CONJECTURE OF COLLIOT-THÉLÈNE ON ZERO-CYCLES OVER LOCAL FIELDS: ABSTRACT

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Let X be a smooth projective variety over a field k. Let $\mathrm{CH}_0(X)$ be the Chow group of the zero-cycles on X modulo rational equivalence and let $A_0(X) \subset \mathrm{CH}_0(X)$ be the subgroup of cycle classes of degree 0. Let

$$alb_X \;:\; A_0(X) \to Alb_X(k)$$

be the map induced by the Albanese map $X \to Alb_X$, where $Alb_X(k)$ is the group of the k-rational points of Alb_X.

By Abel's theorem alb_X is injective if $\dim(X) = 1$. In higher dimension, Mumford discovered that the situation is rather chaotic: $\mathrm{Ker}(alb_X)$ may be too large to be understood by a standard geometric structure.

On the other hand, the situation in case k is a field of arithmetic nature presents a striking contrast to the above case. If k is finite, $\mathrm{Ker}(alb_X)$ has been explicitly determined by geometric class field theory. One of the most fascinating and challenging conjectures in arithmetic geometry is a conjecture of Bloch and Beilinson that $\mathrm{Ker}(alb_X)$ is finite in case k is a number field.

In this talk we focus on the case where k is a p-adic field. In this case Colliot-Thélène posed the following questions ([1]):

$$(CT1) \qquad \mathrm{Ker}(alb_X) = D(X) \bigoplus \text{ (finite) } ??$$

Here $D(X)$ is the maximal divisible subgroup of $A_0(X)$. By a known result on the structure of $Alb_X(k)$, (CT1) implies

$$(CT1)' \qquad A_0(X) = D(X) \bigoplus \text{ (finite) } ??$$

where $D'(X)$ is the maximal subgroup of $A_0(X)$ divisible by all integers prime to p.

Another question of Colliot-Thélène is the following.

$$(CT2) \qquad D(X)_{tor} = 0 \;??$$

Note

$$(CT1) \;+\; (CT2) \;\Rightarrow\; \mathrm{CH}_0(X)_{tor} \text{ is finite}$$

There have been several affirmative results on the questions for special varieties; surfaces, varieties fibered over curves in quadrics or Severi-Brauer varieties, products of curves, rationally connected varieties (see [1]. Some of them are mentioned in the talk).

In this talk I present an affirmative result on (CT1) and a negative result on (CT2).

The affirmative result (joint work with K. Sato, [3]) affirms that (CT1)′ is true if X has a regular projective flat model $\mathcal{X}$ over the ring $\mathcal{O}_k$ of integers in k such that the reduced part of its special fiber is a simple normal crossing divisor on $\mathcal{X}$.

The negative result (joint work with M. Asakura, [2]) provides an example of a smooth surface $X \subset \mathbb{P}^3_k$ such that $\mathrm{CH}_0(X)\{\ell\}$, the ℓ-primary torsion part of $\mathrm{CH}_0(X)$, is infinite for all prime $\ell \neq p$.

The affirmative result is deduced from a theorem which claims that the étale cycle class map for Chow group of 1-cycles on the model $\mathcal{X}$ of X over $\mathcal{O}_k$ is an isomorphism.
Main ingredients in its proof are:
the Bertini theorem for schemes over a discrete valuation ring by Jannsen-Saito,
the absolute purity theorem of Gabber,
the Weil conjecture proved by Deligne,
an affine vanishing theorem of Artin-Gabber and its refinement.

A key step in the proof of the negative result is disproving a variant of the Bloch-Kato conjecture which characterizes the image of an ℓ-adic regulator map from a higher Chow group to a continuous étale cohomology of X by using p-adic Hodge theory. With the aid of the theory of mixed Hodge modules, we reduce the problem to showing the exactness of the de Rham complex associated to a variation of Hodge structure, which is proved by the infinitesimal method in Hodge theory.

References

[1] J.-L. Colliot-Thélène, *L'arithmétique des zéro-cycles* (*exposé aux Journées arithmétiques de Bordeaux, 1993*), J. Théor. Nombres Bordeaux **7** (1995), 51–73.

[2] M. Asakura and S. Saito: *Surfaces over a p-adic field with infinite torsion in the Chow group of* 0*-cycles*, to appear in Algebra and Number Theory.

[3] S. Saito and K. Sato: *A finiteness theorem on zero-cycles over p-adic fields*, to appear in Ann. of Math.