohomologiegruppe glatter affinoider Räume*, *Nederl. Akad. Wetensch. Proc. Ser. A* 40 (1978), no. 1, 1–14.
[5] Bartenwerfer, W. *Die höheren metrischen Kohomologiegruppen affinoider Räume*, *Math. Ann.* 241 (1979), no. 1, 11–34.

---

[9] i.e. the inclusion map $U \to X$ is quasi-compact.

[10] i.e. a finite union of subsets of the form $U \cap V^c$ where $U, V \subset X$ are open and retrocompact in X.

[11] sending (y, w, ε') to $(x = \varphi(y), v = w_{|k(x)}, \varepsilon)$ with $\varepsilon : \mathrm{Spec}(\mathcal{O}_v) \to \tilde{X}$ induced by $\mathrm{Spec}(k(x)) \to X \to \tilde{X}$, $\mathrm{Spec}(k(y)) \to Y \to \tilde{X}$ and $\mathrm{Spec}(\mathcal{O}_w) \to \tilde{Y} \to \tilde{X}$ noting $\mathrm{Spec}(\mathcal{O}_v) = \mathrm{Spec}(\mathcal{O}_w) \sqcup_{\mathrm{Spec}(k(y))} \mathrm{Spec}(k(x))$.

[12] This dictates that both $\{(x, v, \varepsilon)|\ v(f) \leq 1\}$ and $\{(x, v, \varepsilon)|\ v(f) \neq 0\}$ be open for $f \in A$.

[6] F. Binda, D. Park, Doosung and P.A. Østvær, *Triangulated categories of logarithmic motives over a field*, Astérisque, **433** (2022).
[7] B. Bhatt and P. Scholze, *The pro-étale topology for schemes*, Astérisque **369** (2019), 99–201.
[8] A. Chatzistamatiou and K. Rülling, *Vanishing of the higher direct images of the structure sheaf*, Compos. Math. **151** (2015), no. 11, 2131–2144.
[9] de Jong, J., van der Put, M. *Etale cohomology of rigid analytic spaces*, *Doc. Math.* 1 (1996), No. 01, 1–56
[10] K. Fujiwara and F. Kato, *Foundations of rigid geometry I.* Monographs in Mathematics, European Mathematical Society Publishing House.
[11] O. Gabber and L. Ramero, *Almost ring theory*, Lecture Notes in Mathematics, **1800**, Springer-Verlag, Berlin.
[12] Gerritzen, L. and van der Put, M. *Schottky groups and Mumford curves*, Lecture Notes in Math. **817**.
[13] L. Hesselholt, *The big de Rham-Witt complex*, Acta Math., **214** (2015) no 1, 135–207.
[14] R. Huber, *Étale Cohomology of Rigid Analytic Varieties and Adic Spaces*, Aspects of Math., **30** (1996), Vieweg+Teubner Verlag Wiesbaden.
[15] K. Hübner, *The Adic Tame Site*, Documenta Math., **26** (2021), 873–945.
[16] K. Hübner, *Logarithmic differentials on discretely ringed adic spaces*, https://arxiv.org/abs/2009.14128.
[17] K. Hübner A. Schmidt, *Tha tame site of a scheme*, Invent. Math., **223** (2020), 397–443.
[18] O. Hyodo and K. Kato, *Semi-stable reduction and cystalline cohomology with logarithmic poles*, Astérisque, **223** (1994).
[19] S. Kelly and H. Miyazaki, *Modulus sheaves with transfers*, https://arxiv.org/pdf/2106.12837.
[20] S. Kelly and S. Saito, *A Pro-cdh toplogy*, arxiv.org/pdf/2401.02699.
[21] M. Kerz, S. Saito and G. Tamme, *Towards a non-archimedean analytic analog of the Bass-Quillen conjecture*, *J. Inst. Math. Jussieu* 19 (2020), no. 6, 1931–1946
[22] J.S. Milne, *Étale cohomology*, Princeton Mathematical Series **33** (1980).
[23] C. Mazza, V. Voevodsky and C. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematical Monographs **2** (2006), American Mathematical Society.
[24] A forthcoming paper by A. Merici, K. Rülling and S. Saito.
[25] M. Raynaud and L. Gruson, *Critère de plâtitude et de projectivité*, Invent. Math., **13** (1971), 1–89.
[26] van der Put, M. *Cohomology on affinoid spaces*, *Compositio Math.* 45 (1982), no. 2, 165–198.
[27] P.T. Johnstone, *Topos theory*, London Mathematical Society Monographs **10** (1977), Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York.
[28] S. MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics **5** (1971), Springer-Verlag, New York-Berlin.
[29] J.S. Milne, *Étale cohomology*, Princeton Mathematical Series **33** (1980), Princeton Univ. Press.
[30] N. Bourbaki, *Algèbre Commutative*, Hermann (1961).
[31] H. Matsumura, *Commutative Algebra*, W.A. Benjamin, INC, New York (1970)
[32] M. Artin and A. Grothendieck and J. L. Verdier, *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas. (SGA 4). Un séminaire dirigé par M. Artin, A. Grothendieck, J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint-Donat. Tome 1: Théorie des topos. Exposés I à IV. 2e éd.*, Lecture Notes in Mathematics **269** (1972), Springer, Cham.
[33] M. Artin and A. Grothendieck and J. L. Verdier, *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas. (SGA 4). Un séminaire dirigé par M. Artin, A. Grothendieck, J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint-Donat. Tome 2: Théorie des topos. Exposés V à VIII. 2e éd.*, Lecture Notes in Mathematics **270** (1972), Springer, Cham.
[34] M. Artin and A. Grothendieck and J. L. Verdier, *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas. (SGA 4). Un séminaire dirigé par M. Artin, A. Grothendieck, J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint-Donat. Tome 2: Théorie des topos. Exposés IX à XIX. 2e éd.*, Lecture Notes in Mathematics **305** (1972), Springer, Cham.
[35] *Stacks Project*, https://stacks.math.columbia.edu (2018).