A pro-cdh topology and motivic cohomology of schemes (Lectures at University of Tokyo in November, 2024)

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1 Introduction

A quest for theory of *motivic cohomology* dates back to work of Grothendieck and early days of algebraic geometry. As a motivation, we recall Riemann-Roch theorem of Baum-Fulton-McPherson. Let X be a smooth scheme over a field k. The relation between the category of vector bundles and the group of algebraic cycles on X is expressed by natural isomorphisms

$$\bigoplus_n \mathrm{gr}^n_\gamma K_0(X)_{\mathbb{Q}} \simeq K_0(X)_{\mathbb{Q}} \xrightarrow[\simeq]{\tau} \bigoplus_n \mathrm{CH}^n(X)_{\mathbb{Q}}, \tag{1}$$

where gr^n_γ is the n-th graded quotient of the γ-filtration on $K_0(X)$. The composite of the isomorphisms is compatible with the grading so that it induces

$$\mathrm{gr}^n_\gamma K_0(X)_{\mathbb{Q}} \xrightarrow{\simeq} \mathrm{CH}^n(X)_{\mathbb{Q}}$$

and it is known that the inverse is given by a cycle class map. The isomorphism on LHS is a formal consequence of the existence of a λ-structure on $K_0(X)$ while the existence of τ is a part of BFM's RR theorem.

Remark 1.1. *Even for every X of finite type over k, we have isomorphisms*

$$\bigoplus_n \mathrm{gr}^n_\gamma G_0(X)_{\mathbb{Q}} \simeq G_0(X)_{\mathbb{Q}} \underset{\simeq}{\xrightarrow{\tau}} \bigoplus_n \mathrm{CH}^n(X)_{\mathbb{Q}}, \tag{2}$$

where $\mathrm{CH}^n(X)$ is Fulton's Chow group. (1) *follows from* (2) *and the fact $G_0(X) = K_0(X)$ in case X is smooth.*

S. Bloch upgraded (2) to

$$\bigoplus_n \mathrm{gr}^n_\gamma G_q(X)_{\mathbb{Q}} \simeq G_q(X)_{\mathbb{Q}} \underset{\simeq}{\xrightarrow{\tau}} \bigoplus_n \mathrm{CH}^n(X,q)_{\mathbb{Q}}, \tag{3}$$

where $\mathrm{CH}^n(X,q)$ is the higher Chow group defined by himself. In particular, for X smooth over k, we get an isomorphism

$$\mathrm{gr}^n_\gamma K_q(X)_{\mathbb{Q}} \simeq \mathrm{CH}^n(X,q)_{\mathbb{Q}}. \tag{4}$$

Question 1.2. (1) Is there a similar filtration on $K_q(X)$ for X singular?
(2) Is there a filtration with integral coefficient?

1.1 Motivic complexes

In 1980's, Beilinson and Lichtenbaum predicted there exists

$$\mathbb{Z}(n)^{\mathrm{mot}} \in \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sch}, \mathcal{D}(\mathbb{Z})) \text{ for every } n \geq 0,$$

a complex of Zariski sheaves on a category Sch of reasonable schemes called *motivic complex of weight n*. The motivic cohomology of $X \in \mathrm{Sch}$ defined as

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) = H^i(\mathbb{Z}(n)^{\mathrm{mot}}(X))$$

is expected to play a role of the universal cohomology theory for schemes and to play some important roles in algebraic and arithmetic geometry. For example, there are several conjectures expressing special values of L-functions of arithmetic schemes in terms of motivic cohomology or its related invariants. Here is an example.

Theorem 1.3. *(Kerz-Saito [KeS12]) Let X be a smooth projective variety over a finite filed $\mathbb{F}_q$ with $d = \dim(X)$. We have the equality up to a power of $p = ch(\mathbb{F}_q)$:*

$$\zeta(X,0)^* = \prod_{1 \leqq i \leqq 2d} \left| H^i_{\mathcal{M}}(X, \mathbb{Z}(d))_{\mathrm{tor}} \right|^{(-1)^i}.$$

The equality holds also for the p-part if $d \leq 4$. Here

$$\zeta(X,s) = Z(X, q^{-s}), \quad Z(X,t) = \exp\Big(\sum_{m=1}^{\infty} |X(\mathbb{F}_{q^m})| \, \frac{t^m}{m}\Big)$$

$$\zeta(X,r)^* := \lim_{s \to r} \zeta(X,s) \cdot (1 - q^{r-s})^{\rho_r} \quad (\rho_r := -\mathrm{ord}_{s=r} \zeta(X,s))$$

Here, we define the motivic cohomology of X as

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(d)) = H^{2d-i}(z^d(X, \bullet)),$$

where $z^d(X,\bullet)$ is Bloch's cycle complex *(see §6.1).*

There is a list of properties expected for $\mathbb{Z}(n)^{\mathrm{mot}}$:

1. Projective bundle formula, Blowup formula,
2. Relation of $\mathbb{Z}(n)^{\mathrm{mot}} \otimes^{\mathbb{L}} \mathbb{Z}/\ell$ to étale/syntomic cohomology, etc.

Beside these, the most important is the following relation to algebraic K-theory.

Conjecture 1.4 (Beilinson (1985))**.** For $X \in \mathrm{Sch}$, there is a functorial spectral sequence:

$$E_2^{p,q} = H^{p-q}_{\mathcal{M}}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X) = \pi_{-p-q}(K(X)), \tag{5}$$

where $K(X)$ is the non-connective algebraic K-theory spectrum of X.

Remark 1.5. *(1) The conjectural motivic spectral sequence*

$$E_2^{p,q} = H^{p-q}_{\mathcal{M}}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X) = \pi_{-p-q}(K(X))$$

is viewed as an algebraic analogue of the Atiyah-Hirzebruch spectral sequence for a CW complex X

$$E_2^{p,q} = H^{p-q}_{sing}(X, \mathbb{Z}) \Rightarrow K^{top}_{-p-q}(X),$$

where $K^{top}_*(X)$ is the topological K-theory and $H^*_{sing}(X,\mathbb{Z})$ is the singular cohomology.

(2) The motivic spectral sequence is expected to degenerate rationally inducing natural isomorphisms

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq K_{2n-i}(X)^{(n)}_{\mathbb{Q}},$$

where the RHS are Adams eigenspaces of rationalized K-theory.

Conjecture 1.4 is a consequence of the following.

Conjecture 1.6. Let $\mathrm{PSh}(\mathrm{Sch}, \mathrm{Sp})$ be the category of presheaves of spectra on Sch. There exists a tower in $\mathrm{PSh}(\mathrm{Sch}, \mathrm{Sp})$

$$\cdots \to F^{n+1}_{\mathrm{mot}} K \to F^n_{\mathrm{mot}} K \to \cdots \to F^0_{\mathrm{mot}} K = K,$$

with identifications (via Eilenberg-Maclane functor $\mathcal{D}(\mathbb{Z}) \to \mathrm{Sp}$)

$$\mathrm{gr}^n_{F_{\mathrm{mot}}} K := \mathrm{cofib}(F^{n+1}_{\mathrm{mot}} K \to F^n_{\mathrm{mot}} K) \simeq \mathbb{Z}(n)^{\mathrm{mot}}[2n] \tag{6}$$

In case X is smooth over a field k, it has been known (Friedlander-Suslin, Levine, Voevodsky) that $F^n_{\mathrm{mot}} K(X)$ exists and agrees with the γ-filtration after $\otimes \mathbb{Q}$ and moreover that $\mathrm{gr}^n_{F_{\mathrm{mot}}} K(X)$ is equivalent to $z^n(X, -\bullet)$. Thus it gives an integral refinement of (4).

2 Motivic complexes for smooth schemes

The first major progress toward Conjecture 1.4 took place around twenty year after the formulation of the conjecture. It gave a satisfactory answer for smooth schemes over a field.

Let Sm_k be the category of smooth schemes over a field k.

Theorem 2.1 (Friedlander-Suslin [FS02], Levine [Le08], Voevodsky [V-CMAMS])**.** *There exists a tower in* $\mathrm{PSh}(\mathrm{Sm}_k, \mathrm{Sp})$

$$\cdots \to F^{n+1}_{\mathrm{mot}} K \to F^n_{\mathrm{mot}} K \to \cdots \to F^0_{\mathrm{mot}} K = K,$$

and equivalences

$$\mathrm{gr}^n_{F_{\mathrm{mot}}} K \simeq z^n(-, \bullet),$$

where $z^n(X, \bullet)$ *for* $X \in \mathrm{Sm}_k$ *is Bloch's cycle complex.*

We define the motivic complex $\mathbb{Z}(n)^{\mathrm{sm}} \in \mathrm{PSh}(\mathrm{Sm}_k, \mathcal{D}(\mathbb{Z}))$ as

$$\mathbb{Z}(n)^{\mathrm{sm}} := \mathrm{gr}^n_{F_{\mathrm{mot}}} K[2n] \simeq z^n(-, 2n - \bullet).$$

Remark 2.2. *Bloch's cycle complex is a priori only functorial for flat morphisms in* Sm_k*, which is not sufficient for later purposes (e,g. for left Kan extending along* $\mathrm{Sm}_k \to \mathrm{Sch}_k$*), and its multiplicative properties are unclear. The problems are resolved via using*

$$\mathbb{Z}(n)^{\mathrm{sm}}(X) = \underline{C}_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}))(X)[-n] \quad \text{for } X \in \mathrm{Sm}_k, \tag{7}$$

where $\underline{C}_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}))[-q]$ *is Voevodsky's* $\mathbb{A}^1$*-invariant motivic complex defined in [SV00]. This is strictly functorial in* Sm_k *in the sense that it defines a functor between the 1-category* Sm_k *and the 1-category of chain complexes of abelian groups. Scheme-wise this is shown to be quasi-isomorphic to Bloch's cycle complex* $z^n(X, 2n - \bullet)$ *as shown in [V02, Cor.2].*

In §2.1 below, we recall a definition of F^n_{mot} *using the homotopy coniveau tower due to Levine [Le08] but there seem to be a similar technical issue on functoriality and multiplicativity. This is solved by using Voevodsky's slice filtration [V-CMAMS]. There is also a related work [DFJ23].*

Remark 2.3. *It is known that* $\mathbb{Z}(n)^{\mathrm{sm}} \in \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathcal{D}(\mathbb{Z}))$ *(see [Bl86, Th.3.1] and Remark 2.2). We let* $\mathcal{H}^i_{\mathrm{Zar}}(\mathbb{Z}(n)^{\mathrm{sm}})$ *be the Zariski cohomology sheaf of* $\mathbb{Z}(n)^{\mathrm{sm}}$.

Remark 2.4. *By [Bl86, Th.6.1], we have* $\mathcal{H}^1_{\mathrm{Zar}}(\mathbb{Z}(n)^{\mathrm{sm}}) \simeq \mathcal{O}^\times$ *and* $\mathcal{H}^i_{\mathrm{Zar}}(\mathbb{Z}(n)^{\mathrm{sm}}) = 0$ *for* $i \neq 1$*. In view of Remark 2.3, this implies an equivalence in* $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathcal{D}(\mathbb{Z})$*:*

$$\mathbb{Z}(1)^{\mathrm{sm}} \simeq R\Gamma_{\mathrm{Zar}}(-, \mathcal{O}^\times)[-1]. \tag{8}$$

Remark 2.5. *By [Bl86, Th.10.1], $\mathcal{H}^i_{\mathrm{Zar}}(\mathbb{Z}(n)^{\mathrm{sm}})(X) \to \mathcal{H}^i_{\mathrm{Zar}}(\mathbb{Z}(n)^{\mathrm{sm}})(k(X))$ is injective for an integral $X \in \mathrm{Sm}_k$. This implies*

$$\mathcal{H}^i_{\mathrm{Zar}}(\mathbb{Z}(n)^{\mathrm{sm}}) = 0 \quad \text{for } i > n, \tag{9}$$

which implies that for $X \in \mathrm{Sm}_k$,

$$H^i(\mathbb{Z}(n)^{\mathrm{sm}}(X)) = 0 \quad \text{for } i > \dim(X) + n, \tag{10}$$

In particular, $F^n_{\mathrm{mot}}K(X)$ is supported in cohomological degrees $\leq \dim(X) - n$ for each $n \in \mathbb{N}$, so the induced spectral sequence (5) is bounded.

Remark 2.6. *By [NS89], [To92] and [K09, Th. 7.6], there exists a natural isomorphism if k is infinite:*

$$\mathcal{H}^n_{\mathrm{Zar}}(\mathbb{Z}(n)^{\mathrm{sm}}) \simeq \mathcal{K}^M_n, \tag{11}$$

where the right hand side is the Zariski sheaf of Milnor K-theory[1]*. This gives rise to a natural map in $\mathrm{Shv}_{Zar}(\mathrm{Sm}_k, \mathcal{D}(\mathcal{Z}))$:*

$$\mathbb{Z}(n)^{\mathrm{sm}} \to \mathcal{H}^n(\mathbb{Z}(n)^{\mathrm{sm}})[-n] \simeq \mathcal{K}^M_n[-n], \tag{12}$$

where the second map comes from (9).

Remark 2.7. *Combined with some deep results on $\mathbb{Z}(n)^{\mathrm{sm}} \otimes^{\mathbb{L}} \mathbb{Z}/\ell^r$ for a prime ℓ and an integer $r > 0$, Theorem 2.1 provides substantial information on K-theory $K(X, \mathbb{Z}/\ell^r)$ of finite coefficient: If $\ell \neq ch(k)$, the Beilinson-Lichtenbaum conjecture proved by Voevodsky-Rost [V11] implies an equivalence:*

$$\mathbb{Z}(n)^{\mathrm{sm}} \otimes^{\mathbb{L}} \mathbb{Z}/\ell^r \simeq \tau^{\leq n} R\epsilon_* \mu_{\ell^r}^{\otimes n} \text{ in } \mathrm{Shv}_{Zar}(\mathrm{Sm}_k, \mathcal{D}(\mathbb{Z}/\ell^r)), \tag{13}$$

where $\epsilon : (\mathrm{Sm}_k)_{\acute{e}t} \to (\mathrm{Sm}_k)_{Zar}$ is the natural map of sites. If $p = ch(k) > 0$, a theorem of Geisser-Levine [GL00] and Bloch-Gabber-Kato [BK86] gives an equivalence:

$$\mathbb{Z}(n)^{\mathrm{sm}} \otimes^{\mathbb{L}} \mathbb{Z}/p^r \simeq W_r\Omega^n_{\log}[-n] \text{ in } \mathrm{Shv}_{Zar}(\mathrm{Sm}_k, \mathcal{D}(\mathbb{Z}/p^r)), \tag{14}$$

where $W_r\Omega^n_{\log}$ is the logarithmic part of de Rham-Witt sheaf $W_r\Omega^n_{-/k}$ [Il79][2]*. Thanks to Theorem 2.1, this implies for $X \in \mathrm{Sm}_k$ that $K_i(X, \mathbb{Z}/p^r) = 0$ for $i > \dim(X)$.*

Remark 2.8. *The proof of Theorem 1.1 uses the $\mathbb{A}^1$-invariance of K-theory, which is valid only for regular schemes. The higher Chow groups of a singular variety over a field are $\mathbb{A}^1$-invariant and nilinvariant, while algebraic K-theory does not satisfy the properties. It has been an open problem to give a motivic filtration on K-theory (and a motivic cohomology) for singular schemes.*

[1] The hypothesis on the infiniteness of k can be removed if the Milnor K-theory is replaced by the improved Milnor K-theory of Gabber and Kerz [K10].

[2] It is the subsheaf of the de Rham-Witt sheaf $W_r\Omega^n$ generated étale locally by $d\log[x_1] \wedge \cdots \wedge d\log[x_n]$ for local units x_i with its Teichmüller lifts $[x_i]$ in the Witt vectors.

2.1 Motivic filtrations via the homotopy coniveau tower

In this subsection, we explain a construction of the motivic filtration $F^n_{\mathrm{mot}}K$ in Theorem 2.1 following [Le08]. It is given as $F^n_{\mathrm{mot}}K = F^n_{\mathrm{con}}K$, where $\{F^n_{\mathrm{con}}K\}_{n\in\mathbb{N}}$ is the homotopy coniveau tower defined below.

Recall the algebraic q-simplex

$$\Delta^q = \mathrm{Spec}\left(\mathbb{Z}[t_0, \cdots, t_q]/(\sum_{i=0}^{q} t_i - 1)\right)$$

with faces $\Delta^s = \{t_{i_1} = \cdots = t_{i_{q-s}} = 0\} \subset \Delta^q$. The association

$$\Delta^{op} \to \mathrm{Sch}\ ;\ [q] \to \Delta^q$$

gives a cosimplicial scheme $\Delta^\bullet$.

For $X \in \mathrm{Sm}_k$, let $S^n_X(q)$ be the set of closed subsets $W \subset X \times \Delta^q$ such that

$$\mathrm{codim}_{X\times F}(W \cap (X \times F)) \geq n$$

for all faces $T \subset \Delta^q$. Let $X^n(q)$ is the set of codimension n points $x \in X \times \Delta^q$ whose closures $\overline{\{x\}}$ lie in $S^n_X(q)$.

For $E \in \mathrm{PSh}(\mathrm{Sm}_k, \mathrm{Sp})$ and $q \geq 0$, put

$$F^n_{\mathrm{con}}E(X, q) := \underset{W\in S^n_X(q)}{\mathrm{colim}}\ E^W(X \times \Delta^q),$$

$$E^W(X \times \Delta^q) = \mathrm{fib}\big(E(X \times \Delta^q) \to E((X \times \Delta^q) \setminus W)\big).$$

It gives a simplicial spectrum $F^n_{\mathrm{con}}E(X, -)$ and a tower

$$\cdots \to F^{n+1}_{\mathrm{con}}E(X, -) \to F^n_{\mathrm{con}}E(X, -) \to \cdots \to F^0_{\mathrm{con}}E(X, -) = E(X, -). \tag{15}$$

Put

$$\mathrm{gr}^n_{F_{\mathrm{con}}}E(X, -) = \mathrm{cofib}\big(F^{n+1}_{\mathrm{con}}E(X, -) \to F^{n+1}_{\mathrm{con}}E(X, -)\big).$$

Write

$$F^n_{\mathrm{con}}E(X) = |F^n_{\mathrm{con}}E(X, -)| = \mathrm{colim}_{[q]\in\Delta^{op}}\, F^n_{\mathrm{con}}E(X, q).$$

$$\mathrm{gr}^n_{F_{\mathrm{con}}}E(X) = \mathrm{cofib}\big(F^{n+1}_{\mathrm{con}}E(X) \to F^{n+1}_{\mathrm{con}}E(X)\big) \simeq |\mathrm{gr}^n_{F_{\mathrm{con}}}E(X, -)|.$$

Lemma 2.9. *$F^0_{\mathrm{con}}E(X) \to E(X)$ is an equivalence if E is $\mathbb{A}^1$-invariant.*

Proof. $\mathbb{A}^1$-invariance implies $E(X) \simeq E(X \times \Delta^q)$. □

The equivalence $\mathrm{gr}^n_{F_{\mathrm{con}}} K(X) \simeq z^n(X, \bullet)$ in Theorem 2.1 follows from the following:

Theorem 2.10. *(1) $\mathrm{gr}^n_{F_{\mathrm{con}}} E(X,-)$ is equivalent to another simplicial spectrum whose q-simplices is*

$$\bigoplus_{x\in X^n(q)} \mathrm{gr}^0_{F_{\mathrm{con}}}(\Omega^n_{\mathbb{P}^1} E)(x),$$

where $\Omega_{\mathbb{P}^1} E \in \mathrm{PSh}(\mathrm{Sm}_k, \mathrm{Sp})$ is the $\mathbb{P}^1$-loop spectrum of E defined as

$$\Omega_{\mathbb{P}^1} E(X) = \mathrm{fib}(E(X \times \mathbb{P}^1) \to E(X \times \infty)),$$

and $\Omega^n_{\mathbb{P}^1} E$ is defined inductively as $\Omega^n_{\mathbb{P}^1} E = \Omega_{\mathbb{P}^1}(\Omega^{n-1}_{\mathbb{P}^1} E)$.

(2) If $E = K$, we have for a field F

$$\mathrm{gr}^0_{F_{\mathrm{con}}}(\Omega^n_{\mathbb{P}^1} K)(F) \overset{(*1)}{\simeq} \mathrm{gr}^0_{F_{\mathrm{con}}} K(F) \overset{(*2)}{\simeq} K(\mathbb{Z}, 0),$$

where $K(\mathbb{Z}, 0)$ is Eilenberg-Maclane spectrum and $(*1)$ follows from $\Omega^n_{\mathbb{P}^1} K = K$ by the $\mathbb{P}^1$-bundle formula (see Theorem 6.8).

We recall some ingredients of the proof of Theorem 2.10. The basic inputs are Localization Theorem 2.13 and Purity Theorem 2.14. In what follows, we assume that k is infinite while the assumption can be removed under an additional assumption on $E \in \mathrm{PSh}(\mathrm{Sm}_k, \mathrm{Sp})$ which is satisfied by $E = K$.

We let $\mathrm{Shv}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Sp}) \subset \mathrm{PSh}(\mathrm{Sm}_k, \mathrm{Sp})$ denote the full subcategory of those Nisnevich sheaves F of spectra on Sm_k that $F(X) \simeq F(X \times_k \mathbb{A}^1_k)$. Note that K belongs to $\mathrm{Shv}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Sp})$.

Remark 2.11. *For $E \in \mathrm{PSh}(\mathrm{Sm}_k, \mathrm{Sp})$, the association $X \to F^n_{\mathrm{con}} E(X)$ is functorial only for a flat morphism $Y \to X$ in Sm_k. In particular, it gives an object of $\mathrm{PSh}(\mathrm{Sm}//k, \mathrm{Sp})$, where $\mathrm{Sm}//k$ is the category of the same objects as Sm_k with smooth morphisms $Y \to X$. Thus, (15) gives a tower in $\mathrm{PSh}(\mathrm{Sm}//k, \mathrm{Sp})$:*

$$\cdots \to F^{n+1}_{\mathrm{con}} E//k \to F^n_{\mathrm{con}} E//k \to \cdots \to F^0_{\mathrm{con}} E//k \to E \tag{16}$$

such that $F^n_{\mathrm{con}} E//k(X) = F^n_{\mathrm{con}} E(X)$. The following theorem refines this.

For a category $\mathcal{C}$, we write $\mathcal{H}\,\mathrm{Sp}(\mathcal{C}) = \mathrm{PSh}(\mathcal{C}, \mathrm{Sp})[W^{-1}]$, where W is the class of morphisms which are point-wise equivalences[3].

[3] To be more precise, we equip $\mathrm{PSh}(\mathcal{C}, \mathrm{Sp})$ with the model structure whose cofibrations and weak equivalences are point-wise and fibrations are characterized by the RLP with respect to the trivial cofibrations.

Theorem 2.12. *(Functoriality For $E \in \mathrm{Shv}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Sp})$, there exists a tower in* $\mathrm{PSh}(\mathrm{Sm}_k, \mathrm{Sp})$

$$\cdots \to F^{n+1}_{\mathrm{con}} E \to F^n_{\mathrm{con}} E \to \cdots \to F^0_{\mathrm{con}} E \to E \tag{17}$$

whose restriction to $\mathrm{Sm}//k$ *is isomorphic to* (16) *in* $\mathcal{H}\,\mathrm{Sp}(\mathrm{Sm}//k)$*.*

Let $X \in \mathrm{Sm}_k$ and $Z \subset X$ be a closed subset and $U = X \backslash Z$. Let $S^n_{X,Z}(q) \subset S^n_X(q)$ be the subset consisting of those W such that $W \subset Z \times \Delta^q$ and put

$$F^n_{\mathrm{con}} E_Z(X, q) := \operatorname*{colim}_{W \in S^n_{X,Z}(q)} E^W(X \times \Delta^q),$$

which gives a simplicial spectrum $F^n_{\mathrm{con}} E_Z(X, -)$. Write

$$F^n_{\mathrm{con}} E_Z(X) = |F^n_{\mathrm{con}} E_Z(X, -)|.$$

Theorem 2.13. *(Localization, [Le08, Cor.3.2.2]) For $E \in \mathrm{Shv}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Sp})$, the sequences in* Sp

$$F^n_{\mathrm{con}} E_Z(X, q) \to F^n_{\mathrm{con}} E(X, q) \to F^n_{\mathrm{con}} E(U, q),$$

$$\mathrm{gr}^n_{F_{\mathrm{con}}} E_Z(X, q) \to \mathrm{gr}^n_{F_{\mathrm{con}}} E(X, q) \to \mathrm{gr}^n_{F_{\mathrm{con}}} E(U, q),$$

extends canonically to distinguished triangles in SH*, the homotopy category of* Sp*. Hence, we have equivalences in* Sp*:*

$$F^n_{\mathrm{con}} E_Z(X, q) \simeq \mathrm{fib}\big(F^n_{\mathrm{con}} E(X, q) \to F^n_{\mathrm{con}} E(U, q)\big),$$

$$\mathrm{gr}^n_{F_{\mathrm{con}}} E_Z(X, q) \simeq \mathrm{fib}\big(\mathrm{gr}^n_{F_{\mathrm{con}}} E(X, q) \to \mathrm{gr}^n_{F_{\mathrm{con}}} E(U, q)\big).$$

Theorem 2.14. *(Purity, [Le08, Pr.4.2.2]) Let $i : Z \to X$ be a closed immersion of codimension d in Sm_k. Assume there exists a trivialization $\phi : N_{Z/X} \simeq Z \times \mathbb{A}^d$. For $E \in \mathrm{Shv}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Sp})$ and $n \geq 0$, there are equivalences [4] in* Sp

$$F^n_{\mathrm{con}} E_Z(X, q) \simeq F^{n-d}_{\mathrm{con}}(\Omega^d_{\mathbb{P}^1} E)(Z, q),$$

$$\mathrm{gr}^n_{F_{\mathrm{con}}} E_Z(X, q) \simeq \mathrm{gr}^{n-d}_{F_{\mathrm{con}}}(\Omega^d_{\mathbb{P}^1} E)(Z, q).$$

Here, for $m < 0$, we set $F^m_{\mathrm{con}}(\Omega^d_{\mathbb{P}^1} E) = F^0_{\mathrm{con}}(\Omega^d_{\mathbb{P}^1} E)$ and $\mathrm{gr}^m_{F_{\mathrm{con}}}(\Omega^d_{\mathbb{P}^1} E) = *$.

Corollary 2.15. *Let the assumption be as in Theorem 2.14 and put $U = X - \backslash Z$. Then, there exists a distinguished triangles in* SH

$$F^{n-d}_{\mathrm{con}}(\Omega^d_{\mathbb{P}^1} E)(Z, q) \to F^n_{\mathrm{con}} E(X, q) \to F^n_{\mathrm{con}} E(U, q).$$

Thus, there exists a long exact sequence

$$\cdots \to \pi_{i+1}(F^n_{\mathrm{con}} E(U, q)) \to \pi_i(F^{n-d}_{\mathrm{con}}(\Omega^d_{\mathbb{P}^1} E)(Z, q)) \to \pi_i(F^n_{\mathrm{con}} E(X, q)) \to \pi_i(F^n_{\mathrm{con}} E(U, q)) \to \cdots.$$

The same results hold by replacing F^n_{con} by $\mathrm{gr}^n_{F_{\mathrm{con}}}$.

[4] These may depend on the choice of trivialization of $N_{Z/X}$. but are natural in the category of closed embeddings i with trivialization of N_i.

Corollary 2.16. *Suppose k is perfect. Let the assumption be as in Theorem 2.14. Let $W \subset X$ be a closed subset with $\mathrm{codim}_X(W) \geq d$ and $V \subset W$ be a regular open subset containing $W \cap X^{(d)}$. Then, for $n \geq 0$, there is an equivalence in* Sp

$$(\mathrm{gr}^d_{F_{\mathrm{con}}} E)^W(X,q) \simeq \mathrm{gr}^0_{F_{\mathrm{con}}}(\Omega^d_{\mathbb{P}^1} E)(V,q),$$

where $(\mathrm{gr}^d_{F_{\mathrm{con}}} E)^W(X,q) = \mathrm{fib}\big(\mathrm{gr}^d_{F_{\mathrm{con}}} E(X,q) \to \mathrm{gr}^d_{F_{\mathrm{con}}} E(X-W,q)\big)$.

Proof. We prove the corollary assuming that $N_{W/X}$ is trivial and omit a reduction to this case. By Theorem 2.13, we have an equivalence

$$(\mathrm{gr}^d_{F_{\mathrm{con}}} E)^W(X,q) \simeq (\mathrm{gr}^d_{F_{\mathrm{con}}} E_W)(X,q).$$

So, if $W \in \mathrm{Sm}_k$, the corollary follows from Theorem 2.14. To show the general case, let $T = W \setminus V$ and $X^o = X \setminus T$. Note $\mathrm{codim}_X(T) \geq d+1$ and also that $V \in \mathrm{Sm}_k$ thanks to the assumption that k is perfect. Thus, it suffices to show the equivalence

$$(\mathrm{gr}^d_{F_{\mathrm{con}}} E)^W(X,q) \simeq (\mathrm{gr}^d_{F_{\mathrm{con}}} E)^V(X^o,q). \tag{18}$$

By the fiber sequence

$$(\mathrm{gr}^d_{F_{\mathrm{con}}} E)^T(X,q) \to (\mathrm{gr}^d_{F_{\mathrm{con}}} E)^W(X,q) \to (\mathrm{gr}^d_{F_{\mathrm{con}}} E)^V(X^o,q),$$

it suffices to show $(\mathrm{gr}^d_{F_{\mathrm{con}}} E)^T(X,q) \simeq 0$. By Theorem 2.13, it suffices to show $\mathrm{gr}^d_{F_{\mathrm{con}}} E_T(X,q) \simeq 0$. By definition,

$$\mathrm{gr}^d_{F_{\mathrm{con}}} E_T(X,q) = \mathrm{cofib}\big(F^{d+1}_{\mathrm{con}} E_T(X,q) \xrightarrow{\iota} F^d_{\mathrm{con}} E_T(X,q)\big).$$

So, we are reduced to showing ι is an equivalence. For this, it suffices to show $S^d_{X,T}(q) = S^{d+1}_{X,T}(q)$. Recall $S^n_{X,T}(q)$ is the set of closed subsets $W \subset T \times \Delta^q$ satisfying the condition

$$\mathrm{codim}_{X\times F}(W \cap (X \times F)) \geq n \ \text{ for all faces } F \subset \Delta^q. \tag{19}$$

If $W \subset T \times \Delta^q$, $W \cap (X \times F) \subset T \times F$ so

$$\mathrm{codim}_{X\times F}(W \cap (X \times F)) \geq \mathrm{codim}_{X\times F}(T \times F) = \mathrm{codim}_X(T) \geq d+1.$$

Hence, (19) is automatic if $n \leq d+1$. This completes the proof of (18).

□

Corollary 2.17. *Suppose k is perfect. Let the assumption be as in Theorem 2.14. For an integer $d \geq 0$, there is a natural equivalence*

$$F^d_{\mathrm{con}}(\mathrm{gr}^d_{F_{\mathrm{con}}} E)(X,q) \simeq \mathrm{gr}^d_{F_{\mathrm{con}}}(\mathrm{gr}^d_{F_{\mathrm{con}}} E)(X,q) \simeq \bigoplus_{x \in X^d(q)} \mathrm{gr}^0_{F_{\mathrm{con}}}(\Omega^d_{\mathbb{P}^1} E)(x).$$

We also have

$$F^n_{\mathrm{con}}(\mathrm{gr}^d_{F_{\mathrm{con}}} E)(X,q) \simeq 0 \quad \textit{for } n > d. \tag{20}$$

Proof. By Corollary 2.16, for $X \in \mathrm{Sm}_k$ and a closed subset $W \subset X$ of codimension$\geq d$, we have

$$(\mathrm{gr}^d_{F_{\mathrm{con}}} E)^W(X) \simeq \bigoplus_{x \in W \cap X^{(d)}} \mathrm{gr}^0_{F_{\mathrm{con}}}(\Omega^d_{\mathbb{P}^1} E)(x).$$

Applying this to a closed subset $W \subset X \times \Delta^q$ with $\mathrm{codim}_{X\times\Delta}(W) \geq d$, we get an equivalence in Sp

$$(\mathrm{gr}^d_{F_{\mathrm{con}}} E)^W(X \times \Delta^q) \simeq \bigoplus_{x \in W \cap (X \times \Delta^q)^{(d)}} \mathrm{gr}^0_{F_{\mathrm{con}}}(\Omega^d_{\mathbb{P}^1} E)(x).$$

This implies an equivalence

$$F^d_{\mathrm{con}}(\mathrm{gr}^d_{F_{\mathrm{con}}} E)(X, q) \simeq \bigoplus_{x \in X^d(q)} \mathrm{gr}^0_{F_{\mathrm{con}}}(\Omega^d_{\mathbb{P}^1} E)(x).$$

and $F^n_{\mathrm{con}}(\mathrm{gr}^d_{F_{\mathrm{con}}} E)(X, q) \simeq 0$ for $n \geq d+1$. □

Thanks to Corollary 2.17, Theorem 2.10(1) follows from the following.

Theorem 2.18. *For $E \in \mathrm{Shv}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Sp})$ and $d \geq 0$, there is a natural equivalence*

$$\mathrm{gr}^d_{F_{\mathrm{con}}}(\mathrm{gr}^d_{F_{\mathrm{con}}} E) \simeq \mathrm{gr}^d_{F_{\mathrm{con}}} E.$$

Proof. In view of a fiber sequence

$$\mathrm{gr}^d_{F_{\mathrm{con}}}(F^{d+1}_{\mathrm{con}} E) \to \mathrm{gr}^d_{F_{\mathrm{con}}}(F^d_{\mathrm{con}} E) \to \mathrm{gr}^d_{F_{\mathrm{con}}}(\mathrm{gr}^d_{F_{\mathrm{con}}} E),$$

Proposition 2.19 below implies $\mathrm{gr}^d_{F_{\mathrm{con}}}(F^d_{\mathrm{con}} E) \simeq \mathrm{gr}^d_{F_{\mathrm{con}}}(\mathrm{gr}^d_{F_{\mathrm{con}}} E)$. On the other hand, (20) implies $\mathrm{gr}^d_{F_{\mathrm{con}}}(\mathrm{gr}^n_{F_{\mathrm{con}}} E) \simeq 0$ for $n < d$ and it implies $\mathrm{gr}^d_{F_{\mathrm{con}}}(F^d_{\mathrm{con}} E) \simeq \mathrm{gr}^d_{F_{\mathrm{con}}}(E)$. □

The following vanishing is compared with the vanishing in Corollary 2.17.

Proposition 2.19. *For $E \in \mathrm{Shv}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Sp})$ and integers $0 \leq q < p$, we have*

$$\mathrm{gr}^q_{F_{\mathrm{con}}}(F^p_{\mathrm{con}} E) \simeq \mathrm{gr}^q_{F_{\mathrm{con}}}(\mathrm{gr}^p_{F_{\mathrm{con}}} E) \simeq 0.$$

Proof. Omitted. □

To prove Theorem 2.10(2), we need to show

Proposition 2.20. *We have an equivalence $\mathrm{gr}^0_{F_{\mathrm{con}}} K(F) \simeq K(\mathbb{Z}, 0)$.*

Lemma 2.21. *For $E \in \mathrm{PSh}(\mathrm{Sm}_k, \mathrm{Sp})$ and a field F and $q \geq 0$, we have*

$$\mathrm{gr}^0_{F_{\mathrm{con}}} E(F, q) \simeq E(\Delta^q_{0,F}),$$

where $\Delta^\bullet_{0,F}$ is the semi-localization of $\Delta^\bullet_F$ at the 0-dimensional vertices. Hence, we have an equivalence $\mathrm{gr}^0_{F_{\mathrm{con}}} E(F) \simeq |E(\Delta^\bullet_{0,F})| = \mathrm{colim}_{[q]\in\Delta^{op}} E(\Delta^q_{0,F})$.

Proof. Exercise. □

Recall the category Δ whose objects are the finite ordered sets $[n] := \{0 < 1 < \cdots < n\}$ and whose morphisms are the order-preserving maps of sets. For $n \geq 0$, let $([n], \partial)$ be the category whose objects are the injective map $g : [m] \to [n]$ in Δ with $m < n$ (plus $\emptyset \to [n]$) and whose morphisms are the commutative triangles of injective maps. We have a functor $\pi_n : ([n], \partial) \to \Delta$ sending $[m] \to [n]$ to $[m]$.

For a simplicial spectrum $E : \Delta^{op} \to \mathrm{Sp}$, let $E([n], \partial)$ denote the iterated homotopy fiber of $E \circ \pi_n$ over $([n], \partial)$ defined as

$$E([n], \partial) = \lim_{g:[m]\to[n]\in([n],\partial)} \mathrm{fib}\big(E(g) : E([n]) \to E([m])\big).$$

If $E : \Delta^{op} \to \mathcal{D}(\mathbb{Z})$ is a simplicial object in $\mathcal{D}(\mathbb{Z})$, $E([n], \partial)$ is the total complex of the double complex:

$$E([n]) \to \bigoplus_{g:[n-1]\to[n]} E([n-1]) \to \bigoplus_{g:[n-2]\to[n]} E([n-2]) \to \cdots,$$

where $E([n-m])$ is in homological degree $-m$ and g in degree $-m$ range over all injective maps $[n-m] \to [n]$ in Δ and the differential d_{-m} is the signed sum of the maps

$$E(f) : (E([n-m+1]), g) \to (E([n-m], g \circ f) \text{ for } f : [n-m] \to [n-m+1] \in \Delta^{inj}.$$

Applying the above construction to the simplicial spepctrum $[n] \to E(\Delta^n_{0,F})$, we get

$$E(\Delta^n_{0,F}, \partial) = \lim_{g:[m]\to[n]\in([n],\partial)} \mathrm{fib}\big(E(g) : E(\Delta^n_{0,F}) \to E(\Delta^m_{0,F})\big).$$

Proposition 2.22. *([Le08, Pr.6.3.4]) Take $E \in \mathrm{Shv}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Sp})$ and assume*

(i) For any $X \in \mathrm{Sm}_k$ and open $U \subset X$, we have $\pi_i(E(X)) = 0$ for $i < 0$ and $\pi_0(E(X)) \to \pi_0(E(U))$ is surjective.
(ii) $\pi_0(E(\Delta^n_{0,F}, \partial)) = 0$ for all $n \geq 0$.

Then, we have $\pi_0(|E(\Delta^\bullet_{0,F})|) \simeq \pi_0(E(F))$ and $\pi_i(|E(\Delta^\bullet_{0,F})|) = 0$ for all $i > 0$.

Proof. A formal argument in homotopical algebra using the spectral sequence

$$E^1_{p,q} = \pi_{p+q}(E(\Delta^p_{0,F})) \Rightarrow \pi_{p+q}(|E(\Delta^\bullet_{0,F})|).$$

□

Proof of Proposition 2.20: We check the conditions of Proposition 2.22. (i) follows from the fact that $K_i(X) = 0$ for $i < 0$ and $K_0(X) \to K_0(U)$ is surjective if $X \in \mathrm{Sm}_k$ and $U \subset X$ is open. We now show (ii).

Claim 2.23. There is an exact sequence

$$K_1(\Delta^n_{0,F}) \xrightarrow{f} K_1(\partial\Delta^n_{0,F}) \to \pi_0(K(\Delta^n_{0,F}, \partial)) \to K_0(\Delta^n_{0,F}) \xrightarrow{g} K_0(\partial\Delta^n_{0,F})$$

where $\partial\Delta^n_{0,F}$ is the union of the faces of codimension one in $\Delta^n_{0,F}$.

Proof. This follows from the fact that $\partial\Delta^n_{0,F}$ is K_1-regular so that $K_i(\partial\Delta^n_{0,F}) \simeq \mathrm{KH}_i(\partial\Delta^n_{0,F})$ for $i \leq 1$ (see Definition 3.9 for KH), and the fact that KH satisfies Mayer-Vietoris property for unions of closed subschemes (see [W13, IV Cor.12.6]). The detail is left as an exercise. □

Remark 2.24. *A ring A is called K_i-regular if $K_i(A) \simeq K_i(A[T_1, \ldots, T_r])$ for all r. It is known that a regular noetherian ring is K_n-regular for all n. Vorst conjectured that for an algebra A of essentially finite type over a field k with* $\dim(A) \leq n$*, A is K_{n+1}-regular implies A is regular. For example,*

- *(i) A is K_0-regular if and only if A is seminormal* [5].
- *(ii) A is K_1-regular if and only if A is seminormal and for every $x \in \mathrm{Spec}(A)$ and every point y of the normalization lying above x, $k(y)/k(x)$ is separable.*

If $ch(k) = 0$, the conjecture was proved by Cortiñas, Haesemeyer, and Weibel [CHW08]. In case $ch(k) = p > 0$, Geisser and Hesselholt [GH12] proved the conjecture assuming resolution of singularities. Kerz-Strunk-Tamme [KST21] proved it by replacing $\dim(A)$ *by its variant called the p-dimension.*

We are now reduced to showing the surjectivity of f and the injectivity of g in Claim 2.23. Let R be the affine ring of the semi-local scheme $\Delta^n_{0,F}$ and $I \subset R$ be the ideal defining $\partial\Delta^n_{0,F}$. The injectivity of g follows from $K_0(R) = K_0(R/I) = \mathbb{Z}$ since R is semi-local (see math.stackexchange.com/questions/150944/). To show

[5] i.e. for all $x, y \in A$ with $x^3 = y^2$, there is a unique $a \in A$ with $x = a^2$ and $y = a^3$.

the surjectivity of f, consider a commutative diagram

$$\begin{array}{ccc} \mathrm{GL}(R) & \longrightarrow & K_1(R) \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow \\ \mathrm{GL}(R/I) & \longrightarrow & K_1(R/I) \end{array}$$

The horizontal maps are surjective since $K_1(S) = \mathrm{GL}(S)/[\mathrm{GL}(S), \mathrm{GL}(S)]$ for any associative ring S with unit. The map π is surjective since $I \subset R$ is a radical ideal (see [W13, I, Exc.1.1.12]). This proves the desired surjectivity.

3 Cdh-local motivic complex $\mathbb{Z}(n)^{\mathrm{cdh}}$

This is an extension of $\mathbb{Z}(n)^{\mathrm{sm}} \simeq z^n(-, 2n - \bullet)$ on Sm_k to a motivic complex on the category Sch_k of schemes of finite type over k, first considered by Voevodsky and studied by Cisinski-Déglise and more recently by Bachmann-Elmanto-Morrow.

3.1 Left Kan extension

We use the *left Kan extension* along $\mathrm{Sm}_k \to \mathrm{Sch}_k$:

$$L^{sm} : \mathrm{PSh}(\mathrm{Sm}_k, \mathcal{C}) \to \mathrm{PSh}(\mathrm{Sch}_k, \mathcal{C}) \ (\mathcal{C} = \mathrm{Set}, \mathrm{Sp}, \mathcal{D}(\mathbb{Z}), etc),$$

which is a left adjoint to the restriction $\mathrm{PSh}(\mathrm{Sch}_k, \mathcal{C}) \to \mathrm{PSh}(\mathrm{Sm}_k, \mathcal{C})$. For $F \in \mathrm{PSh}(\mathrm{Sm}_k, \mathcal{C})$ and $X \in \mathrm{Sch}_k$, $L^{sm}F(X)$ is calculated as the colimit in $\mathcal{C}$ of the diagram

$$(\mathrm{Sm}_{X/})^{op} \to \mathcal{C};\ (Y, \phi) \to F(Y),$$

where $\mathrm{Sm}_{X/}$ is the category of pairs (Y, ϕ) with $Y \in \mathrm{Sm}_k$ and a map $\phi : X \to Y$ and a morphism $(Y, \phi) \to (Y', \phi')$ is given by a map $f : Y \to Y'$ such that $\phi' = f\phi$.

To compute $L^{sm}F(X)$ more explicitly for $X = \mathrm{Spec}(A) \in \mathrm{Sch}_k$, pick a simplicial resolution $P_\bullet \to A$ such that P_n ind-smooth over k and $\mathrm{Ker}(P_n \to A)$ henselian ideal. Then,

$$(L^{sm}F)(\mathrm{Spec}(A)) = \mathrm{colim}_{[n] \in \Delta^{op}} F(P_n),$$

where the colimit is taken in $\mathcal{D}(\mathbb{Z})$. If $F = \mathbb{Z}(n)^{\mathrm{sm}}$, this is computed more explicitly as $[-2n]$-shift of totalisation of bicomplex

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & z^n(P_2, 2) & \longrightarrow & z^n(P_2, 1) & \longrightarrow & z^n(P_2, 0) \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & z^n(P_1, 2) & \longrightarrow & z^n(P_1, 1) & \longrightarrow & z^n(P_1, 0) \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & z^n(P_0, 2) & \longrightarrow & z^n(P_0, 1) & \longrightarrow & z^n(P_0, 0)
\end{array}$$

Remark 3.1. *It should be warned that basic properties of* $\mathbb{Z}(n)^{\mathrm{sm}}$ *(projective bundle formula, Zariski descent, etc) are not genetic to* $L^{sm}\mathbb{Z}(n)^{\mathrm{sm}}$*. For example,* $L^{sm}\mathbb{Z}(1)^{\mathrm{sm}}$ *is not in* $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sch}_k, \mathcal{D}(\mathbb{Z}))$*: Recall an equivalence in* $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sm}_k, \mathcal{D}(\mathbb{Z})$ *(cf.* (8)*)*

$$\mathbb{Z}(1)^{\mathrm{sm}} \simeq R\Gamma_{\mathrm{Zar}}(-, \mathcal{O}^\times)[-1].$$

Using the Gersten resolution of $\mathcal{O}^\times$, we see

$$R\Gamma_{\mathrm{Zar}}(-,\mathcal{O}^\times)[-1] \simeq (\tau_{\leq 1} R\Gamma_{\mathrm{Zar}}(-,\mathcal{O}^\times))[-1] \ \ \textit{on } \mathrm{Sm}_k.$$

It is known that $\mathcal{O}^\times$ and $\mathrm{Pic} = H^1_{\mathrm{Zar}}(-,\mathcal{O}^\times)$ as functors on the category CAlg_k of k-algebras are left Kan extended from the full subcategory CAlg_k^{sm} of smooth k-algebras[6]. This implies that for a k-algebras A, there is a natural equivalence

$$L^{sm}\mathbb{Z}(1)^{\mathrm{sm}}(A) \simeq (\tau_{\leq 1} R\Gamma_{\mathrm{Zar}}(\mathrm{Spec}(A),\mathcal{O}^\times))[-1].$$

On the other hand, $\tau_{\leq 1} R\Gamma_{\mathrm{Zar}}(-,\mathcal{O}^\times)$ is not in $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Sch}_k,\mathcal{D}(\mathbb{Z}))$ with its sheafification given by $R\Gamma_{\mathrm{Zar}}(-,\mathcal{O}^\times)$.

Exercise 3.2. Give a k-algebra A such that $H^2_{\mathrm{Zar}}(\mathrm{Spec}(A),\mathcal{O}^\times) \neq 0$.

Lemma 3.3. *Let $\mathsf{CAlg}_k^{sm} \subset \mathsf{CAlg}_k^{sm}$ be as in Remark 3.1. Let $F : \mathsf{CAlg}_k \to \mathcal{S}$ be a functor satisfying the conditions:*

(1) F preserves filtered colimits.
(2) For every henselian surjection $A \to B$, the map $\pi_0(F(A)) \to \pi_0(F(B))$ is surjective.
(3) For every henselian surjections $A \to C \leftarrow B$, the diagram

$$\begin{array}{ccc} F(A \times_C B) & \longrightarrow & F(B) \\ \downarrow & & \downarrow \\ F(A) & \longrightarrow & F(C) \end{array}$$

is a pullback square in $\mathcal{S}$.

Then, F is left Kan extended from CAlg_k^{sm}.

Despite of Remark 3.1, we have the following.

Theorem 3.4. *There exists a tower in $\mathrm{PSh}(\mathrm{Sch}_k,\mathrm{Sp})$*

$$\cdots \to F^{n+1}_{\mathrm{lke}} K_{\geq 0} \to F^{n}_{\mathrm{lke}} K_{\geq 0} \to \cdots \to F^{0}_{\mathrm{lke}} K_{\geq 0} = K_{\geq 0},$$

and equivalences

$$\mathrm{gr}^n_{F_{\mathrm{lke}}} K_{\geq 0} \simeq L^{sm}\mathbb{Z}(n)^{\mathrm{sm}}[2n].$$

Proof. By Bhatt-Lurier (see [EHKSY22, Ex. A.0.6]), there is a natural equivalence

$$K_{\geq 0} \simeq L^{sm} K_{|\mathrm{Sm}_k} \ \text{ in } \mathrm{PSh}(\mathrm{Sch}_k,\mathrm{Sp}),$$

where the right hand side is the left Kan extension of $K_{|\mathrm{Sm}_k}$ along $\mathrm{Sm}_k \to \mathrm{Sch}_k$. Using Theorem 2.1, the desired filtration on K is obtained by left-Kan extending $F^\bullet_{\mathrm{mot}} K_{|\mathrm{Sm}_k}$ along $\mathrm{Sm}_k \to \mathrm{Sch}_k$. □

[6] This follows from Lemma 3.3 below.

3.2 Cdh topology

As is remarked in Remark 3.1, $L^{sm}\mathbb{Z}(n)^{\mathrm{sm}}$ does not carry over basic properties of $\mathbb{Z}(n)^{\mathrm{sm}}$. Voevodsky's idea is to *cdh-sheafified* $L^{sm}\mathbb{Z}(n)^{\mathrm{sm}}$ with respect to the cdh topology to retrieve those properties.

A *distinguished Nisnevich square* is a cartesian square of schemes:

$$\begin{array}{ccc} W & \xrightarrow{j} & V \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{i} & X \end{array} \tag{21}$$

where i is a quasi-compact open immersion and f is étale inducing an isomorphism over $X \setminus U$. An *abstract blowup square* is a cartesian square of schemes:

$$\begin{array}{ccc} E & \xrightarrow{j} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array} \tag{22}$$

where i is a closed immersion locally of finite presentation and f is proper inducing an isomorphism over $X \setminus Z$.

Definition 3.5. *Let S be a qcqs scheme and Sch_S be the category of schemes of finite presentation over S. The Nisnevich topology on Sch_S is the Grothendieck topology generated by coverings families of the form*

$$\{U \to X\} \sqcup \{V \to X\},$$

where X, U, V are from (21)*. The cdh topology on Sch_S is the Grothendieck topology generated by Nisnevich topology and coverings families of the form*

$$\{Z \to X\} \sqcup \{Y \to X\},$$

where X, Z, Y are from (22)*.*

For $\mathcal{C} = \mathrm{Set}, \mathcal{S}, \mathrm{Sp}, \mathcal{D}(\mathbb{Z})$, , let $\mathsf{Shv}_{cdh}(\mathsf{Sch}_S, \mathcal{C}) \subset \mathrm{PSh}(\mathsf{Sch}_S, \mathcal{C})$ denote the full subcategory of sheaves for the cdh topology with the sheafication functor

$$a_{cdh} : \mathrm{PSh}(\mathsf{Sch}_S, \mathcal{C}) \to \mathsf{Shv}_{cdh}(\mathsf{Sch}_S, \mathcal{C}).$$

Exercise 3.6. Using Proposition 3.14 below, show $R\Gamma_{\text{ét}}(-, \mathbb{Z}/n) \in \mathsf{Shv}_{cdh}(\mathsf{Sch}_S, \mathcal{D}(\mathbb{Z}))$ for any integer $n > 0$.

Remark 3.7. If one replaces Sch_S with the category $\mathrm{Sch}^{\mathrm{qcqs}}$ of qcqs schemes, then one gets the coarsest topology such that for all $X \in \mathrm{Sch}^{\mathrm{qcqs}}$ the functor $\mathrm{Sch}_X \to \mathrm{Sch}^{\mathrm{qcqs}}$ is a continuous morphism of sites, [Sta18, 00WV], [SGA41, Def.III.1.1]. We also have the sheafification functor (see [KS23, Rem.8.1])

$$a_{cdh} : \mathrm{PSh}(\mathrm{Sch}^{\mathrm{qcqs}}, \mathcal{C}) \to \mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}^{\mathrm{qcqs}}, \mathcal{C}).$$

Remark 3.8. *(1) In* (22)*, we are allowed to take $Z = X_{\mathrm{red}}$ the reduced part of X and $Y, E = \emptyset$. Hence, we have an equivalence $F(X) \simeq F(X_{\mathrm{red}})$ for any $F \in \mathsf{Shv}_{cdh}(\mathrm{Sch}, \mathcal{C})$.*

(2) If one works with the cdh topology on the category Sch_k of schemes of finite type over a field of characteristic zero[7]*, one can show that any $X \in \mathrm{Sch}_k$ is* cdh-locally smooth*, i.e. there exist a cdh-covering $Y \to X$ with $Y \in \mathrm{Sm}_k$. Thus, some property on the cdh cohomology of X may be deduced from the smooth case.*

An important result on the cdh tooplogy is the following theorem [KST18, Th.6.3], which characterizes Weibel's homotopy invariant K-theory KH as the cdh sheafiation of K. Recall that $\mathrm{KH} \in \mathrm{PSh}(\mathrm{Sch}, \mathrm{Sp})$ is obtained by "forcing the $\mathbb{A}^1$-invariance to the algebraic K-theory". More precisely, we have the following.

Definition 3.9. *For a scheme $X \in \mathrm{Sch}$, we define $\mathrm{KH}(X) \in \mathrm{Sp}$ as the geometric realization of the simplicial spectrum $K(X \times \Delta^\bullet)$, where $\Delta^\bullet$ is the cosimplicial scheme from* (53)*. By the definition,*

$$\mathrm{KH}(X) = \mathrm{colim}_{[q] \in \Delta^{op}} K(X \times \Delta^q).$$

Theorem 3.10. *([KST18, Th.6.3] and [KM21]) There exists an equivalence*

$$a_{cdh} K \simeq \mathrm{KH}$$

in $\mathrm{PSh}(\mathsf{Sch}^{\mathrm{qcqs}}, \mathrm{Sp})$ (see Remark 3.7 for a_{cdh}).

3.3 Cdh-local motivic complex

Now we fix a field k and let $\mathsf{Sch}_k^{\mathrm{qcqs}}$ denote the category of qcqs schemes over k.

Definition 3.11. *For integers $n \geq 0$, we define*

$$\mathbb{Z}(n)^{\mathrm{cdh}} = a_{cdh} L^{sm} \mathbb{Z}(n)^{\mathrm{sm}} \in \mathsf{Shv}_{cdh}(\mathsf{Sch}_k^{\mathrm{qcqs}}, \mathcal{D}(\mathbb{Z})).$$

[7] or assuming resolution of singularities over the field k

Remark 3.12. *By [FV00, Pr.5.9], [K17, Pr.5.2.5] and [BEM23], $\mathbb{Z}(n)^{\mathrm{cdh}}$ is $\mathbb{A}^1$-invariant, i.e.*

$$\Gamma(X, \mathbb{Z}(n)^{\mathrm{cdh}}) \simeq \Gamma(X \times_k \mathbb{A}^1_k, \mathbb{Z}(n)^{\mathrm{cdh}}) \quad \text{for } X \in \mathrm{Sch}_k.$$

In view of Remark 3.8, it is also nilinvariant, i.e. $\mathbb{Z}(n)^{\mathrm{cdh}}(X) \simeq \mathbb{Z}(n)^{\mathrm{cdh}}(X_{\mathrm{red}})$. *On the other hand, K-theory does not satisfy these properties. Hence, $\mathbb{Z}(n)^{\mathrm{cdh}}$ cannot be a hoped-for motivic complex for $X \in \mathrm{Sch}_k$ providing the spectral sequence* (5). *Instead, it gives such a spectral sequence after replacing K by* KH.

Theorem 3.13. *There exists a tower in* $\mathrm{PSh}(\mathsf{Sch}_k^{\mathrm{qcqs}}, \mathrm{Sp})$

$$\cdots \to F^{n+1}_{\mathrm{mot}}\mathrm{KH} \to F^{n}_{\mathrm{mot}}\mathrm{KH} \to \cdots \to F^{0}_{\mathrm{mot}}\mathrm{KH} = \mathrm{KH},$$

and equivalences

$$\mathrm{gr}^n_{F_{\mathrm{mot}}}\mathrm{KH} \simeq \mathbb{Z}(n)^{\mathrm{cdh}}[2n].$$

Proof. Using Theorem 3.10, the filtration $F^{\bullet}_{\mathrm{cdh}}\mathrm{KH}$ is obtained by the cdh-sheafifying the left Kan extension along $\mathrm{Sm}_k \to \mathsf{Sch}_k^{\mathrm{qcqs}}$ of $F^{\bullet}_{\mathrm{mot}}K_{|\mathrm{Sm}_k}$ from Theorem 2.1. □

3.4 Recollection of basic facts on the cdh topology

In what follows, we list some basic facts on $\mathsf{Shv}_{cdh}(\mathsf{Sch}_S, \mathcal{C})$.

Proposition 3.14. *$F \in \mathrm{PSh}(\mathsf{Sch}_S, \mathcal{D}(\mathbb{Z}))$ is a cdh sheaf if and only if $F(\emptyset) = 0$ and for any distinguished Nisnevich square* (21) *and abstract blowup square* (22), *the squares*

$$\begin{array}{ccc} F(X) & \xrightarrow{i^*} & F(U) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle g^*} \\ F(V) & \xrightarrow{j^*} & F(W) \end{array} \qquad \begin{array}{ccc} F(X) & \xrightarrow{i^*} & F(Z) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle g^*} \\ F(Y) & \xrightarrow{j^*} & F(E) \end{array}$$

are cartesian in $\mathcal{D}(\mathbb{Z})$, or equivalently that the following sequences are exact:

$$\cdots \to H^i(F(X)) \to H^i(F(U)) \oplus H^i(F(V)) \to H^i(F(W)) \to H^{i+1}(F(X)) \to \cdots$$

$$\cdots \to H^i(F(X)) \to H^i(F(Y)) \oplus H^i(F(Z)) \to H^i(F(E)) \to H^{i+1}(F(X)) \to \cdots$$

The same statement holds replacing $\mathcal{D}(\mathbb{Z})$ by Sp *and cohomology H^i by homotopy π_i respectively*

Proof. This follows from Corollary 6.11. □

Theorem 3.15 ([GK15],[GL01]). *(1) A map $\phi : F \to G$ in $\mathsf{Shv}_{cdh}(\mathrm{Sch}_S, \mathrm{Set})$ is an isomorphism if and only if so is $F(R) \to G(R)$ for every henselian valuation ring R over S, where by definition, $F(R) = \varinjlim_\lambda F(R_\lambda)$ where the colimit is over factorizations $\mathrm{Spec}(R) \to \mathrm{Spec}(R_\lambda) \to S$ of $\mathrm{Spec}(R) \to S$ with $\mathrm{Spec}(R_\lambda) \in \mathsf{Sch}_S$.*

(2) A collection of maps $\{Y_i \to X\}_{i\in I}$ in Sch_S is a covering for the cdh topology if and only if

$$\coprod_{i\in I} \mathrm{Hom}_{\mathsf{Sch}_S^{\mathrm{qcqs}}}(\mathrm{Spec}(R), Y_i) \to \mathrm{Hom}_{\mathsf{Sch}_S^{\mathrm{qcqs}}}(\mathrm{Spec}(R), X)$$

is surjective for every henselian valuation ring R over S, where $\mathsf{Sch}_S^{\mathrm{qcqs}}$ is the category of qcqs schemes over S.

Theorem 3.16. *[[EHIK21]] Let S be a qcqs scheme of finite valuative dimension $\dim_v(X)$[8]. Then, $\mathrm{Shv}_{\mathrm{cdh}}(\mathrm{Sch}_S, \mathcal{S})$ has homotopy dimension $\leq d$.*

Definition 3.17. *The* valuative dimension *of a scheme X is the supremum of the ranks of all valuation rings of residue fields of X centred on X:*

$$\dim_v(X) = \sup \left\{ \dim R \;\middle|\; \begin{array}{c} \exists\, x \in X; R \text{ is a valuation ring of } k(x) \text{ such that} \\ \mathrm{Spec}(k(x)) \to X \text{ factors through } \mathrm{Spec}(R) \end{array} \right\}.$$

Corollary 3.18. *[CM21, Cor.3.11, Thm.3.18] Let X be a qcqs scheme of finite valuative dimension $\dim_v(X)$ and F be a cdh sheaf of abelian groups on Sch_X. Then, we have*

$$H^i_{cdh}(X, F) = 0 \quad \text{for } i > \dim_v(X). \tag{23}$$

Proof. This follows from Theorem 3.16 and Lemma 6.22. □

Remark 3.19. (9) *implies that for any local k-algebra A, we have*

$$H^i((L^{sm}\mathbb{Z}(n)^{\mathrm{sm}})(A)) = 0 \quad \text{for } i > n. \tag{24}$$

In view of Corollary 3.18, this implies that for $X \in \mathrm{Sch}_k$

$$H^i(\mathbb{Z}(n)^{\mathrm{cdh}}(X)) = 0 \quad \text{for } i > \dim(X) + n. \tag{25}$$

In particular, $F^n_{\mathrm{cdh}}\mathrm{KH}(X)$ from Theorem 3.13 is supported in cohomological degrees $\leq \dim(X) - n$ for each $n \in \mathbb{N}$, so the induced spectral sequence (5) *is bounded.*

[8] We have $\dim(X) \leq \dim_v(X)$.

4 Pro-cdh-local motivic complex

The content of this section is a joint work [KS23] of Shane Kelly and the author. A main aim is to introduce a new Grothendieck topology on schemes called the *pro-cdh topology* and to define the *pro-cdh-local motivic complex* $\mathbb{Z}(n)^{\mathrm{procdh}}$ as the pro-cdh sheafication of the left Kan extension of $\mathbb{Z}(n)^{\mathrm{sm}}$ (see Definition 4.7 and Theorem 4.8). It is motivated by the following facts: Let

$$\begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array} \tag{26}$$

be an abstract blowup square in Sch from (22) and let $Z_r \hookrightarrow X$ (resp. $E_r \hookrightarrow Y$) be the r-the infinitesimal thickening of $Z \hookrightarrow X$ (resp. $E \hookrightarrow Y$) for integers $r > 0$. If X is noetherian, the square

$$\begin{array}{ccc} K(X) & \longrightarrow & \varprojlim_r K(Z_r) \\ \downarrow & & \downarrow \\ K(Y) & \longrightarrow & \varprojlim_r K(E_r) \end{array} \tag{27}$$

is cartesian in Sp. This is proved in [KST18]. If X is a notherian scheme over a noetherian ring k, for every integer $i \geq 0$, the square

$$\begin{array}{ccc} L\Omega^i_{X/k} & \longrightarrow & \varprojlim_r L\Omega^i_{Z_r/k} \\ \downarrow & & \downarrow \\ L\Omega^i_{Y/k} & \longrightarrow & \varprojlim_r L\Omega^i_{E_r/k} \end{array} \tag{28}$$

is cartesian in $\mathcal{D}(k)$, where $L\Omega^i_{-/k} = \wedge^i L_{-/k}$ for the cotangent complex $L_{-/k}$ [Il71]. This is proved in [Mor16, Th. 2.10] and deduced from Grothendieck's formal functions theorem on cohomology of coherent sheaves [9].

An idea to define the pro-cdh topology is to modify the cdh topology to make it sensitive to nil-immersions.

Definition 4.1. *Let S be a qcqs scheme and Sch_S be the category of schemes of finite presentation over S. The pro-cdh topology on Sch_S is the Grothendieck topology generated by Nisnevich topology and coverings families of the form*

$$\{Z_r \to X\}_{r\in\mathbb{N}} \sqcup \{Y \to X\},$$

[9][Mor16, Th. 2.10] requires the assumption of finite Krull dimension but it is noted in [EM23] that it can be removed by using the general formal function theorem of [Lur17b, Lem. 8.5.1.1].

for all squares (26)*, where* $Z_r = \underline{\mathrm{Spec}}(\mathcal{O}_X/\mathcal{I}_Z^r)$ *is the* r*-th thickening of* $Z_0 \hookrightarrow X$*.*

For $\mathcal{C} = \mathrm{Set}, \mathcal{S}, \mathrm{Sp}, \mathcal{D}(\mathbb{Z})$*, let* $\mathsf{Shv}_{procdh}(\mathsf{Sch}_S, \mathcal{C})$ *denote the full subcategory of* $\mathrm{PSh}(\mathrm{Sch}, \mathcal{C})$ *consisting of sheaves for the pro-cdh topology. Let*

$$a_{procdh} : \mathrm{PSh}(\mathsf{Sch}_S, \mathcal{C}) \to \mathsf{Shv}_{procdh}(\mathsf{Sch}_S, \mathcal{C})$$

be the sheafication functor (cf. Notations and Conventions). We also have the sheafification functor (see Remark 3.7)

$$a_{procdh} : \mathrm{PSh}(\mathrm{Sch}^{\mathrm{qcqs}}, \mathcal{C}) \to \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}^{\mathrm{qcqs}}, \mathcal{C}).$$

Example 4.2. Thanks to (27) and (28) and Proposition 4.3 below, we have the following facts.

(1) Algebraic K-theory belongs to $\mathsf{Shv}_{procdh}(\mathsf{Sch}_S, \mathrm{Sp})$ if S is noetherian.
(2) $L\Omega^i_{-/k}$ belongs to $\mathsf{Shv}_{procdh}(\mathsf{Sch}_S, \mathcal{D}(k))$ for $S = \mathrm{Spec}(k)$ with a noetherian ring k.

We give a list of basic properties of the procdh topology.

Proposition 4.3 ([KS23])**.** $F \in \mathrm{PSh}(\mathrm{Sch}, \mathcal{C})$ *is a pro-cdh sheaf if and only if it is a Nisnevich sheaf and for any abstract blowup square* (26)*, the following squares is cartesian in* $\mathcal{C}$*:*

$$\begin{array}{ccc} F(X) & \longrightarrow & \varprojlim_r F(Z_r) \\ \downarrow & & \downarrow \\ F(Y) & \longrightarrow & \varprojlim_r F(E_r). \end{array}$$

Proof. This follows from Corollary 6.11. □

Theorem 4.4.

(1) A map $\phi : F \to G$ *in* $\mathsf{Shv}_{procdh}(\mathrm{Sch}_S)$ *is an isomorphism if and only if so is* $F(R) \to G(R)$ *for every pro-cdh local ring* R *over* S *(cf.* (32)*).*
(2) A collection of maps $\{Y_i \to X\}_{i \in I}$ *in* Sch_S *is a covering for the pro-cdh topology if and only if*

$$\coprod_{i \in I} \mathrm{Hom}(\mathrm{Spec}(R), Y_i) \to \mathrm{Hom}(\mathrm{Spec}(R), X)$$

is surjective for every pro-cdh local ring R *over* S*.*

Theorem 4.5 ([KS23])**.** *Let* X *be a noetherian scheme and* F *be a pro-cdh sheaf of abelian groups on* Sch_X*. Then, we have a vanishing of the pro-cdh cohomology:*

$$H^i_{procdh}(X, F) = 0 \quad \text{for } i > 2\dim(X). \tag{29}$$

4.1 Applications to K-theory and motivic cohomology

In what follows, we give an application of the pro-cdh topology to the algebraic K-theory, which can be viewed as an analog of Theorem 3.10. Recall that the algebraic K-theory gives an object $K \in \mathsf{Shv}_{procdh}(\mathrm{Sch}^{noe}, \mathrm{Sp})$ (cf. Example 4.2). Let $K_{\geq 0} \in \mathrm{PSh}(\mathrm{Sch}, \mathrm{Sp})$ be the connective cover of K (cf. §6.3).

Theorem 4.6 ([KS23])**.** *For $X \in \mathrm{Sch}^{noe}$ with $\dim(X) < \infty$, there exists a natural equivalence*

$$(a_{procdh}K_{\geq 0})(X) \simeq K(X).$$

Proof. For X as above, we have the descent spectral sequences

$$E_2^{p,q} = H^p_{pcdh}(X, \widetilde{K}_{-q}) \Rightarrow K_{-p-q}(X),$$

$$E_2^{p,q} = H^p_{pcdh}(X, \tau_{\geq 0}\widetilde{K}_{-q}) \Rightarrow \pi_{-p-q}(a_{procdh}K_{\geq 0}(X)),$$

where $\widetilde{K}_i = a_{procdh}K_i$ is the pro-cdh sheafication of the presheaf $K_i = \pi_i K$ of abelian groups, and $\tau_{\geq 0}\widetilde{K}_i = \widetilde{K}_i$ for $i \geq 0$ and $\tau_{\geq 0}\widetilde{K}_i = 0$ for $i < 0$. By Theorem 4.5, the spectral sequences are bounded so strongly convergent. So, it suffices to show that $\widetilde{K}_i = 0$ for $i < 0$. By Theorem 4.4(1), this is reduced to showing $K_i(R) = 0$ for $i < 0$ and for a pro-cdh local ring R [10]. Then, for $i < 0$, we get $K_i(R) = K_i(R/\mathfrak{N}) = 0$, where the first equality follow from the nil-invariance of the negative K-theory and the last equality follows from [KM21, Th.1.3] since $R/\mathfrak{N}$ is a valuation ring for $R = V \times_K Q$ as in Theorem 4.14. This completes the proof. □

In the rest of this section, we let $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$ and let $\mathrm{Sch}^{noe}_{\mathbb{F}}$ be the category of noetherian schemes over $\mathbb{F}$.

Definition 4.7. *For integers $n \geq 0$, we define the* pro-cdh-local motivic complex

$$\mathbb{Z}(n)^{\mathrm{procdh}} := a_{procdh}L^{sm}\mathbb{Z}(n)^{\mathrm{sm}} \in \mathsf{Shv}_{procdh}(\mathrm{Sch}^{noe}_{\mathbb{F}}, \mathcal{D}(\mathbb{Z})),$$

as the pro-cdh-sheafication of the left Kan extension of $\mathbb{Z}(n)^{\mathrm{sm}}$ along $\mathrm{Sm}_{\mathbb{F}} \to \mathrm{Sch}^{noe}_{\mathbb{F}}$.

Theorem 4.8 ([KS23])**.** *There exists a tower in $\mathrm{PSh}(\mathrm{Sch}^{noe}_{\mathbb{F}}, \mathrm{Sp})$*

$$\cdots \to F^{n+1}_{\mathrm{procdh}}K \to F^n_{\mathrm{procdh}}K \to \cdots \to F^0_{\mathrm{procdh}}K = K,$$

called the procdh-local motivic filtration on K *and equivalences*

$$\mathrm{gr}^n_{F_{\mathrm{procdh}}}K \simeq \mathbb{Z}(n)^{\mathrm{procdh}}[2n].$$

[10] Here, we used the fact that K is finitary.

So, we get an Atiyah-Hirzebruch spectral sequence:

$$E_2^{p,q} = H_{\mathcal{M}}^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X) \quad \textit{for } X \in \mathrm{Sch}_{\mathbb{F}}^{noe}$$

defining $H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) := H^i(\mathbb{Z}(n)^{\mathrm{procdh}}(X))$*.*

Proof. By Bhatt-Lurier (see [EHKSY22, Ex. A.0.6]), there is a natural equivalence

$$K_{\geq 0} \simeq L^{sm} K_{|\mathrm{Sm}_{\mathbb{F}}},$$

where the right hand side is the left Kan extension of $K_{|\mathrm{Sm}_{\mathbb{F}}}$ along $\mathrm{Sm}_{\mathbb{F}} \to \mathrm{Sch}_{\mathbb{F}}^{noe}$. Using Theorem 4.6, the desired filtration on K is obtained by the pro-cdh-sheafifying the left Kan extension along $\mathrm{Sm}_{\mathbb{F}} \to \mathrm{Sch}_{\mathbb{F}}$ of $F^{\bullet}_{\mathrm{mot}} K_{|\mathrm{Sm}_{\mathbb{F}}}$ from Theorem 2.1. □

Remark 4.9. *In view of* (24)*, Theorem 4.5 implies that for* $X \in \mathrm{Sch}_{\mathbb{F}}^{noe}$ *with* $\dim(X) < \infty$*, we have*

$$H^i(\mathbb{Z}(n)^{\mathrm{procdh}}(X)) = 0 \quad \textit{for } i > 2\dim(X) + n. \tag{30}$$

In particular, $F^n_{\mathrm{procdh}} K(X)$ *is supported in cohomological degrees* $\leq 2\dim(X) - n$ *for each* $n \in \mathbb{N}$*, so the induced spectral sequence* (5) *is bounded.*

4.2 Fiber functors of the pro-cdh topos

In this subsection, we prove Theorem 4.4 except the part of enough points. Let $\mathrm{Shv}_\tau(C)$ be the category of τ-sheaves of sets on a site (C, τ).

Definition 4.10. *A* fibre functor *of* $\mathrm{Shv}_\tau(C)$ *is a continuous morphism of topoi* $\phi^* : \mathrm{Shv}_\tau(C) \rightleftarrows \mathrm{Set} : \phi_*$*, or equivalently, a functor* $\phi^* : \mathrm{Shv}_\tau(C) \to \mathrm{Set}$ *which preserves colimits and finite limits.*

Definition 4.11. *([SGA41, Thm.III.4.1], [SGA41, I.8.10.14]) Let* τ *be a topology on* Sch_S *such that every scheme is covered by affine ones. An affine* S*-scheme* $\mathrm{Spec}(R) \to S$ *is said to be* τ-local *if for every* τ*-covering* $\{Y_i \to X\}_{i \in I}$*, the map*

$$\coprod_{i \in I} \mathrm{Hom}(\mathrm{Spec}(R), Y_i) \to \mathrm{Hom}(\mathrm{Spec}(R), X) \tag{31}$$

is surjective.

For a topology τ on Sch_S as in Definition 4.11, there is a bijection between fibre functors of $\mathrm{Shv}_\tau(\mathrm{Sch}_S)$ and affine S-schemes $\mathrm{Spec}(R) \to S$ which are τ-local. To get the fibre functor associated to a τ-local ring R, one first replaces R with the pro-object $\underset{\mathrm{Spec}(R)\to\mathrm{Spec}(R_\lambda)\to S}{\text{“}\lim\text{”}}\mathrm{Spec}(R_\lambda)$ in Sch_S, where the limit is over factorisations with $\mathrm{Spec}(R_\lambda)\in\mathrm{Sch}_S$. Then the fibre functor is given by

$$\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S)\to\mathrm{Set}\ ;\ F\mapsto F(R):=\underset{\mathrm{Spec}(R)\to\mathrm{Spec}(R_\lambda)\to S}{\mathrm{colim}}F(R_\lambda). \tag{32}$$

Example 4.12. (1) The Nisnevich local S-schemes are those $\mathrm{Spec}(R)\to S$ such that R is a henselian local ring.
(2) The cdh local S-schemes are those $\mathrm{Spec}(R)\to S$ such that R is a henselian valuation ring, [GK15], [GL01].

Definition 4.13. *([SGA41, Exposé IV, Déf.6.4.1], [SGA41, Exposé IV, Prop.6.5(a)]) A topos* $\mathrm{Shv}_\tau(C)$ *has* enough points *when a morphism f in* $\mathrm{Shv}_\tau(C)$ *is an isomorphism if and only if $\phi(f)$ is an isomorphism for all fibre functors ϕ, Def.4.10. This is equivalent to that a family $\{Y_i\to X\}_{i\in I}$ in C is a covering family if and only if $\sqcup_{i\in I}\phi(Y_i)\to\phi(X)$ is surjective for all fibre functors ϕ.*[11]

By Deligne's completeness theorem, if C is an essentially small category with fibre products, and every τ-covering is refinable by a finite one, then $\mathrm{Shv}_\tau(C)$ has enough points, [SGA42, Prop.VI.9.0] or [Joh77, Thm.7.44, 7.17]. Since the pro-cdh toplogy is not finitary, the last statement of Theorem 4.14 does not follows from the Deligne theorem.

Now, Theorem 4.4 is a consequence of the following.

Theorem 4.14 ([KS23])**.** *The pro-cdh local S-rings R are those* $\mathrm{Spec}(R)\to S$ *such that R is a henselian local ring and $R=\mathcal{O}\times_K A$, where A is a local ring of Krull dimension* 0*, K is the residue field of A and $\mathcal{O}$ is a henselian valuation ring of K*[12]*. If the underlying topological space of S is noetherian of finite Krull dimension,* $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S)$ *has enough points.*

In what follows, we prove the first statement of Theorem 4.14. The second statement will be proved in §4.4.

First, we show that procdh local rings R are of the form described in Theorem 4.14. The following composition of a Zariski covering and the procdh covering

$$\left\{\underline{\mathrm{Spec}}\,\frac{\mathcal{O}_S[x,y]}{\langle x^n,y^n\rangle}\to\mathbb{A}^2_S\right\}_{n\in\mathbb{N}}\sqcup\left\{\underline{\mathrm{Spec}}\,\mathcal{O}_S[x,\tfrac{y}{x}]\sqcup\underline{\mathrm{Spec}}\,\mathcal{O}_S[\tfrac{x}{y},y]\to\mathbb{A}^2_S\right\} \tag{33}$$

shows that procdh local rings satisfy:

[11]Here, we have used the same symbol for an object X of C and the sheafification of the presheaf $\hom(-,X)$ it represents.

[12]For example, $R=\mathcal{O}$ and $R=A$ are both pro-cdh local rings.

(∗) $\forall a, b \in R$; we have $a|b$ or $b|a$ or a and b are both nilpotent.

This implies that R_{red} is a valuation ring, and in particular, R has a unique minimal prime ideal $\mathfrak{n}$, which equals the set of nilpotents. Moreover, all zero divisors of R are nilpotent by virtue of the procdh covering

$$\left\{\underline{\mathrm{Spec}}\,\frac{\mathcal{O}_S[x,y]}{\langle x^n, xy\rangle} \to \underline{\mathrm{Spec}}\,\frac{\mathcal{O}_S[x,y]}{\langle xy\rangle}\right\}_{n\in\mathbb{N}} \sqcup \left\{\underline{\mathrm{Spec}}\,\frac{\mathcal{O}_S[x,y]}{\langle y\rangle} \to \underline{\mathrm{Spec}}\,\frac{\mathcal{O}_S[x,y]}{\langle xy\rangle}\right\} \tag{34}$$

Hence, the map $R \to R_{\mathfrak{n}}$, and therefore $R \to (R/\mathfrak{n}) \times_{k(\mathfrak{n})} R_{\mathfrak{n}}$ is injective. We claim that the latter is also surjective. Consider a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{n} & \longrightarrow & R & \longrightarrow & R/\mathfrak{n} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{n}R_{\mathfrak{n}} & \longrightarrow & R_{\mathfrak{n}} & \longrightarrow & R_{\mathfrak{n}}/\mathfrak{n}R_{\mathfrak{n}} & \longrightarrow & 0 \end{array} \tag{35}$$

By a diagram chase, $\mathfrak{n} \to \mathfrak{n}R_{\mathfrak{n}}$ being surjective implies $R \to (R/\mathfrak{n}) \times_{k(\mathfrak{n})} R_{\mathfrak{n}}$ being surjective. For $a \in \mathfrak{n}$ and $s \in R \setminus \mathfrak{n}$, (∗) implies that there is $b \in \mathfrak{n}$ such that $b/1 = a/s$, which proves the desired surjectivity.

So we have shown $R \to (R/\mathfrak{n}) \times_{k(\mathfrak{n})} R_{\mathfrak{n}}$ is both injective and surjective. The Krull dimension of $R_{\mathfrak{n}}$ is zero because $\mathfrak{n}$ is a minimal prime, and we have already observed that $R/\mathfrak{n} = R_{\mathrm{red}}$ is a valuation ring, so it suffices to show that $R/\mathfrak{n}$ is henselian. But procdh local rings are Nisnevich local rings, also known as henselian local rings, and quotients of henselian local rings are henselian local rings.

Next, we show that for $R = \mathcal{O} \times_K A$ as in Theorem 4.14, R is a pro-cdh local. We want to show that (31) is an epimorphism for all procdh coverings. It suffices to consider the generator coverings described in the definition. Noting that R is henselian local, the desired lifting condition with respect to Nisnevich coverings follows from [Sta18, 04GG, Item(7)].

Suppose we have a proabstract blowup square $\{Z_n \to X\}_{n\in\mathbb{N}} \sqcup \{Y \to X\}$ and a morphism $f : \mathrm{Spec}(R) \to X$. If the image of the induced map $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}) \to X$ does not lie in Z_0, then it lifts through Y because $Y \to X$ is an isomorphism over $X \setminus Z_0$. By the valuative criterion for properness, the lifting extends to $g : \mathrm{Spec}(\mathcal{O}) \to Y$. On the other hand, the morphism $\mathrm{Spec}(A) \to X$ induced by f factors through the open $X \setminus Z_0 \subset X$ since A is local and the composite $\mathrm{Spec}(K) \to \mathrm{Spec}(A) \to X$ factors through $X \setminus Z_0$ by the assumption. So, it lifts to a morphism $h : \mathrm{Spec}(A) \to Y$. These morphisms g and h factor through some open affine of Y, so they glue to give a lifting $\mathrm{Spec}(R) \to Y \to X$ since $\mathrm{Spec}(\mathcal{O} \times_K A) = \mathrm{Spec}(\mathcal{O}) \sqcup_{\mathrm{Spec}(K)} \mathrm{Spec}(A)$ is the categorical pushout in the category of affine schemes.

On the other hand, if $\mathrm{Spec}(K) \to X$ factors through Z_0, then $\mathrm{Spec}(\mathcal{O}) \to X$ also factors through Z_0. The morphism $\mathrm{Spec}(A) \to X$ doesn't necessarily factor through

Z_0 but $\mathcal{I}_{Z_0}$ is sent into the maximal ideal of A, which consists of nilpotent elements of A. Since $\mathcal{I}_{Z_0}$ is finitely generated, this implies that $\operatorname{Spec}(A) \to X$ factors through Z_n for some $n > 0$. Then, we glue these morphisms as in the previous case to get a morphism $\operatorname{Spec}(R) \to Z_n$, which factors f.

4.3 Homotopy dimension of the pro-cdh topos

Theorem 4.15 ([KS23])**.** *Let S be a qcqs scheme of finite valuative dimension $d \geq 0$ with Noetherian underlying topological space. Then,* $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S, \mathcal{S})$ *has homotopy dimension $\leq 2d$ (see Definition 3.17 for valuative dimension).*

Remark 4.16. *There exists a Noetherian scheme of dimension one with procdh homotopy dimension two.*

Corollary 4.17 ([KS23])**.** *Let X be a noetherian scheme and F be a pro-cdh sheaf of abelian groups on* Sch_X*. Then, we have a vanishing of the pro-cdh cohomology:*

$$H^i_{procdh}(X, F) = 0 \quad \textit{for } i > 2\dim(X). \tag{36}$$

Proof. This follows from Theorem 4.15 and Lemma 6.22. □

In what follows, we give a proof of Theorem 4.15 in case S is noetherian.

Definition 4.18. *Let X be a qcqs scheme. By a* modification *of X, we mean a morphism of schemes $Y \to X$ which is proper, of finite presentation, and an isomorphism over a dense qc open $D \subset X$. We let* $\mathrm{Mod}_X \subset \mathrm{Sch}_X$ *denote the full subcategory of modifications of X. We call a morphism in* Mod_X *a modification.*

Remark 4.19. (1) If $Y' \to Y$, $Y'' \to Y$ are morphisms in Mod_X then $Y' \times_Y Y''$ is again in Mod_X. In particular, Mod_X admits finite limits, calculated in Sch_X, and is therefore is filtered.

(2) We do not ask modifications to be birational, i.e. $Y^{gen} = X^{gen}$ so that finite limits in Mod_X are more nicely behaved. But, we can refine any object in $Y \in \mathrm{Mod}_X$ by Y' which is birational to X. To see this in case X is noetherian, let $Y \to X$ be a modification which is an isomorphism over a dense open $D \subseteq X$. Then, letting Y' be the closure of the image of $D \to Y$, $Y' \to X$ is as is wanted.

Definition 4.20. *Let S be a qcqs scheme. For $X \in \mathrm{Sch}_S$ we define*

$$\mathrm{RZ}(X_{\mathrm{Nis}}) = \int_{Y \in \mathrm{Mod}_X} Y_{\mathrm{Nis}}.$$

Explicitly, $\mathrm{RZ}(X_{\mathrm{Nis}}) \subseteq \mathrm{Arr}(\mathrm{Sch}_X)$ *is the category whose objects are morphisms* $U \to Y$ *such that* $U \in Y_{\mathrm{Nis}}$ *and* $Y \in \mathrm{Mod}_X$*, and morphisms are commutative squares*

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \big\downarrow & & \big\downarrow \\ Y' & \longrightarrow & Y \end{array}$$

We abbreviate $U \to Y$ *to* (U/Y)*.*

Remark 4.21. As it is a category of arrows in a category admitting finite limits, $\mathrm{Arr}(\mathrm{Sch}_X)$ admits finite limits and they are calculated component wise: $\lim(A_i/B_i) = (\lim A_i/\lim B_i)$. If each (A_i/B_i) is in $\mathrm{RZ}(X_{\mathrm{Nis}})$, then one checks that $\lim(A_i/B_i)$ is again in $\mathrm{RZ}(X_{\mathrm{Nis}})$. Thus, $\mathrm{RZ}(X_{\mathrm{Nis}})$ admits finite limits, and they are calculated termwise.

Definition 4.22. *We equip* $\mathrm{RZ}(X_{\mathrm{Nis}})$ *with a Grothendieck topology generated by:*

1. *families of the form*
$$\{(U_i/Y) \to (U/Y)\}_{i\in I} \tag{Nis}$$
such that $\{U_i \to U\}$ *is a Nisnevich covering, and*
2. *families of the form*
$$\{(Y' \times_Y U/Y') \to (U/Y)\} \tag{Car}$$
for morphisms $Y' \to Y$ *in* Mod_X*.*

We will write $\mathrm{Shv}(\mathrm{RZ}(X_{\mathrm{Nis}}))$ *for the topos associated to the topology generated by coverings of the form* (Nis) *and* (Car)*.*

Remark 4.23. $F \in \mathrm{PSh}(\mathrm{RZ}(X_{\mathrm{Nis}}), \mathcal{S})$ satisfies descent for families of the form (Car) if and only if it sends each $(Y' \times_Y U/Y') \to (U/Y)$ to an equivalence Indeed, assume F satisfies descent for (Car). If $Y' \to Y$ in Mod_X is a closed immersion, then $(Y')^{\times_Y n} = Y'$ so we have for $U' = Y' \times_Y U$

$$F(U/Y) \simeq \lim_n F((U'/Y')^{\times_{(U/Y)}(n+1)}) \simeq \lim_n F(U'/Y') = F(U'/Y') \tag{37}$$

where the first equivalence holds by the assumption. For a general $Y' \to Y$ in Mod_X, (37) also holds since each diagonal $Y' \to (Y')^{\times_Y n}$ is a closed immersion in Mod_X so $F((U'/Y')^{\times_{(U/Y)} n}) \simeq F(U'/Y')$ by what we have seen. Conversely, if F sends families of the form (Car) to equivalences, then it clearly satisfies Čech descent for such families. Consequently, we have

$$\mathrm{Shv}(\mathrm{RZ}(X_{\mathrm{Nis}}), \mathcal{S}) = \lim_{Y \in \mathrm{Mod}_X} \mathrm{Shv}_{\mathrm{Nis}}(Y_{\mathrm{Nis}}, \mathcal{S}), \tag{38}$$

where the limit is along pushforwards $f_* : \mathrm{Shv}_{\mathrm{Nis}}(Y'_{\mathrm{Nis}}, \mathcal{S}) \to \mathrm{Shv}_{\mathrm{Nis}}(Y_{\mathrm{Nis}}, \mathcal{S})$ for morphisms $f : Y' \to Y$ in Mod_X.

Proposition 4.24. *Let X be a qcqs scheme and suppose $F \in \mathrm{PSh}(\mathrm{RZ}(X_{\mathrm{Nis}}), \mathcal{S})$ has descent for the coverings* (Nis). *Then the sheafification $aF \in \mathrm{Shv}(\mathrm{RZ}(X_{\mathrm{Nis}}), \mathcal{S})$ satisfies*

$$aF(U/Y) = \mathrm{colim}_{Y' \in (\mathrm{Mod}_X)_{/Y}} F(Y' \times_Y U/Y'). \tag{39}$$

Proof. First we show that the presheaf aF defined via (39) is a sheaf. By definition, a presheaf on $\mathrm{RZ}(X_{\mathrm{Nis}})$ is a sheaf if and only if it has descent for coverings of the form (Nis) and (Car) in Definition 4.22. The presheaf aF in the statement certainly sends modifications to equivalences, so it has descent for coverings of the form (Car) by remarks 4.23.

For Nisnevich coverings, we notice that a presheaf F has descent for coverings of the form (Nis) if and only if the restriction to the small Nisnevich site Y_{Nis} for each $Y \in \mathrm{Mod}_X$ has Nisnevich descent if and only if it sends distinguished Nisnevich squares to cartesian squares. Take $(U/Y) \in \mathrm{RZ}(X_{\mathrm{Nis}})$ and a distinguished Nisnevich square $\{U_0 \to U, U_1 \to U\}$ with $U_{01} = U_0 \times_U U_1$. Then, for any $Y' \to Y$ in Mod_X, we have

$$F(Y' \times_Y U/Y') = F(Y' \times_Y U_0/Y') \times_{F(Y' \times_Y U_{01}/Y')} F(Y' \times_Y U_1/Y')$$

by the assumption that F has descent for (Nis). Taking the colimit over Y' and using the fact that filtered colimits commute with fibre products we find

$$aF(U/Y) = aF(U_0/Y) \times_{aF(U_{01}/Y)} aF(U_1/Y),$$

which proves that aF is a sheaf.

To conclude that a is the sheafification functor, it suffices to show that if F is already a sheaf, then $F \to aF$ is an equivalence. But this is clear, since sheaves send modifications to isomorphisms, resp. equivalences, by Rem.4.23. □

Proposition 4.25. *If X is a qcqs scheme of finite valuative dimension $d \geq 0$ with Noetherian underlying topological space,* $\mathrm{RZ}(X_{\mathrm{Nis}}, \mathcal{S})$ *has homotopy dimension $\leq d$.*

Proof. This follows from [CM21, Cor.3.11, Thm.3.18] (see also Example 6.21) and Remark 4.19(2). □

Definition 4.26. *Let S be a qcqs scheme. For $X \in \mathrm{Sch}_S$, we consider the canonical projection functor*

$$\rho_X : \mathrm{RZ}(X_{\mathrm{Nis}}, \mathcal{S}) \to \mathrm{Sch}_S; \quad (U/Y) \mapsto U$$

and the functor induced by restriction

$$\mathrm{PSh}(\mathrm{Sch}_S, \mathcal{S}) \to \mathrm{PSh}(\mathrm{RZ}(X_{\mathrm{Nis}}), \mathcal{S}); \qquad F \mapsto F \circ \rho_X.$$

By composing this with the sheafification functor $\mathrm{PSh}(\mathrm{RZ}(X_{\mathrm{Nis}}),\mathcal{S}) \to \mathrm{Shv}(\mathrm{RZ}(X_{\mathrm{Nis}}),\mathcal{S})$, *we get*

$$\rho_X^* : \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S,\mathcal{S}) \to \mathrm{Shv}(\mathrm{RZ}(X_{\mathrm{Nis}}),\mathcal{S}). \tag{40}$$

Remark 4.27. Using Proposition 4.24 we have the following concrete description.

$$(\rho_X^* F)(U/Y) = \mathrm{colim}_{Y' \in \mathrm{Mod}_X} F(Y' \times_X U).$$

Recall that a morphism of sites $\phi : C \to D$ is *cocontinuous* if for every $U \in C$ and covering family $\mathcal{U} = \{U_i \to \phi U\}_{i \in I}$ there is a covering family $\{V_i \to U_i\}$ such that $\{\phi V_i \to \phi U\}_{i\in I}$ refines $\mathcal{U}$, [SGA41, Def.III.2.1], [Sta18, 00XJ].

Proposition 4.28. *Let X be a qcqs scheme of finite valuative dimension with Noetherian underlying topological space. Then, ρ_X is cocontinuous.*

Proof of Theorem 4.15 in case S is noetherian: The proof is by induction on the Krull dimension d of S. Suppose $F \in \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S,\mathcal{S})$ has $F_{\leq 2d-1} = *$. We want to show that $F(S)$ is non-empty. By Proposition 4.28 and Example 6.17,

$$\rho^* = \rho_S^* : \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S,\mathcal{S}) \to \mathrm{Shv}(\mathrm{RZ}(S_{\mathrm{Nis}}),\mathcal{S})$$

is a left adjoint of a morphism of ∞-topoi and preserves n-connective objects so that $(\rho^* F)_{\leq 2d-1} \cong *$. Since the homotopy dimension of $\mathrm{Shv}(\mathrm{RZ}(S_{\mathrm{Nis}}),\mathcal{S})$ is $\leq d$ by Proposition 4.25, the space $(\rho^* F)(S)$ is non-empty. Since

$$(\rho^* F)(S) = \mathrm{colim}_{Y \in \mathrm{Mod}_S} F(Y)$$

by Remark 4.27, we can find a modification $Y \to S$ such that $F(Y)$ is non-empty. Up to refining Y, we can assume that $Y^{\mathrm{gen}} = S^{\mathrm{gen}}$ by Remark 4.19(2).

If $d = 0$, we have $Y = S$ and we are done with this step. If $d > 0$, there exists a nowhere dense non-empty closed subscheme of finite presentation $Z_0 \subseteq S$ such that $Y \to S$ is an isomorphism over $S \setminus Z_0$ and $E_0 := Z_0 \times_S Y$ is also a nowhere dense closed subscheme of finite presentation in Y. Note we now have $0 \leq \dim Z_0 \leq d-1$ and similar for E_0. We continue to have $(F|_{\mathrm{Sch}_{Z_n}})_{\leq 2d-1} \cong *$ and $(F|_{\mathrm{Sch}_{E_n}})_{\leq 2d-1} \cong *$ by Exam.6.17. By the induction hypothesis, $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_{Z_n},\mathcal{S})$ and $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_{E_n},\mathcal{S})$ have homotopy dimension $\leq 2d-2$, so by Rem.6.19,

$$F(Z_n)_{\leq 1} \simeq F(E_n)_{\leq 1} \cong * \quad \text{for all } n.$$

Since $\mathrm{PSh}(\mathbb{N},\mathcal{S})$ has homotopy dimension ≤ 1 by Exam.6.20, we get

$$(\lim_{n\in\mathbb{N}} F(Z_n))_{\leq 0} \cong (\lim_{n\in\mathbb{N}} F(E_n))_{\leq 0} \cong *,$$

which means that these spaces are non-empty and connected. By Prop.4.3, we have a cartesian square

$$\begin{array}{ccc} F(S) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ \lim_{n\in\mathbb{N}} F(Z_n) & \longrightarrow & \lim_{n\in\mathbb{N}} F(E_n) \end{array}$$

where $F(Y)$ is non-empty and both $\lim_{n\in\mathbb{N}} F(Z_n)$ and $\lim_{n\in\mathbb{N}} F(E_n)$ are non-empty connected. This implies that $F(S)$ is non-empty as desired. This completes the proof of Theorem4.15.

Proof of Proposition 4.28 in case S is noetherian: For $(U/Y) \in \mathrm{RZ}(X_{\mathrm{Nis}})$ and a procdh covering $\{V_i \to U\}_{i\in I}$ in Sch_S, we want to find a morphism $Y' \to Y$ in Mod_X and a Nisnevich covering $\{W_j \to U \times_X Y'\}_{j\in J}$, a function $J \to I$; $j \mapsto i_j$, and commutative triangles

$$\begin{array}{ccc} W_j & \dashrightarrow & V_{i_j} \\ & \searrow & \downarrow \\ & & U. \end{array}$$

Since procdh coverings are refined by finite length compositions of generator procdh coverings, it suffices to prove the claim for distinguished Nisnevich coverings and pro-abstract blowup coverings.

For Nisnevich coverings the statement is obvious since for any $(U/Y) \in \mathrm{RZ}(X_{\mathrm{Nis}})$, a Nisnevich covering $\{U_i \to U\}_{i\in I}$ gives rise to a Nisnevich covering $\{(U_i/Y) \to (U/Y)\}_{i\in I}$ of (U/Y).

Consider $(U/Y) \in \mathrm{RZ}(X_{\mathrm{Nis}})$ and a pro-abstract blowup covering

$$\mathcal{U} = \{Z_n \to U\}_{n\in\mathbb{N}} \sqcup \{W \to U\}.$$

We will find a morphism $Y' \to Y$ in Mod_X such that letting $U' = U \times_Y Y'$, the morphism $U' \to U$ factors through either W or Z_n for some n. By the noetherian assumption, U has finitely many generic points so by Lemma 4.31, we can assume that U is irreducible with the generic point η. As such it suffices to treat the following two cases.

Case 1: $\eta \in Z_0$. In this case, $(Z_0)_{\mathrm{red}} = U_{\mathrm{red}}$. This means the (finitely many) generators of $\mathcal{I}_{Z_0}$ are nilpotent, so $Z_n = U$ for some n. Hence, $Y' = Y$ and the trivial covering $\{U \to U\}$ give a square on the left of (**??**).

Case 2: $\eta \notin Z_0$. We will build a square as on the right of (**??**) with $V_j = U'$. By the assumption $\eta \notin Z_0$, the morphism $W \to U$ is an isomorphism over a dense open subset of U. Since $U \to Y$ is étale, $W \to U \to Y$ is generically flat. More precisely, letting $T \subset Y$ be the closure of the image of Z_0 in Y, the morphism $W \to Y$ is flat over $Y \backslash T$ and T is nowhere dense in Y by the assumption $\eta \notin Z_0$.

By Raynaud-Gruson [RG71, Th.5.2.2], [Sta18, 081R], there is a blowup $Y' \to Y$ with a center contained in T such that the strict transform $W' \to Y'$ of $W \to Y$ is flat. Since $U' := Y' \times_Y U \to Y'$ is étale, this implies that $W' \to U'$ is flat by Lemma 4.29. So now we have a flat proper morphism which is generically an isomorphism. This implies it is globally an isomorphism by Lemma 4.30. So we obtain a factorisation $U' \cong W' \to W \to U$, which completes the proof of Proposition 4.28.

Here are some lemmas that were used above.

Lemma 4.29. *Suppose that $W \to U$ is any morphism of schemes, $U \to Y$ is étale and $W \to Y$ is flat. Then $W \to U$ is also flat.*

Proof. Exercise. □

Lemma 4.30. *Suppose that $f : W \to U$ is a flat, proper morphism of schemes, and $D \subseteq U$ is a schematically dense open such that $D \times_U W \to D$ is an isomorphism. Then $W \to U$ is an isomorphism.*

Proof. Exercise.

□

Lemma 4.31. *Let Y be a noetherian scheme and $U \to Y$ be an étale morphism, and suppose $U^{\mathrm{gen}} = \eta_0 \sqcup \eta_1$ is a decomposition of the space of generic points of U into clopens. Then there exists a* cartesian *square*

$$\begin{array}{ccc} U_0' \sqcup U_1' & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

such that

1. *$Y' \to Y$ is a proper morphism which is an isomorphism over a dense qc open of Y, and*
2. *there are identifications $\eta_0 = (U_0')^{\mathrm{gen}}$ and $\eta_1 = (U_1')^{\mathrm{gen}}$.*

Proof. Let $U_0 \subseteq U$ and $U_1 \subseteq U$ be the closures of η_0 and η_1 respectively. Since $U_0 \sqcup U_1 \to U \to Y$ is generically étale, there is some dense open $D \subseteq Y$ over which it is étale, [Sta18, 07RP], and in particular, flat and of finite presentation. So we can apply Raynaud-Gruson platification, [Sta18, 081R], to find a blowup $Y' \to Y$ which is an isomorphism over $D \subseteq Y$, and for which the strict transform $U_0' \sqcup U_1' \to Y'$ of $U_0 \sqcup U_1 \to Y$ is flat. By Lemma 4.29 this implies $U_0' \sqcup U_1' \to Y' \times_Y U$ is also flat. It is proper and an isomorphism over a dense open by construction, so it is in fact an isomorphism, Lem.4.30. Now the second condition is satisfied, since $U_0' \to U$ factors through $U_0' \to U_0$ and this latter is an isomorphism generically by construction. □

4.4 Conservativity of the fiber functors

Theorem 4.32. *Suppose S is a qcqs scheme with Noetherian topological space of finite Krull dimension. Then,* $\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S)$ *has enough points.*

Proof. Suppose that $\mathcal{Y} = \{Y_i \to Y\}_{i\in I}$ is a family of morphisms in Sch_S such that the morphism of sets $\sqcup_i \phi(Y_i) \to \phi(Y)$ is surjective for every fibre functor ϕ. We want to show that $\mathcal{Y}$ is refinable by a procdh-covering. We work by induction on the Krull dimension of Y, the base case being $Y = \varnothing$ with $\dim Y = -1$. In this base case, either I is empty, or I is nonempty and each $Y_i \to Y$ is an isomorphism. Both of these are already covering families, so no refinement is necessary.

Now we do the induction step. The functor $\rho_Y^* : \mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S) \to \mathrm{Shv}\,\mathrm{RZ}(Y_{\mathrm{Nis}})$ from (40) preserves colimits and finite limits by Proposition 4.28. So by composition, every fibre functor ϕ of $\mathrm{Shv}\,\mathrm{RZ}(Y_{\mathrm{Nis}})$ induces a fibre functor

$$\mathrm{Shv}_{\mathrm{procdh}}(\mathrm{Sch}_S) \overset{\rho_Y^*}{\to} \mathrm{RZ}(Y_{\mathrm{Nis}}) \overset{\phi}{\to} \mathrm{Set}.$$

By assumption, our family $\mathcal{Y}$ is sent to a surjection of sets under each such fibre functor $\phi \circ \rho_Y^*$. Since the site $\mathrm{RZ}(Y_{\mathrm{Nis}})$ is finitary so it has enough points by Deligne's completeness theorem, it follows that $\rho_Y^* \mathcal{Y}$ is a surjective family of sheaves in $\mathrm{Shv}(\mathrm{RZ}(Y_{\mathrm{Nis}}))$. This means that, locally, we can lift the section id_Y of $(\rho_Y^* Y)((Y/Y)) = \hom_{\mathrm{Sch}_S}(\rho((Y/Y)), Y) = \hom_{\mathrm{Sch}_S}(Y, Y)$. Explicitly, this means that there exists a covering $\{(U_j/Y') \to (Y/Y)\}_{j\in J}$ such that the family $\{U_j \to Y' \to Y\}_{j\in J}$ refines $\mathcal{Y}$. Since $Y' \to Y$ is a modification, there is a nowhere dense closed subscheme of finite presentation $Z_0 \to Y$ outside of which $Y' \to Y$ is an isomorphism. Since Y has finite Krull dimension and $Z_0 \to Y$ is nowhere dense, $\dim Z_0 < \dim Y$. So by the induction hypothesis, the pullbacks $Z_n \times_Y \mathcal{Y}$ of $\mathcal{Y}$ to each Z_n also admit refinements by procdh-coverings (here we are using that fibre functors preserve finite limits to know that each $\phi(Z_n \times_Y \mathcal{Y})$ is a surjective morphism of sets). Composing all these procdh coverings produces a procdh-covering of Y which refines the original $\mathcal{Y}$. □

5 Elmanto-Morrow's motivic complex

In this section, we let $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$ and $\mathsf{Sch}_{\mathbb{F}}^{\mathrm{qcqs}}$ be the category of qcqs schemes over $\mathbb{F}$ and review the construction of Elmanto-Morrow's motivic complex $\mathbb{Z}(n)^{\mathrm{EM}}$ as an object of $\mathrm{PSh}(\mathsf{Sch}_{\mathbb{F}}^{\mathrm{qcqs}}, \mathcal{D}(\mathbb{Z}))$. It is constructed by modifying the cdh-local motivic complex $\mathbb{Z}(n)^{\mathrm{cdh}}$ using other cohomology theories: Hodge-completed derived de Rham complexes in case $\mathbb{F} = \mathbb{Q}$ and syntomic complexes in case $\mathbb{F} = \mathbb{F}_p$. The construction is motivated by a pullback square in $\mathrm{PSh}(\mathsf{Sch}^{\mathrm{qcqs}}, \mathrm{Sp})$:

$$\begin{array}{ccc} K & \xrightarrow{\mathrm{tr}} & \mathrm{TC} \\ \downarrow & & \downarrow \\ KH & \xrightarrow{\mathrm{tr}^{cdh}} & a_{\mathrm{cdh}}\,\mathrm{TC} \end{array} \quad , \tag{41}$$

where KH is the homotopy K-theory, Definition 3.9, and

$$\mathrm{tr} : K \to \mathrm{TC}, \tag{42}$$

is the cyclotomic trace ([BHM93], [DGM13]) and TC is the *integral topological cyclic homology*, which is a sort of a 'linearlisation' of K and provides a more computable invariant built from Hochschild homology (see [HN19] for a survey). The map tr^{cdh} is induced by tr via the equivalence $\mathrm{KH} \simeq a_{\mathrm{cdh}} K$, Theorem 3.10. The pullback square follows from the latter equivalence and the fact that the fiber of tr is a cdh sheaf by [LT19, Th. A.3].

For $X/\mathbb{Q}$, $\mathrm{TC}(X)$ agrees with $\mathrm{HC}^-(X/\mathbb{Q}) = \mathbb{H}(X/\mathbb{Q})^{hS^1}$, the negative cyclic homology. Antieau [An19] defined a complete filtration,

$$\left\{F^n_{\mathrm{HKR}}\,\mathrm{HC}^-(X/\mathbb{Q})\right\}_{n\in\mathbb{Z}} \text{ on } \mathrm{HC}^-(X/\mathbb{Q})$$

and natural equivalences

$$\mathrm{gr}^n_{F_{\mathrm{HKR}}}\,\mathrm{HC}^-(X/\mathbb{Q}) \simeq \widehat{L\Omega}^{\geq n}_{X/\mathbb{Q}}[2n] := \varprojlim_{m\geq n} L\Omega^{\geq n}_{X/\mathbb{Q}}/L\Omega^{\geq m}_{X/\mathbb{Q}}\,[2n].$$

Here, $L\Omega^{\bullet}_{X/\mathbb{Q}}$ is the derived de Rham complex equipped with the Hodge-filtration $\left\{L\Omega^{\geq n}_{X/\mathbb{Q}}\right\}_{n\in\mathbb{N}}$. If $X = \mathrm{Spec}(A)$, picking a simplicial polynomial resolution $P_\bullet \to A$,

$$L\Omega^{\geq n}_{X/\mathbb{Q}} = \mathrm{colim}_{[q]\in\Delta^{op}}\,\Omega^{\geq n}_{P_q/\mathbb{Q}},$$

computed as the total complex of an associated bicomplex. In general $L\Omega^{\geq n}_{-/\mathbb{Q}}$ is the Zariski sheafification of the presheaf $\mathrm{Sch} \to \mathcal{D}(\mathbb{Q}); X \to L\Omega^{\geq n}_{\Gamma(X,\mathcal{O})/\mathbb{Q}}$ (see §5.1).

For $X/\mathbb{F}_p$, BMS defined a complete filtration

$$\left\{F^n_{\mathrm{BMS}}\,\mathrm{TC}(X)\right\}_{n\in\mathbb{N}} \text{ on } \mathrm{TC}(X)$$

with associated graded quotients

$$\mathrm{gr}^n_{F_{\mathrm{BMS}}}\,\mathrm{TC}(X) \simeq \mathbb{Z}(n)^{\mathrm{syn}}(X)[2n]$$

for a natural object $\mathbb{Z}(n)^{\mathrm{syn}} \in \mathrm{PSh}(\mathsf{Sch}^{\mathrm{qcqs}}_{\mathbb{F}_p}, \mathcal{D}(\mathbb{Z}_p))$ called the syntomic complex. There exists another definition of $\mathbb{Z}(n)^{\mathrm{syn}}$ using the prismatic cohomology theory of Bhatt-Scholze (see §5.2).

Elmanto-Morrow defined $\mathbb{Z}(n)^{\mathrm{EM}}$ so to fit into a pullback square in $\mathrm{PSh}(\mathsf{Sch}^{\mathrm{qcqs}}_{\mathbb{Q}}, \mathcal{D}(\mathbb{Z}))$

$$\begin{array}{ccc} \mathbb{Z}(n)^{\mathrm{EM}} & \longrightarrow & \widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}} \\ \downarrow & & \downarrow \\ \mathbb{Z}(n)^{\mathrm{cdh}} & \longrightarrow & a_{cdh}\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}} \end{array}$$

if $\mathbb{F} = \mathbb{Q}$, and a pullback square

$$\begin{array}{ccc} \mathbb{Z}(n)^{\mathrm{EM}} & \longrightarrow & \mathbb{Z}(n)^{\mathrm{syn}} \\ \downarrow & & \downarrow \\ \mathbb{Z}(n)^{\mathrm{cdh}} & \longrightarrow & a_{cdh}\mathbb{Z}(n)^{\mathrm{syn}} \end{array}$$

if $\mathbb{F} = \mathbb{F}_p$, and proved the following..

Theorem 5.1 ([EM23])**.** *There exists a tower in* $\mathrm{PSh}(\mathsf{Sch}^{\mathrm{qcqs}}_{\mathbb{F}}, \mathrm{Sp})$

$$\cdots \to F^{n+1}_{\mathrm{EM}}K \to F^n_{\mathrm{EM}}K \to \cdots \to F^0_{\mathrm{EM}}K = K,$$

and equivalences

$$\mathrm{gr}^n_{F_{\mathrm{EM}}}K \simeq \mathbb{Z}(n)^{\mathrm{EM}}[2n].$$

So, we get an Atiyah-Hirzebruch spectral sequence:

$$E_2^{p,q} = H^{p-q}_{\mathcal{M}}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X) \quad \textit{for } X \in \mathrm{Sch}^{noe}_{\mathbb{F}}$$

defining $H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) := H^i(\mathbb{Z}(n)^{\mathrm{EM}}(X))$.

In view of Theorems 5.1 and 4.8, it is a natural question if two constructions $\mathbb{Z}(n)^{\mathrm{EM}}$ and $\mathbb{Z}(n)^{\mathrm{procdh}}$ coincide.

Theorem 5.2 (Kelly-S)**.** *For $X \in \mathrm{Sch}_{\mathbb{F}}^{noe}$ of finite dimension, there is a natural equivalence*

$$\mathbb{Z}(n)^{\mathrm{procdh}}(X) \simeq \mathbb{Z}(n)^{\mathrm{EM}}(X)$$

functorial in X.

Strategy of the proof: Using Theorem 4.4 (a description of fiber functors of $\mathsf{Shv}_{procdh}(\mathsf{Sch}_S, \mathrm{Set})$) and Theorem 4.5 (the finiteness of its cohomological dimension), we give a characterization of $\mathbb{Z}(n)^{\mathrm{procdh}}$ by a list of properties (see Theorem 5.3 below). Elmanto-Morrow showed that $\mathbb{Z}(n)^{\mathrm{EM}}$ satisfies those properties.

Theorem 5.3. *Assume given $\mathcal{Z}(n) \in \mathrm{PSh}(\mathsf{Sch}_{\mathbb{F}}^{\mathrm{qcqs}}, \mathcal{D}(\mathbb{Z}))$ satisfying the following conditions.*

(a) There is an equivalence $\psi : \mathbb{Z}(n)^{\mathrm{sm}} \simeq \mathcal{Z}(n)_{|\mathrm{Sm}_{\mathbb{F}}}$ in $\mathrm{PSh}(\mathrm{Sm}_{\mathbb{F}}, \mathcal{D}(\mathbb{Z}))$.
(b) $\mathcal{Z}(n)$ is finitary, i.e. commutes with filtered colimits of rings.
(c) $\mathcal{Z}(n) \in \mathsf{Shv}_{procdh}(\mathrm{Sch}_{\mathbb{F}}^{noe}, \mathcal{D}(\mathbb{Z}))$.
(d) Let $\phi : L^{sm}\mathbb{Z}(n)^{\mathrm{sm}} \to \mathcal{Z}(n)$ be the map in $\mathrm{PSh}(\mathsf{Sch}_{\mathbb{F}}^{\mathrm{qcqs}}, \mathcal{D}(\mathbb{Z}))$ induced from ψ by adjunction. Then, $\phi(R)$ is an equivalence for all pro-cdh local rings R.

Then, ϕ induces an equivalence $\mathbb{Z}(n)^{\mathrm{procdh}}(X) \simeq \mathcal{Z}(n)(X)$ for any $X \in \mathrm{Sch}_{\mathbb{F}}^{noe}$.

Proof. We apply Theorem 4.4(1) to $F = L^{sm}\mathbb{Z}(n)^{\mathrm{sm}}$ and $G = \mathcal{Z}(n)$. Note that $L^{sm}\mathbb{Z}(n)^{\mathrm{sm}}$ is finitary by a formal reason and so is $\mathcal{Z}(n)$ by (b). Moreover, (c) implies $a_{procdh}\mathcal{Z}(n)(X) = \mathcal{Z}(n)(X)$ for $X \in \mathrm{Sch}_{\mathbb{F}}^{noe}$. Hence, the theorem follows from (d) and Corollary 4.4(1). □

Now, Theorem 5.2 is a consequence of the following.

Theorem 5.4 (Elmanto-Morrow [EM23])**.** $\mathbb{Z}(n)^{\mathrm{EM}}$ *satisfies the conditions (a), (b), (c), (d) of Theorem 5.3.*

We will give a proof of Theorem 5.4 in what follows. We start with a brief review on some basic definitions and facts.

5.1 Hodge-completed derived de Rham complexes

Let CAlg_k be the category of k-algebras and $\mathsf{CAlg}_k^{poly} \subset \mathsf{CAlg}_k$ be the full subcategory of polynomial k-algebras .

Definition 5.5. *We define a functor* $L\Omega_{-/k} : \mathsf{CAlg}_k \to \mathcal{D}(k)$ *as the left Kan extension along* $\mathsf{CAlg}_k^{poly} \to \mathsf{CAlg}_k$ *of the functor*

$$\mathsf{CAlg}_k^{poly} \to \mathcal{D}(k)\ ;\ R \to \Omega^\bullet_{R/k},$$

where $\Omega^\bullet_{R/k}$ *is the de Rham complex. It is equipped with a decreasing filtration called the derived Hodge filtration* $\left\{L\Omega^{\geq n}_{-/k}\right\}_{n\in\mathbb{N}}$ *by left-Kan extending the Hodge filtration on* $\Omega^\bullet_{R/k}$ *for* $R \in \mathsf{CAlg}_k^{poly}$*. The graded pieces are computed as*[13]

$$L\Omega^{\geq n}_{R/k}/L\Omega^{\geq n+1}_{R/k} \simeq \wedge^n L_{R/k}[-n] \quad \textit{for } R \in \mathsf{CAlg}_k,$$

where $L_{R/k}$ *is the cotangent complex of* R/k[14] *[Il71]. The Hodge-completed derived de Rham complex* $\widehat{L\Omega}_{R/k}$ *for* $R \in \mathsf{CAlg}_k$ *is defined as the limit of the diagram:*

$$\mathbb{N} \to \mathcal{D}(k)\ ;\ n \to L\Omega_{R/k}/L\Omega^{\geq n}_{R/k}.$$

We define $\widehat{L\Omega}_{-/k} \in \mathrm{PSh}(\mathrm{Sch}_k, \mathcal{D}(k))$ *as the Zariski sheafication of the presheaf*

$$\mathrm{Sch}_k^{op} \to \mathcal{D}(k)\ ;\ X \to \widehat{L\Omega}_{\Gamma(X,\mathcal{O})/k}.$$

The derived Hodge filtration induces a complete decreasing $\mathbb{N}$*-indexed filtration* $\left\{\widehat{L\Omega}^{\geq n}_{-/k}\right\}_{n\in\mathbb{N}}$ *on* $\widehat{L\Omega}_{-/k}$*.*

Example 5.6. (i) ([Il71]) For $X \in \mathrm{Sm}_k$, $\widehat{L\Omega}_{X/k} = \Omega^\bullet_{X/k}$.

(ii) ([Bha12]) If $k = \mathbb{C}$ and X is a scheme of finite type over $\mathbb{C}$, $H^*(\widehat{L\Omega}_{X/\mathbb{C}})$ is canonically isomorphic to the singular cohomology with $\mathbb{C}$-coefficients of the associated $\mathbb{C}$-points of X.

(iii) ([Bha12]) Assume $\mathrm{ch}(k) = 0$. If X is of finite type over k, then $H^*(\widehat{L\Omega}_{X/k})$ is canonically isomorphic to Hartshorne's algebraic de Rham cohomology [Har75][15].

An important result relevant to the construction of $\mathbb{Z}(n)^{\mathrm{EM}}$ is the following result due to Antieau [An19].

Theorem 5.7. *There exists a functorial complete decreasing* $\mathbb{Z}$*-indexed filtration* $\left\{F^n_{\mathrm{HKR}}\,\mathrm{HC}^-(X/k)\right\}_{n\in\mathbb{Z}}$ *on negative cyclic homology* $\mathrm{HC}^-(X/k)$ *for* $X \in \mathrm{Sch}_k$ *with*

$$\mathrm{gr}^n_{F_{\mathrm{HKR}}}\,\mathrm{HC}^-(X/k) \simeq \widehat{L\Omega}^{\geq n}_{X/k}[2n].$$

Remark 5.8. *If* $ch(k) = 0$*,* $\widehat{L\Omega}_{-/k}$ *and* $\mathrm{HC}^-(-/k)/F^0_{\mathrm{HKR}}$ *belong to* $\mathrm{Shv}_{cdh}(\mathrm{Sch}_k, \mathcal{D}(k))$*.*

[13] The left hand side is the cofiber in $\mathcal{D}(k)$ of $L\Omega^{\geq n+1}_{R/k} \to L\Omega^{\geq n}_{R/k}$.

[14] It is the left Kan extension along $\mathsf{CAlg}_k^{poly} \to \mathsf{CAlg}_k$ of $\mathsf{CAlg}_k^{poly} \to \mathcal{D}(k)\ ;\ R \to \Omega^1_{R/k}$.

[15] Choosing a closed immersion $i : X \hookrightarrow Y$ with $Y \in \mathrm{Sm}_k$, it is defined as the cohomology of the formal completion along X of $\Omega^\bullet_{Y/k}$, which is shown to be independent of i.

5.2 Syntomic complexes $\mathbb{Z}(n)^{\mathrm{syn}}$

By [BMS19], for any qcqs $\mathbb{F}_p$-scheme X, there exists a functorial complete decreasing $\mathbb{N}$-indexed filtration[16]

$$\left\{F^n_{\mathrm{BMS}}\,\mathrm{TC}(X)\right\}_{n\in\mathbb{N}} \text{ on } \mathrm{TC}(X) \tag{43}$$

with associated graded quotients

$$\mathrm{gr}^n_{F_{\mathrm{BMS}}}\,\mathrm{TC}(X) \simeq \mathbb{Z}(n)^{\mathrm{syn}}(X)[2n]$$

for a natural object $\mathbb{Z}(n)^{\mathrm{syn}} \in \mathrm{PSh}(\mathrm{Sch}_{\mathbb{F}_p}, \mathcal{D}(\mathbb{Z}_p))$ called the syntomic complex. It is an analog of the motivic filtration on algebraic K-theory and of deep interest in arithmetic geometry. For a regular $\mathbb{F}_p$-algebra R, we have

$$\mathbb{Z}(n)^{\mathrm{syn}}(R) = R\Gamma_{\mathrm{pro\acute{e}t}}(R, W\Omega^n_{\log})[-n] \in \mathcal{D}(\mathbb{Z}_p), \tag{44}$$

where $R\Gamma_{\mathrm{pro\acute{e}t}}(R, W\Omega^n_{\log})$ is a complex which computes the pro-étale cohomology of $\mathrm{Spec}(R)$ with coefficient $W\Omega^n_{\log} = \varprojlim_r W_r\Omega^n_{\log}$, where $W_r\Omega^n_{\log}$ are from (14). The syntomic complex $\mathbb{Z}(n)^{\mathrm{syn}} \in \mathrm{PSh}(\mathrm{Sch}_{\mathbb{F}_p}, \mathcal{D}(\mathbb{Z}_p))$ is recovered from (44) as follows. For each integer $r > 0$, we define a functor

$$\mathbb{Z}(n)^{\mathrm{syn}}(-)/p^r : \mathsf{CAlg}_{\mathbb{F}_p} \to \mathcal{D}(\mathbb{Z}/p^r\mathbb{Z})$$

as the left Kan extension along $\mathsf{CAlg}^{poly}_{\mathbb{F}_p} \to \mathsf{CAlg}_{\mathbb{F}_p}$ of the functor

$$\mathsf{CAlg}^{poly}_{\mathbb{F}_p} \to \mathcal{D}(\mathbb{Z}/p^r\mathbb{Z})\ ;\ R \to R\Gamma_{\mathrm{\acute{e}t}}(R, W_r\Omega^n_{\log})[-n],$$

and define for $A \in \mathsf{CAlg}_{\mathbb{F}_p}$

$$\mathbb{Z}(n)^{\mathrm{syn}}(A) := \varprojlim_r \mathbb{Z}(n)^{\mathrm{syn}}(A)/p^r \in \mathcal{D}(\mathbb{Z}_p).$$

Then, $\mathbb{Z}(n)^{\mathrm{syn}}$ is the Zariski sheafication of the presheaf

$$\mathrm{Sch}^{op}_{\mathbb{F}_p} \to \mathcal{D}(\mathbb{Z}_p)\ ;\ X \to \mathbb{Z}(n)^{\mathrm{syn}}(\Gamma(X, \mathcal{O})).$$

5.3 Motivic complexes $\mathbb{Z}(n)^{\mathrm{EM}}$

A key ingredient is the following pullback square in $\mathrm{PSh}(\mathrm{Sch}, \mathrm{Sp})$

$$\begin{array}{ccc} K & \xrightarrow{\mathrm{tr}} & \mathrm{TC} \\ \downarrow & & \downarrow \\ \mathrm{KH} & \xrightarrow{\mathrm{tr}^{cdh}} & a_{cdh}\,\mathrm{TC} \end{array}\quad , \tag{45}$$

[16] [BMS19] treats quasi-syntomic rings and it is extended to all p-complete rings in [AMMN20].

where tr^{cdh} is induced by tr (42) via the equivalence $\mathrm{KH} \simeq a_{cdh}K$ (Theorem 3.10). The pullback square follows from the latter equivalence and the fact that the fiber of tr is a cdh sheaf by [LT19, Th. A.3]. A key result in [EM23] is the following.

Theorem 5.9 ([EM23])**.** *(1) Recall $TC(X) = \mathrm{HC}^-(X/\mathbb{Q})$ for $X \in \mathrm{Sch}_{\mathbb{Q}}$. The cdh-local trace map $\mathrm{tr}^{cdh} : \mathrm{KH} \to a_{cdh}\,\mathrm{HC}^-(-/\mathbb{Q})$ as a map in $\mathrm{PSh}(\mathrm{Sch}_{\mathbb{Q}}, \mathrm{Sp})$ admits a unique extension to a map of filtered presheaves*

$$F^{\bullet}_{\mathrm{cdh}}\mathrm{KH} \to F^{\bullet}_{\mathrm{HKR}} a_{cdh}\,\mathrm{HC}^-(-/\mathbb{Q}),$$

where the filtration on the left is from Theorem 3.13 and that on the right is induced from $F^{\bullet}_{\mathrm{HKR}}\,\mathrm{HC}^-(-/\mathbb{Q})$ in Theorem 5.7 by the cdh sheafication.

(2) The cdh-local trace map $\mathrm{tr}^{cdh} : \mathrm{KH} \to a_{cdh}\,\mathrm{TC}$ as a map in $\mathrm{PSh}(\mathrm{Sch}_{\mathbb{F}_p}, \mathrm{Sp})$ admits a unique extension to a map of filtered presheaves

$$F^{\bullet}_{\mathrm{cdh}}\mathrm{KH} \to F^{\bullet}_{\mathrm{BMS}} a_{cdh}\,\mathrm{TC},$$

where the filtration on the right is induced from (43) by the cdh sheafication.

Definition 5.10 ([EM23])**.** *(1) For a qcqs $\mathbb{Q}$-scheme X, define a decreasing $\mathbb{Z}$-indexed filtered spectrum $F^{\bullet}_{\mathrm{EM}}K(X)$ as the pullback in the category of filtered spectra of the diagram*

$$\begin{array}{ccc} F^{\bullet}_{\mathrm{EM}}K(X) & \longrightarrow & F^{\bullet}_{\mathrm{HKR}}\,\mathrm{HC}^-(X/\mathbb{Q}) \\ \downarrow & & \downarrow \\ F^{\bullet}_{\mathrm{cdh}}\mathrm{KH} & \xrightarrow{\mathrm{tr}^{cdh}} & F^{\bullet}_{\mathrm{HKR}} a_{cdh}\,\mathrm{HC}^-(X/\mathbb{Q}) \end{array}$$

(2) For a qcqs $\mathbb{F}_p$-scheme X, define a decreasing $\mathbb{Z}$-indexed filtered spectrum $F^{\bullet}_{\mathrm{EM}}K(X)$ as the pullback in the category of filtered spectra of the diagram

$$\begin{array}{ccc} F^{\bullet}_{\mathrm{EM}}K(X) & \longrightarrow & F^{\bullet}_{\mathrm{BMS}}\,\mathrm{TC}(X) \\ \downarrow & & \downarrow \\ F^{\bullet}_{\mathrm{cdh}}\mathrm{KH} & \xrightarrow{\mathrm{tr}^{cdh}} & F^{\bullet}_{\mathrm{BMS}} a_{cdh}\,\mathrm{TC}(X) \end{array}$$

Let $\mathbb{F}$ denote $\mathbb{Q}$ or $\mathbb{F}_p$. For $n \in \mathbb{Z}$, Elmanto-Morrow's weight-n motivic complex of a qcqs $\mathbb{F}$-scheme X is defined as

$$\mathbb{Z}(n)^{\mathrm{EM}}(X) = (\mathrm{gr}^n_{F_{\mathrm{EM}}} K(X))[-2n]. \tag{46}$$

Theorem 5.11 ([EM23])**.** *For any qcqs $\mathbb{F}$-scheme X, the following hold.*

(1) $\mathbb{Z}(n)^{\mathrm{EM}}(X) = 0$ for $n < 0$ and $F^0_{\mathrm{EM}}K(X) = K(X)$, so $F^{\bullet}_{\mathrm{EM}}K(X)$ is $\mathbb{N}$-indexed. If $\dim_v(X) < \infty$, there exists an integer N such that $F^n_{\mathrm{EM}}K(X)$ is supported in cohomological degrees $\leq N - n$ for each $n \in \mathbb{N}$, so it induces a bounded Atiyah-Hirzebruch spectral sequence (5).

(2) *If $\mathbb{F} = \mathbb{Q}$, there exists a pullback square*

$$\begin{array}{ccc} \mathbb{Z}(n)^{\mathrm{EM}}(X) & \longrightarrow & \widehat{L\Omega}^{\geq n}_{X/\mathbb{Q}} \\ \downarrow & & \downarrow \\ \mathbb{Z}(n)^{\mathrm{cdh}}(X) & \xrightarrow{c^H} & (a_{cdh}\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}})(X) \end{array} \tag{47}$$

In particular, $\mathbb{Z}(n)^{\mathrm{EM}}(X) \in \mathcal{D}(\mathbb{Z})$.

(2) *If $\mathbb{F} = \mathbb{F}_p$, there exists a pullback square*

$$\begin{array}{ccc} \mathbb{Z}(n)^{\mathrm{EM}}(X) & \longrightarrow & \mathbb{Z}(n)^{\mathrm{syn}}(X) \\ \downarrow & & \downarrow \\ \mathbb{Z}(n)^{\mathrm{cdh}}(X) & \xrightarrow{c^{syn}} & (a_{cdh}\mathbb{Z}(n)^{\mathrm{syn}})(X) \end{array} \tag{48}$$

In particular, $\mathbb{Z}(n)^{\mathrm{EM}}(X) \in \mathcal{D}(\mathbb{Z})$.

(3) *$\mathbb{Z}(n)^{\mathrm{EM}}$ is finitary*[17]

(4) *There is an equivalence $\mathbb{Z}(n)^{\mathrm{sm}} \simeq \mathbb{Z}(n)^{\mathrm{EM}}{}_{|\mathrm{Sm}_{\mathbb{F}}}$ in $\mathrm{PSh}(\mathrm{Sm}_{\mathbb{F}}, \mathcal{D}(\mathbb{Z}))$.*

(5) *For any local $\mathbb{F}$-algebra A, we have an equivalence*

$$L^{sm}\mathbb{Z}(n)^{\mathrm{sm}}(A) \simeq \tau^{\leq n}\mathbb{Z}(n)^{\mathrm{EM}}(A),$$

where $\tau^{\leq n}$ is the truncation in cohomological degrees $\leq n$.

(6) *$\mathbb{Z}(n)^{\mathrm{EM}}$ restricted to the category $\mathrm{Sch}^{noe}_{\mathbb{F}}$ of noetherian $\mathbb{F}$-schemes is a pro-cdh sheaf (see Definition 4.1 below).*

Remark 5.12. (1) *There is a map in $\mathrm{Shv}_{Zar}(\mathrm{Sm}_{\mathbb{Q}}, \mathcal{D}(\mathbb{Z}))$:*

$$\mathbb{Z}(n)^{\mathrm{sm}} \to \mathcal{K}^M_n[-n] \xrightarrow{d\log} \Omega^n_{-/\mathbb{Q}}[-n] \to \Omega^{\geq n}_{-/\mathbb{Q}}, \tag{49}$$

where the first map is from (12) and the second map sends a local section $\{x_1 \cdots, x_n\}$ of K^M_n to $d\log x_1 \wedge \cdots \wedge d\log x_n$. The map c^H in (47) is identified with the cdh-sheafication of the left Kan extension of (49) along $\mathrm{Sm}_{\mathbb{Q}} \to \mathrm{Sch}_{\mathbb{Q}}$.

(2) *For a smooth $\mathbb{F}_p$-algebra R and an integer $r > 0$, we have a map*

$$\phi_r : \mathbb{Z}(n)^{\mathrm{sm}}(R) \to R\Gamma_{\mathrm{Zar}}(R, W_r\Omega^n_{\log})[-n] \to R\Gamma_{\acute{e}t}(R, W_r\Omega^n_{\log})[-n] = \mathbb{Z}(n)^{\mathrm{syn}}(R)/p^r,$$

where the first map comes from (14). The map c^{syn} in (48) is identified with the cdh-sheafication of $\varprojlim_r L^{sm}\phi_r$, where $L^{sm}\phi_r$ is the left Kan extension of ϕ_r along $\mathsf{CAlg}^{sm}_{\mathbb{F}_p} \to \mathsf{CAlg}_{\mathbb{F}_p}$, where $\mathsf{CAlg}^{sm}_{\mathbb{F}_p}$ is the category of smooth $\mathbb{F}_p$-algebras.

[17] $F \in \mathrm{PSh}(\mathsf{Sch}^{\mathrm{qcqs}}, \mathcal{C})$ is finitary if for any cofiltered diagram $\lambda \to X_\lambda$ in $\mathsf{Sch}^{\mathrm{qcqs}}$ with affine transition maps, we have an equivalence $F(\varprojlim_\lambda X_\lambda) \simeq \varinjlim_\lambda F(X_\lambda)$.

Remark 5.13. *Assume $\mathbb{F} = \mathbb{F}_p$. From* (14)*, we obtain an equivalence*

$$\mathbb{Z}(n)^{\mathrm{procdh}}/p^r \simeq a_{procdh} L^{sm} W_r \Omega^n_{\log}.$$

By [EM23, Cor. 4.31] and Corollary 5.2, this implies a fiber sequence

$$(a_{procdh} L^{sm} W_r \Omega^n_{\log})(X) \to \mathbb{Z}(n)^{\mathrm{syn}}(X)/p^r \to (a_{cdh} \tilde{\nu}_r(n))(X)[-n-1],$$

for $X \in \mathrm{Sch}^{noe}_{\mathbb{F}_p}$, where $\tilde{\nu}_r(n)$ is a presheaf of abelina groups given by

$$\tilde{\nu}_r(n)(A) := \mathrm{Coker}\big(C^{-1} - 1 : W_r\Omega^n_A \to W_r\Omega^n_A / dV^{r-1}\Omega^{n-1}_A\big) \text{ for } \mathbb{F}_p\text{-algebras } A.$$

Remark 5.14. *The same argument as the proof of Theorem 4.6 proves the following: Take $F \in \mathrm{PSh}(\mathrm{Sch}, \mathcal{C})$ for $\mathcal{C} = \mathrm{Sp}$ or $\mathcal{D}(\mathbb{Z})$ satisfying the conditions:*

(i) $F \in \mathsf{Shv}_{procdh}(\mathrm{Sch}^{noe}, \mathcal{C})$.
(ii) F *is finitary.*
(iii) *There is $N \in \mathbb{Z}$ such that $F(R) \in \mathcal{C}_{\geq -N}$*[18] *for any pro-cdh local ring R.*

Then, there exists a natural equivalence

$$a_{procdh}(F_{\geq -N})(X) \simeq F(X) \quad \text{for } X \in \mathrm{Sch}^{noe} \text{ with } \dim(X) < \infty. \tag{50}$$

We give some examples of F as in Remark 5.14. For $F \in \mathrm{PSh}(\mathrm{Sch}, \mathcal{C})$, put

$$\mathrm{Nil}F := \mathrm{fib}(F \to a_{cdh}F) \in \mathrm{PSh}(\mathrm{Sch}, \mathcal{C}), \tag{51}$$

and consider

$$\mathrm{Nil}\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}} \in \mathrm{PSh}(\mathrm{Sch}_{\mathbb{Q}}, \mathcal{D}(\mathbb{Q})) \text{ and } \mathrm{Nil}\,\mathrm{HC}^-(-/\mathbb{Q}) \in \mathrm{PSh}(\mathrm{Sch}_{\mathbb{Q}}, \mathrm{Sp}),$$

$$\mathrm{Nil}\mathbb{Z}(n)^{\mathrm{syn}} \in \mathrm{PSh}(\mathrm{Sch}_{\mathbb{F}_p}, \mathcal{D}(\mathbb{Z}_p)) \text{ and } \mathrm{Nil}\,\mathrm{TC} \in \mathrm{PSh}(\mathrm{Sch}_{\mathbb{F}_p}, \mathrm{Sp}).$$

Theorem 5.15. *(1) $\mathrm{Nil}\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}}$ and $\mathrm{Nil}\,\mathrm{HC}^-(-/\mathbb{Q})$ satisfy (i), (ii), and (iii) with $N = n$ and $N = 0$ respectively. So, we have for $X \in \mathrm{Sch}^{noe}_{\mathbb{Q}}$ with $\dim(X) < \infty$,*

$$a_{procdh}(\mathrm{Nil}\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}})_{\geq -n}(X) \simeq \mathrm{Nil}\widehat{L\Omega}^{\geq n}_{X/\mathbb{Q}},$$
$$a_{procdh}(\mathrm{Nil}\,\mathrm{HC}^-(-/\mathbb{Q}))_{\geq 0}(X) \simeq \mathrm{Nil}\,\mathrm{HC}^-(X/\mathbb{Q}).$$

(2) $\mathrm{Nil}\mathbb{Z}(n)^{\mathrm{syn}}$ and $\mathrm{Nil}\,\mathrm{TC}$ satisfy (i), (ii), and (iii) with $N = n$ and $N = 0$ respectively. So, we have for $X \in \mathrm{Sch}^{noe}_{\mathbb{F}_p}$ with $\dim(X) < \infty$

$$a_{procdh}(\mathrm{Nil}\mathbb{Z}(n)^{\mathrm{syn}})_{\geq -n}(X) \simeq \mathrm{Nil}\mathbb{Z}(n)^{\mathrm{syn}}(X),$$
$$a_{procdh}(\mathrm{Nil}\,\mathrm{TC})_{\geq 0}(X) \simeq \mathrm{Nil}\,\mathrm{TC}(X).$$

[18] i.e. $\pi_i F(R) = 0$ for any $i < -N$ in case $\mathcal{C} = \mathrm{Sp}$ and $H^i(F(R)) = 0$ for $i > N$ in case $\mathcal{C} = \mathcal{D}(\mathbb{Z})$.

Proof. The idea of the proof is borrowed from [EM23]. First we prove (1) for $\mathrm{Nil}\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}}$. (i) follows from Example 4.2(2). To show (ii) and (iii) with $N = n$, we use a fiber sequence in $\mathrm{PSh}(\mathrm{Sch}_{\mathbb{Q}}, \mathcal{D}(\mathbb{Q}))$

$$\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}} \to \widehat{L\Omega}_{-/\mathbb{Q}} \to L\Omega^{<n}_{-/\mathbb{Q}},$$

where the middle term is a cdh sheaf by Remark 5.8. It implies an equivalence

$$\mathrm{Nil}\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}} \simeq \mathrm{Nil}L\Omega^{<n}_{-/\mathbb{Q}}[-1].$$

So, it suffices to show that $\mathrm{Nil}L\Omega^{<n}_{-/\mathbb{Q}}$ is finitary and that $\mathrm{Nil}L\Omega^{<n}_{R/\mathbb{Q}} \in \mathcal{D}(\mathbb{Q})^{\leq n-1}$ for any pro-cdh-local ring R. The first assertion follows from the fact that $L\Omega^{i}_{-/\mathbb{Q}} = \wedge^i L_{-/\mathbb{Q}}$ is finitary and that the cdh sheafication of a finitary presheaf is finitary. As for the second assertion, by the finitarity and Lemma 5.16 below, it suffices to show that $\mathrm{Nil}L\Omega^{<n}_{R/\mathbb{Q}} \in \mathcal{D}(\mathbb{Q})^{\leq n-1}$ for any local rings R such that the ideal $\mathfrak{N} \subset R$ of nilpotent elements is finitely generated and $R/\mathfrak{N}$ is a valuation ring. Then, we have

$$(a_{cdh}L\Omega^{<n}_{-/\mathbb{Q}})(R) = (a_{cdh}L\Omega^{<n}_{-/\mathbb{Q}})(R/\mathfrak{N}) = L\Omega^{<n}_{(R/\mathfrak{N})/\mathbb{Q}}, \tag{52}$$

where the first equality follows from Remark 3.8(1) and the second from Theorem 3.15. Since $L\Omega^{<n}_{A/\mathbb{Q}}$ for a local $\mathbb{Q}$-algebra A is supported in degrees $\leq n-1$, we are reduced to showing the surjectivity of the map

$$H^{n-1}(L\Omega^{<n}_{R/\mathbb{Q}}) \to H^{n-1}(L\Omega^{<n}_{(R/\mathfrak{N})/\mathbb{Q}}).$$

This holds since the map is identified with $\Omega^{n-1}_{R/\mathbb{Q}} \to \Omega^{n-1}_{(R/\mathfrak{N})/\mathbb{Q}}$.

We deduce (1) for $\mathrm{Nil}\,\mathrm{HC}^{-}(-/\mathbb{Q})$ from (1) for $\mathrm{Nil}\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}}$. By Remark 5.8, we have $\mathrm{Nil}F^0_{\mathrm{HKR}}\,\mathrm{HC}^{-}(-/\mathbb{Q}) \simeq \mathrm{Nil}\,\mathrm{HC}^{-}(-/\mathbb{Q})$, so $F^{\bullet}_{\mathrm{HKR}}\,\mathrm{HC}^{-}(-/\mathbb{Q})$ from Theorem 5.7 induces a complete and exhaustive $\mathbb{N}$-indexed filtration $\left\{F^n_{\mathrm{HKR}}\mathrm{Nil}\,\mathrm{HC}^{-}(-/\mathbb{Q})\right\}_{n\in\mathbb{N}}$ on $\mathrm{Nil}\,\mathrm{HC}^{-}(-/\mathbb{Q})$ with identifications

$$\mathrm{gr}^n_{F_{\mathrm{HKR}}}\mathrm{Nil}\,\mathrm{HC}^{-}(-/\mathbb{Q}) \simeq \mathrm{Nil}\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}}[2n].$$

Moreover, for $X \in \mathrm{Sch}_{\mathbb{Q}}$ with $\dim_v(X) < \infty$, (iii) for $\mathrm{Nil}\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}}$ and Theorem 4.5 imply that $F^n_{\mathrm{HKR}}\mathrm{Nil}\,\mathrm{HC}^{-}(X/\mathbb{Q})$ is supported in cohomological degrees $\leq 2\dim_v(X)-n$, so the induced spectral sequence

$$E_2^{i,j} = H^{i-j}(\mathrm{Nil}\widehat{L\Omega}^{\geq -j}_{X/\mathbb{Q}}) \Rightarrow H^{i+j}\mathrm{Nil}\,\mathrm{HC}^{-}(X/\mathbb{Q}),$$

is bounded. Hence, (1) for $\mathrm{Nil}\,\mathrm{HC}^{-}(-/\mathbb{Q})$ follows from (1) for $\mathrm{Nil}\widehat{L\Omega}^{\geq n}_{-/\mathbb{Q}}$.

Next, we prove (2) for $\mathrm{Nil}\mathbb{Z}(n)^{\mathrm{syn}}$. The conditions (i) and (ii) are shown in [EM23] and we recall some arguments here. First, we claim $\mathrm{Nil}\mathbb{Z}(n)^{\mathrm{syn}}[\frac{1}{p}] = 0$. Using the fact that $F^{\bullet}_{\mathrm{BMS}}\,\mathrm{TC}$ from (43) naturally splits after inverting p, the claim is reduced to $\mathrm{Nil}\,\mathrm{TC}[\frac{1}{p}] = 0$. By (45), the latter follows from $\mathrm{fib}(K \to \mathrm{KH})[\frac{1}{p}] = 0$ (see [TT07, Th. 9.6]). Thus, it suffices to show (i) and (ii) for $\mathbb{Z}(n)^{\mathrm{syn}}(-)/p$ noting that the cdh sheafication of a finitary presheaf is finitary. It is shown in [EM23] that $\mathbb{Z}(n)^{\mathrm{syn}}(-)/p$ admits a finite increasing filtration whose graded pieces are some shifts of $L\Omega^i_{-/\mathbb{F}_p}$ with $i \leq n$, so the desired assertion follows from the finitarity of $L\Omega^i_{-/\mathbb{F}_p}$ and (28). To show (iii), by the finitarity and Lemma 5.16 below, it suffices to show that $\mathrm{Nil}\mathbb{Z}(n)^{\mathrm{syn}}(R) \in \mathcal{D}(\mathbb{Q})^{\leq n}$ for any local rings R such that the ideal $\mathfrak{N} \subset R$ of nilpotent elements is finitely generated and $R/\mathfrak{N}$ is a valuation ring. As (52), we have

$$a_{cdh}\mathbb{Z}(n)^{\mathrm{syn}}(R) = a_{cdh}\mathbb{Z}(n)^{\mathrm{syn}}(R/\mathfrak{N}) = \mathbb{Z}(n)^{\mathrm{syn}}(R/\mathfrak{N}).$$

Hence, (iii) with $N = n$ follows from [AMMN20, Th.5.2] noting $(R, \mathfrak{N})$ is a henselian pair. By the same argument as the proof of (1) for $\mathrm{HC}^-(-/\mathbb{Q})$, (2) for $\mathrm{Nil}\,\mathrm{TC}$ follows from (2) for $\mathrm{Nil}\mathbb{Z}(n)^{\mathrm{syn}}$ using a bounded spectral sequence

$$E_2^{i,j} = H^{i-j}(\mathrm{Nil}\mathbb{Z}_p(-j)^{syn}(R)) \Rightarrow \pi_{-i-j}\mathrm{Nil}\,\mathrm{TC}(R),$$

arising from the filtration (43). Here, the boundedness follows from the following fact proved in [EM23]: For $X \in \mathrm{Sch}_{\mathbb{F}_p}$, there exists an integer $d > 0$ such that $F^n_{\mathrm{BMS}}\,\mathrm{TC}(X)$ is supported in homological degrees $\geq n - d$. □

Lemma 5.16 ([KS23]). *(i) Any pro-cdh local ring is a filtered colimit of pro-cdh local rings of $V \times_K Q$ as in Theorem 4.14 where Q is an Artinian local ring.*

(ii) Any pro-cdh local ring $V \times_K Q$ as in (i) is a filtered colimit of henselian local rings R_λ such that the ideal $\mathfrak{N}_\lambda \subset R_\lambda$ of nilpotent elements is finitely generated and $R_\lambda/\mathfrak{N}_\lambda$ is a valuation ring.

We now give a proof of Theorem 5.4. The conditions (a), (b), (c) follow from Theorem 5.11(3), (4), (6). We prove (d). By Theorem 5.11(5), it suffices to prove $\mathbb{Z}(n)^{\mathrm{EM}}(R)$ is supported in cohomological degrees $\leq n$ for any pro-cdh local ring R. By (47) and (48), there are fiber sequences (cf. (51))

$$\mathrm{Nil}\widehat{L\Omega}^{\geq n}_{R/\mathbb{Q}} \to \mathbb{Z}(n)^{\mathrm{EM}}(R) \to \mathbb{Z}(n)^{\mathrm{cdh}}(R) \ \text{ if } \mathbb{F} = \mathbb{Q},$$

$$\mathrm{Nil}\mathbb{Z}(n)^{\mathrm{syn}}(R) \to \mathbb{Z}(n)^{\mathrm{EM}}(R) \to \mathbb{Z}(n)^{\mathrm{cdh}}(R) \ \text{ if } \mathbb{F} = \mathbb{F}_p.$$

By the same argument as the proof of Theorem 5.15, we have

$$\mathbb{Z}(n)^{\mathrm{cdh}}(R) = \mathbb{Z}(n)^{\mathrm{cdh}}(R/\mathfrak{N}) = (L^{sm}\mathbb{Z}(n)^{\mathrm{sm}})(R/\mathfrak{N}).$$

So, $\mathbb{Z}(n)^{\mathrm{cdh}}(R)$ is supported in degrees $\leq n$ by (24) while $\mathrm{Nil}\widehat{L\Omega}^{\geq n}_{R/\mathbb{Q}}$ and $\mathrm{Nil}\mathbb{Z}(n)^{\mathrm{syn}}(R)$ are supported in degrees $\leq n$ by Theorem 5.15. This proves the desired assertion.

6 Appendix: Short reviews on basic notions

6.1 Bloch's higher cycle complexes

Recall the singular homology $H_q(X,\mathbb{Z})$ of a topological space X is the q-th homology of the chain complex

$$\cdots \to s(X,q) \xrightarrow{\partial} s(X,q-1) \xrightarrow{\partial} \cdots \xrightarrow{\partial} s(X,0),$$

$$s(X,q) = \bigoplus_{\Gamma} \mathbb{Z}[\Gamma] \quad (\Gamma : \Delta^q_{top} \to X \text{ continuous}),$$

$$\Delta^q_{top} = \big\{(x_0, x_1, \cdots, x_q) \in \mathbb{R}^{q+1} \;\big|\; \sum_{0\leqq i \leqq q} x_i = 1,\ x_i \geqq 0\big\},$$

∂ is the alternating sum of the restriction maps to faces of Δ^q_{top}.

For integers $r, q \geq 0$, Bloch's higher Chow groups $\mathrm{CH}^r(X,q)$ of a scheme X of finite type over k is defined as an algebraic analog of the singular homology. The algebraic analog of Δ^q_{top} is

$$\Delta^q = \mathrm{Spec}\,\big(\mathbb{Z}[t_0, \cdots, t_q]/(\sum_{i=0}^{q} t_i - 1)\big) \tag{53}$$

with faces $\Delta^s = \{t_{i_1} = \cdots = t_{i_{q-s}} = 0\} \subset \Delta^q$[19], and that of the complex $s(X,\bullet)$ is

$$\cdots \to z^n(X,q) \xrightarrow{\partial} z^n(X,q-1) \xrightarrow{\partial} \cdots \xrightarrow{\partial} z^n(X,0),$$

$$z^n(X,q) = \bigoplus_{x\in X^n(q)} \mathbb{Z}[x],$$

where $X^n(q)$ is the set of codimension n points $x \in X \times \Delta^q$ whose closures $\overline{\{x\}}$ intersect properly with $X \times T$ in $X \times \Delta^q$ for all faces $T \subset \Delta^q$, i.e.

$$\mathrm{codim}_{X\times F}(\overline{\{x\}} \cap (X \times F)) \geq n,$$

and ∂ is alternating sum of restriction maps to faces of codimension one.

Example: $\mathrm{CH}^n(X,0) = \mathrm{CH}^n(X)$, $\mathrm{CH}^1(X,1) = \mathcal{O}(X)^\times$.

---

[19] $q \to \Delta^q$ gives a cosimplicial scheme $\Delta^\bullet$.

6.2 The ∞-category of spaces

Recall the category Δ whose objects are the finite ordered sets $[n] := \{0 < 1 < \cdots < n\}$ and whose morphisms are the order-preserving maps of sets. For a category $\mathcal{C}$, the category of simplicial objects is the functor category

$$\mathcal{C}^{\Delta} = \mathrm{Fun}(\Delta^{op}, \mathcal{C}).$$

and the category sSet of simplicial sets is defined to be $\mathrm{Set}^{\Delta} = \mathrm{Fun}(\Delta^{op}, \mathrm{Set})$. For $K \in \mathsf{sSet}$ and $n \geq 0$, we write $K_n = K([n])$ called the set of n-simplices. By Yoneda, we have $K_n = \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, K)$, where $\Delta^n = \mathrm{Hom}_{\Delta}(-, [n]) \in \mathsf{sSet}$.

For $0 \leq j \leq n$, we have the i-th face map $\delta_j : [n-1] \to [n]$ defined as the unique injective map in Δ which does not have j in its image. For $K \in \mathsf{sSet}$, we have the corresponding map (also called a face map)

$$d_j : K_n \to K_{n-1}.$$

The geometric realization is a functor $|-| : \mathsf{sSet} \to \mathsf{Top}$ that builds from a simplicial set X a topological space $|X|$ obtained by interpreting each element in X_n, i.e. each abstract n-simplex in X, as one copy of the standard topological n-simplex Δ^n_{top} and then gluing together all these along their boundaries to a big topological space, using the information encoded in the face maps of X on how these simplices are supposed to be stuck together.

For $0 \leq j \leq n$, the j*-th horn* is the union $\Lambda^n_j = \bigcup_{0 \leq i \neq j \leq n} \delta_i(\Delta^{n-1})$. By definition, $(\Lambda^n_j)_q \subset (\Delta^n)_q = \mathrm{Hom}_{\Delta}([q], [n])$ is the set of those morphisms $[q] \to [n]$ whose images do not contain $\{0, 1, \ldots, q-1, q+1, \ldots, n\}$. We also define the union of the faces

$$\partial \Delta^n = \bigcup_{0 \leq i \leq n} \delta_i(\Delta^{n-1}).$$

By definition, $(\partial\Delta^n)_q \subset (\Delta^n)_q = \mathrm{Hom}_{\Delta}([q], [n])$ is the set of those morphisms $[q] \to [n]$ which are not surjective.

A simplicial set X is a *Kan complex* if for any $0 \leq j \leq n$ and any diagram

$$\begin{array}{ccc} \Lambda^n_j & \longrightarrow & K \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array} \tag{54}$$

there exists a dotted arrow making a commutative triangle. An example is the singular simplicial complex of a topological space[20].

[20]This follows from the existence of retractions of any geometric simplex to any of its horns.

For a small category C, the *nerve* $N(C)$ of C is a simplicial set given by

$$\Delta^{op} \to \text{Set} \; ; \; [n] \to \text{Fun}([n], C),$$

where $[n] = \{0 < 1 < \cdots < n\}$ is viewed as a category by declaring that for $i, j \in [n]$, there exists a unique morphism $i \to j$ if $i \leq j$ and no morphism otherwise. By [Lur09, 1.1.2.2], $K \in \mathsf{sSet}$ is the nerve of a category if and only if for any $0 < j < n$ and any diagram (54), there exists a *unique* dotted arrow making a commutative triangle.

Definition 6.1. *An ∞-category is a simplicial set C such that for any $0 < j < n$ and any diagram (54), there exists a (not necessarily unique) dotted arrow making a commutative triangle. An element of C_0 is called an* object *of C and that of C_1 called a* morphism. *Given two morphism $f, g \in K_1$ such that $d_1 f = d_0 g$ giving a diagram $\Lambda^2_1 \to C$, there exists $\sigma : \Delta^2 \to C$ making a commutative triangle as (54). Then, $d_1\sigma \in C_1$ is called a* composition *of g and f.*

For a ∞-category C and $x, y \in C_0$, one can define (not easy!) a *mapping space* $\text{Map}_C(x, y)$ which is a Kan complex such that for $x, y, z \in C_0$, there is a morphism of simplicial sets (cf. [DS11])

$$\text{Map}_C(x, y) \times \text{Map}_C(x, y) \to \text{Map}_C(x, y)$$

satisfying the identity and associativity properties. A different model of $\text{Map}_C(x, y)$ is given by relating ∞-categories to simplicial categories (see (55)). The homotopy category hC of C is the 1-category whose objects are the same as those of C and $\text{Hom}_{hC}(x, y) = \pi_0(\text{Map}_C(x, y))$. Note that a morphism $x \to y$ in C as defined in Definition 6.1 is viewed as a point of $\text{Map}_C(x, y)$. A morphism in C is called an *equivalence* if it becomes an isomorphism in hC.

Let Cat_Δ be the category of simplicial categories[21]. Recall $\mathsf{sSet} \in \text{Cat}_\Delta$: For $K, L \in \mathsf{sSet}$, the mapping simplicial set $\text{Map}_{\mathsf{sSet}}(K, L)$ is defined by

$$\Delta^{op} \to \text{Set} \; ; \; [n] \to \text{Hom}_{\mathsf{sSet}}(K \times \Delta^n, L),$$

where for $X, Y \in \mathsf{sSet}$, $X \times Y$ is a simplicial set given by $(X \times Y)_n = X_n \times Y_n$. For $C \in \text{Cat}_\Delta$, its simplicial nerve NC is the simplicial set

$$[n] \mapsto \text{Hom}_{\text{Cat}_\Delta}(\mathfrak{C}[\Delta^n], C),$$

where $\mathfrak{C}[\Delta^n]$ is the simplicial category whose objects are elements of $[n] = \{0 < \cdots < n\}$. For $0 \leq i, j \leq n$, the mapping space is defined as

$$\text{Map}_{\mathfrak{C}[\Delta^n]}(i, j) = N\Big(\{i, j\} \subset J \subset \{i, i+1, \ldots, j\}\Big),$$

[21] i.e. categories enriched over sSet.

the nerve of the partially ordered set consisting of subsets $J \subset [0, n]$ containing $\{i, j\}$ and contained in $\{i, i+1, \ldots, j\}$. Composition

$$\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, j) \times \mathrm{Map}_{\mathfrak{C}[\Delta^n]}(j, k) \to \mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, k)$$

is induced by unions. By [Lur09, 1.1.5.5], it gives a functor

$$\mathfrak{C}[\Delta^\bullet] : \Delta \to \mathrm{Cat}_\Delta \; ; \; [n] \to \mathfrak{C}[\Delta^n],$$

which extends to a functor $\mathfrak{C} : \mathsf{sSet} \to \mathrm{Cat}_\Delta$ via the Yoneda functor $\Delta \to \mathsf{sSet}$.

Theorem 6.2. *([Lur09, §2.2, 1.1.5.10, 2.2.5.1]) We have a pair of adjoint funtors*

$$\mathfrak{C} : \mathsf{sSet} \leftrightarrows \mathrm{Cat}_\Delta : N.$$

The functor N sends fibrant simplicial categories[22] *to ∞-categories. For an ∞-category C and $x, y \in C_0$, there are equivalence in* sSet

$$\mathrm{Map}_{\mathfrak{C}[C]}(x, y) \simeq \mathrm{Map}_C(x, y). \tag{55}$$

Definition 6.3. *The ∞-category $\mathcal{S}$ of spaces is defined as $N(\mathcal{G}\mathrm{pd}_\infty)$, where $\mathcal{G}\mathrm{pd}_\infty$ is the simplicial category of Kan complexes. Note that the objects of $\mathcal{S}$ is the Kan complexes.*

6.3 Algebraic K-theory

Fix a (commutative) ring R and let Proj_R be the groupoid of finitely generated projective R-modules with isomorphisms. We have an equivalence

$$\mathrm{Proj}_R \simeq \coprod_{P \in \mathrm{Proj}_R/\sim} B\mathrm{Aut}(P),$$

where $B\mathrm{Aut}(P)$ is the groupoid with one object $*$ and $\mathrm{Hom}(*, *) = \mathrm{Aut}(P)$. The direst sum $\oplus$ turns Proj_R into a symmetric monodial category [23] and the set $\mathrm{Proj}_R/\sim$ of isomorphism classes is an abelian monoid with product $\oplus$ and the identity 0.

[22] i.e. those $C \in \mathrm{Cat}_\Delta$ such that $\mathrm{Map}_C(x, y)$ are a Kan complex for all $x, y \in C$.

[23] i.e. a category S equipped with a functor $\Box : S \times S \to S$ and a distinguished object $e \in S$ and four basic natural isomorphisms

$$e \Box s \simeq s, \; s \Box e \simeq s, \; s \Box (t \Box u) \simeq (s \Box t) \Box u), \; s \Box t \simeq t \Box s,$$

which are *coherent* in the sense that two natural isomorphisms of products of $s_1, \ldots, s_n$ built up from the four basic ones are the same whenever they have the same source and target. Assume that the isomorphism classes of objects of S form a set denoted by $S/\sim$. Then, $S/\sim$ is an abelian monoid with product $\Box$ and the identity e. The group completion of $S/\sim$ is called the Grothendieck group of S and denoted by $K_0(S)$. In case $S = \mathrm{Proj}_R$. this is $K_0(R)$.

Recall the inclusion of the categories

$$\{\text{commutative groups}\} \hookrightarrow \{\text{commutative monoids}\}$$

admits a left adjoint $M \to M^{gr}$ called the group completion.

Definition 6.4. $K_0(R) = \big(\mathrm{Proj}_R/\sim\big)^{gr}$.

Taking its nerve, every groupoid X is viewed as an object of the ∞-category $\mathcal{S}$ of spaces, which is 1-truncated, i.e. $\pi_i(X, x) = 0$ for $i > 1$ and $x \in X$. The symmetric monoidal structure on Proj_R turns it into a *commutative monoid* in $\mathcal{S}$, commutative up to higher homotopies.

Let $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S})$ be the ∞-category of commutative monoids in $\mathcal{S}$ and $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$ be its full subcategory of group-like objects, i.e. those $M \in \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S})$ such that $\pi_0(M)$ are groups. Similarly as above, the inclusion

$$\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S}) \hookrightarrow \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{S})$$

admits a left adjoint $M \to M^{gr}$.

Definition 6.5. $K(R) = \big(\mathrm{Proj}_R\big)^{gr} \in \mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$. *For* $n \geq 0$, $K_n(R) = \pi_n(K(R))$.

Connective K-theory spectrum: $K(R)$ (in fact any object of $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{S})$) is an infinite loop space, i.e. there exists a sequence $T_0 = K(R), T_1, T_2, \dots$ in $\mathcal{S}_*$[24] such that for $n \geq 1$, T_n are n-connective[25] and $T_{n-1} = \Omega T_n$, so it gives an object $\mathbf{K}(R)$ of the ∞-category Sp of spectra[26]. Moreover, it belongs to the full subcategory $\mathrm{Sp}_{\geq 0}$ of connective spectra $\mathbf{T} = (T_0, T_1, \dots)$, i.e. its stable homotopy group

$$\pi_i(\mathbf{T}) = \varinjlim_n \pi_{i+n}(T_n)$$

vanishes for $i < 0$, where the transition maps in the colimit are

$$\pi_{i+n}(T_n) \to \pi_{i+n}(\Omega T_{n+1}) = \pi_{i+n+1}(T_{n+1}).$$

By the construction, $K_n(R) = \pi_n(\mathbf{K}(R))$ for $n \geq 0$.

Non-connective K-theory spectrum: The map

$$K_0(R[t]) \oplus K_0(R[t^{-1}]) \xrightarrow{\phi} K_0(R[t, t^{-1}])$$

[24] The category of pointed spaces and pointed maps.

[25] i.e. $\pi_i(T_n, *) = *$ for $i < n$.

[26] A spectra is a sequence $\mathbf{T} = (T_0, T_1, \dots)$ in $\mathcal{S}_*$ with structure maps $T_{n-1} \to \Omega T_n$ for $n \geq 1$.

is not surjective in general unless R is regular. H. Bass used this to define the negative K-groups $K_{-1}(R) = \mathrm{Coker}(\phi)$ and $K_{-n}(R)$ for all $n > 0$ inductively as the cokernel of

$$K_{-n+1}(R[t]) \oplus K_{-n+1}(R[t^{-1}]) \xrightarrow{\phi} K_{-n+1}(R[t, t^{-1}]).$$

The definition can be upgraded to a spectrum-level version, a non-connective K-theory spectrum $\mathbf{K}^B(R)$ with a natural map $\mathbf{K}(R) \to \mathbf{K}^B(R)$ inducing an equivalence $\mathbf{K}(R) \simeq \tau_{\geq 0}\mathbf{K}^B(R)$, i.e. $\pi_i(\mathbf{K}(R)) \simeq \pi_i(\mathbf{K}^B(R))$ for $i \geq 0$, and for $n > 0$, $\pi_{-n}(\mathbf{K}^B(R)) = K_{-n}(R)$ defined above.

Exercise 6.6. Let $R = k[x, y]/(y^2 - x^3 + x^2)$. Using the Mayer-Vietoris property below, prove $K_{-1}(R) \simeq \mathbb{Z}$. Construct a projective $R[x^{-1}]$ module M which does not come from a projective R-module via $\otimes_R R[x^{-1}]$.

Theorem 6.7. *([W13, Ch.III, Th.4.3]) Let $f : R \to S$ be a ring map and I be an ideal of R mapped isomorphically into an ideal of S. Then, there exist a long exact sequence*

$$\begin{gathered} K_0(S) \oplus K_0(R/I) \to K_0(S/I) \to K_{-1}(R) \to K_{-1}(S) \oplus K_{-1}(R/I) \to K_{-1}(S/I) \\ \to K_{-2}(R) \to K_{-2}(S) \oplus K_{-2}(R/I) \to K_{-2}(S/I) \to \cdots \end{gathered}$$

Globalisation: Let $\mathsf{Sch}^{\mathrm{qcqs}}$ be the category of qcqs schemes and AffSch be the full subcategory of affine schemes. Thomason proved the presheaf of spectra

$$\mathsf{AffSch}^{op} \to \mathrm{Sp}\,; X = \mathrm{Spec}(R) \to \mathbf{K}^B(R)$$

is a *Zariski sheaf*, i.e. for any Zariski covering $Y \to X$ in AffSch,

$$\mathbf{K}^B(X) \simeq \varprojlim_{\Delta} \Big(\mathbf{K}^B(Y) \rightrightarrows \mathbf{K}^B(Y \times_X Y) \rightrightarrows \mathbf{K}^B(Y \times_X Y \times_X Y) \cdots \Big)$$

We have the sheafification $a_{\mathrm{Zar}} : \mathrm{PSh}(\mathsf{Sch}^{\mathrm{qcqs}}, \mathrm{Sp}) \to \mathrm{Shv}_{\mathrm{Zar}}(\mathsf{Sch}^{\mathrm{qcqs}}, \mathrm{Sp})$. We extend $\mathbf{K}^B$ to $\mathrm{Shv}_{\mathrm{Zar}}(\mathsf{Sch}^{\mathrm{qcqs}}, \mathrm{Sp})$ by applying a_{Zar} to the presheaf

$$\mathsf{Sch}^{\mathrm{qcqs}} \to \mathrm{Sp}; X \to \mathbf{K}^B(\Gamma(X, \mathcal{O})).$$

Notation: In this note, we write K for $\mathbf{K}^B$ and $K_{\geq 0}$ for $\mathbf{K}$.

Projective bundle formula: Let X be a qcqs-scheme and $\mathcal{E}$ be a vector bundle of tank $r + 1$ over X. Consider the projective bundle $\mathbb{P} = \mathbb{P}(\mathcal{E}) \to X$.

Theorem 6.8. *$K_0(\mathbb{P})$ is a free $K_0(X)$-module with basis $\{|\mathcal{O}_{\mathbb{P}}(-i)]|\ i = 0, \ldots, r\}$. The map*

$$K_n(X) \otimes_{K_0(X)} K_0(\mathbb{P}) \to K_n(\mathbb{P})$$

is a ring isomorphism.

6.4 Sheaves with values in ∞-categories

We will use the ∞-category $\mathcal{S}$ of spaces and Sp of spectra. We will use also the ∞-categorie $\mathcal{D}(A)$ of unbounded complexes of A-modules for a commutative ring A. Note that the homotopy category of $\mathcal{D}(A)$ is the derived category $D(A)$ of unbounded complexes of A-modules.

Let T be a small category with fiber products and τ be a Grothendieck topology on T. For an ∞-category $\mathcal{C}$, let $\mathrm{PSh}(T,\mathcal{C}) = \mathrm{Fun}(T^{op},\mathcal{C})$ denote the ∞-category of presheaves on T with values in $\mathcal{C}$ and $\mathrm{Shv}_\tau(T,\mathcal{C})$ denote the full subcategory of $\mathrm{PSh}(T,\mathcal{C})$ consisting of *τ-sheaves*. By definition, $F \in \mathrm{PSh}(T,\mathcal{C})$ is a τ-sheaf if for every $X \in T$ and every τ-sieve $R \subset X$, the restriction map $\mathrm{Map}(X,F) \to \mathrm{Map}(R,F)$ is an equivalence. By [AHW17, Lem.3.1.3], $F \in \mathrm{PSh}(T,\mathcal{C})$ is a τ-sheaf if and only if for any τ-covering family $\{\phi_i : Y_i \to X\}_{i\in I}$ in T, we have an equivalence

$$F(X) \simeq \varprojlim_{[n]\in\Delta} \prod_{(i_0,\ldots,i_n)\in I^{n+1}} F(Y_{i_0} \times_X \cdots \times_X Y_{i_n}),$$

where

$$[n] \in \Delta^{op} \to \bigsqcup_{(i_0,\ldots,i_n)\in I^{n+1}} Y_{i_0} \times_X \cdots \times_X Y_{i_n}$$

is the Čech nerve of the covering family. If $Y = \sqcup_{i\in I} Y_i$ exists in T, the condition can be written as

$$F(X) \simeq \varprojlim_{\Delta} \Big(F(Y) \overset{\longrightarrow}{\underset{\longleftarrow}{\longrightarrow}} F(Y \times_X Y) \overset{\longrightarrow}{\underset{\longleftarrow}{\overset{\longrightarrow}{\underset{\longleftarrow}{\longrightarrow}}}} F(Y \times_X Y \times_X Y) \cdots \Big)$$

By [Lur09, 6.2.2.7] the inclusion $\mathrm{Shv}_\tau(T,\mathcal{C}) \to \mathrm{PSh}(T,\mathcal{C})$ admits a left-exact left adjoint called the τ-sheafication

$$a_\tau : \mathrm{PSh}(T,\mathcal{C}) \to \mathrm{Shv}_\tau(T,\mathcal{C}).$$

In what follows, we recall an alternative definition $\mathrm{Shv}_\tau(T,\mathcal{S})$ using a model category. Let $\mathrm{PSh}^\Delta(T)$ be the category of simplicial presheaves on T equipped with the injective model structure: the weak equivalences are the objectwise weak equivalences, the cofibrations are the monomorphisms, and the fibrations are determined by the RLP. For $F, G \in \mathrm{PSh}^\Delta(T)$, the mapping space $\mathrm{Map}(F,G)$ is defined as

$$\mathrm{Map}(F,G)_n = \mathrm{Hom}_{\mathrm{PSh}^\Delta(T)}(F \times \Delta^n, G),$$

where Δ^n is considered as an object of $\mathrm{PSh}(T)$ as a constant presheaf. The model category $\mathrm{PSh}^\Delta(T)$ is simplicial, proper, and combinatorial [Lur09, A.2.8.2] so that we can use the machinery of left Bousfield localizations [Lur09, A.3.7]: For a set Σ

of morphisms in $\mathrm{PSh}^{\Delta}(T)$, we say $F \in \mathrm{PSh}^{\Delta}(T)$ is Σ-local if for every $f : G \to H$ in Σ, the induced map

$$f : \mathrm{Map}(H, F) \to \mathrm{Map}(G, F) \tag{56}$$

is a weak equivalence. A morphism $f : G \to H$ is an Σ-equivalence if, for every Σ-local F, (56) is a weak equivalence. We define *the Σ-local model structure* $\Sigma^{-1}\mathrm{PSh}^{\Delta}(T)$ on $\mathrm{PSh}^{\Delta}(T)$, whose weak equivalences are Σ-equivalences and whose cofibrations are still the monomorphisms, and whose fibrant objects are the fibrant objects in $\mathrm{PSh}^{\Delta}(T)$ that are Σ-local. The identity functors give a Quillen adjunction $\mathrm{PSh}^{\Delta}(T) \leftrightarrows \Sigma^{-1}\mathrm{PSh}^{\Delta}(T)$. Moreover, the right derived functor

$$\mathrm{Ho}(\Sigma^{-1}\mathrm{PSh}^{\Delta}(T)) \to \mathrm{Ho}(\mathrm{PSh}^{\Delta}(T))$$

is fully faithful and its essential image is the subcategory of Σ-local objects.

Now let τ be a Grothendieck topology on T and Σ_τ be the set of τ-covering sieves viewed as monomorpshims $R \to X$ in $\mathrm{PSh}^{\Delta}(T)$ with X representable. Then, $\mathrm{Shv}_\tau(T, \mathcal{S})$ is identified with the ∞-category $N\big((\Sigma_\tau^{-1}\mathrm{PSh}^{\Delta}(T))^{\mathrm{cf}}\big)$ associated to the full simplicial sub-category of the fibrant-cofibrant objects of $\Sigma_\tau^{-1}\mathrm{PSh}^{\Delta}(T)$.

6.5 *cd*-structure

Definition 6.9. *([V10, Def.2.1], [AHW17, Def.2.1.1]) Let T be a category with an initial object $\emptyset$.*

(1) A cd-structure on T is a collection P of commutative squares:

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array} \tag{57}$$

such that if $Q \in P$ and $Q' \simeq Q$ in C, then $Q' \in P$.

(2) The Grothendieck topology τ_P on T generated by a cd-structure P is the coarsest topology such that the empty sieve covers $\emptyset$ and that for every square (57) in P, the sieve on X generated by $U \to X$ and $V \to X$ is a τ_P-covering.

Theorem 6.10. *([V10], [AHW17, Th.3.2.5]) Let T be a small category with a strict initial object $\emptyset$[27] and P be a cd-structure on C. Assume*

1. Every square in P is cartesian.

[27]i.e. any morphism $X \to \emptyset$ is an isomorphism.

2. The pullback of every square (57) *in P via every morphism $X' \to X$ in T exists and belongs to P.*
3. For every square (57) *in P, $U \to X$ is a monomorphism.*
4. For every square (57) *in P, the following square belongs to P:*

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \\ V \times_U V & \longrightarrow & Y \times_X Y \end{array}$$

Then, $F \in \mathrm{PSh}(T, \mathcal{S})$ is a τ_P-sheaf if and only if $F(\emptyset) \simeq *$ and F send every square in P to a cartesian square in $\mathcal{S}$.

An easy argument upgrades the above theorem to coeffcients in any ∞-category $\mathcal{C}$ admitting all small limits.

Corollary 6.11. *Suppose that $\mathcal{C}$ be an ∞-category admitting small limits, Then, $F \in \mathrm{PSh}(T, \mathcal{C})$ is a τ_P-sheaf if and only if $F(\emptyset) \simeq *$ and F sends every square in P to a cartesian square in $\mathcal{C}$.*

6.6 Homotopy dimension

Truncated spaces. Recall that for $n \geq -2$ one says that a space $K \in \mathcal{S}$ is *n-truncated* if $\mathrm{Map}(D^{n+2}, K) \xrightarrow{\sim} \mathrm{Map}(S^{n+1}, K)$, where D^{n+2} is the $(n+2)$-disc, S^{n+1} is its boundary, the $(n+1)$-sphere, so $S^0 = * \sqcup *$ and $S^{-1} := \varnothing$, [Lur09, Lem.5.5.6.17]. For $n \geq -1$, this is equivalent to asking that $\pi_i(K, k) \cong *$ for all $k \in \pi_0(K)$ and all $i > n$, [Lur09, p.xiv]. This leads to a decreasing sequence of full subcategories

$$\mathcal{S} \hookleftarrow \cdots \hookleftarrow \mathcal{S}_{\leq 1} \hookleftarrow \mathcal{S}_{\leq 0} \hookleftarrow \mathcal{S}_{\leq -1} \hookleftarrow \mathcal{S}_{\leq -2} = \{*\}$$

which in low degrees is

$$\mathcal{S}_{\leq 1} = \text{ 1-groupoids}, \quad \mathcal{S}_{\leq 0} = \text{ discrete spaces}, \quad \mathcal{S}_{\leq -1} = \{\varnothing \to *\}, \quad \mathcal{S}_{\leq -2} = \{*\}.$$

By definition $\mathcal{S}_{\leq n} \subset \mathcal{S}$ is the subcategory of $\{S^{n+1} \to D^{n+2}\}$-local objects, [Lur09, Def.5.5.4.1], so localisation [28] at $\{S^{n+1} \to D^{n+2}\}$ is a left adjoint

$$(-)_{\leq n} : \mathcal{S} \to \mathcal{S}_{\leq n}$$

to inclusion.[29]

[28] We mean localisation in the sense of [Lur09, Def.5.2.7.2]. So $\mathcal{S} \to \mathcal{S}_{\leq n}$ is the universal functor sending every morphism in the *strong saturation*, [Lur09, Def.5.5.4.5], of $\{S^{n+1} \to D^{n+2}\}$ to an equivalence, [Lur09, Prop.5.2.7.12], [Lur09, Prop.5.5.4.15].

[29] Existence of the left adjoint can be deduced from the adjoint functor theorem, [Lur09, Cor.5.5.2.9] or constructed directly using the small object argument applied to the set $\{S^i \to D^{i+1}\}_{i>n}$. See Hatcher's textbook [Hat, Exam.4.17] for an extremely concrete construction.

The notion of n-truncatedness is extended to a general ∞-category T, such as $T = \mathrm{PSh}(C, \mathcal{S})$ or $\mathrm{Shv}_\tau(\mathcal{C}, \mathcal{S})$ for a small category $\mathcal{C}$ with a Grothendieck topology τ, by declaring an object $F \in T$ to be *n-truncated* if the mapping space $\mathrm{Map}_T(G, F)$ is n-truncated for all objects G of T. If T is presentable[30] then the inclusion admits a left adjoint, [Lur09, Prop.5.5.6.18],

$$(-)_{\leq n} : T \to T_{\leq n}.$$

Definition 6.12. *For $n \geq 0$, we say $F \in T$ is n-connective if $F_{\leq n-1} \simeq *$.*

Remark 6.13. *([Lur09, Pr. 6.5.1.12]) $F \in T$ is n-connective if and only if $\tau_{\leq 0} F \neq \emptyset$ and $\pi_i(F) = *$ for all $i < n$, where $\pi_i(F)$ is the categorical homotopy group:*

$$\pi_i(F) := \tau^F_{\leq 0}(F^{S^n} \xrightarrow{i_n^*} F) \in T_{/F},$$

where i_n^* is induced by the canonical morphism $* \to S^n$ and $\tau^F_{\leq 0}$ is the 0-th truncation on the slice category $T_{/F}$.

Remark 6.14. *If $T = \mathrm{Shv}_\tau(\mathcal{C}, \mathrm{Set})$ has enough points, then $F \in \mathrm{Shv}_\tau(\mathcal{C}, \mathcal{S})$ is n-connective if and only if $a_\tau \underline{\pi}_i(F) = *$ for all $i < n$, where $\underline{\pi}_i(F)$ is the homotopy presheaf of F and a_τ is the τ-sheafification. This follows from Lemma 6.15 below.*

Lemma 6.15. *([Lur09, 5.5.6.28 and 6.5.1.4]) If $\phi : T \to T'$ is a geometric morphism of ∞-topoi[31], then for any $F \in T'$, there is a canonical equivalence*

$$\phi^*(F_{\leq i}) \simeq (\phi^* F)_{\leq i},$$

and a canonical isomorphism

$$\phi^*(\pi_i(F)) \simeq \pi_i(\phi^*(F)).$$

The following lemma plays a key role in the proof of Theorem 4.15.

Lemma 6.16. *Let T, T' be ∞-topoi and $\lambda : T \to T'$ be a functor. If λ admits a right adjoint and preserves the final object, then λ preserves n-connective objects for all $n \geq 0$.*

Proof. See [Lur09, 5.5.6.28] and also [KS23, §7.2]. □

Example 6.17. (1) If $\Phi : \mathrm{Shv}_\tau(C, \mathcal{S}) \to \mathcal{S}$ is any fibre functor[32] and $F \in \mathrm{Shv}_\tau(C, \mathcal{S})$, we have $F_{\leq n} \cong * \Rightarrow \Phi(F)_{\leq n} \cong *$.

[30] i.e. T admits all small colimits and is of the form $T = \mathrm{Ind}(T')$ for some small category T'.
[31] i.e. there is a pair of adjoint functors $\phi^* : T' \leftrightarrows T : \phi_*$ such that ϕ^* preserves finite limits.
[32] As in the case of sets, Φ is a fibre functor if it preserves all colimits and finite limits, cf.[Lur09, Rem.6.3.1.2, Cor.5.5.2.9, Thm.6.1.0.6].

(2) If $X \in C$ is any object and $F \in \mathrm{Shv}(C, \mathcal{S})$, we have $F_{\leq n} \cong * \Rightarrow (F|_X)_{\leq n} \cong *$, where $(-)|_X : \mathrm{Shv}(C, \mathcal{S}) \to \mathrm{Shv}(C_{/X}, \mathcal{S})$ is the restriction functor with $C_{/X}$ equipped with the induced topology: coverings in $C_{/X}$ are precisely those families which are sent to coverings in C; the projection $C_{/X} \to C$ is a continuous and cocontinuous morphism of sites.
(3) Recall that a morphism of sites $\phi : (C, \tau) \to (D, \sigma)$ is *cocontinuous* if for every $U \in C$ and σ-covering family $\mathcal{U} = \{U_i \to \phi U\}_{i \in I}$ there is a τ-covering family $\{V_i \to U_i\}$ such that $\{\phi V_i \to \phi U\}_{i \in I}$ refines $\mathcal{U}$, [Sta18, 00XJ]. By [Sta18, 00XL], we have a pair of adjoint functors $\phi^* : \mathrm{Shv}_\sigma(D) \leftrightarrows : \mathrm{Shv}_\tau(C) : \phi_*$. Here, $\phi^* = a_\tau \phi^p$ where $\phi^p : \mathrm{PSh}(D) \to \mathrm{PSh}(C)$ is the restriction along ϕ and $a_\tau : \mathrm{PSh}(C) \to \mathrm{Shv}_\tau(C)$ is the τ-sheafification and $\phi_* = \phi_p$ is the right Kan extension along ϕ which preserves sheaves by [Sta18, 00XK].

As a right adjoint, global sections $\mathrm{Map}(*, -)$ does not preserve n-connectivity in general. Homotopy dimension describes how badly this fails.

Definition 6.18 ([Lur09, Prop.6.5.1.12, Def.7.2.1.1])**.** *One says the ∞-topos T has* homotopy dimension $\leq d$ *if the global section functor* $\mathrm{Map}(*, -) : T \to \mathcal{S}$ *sends d-connective objects to 0-connective objects, i.e. for every $F \in T$, we have*

$$F_{\leq d-1} \cong * \Rightarrow \mathrm{Map}_T(*, F)_{\leq -1} \cong *.$$

Note that the latter condition is equivalent to $\mathrm{Map}_T(*, F)$ *is non-empty.*

Remark 6.19. If $\mathrm{Shv}_\tau(C, \mathcal{S})$ has homotopy dimension $\leq d$, then we have

$$F_{\leq d+n} \cong * \Rightarrow \mathrm{Map}(*, F)_{\leq n} \cong *$$

for all $n \geq -1$, [Lur09, Def.7.2.1.6, Lem.7.2.1.7].

Exercise 6.20. Consider the category $\mathbb{N} = \{0 \to 1 \to 2 \to \dots\}$. An object of $\mathrm{PSh}(\mathbb{N}, \mathcal{S})$ is a diagram $\dots \to K(2) \to K(1) \to K(0)$ and the global sections functor is given by $\{K(n)\}_{n \in \mathbb{N}} \mapsto \lim_{n \in \mathbb{N}} K(n)$. Show that the homotopy dimension of $\mathrm{PSh}(\mathbb{N}, \mathcal{S})$ is ≤ 1 but not ≤ 0

Example 6.21 ([CM21, Cor.3.11, Thm.3.18])**.** If C_λ is a filtered system of finitary[33] excisive[34] sites with colimit C, then Clausen and Mathew show that $\mathrm{Shv}(C, \mathcal{S})$ has homotopy dimension $\leq d$ if all $\mathrm{Shv}(C_\lambda, \mathcal{S})$ do. Using this they show that for any qcqs algebraic space whose underlying topological space has Krull dimension $\leq d$, the ∞-topos $\mathrm{Shv}(X_{\mathrm{Nis}}, \mathcal{S})$ has homotopy dimension $\leq d$. It also follows from this that if X is a qcqs scheme of valuative dimension d then $\mathrm{RZ}(X_{\mathrm{Nis}})$ has homotopy dimension $\leq d$.

[33] A site is finitary if it has finite limits and every covering family is refineable by a finite one.
[34] A site is excisive if for all $U \in C$, the functor $F \mapsto \mathrm{Map}(U, F)$ commutes with filtered colimits.

Lemma 6.22. *If T has homotopy dimension$\leq d$, then it also has cohomological dimension$\leq d$.*

Proof. See [Lur09, Cro.7.2.2.30]. We give a proof in case $T = \mathrm{Shv}_\tau(\mathcal{C}, \mathcal{S})$. For a τ-sheaf of abelian groups $\mathcal{A}$ on $\mathcal{C}$ and an integer $n \geq 0$, there exists an Eilenberg-MacLane object $K(\mathcal{A}, n) \in T$ such that (cf. [Lur09, 7.2.2.17])

1. $K(\mathcal{A}, n)$ is n-connective and n-truncated.
2. There exists an isomorphism $H^n(\mathcal{C}, \mathcal{A}) \simeq \pi_0 \operatorname{Map}(*, K(\mathcal{A}, n))$.

Assume $n > d$. For $a \in H^n(\mathcal{C}, \mathcal{A})$, a nullhomotopy of a is equivalent to a global section of the pullback X of

$$\begin{array}{ccc} * & \xrightarrow{0} & K(\mathcal{A}, n) \\ & & \uparrow{\scriptstyle a} \\ & & * \end{array}$$

The lemma follows from the fact that X is $(n-1)$-connective, which follows from the long exact sequence of categorical homotopy groups (see [Lur09, 6.5.1.5]). $\square$

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