ON PRO-CDH DESCENT ON DERIVED SCHEMES

SHANE KELLY, SHUJI SAITO, AND GEORG TAMME

Abstract. We prove a 'pro-cdh descent' result for suitably connective localizing invariants and the cotangent complex on arbitrary qcqs derived schemes. As an application, we deduce a generalised Weibel vanishing for negative K-groups of non-Noetherian schemes.

Contents

1. Introduction 1
2. Preliminaries on derived algebraic geometry 5
3. Modifications of derived schemes 11
4. Pro-cdh descent for connective localizing invariants 13
5. Pro-cdh descent for the cotangent complex and motivic cohomology 17
6. Generalised Weibel vanishing 20
References 26

1. Introduction

The algebraic K-groups of a blowup $\mathrm{Bl}_Z(X)$ of a quasi-compact, quasi-separated (qcqs, for short) scheme X in a regularly immersed center Z have been computed by Thomason [Tho93] (see also [CHSW08, Prop. 1.5]). In particular, the blowup square

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{Bl}_Z(X) \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & X \end{array}$$

gives rise to a cartesian square of algebraic K-theory spectra and hence to a long exact sequence of algebraic K-groups. If the closed immersion $Z \hookrightarrow X$ is not regular, this will fail in general. However, if X is Noetherian, and one takes infinitesimal information into account, one still gets a cartesian square. More precisely, if $Z(n)$ and $E(n)$ denote the n-th infinitesimal thickenings of Z in X and E in $\mathrm{Bl}_Z(X)$, respectively, the square of pro-spectra

$$\begin{array}{ccc} K(X) & \longrightarrow & \{K(Z(n))\}_n \\ \downarrow & & \downarrow \\ K(\mathrm{Bl}_Z(X)) & \longrightarrow & \{K(E(n))\}_n \end{array}$$

Date: July 5, 2024.

The first author is supported by JSPS KAKENHI Grant (19K14498) and Bilateral Joint Research Projects (120213206). The second author is supported by JSPS Grant-in-aid (B) #20H01791 representative Shuji Saito. The third author is partially supported by the DFG through TRR 326 (Project-ID 444845124).

is weakly cartesian and hence induces a long exact sequence of pro-abelian groups. Note here that the pro-systems only depend on the underlying topological space of Z (and E) as a closed subset of X (and $\mathrm{Bl}_Z(X)$, respectively) but not on the particular subscheme structure. In fact, more generally any abstract blowup square of Noetherian schemes gives rise to a weakly cartesian square of K-theory pro-spectra. In this form, this result, often coined "pro-cdh descent",[1] is first proven in [KST18b], but there are important precursors [KS02, Cor06, Kri10, GH06, GH11, Mor16a, Mor18]. A variant for Noetherian ANS stacks has been considered in [BKRS22]. These pro-cdh descent results play a central role in the resolution of Weibel's K-dimension conjecture [KST18b] (and [BKRS22] for stacks) and the development of a continuous K-theory of rigid spaces [Mor16b, KST18a, KST23].

Given the rising interest in non-Noetherian schemes, which appear naturally, for example, when working over a perfectoid base ring such as $\mathcal{O}_{\mathbb{C}_p}$, it is an obvious question, whether, or in which form, pro-cdh descent also holds in this general setting. The following example, a variant of which was constructed by Dahlhausen and the third author [DT22] precisely for that purpose and which was independently studied by the first two authors [Kel24, Footnote 2] with respect to the procdh topology, shows that pro-cdh descent as formulated above does not hold for general abstract blowup squares of non-Noetherian schemes.

Example 1.1. Let R be a valuation ring of dimension at least 2. Let $0 \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$ be prime ideals in R, and choose $x \in \mathfrak{m} \setminus \mathfrak{p}$ and $y \in \mathfrak{p} \setminus \{0\}$. Then x^n divides y for all $n \geq 1$ and for each n the square

$$\begin{array}{ccc} \operatorname{Spec}(R/(x^n)) & \hookrightarrow & \operatorname{Spec}(R/(y)) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ \operatorname{Spec}(R/(x^n)) & \hookrightarrow & \operatorname{Spec}(R/(xy)) \end{array}$$

is an abstract blowup square. If the induced square of K-theory pro-spectra would be weakly cartesian, the map $K(R/(xy)) \to K(R/(y))$ would be an equivalence. However, as $y^2 = 0$ in $R/(xy)$, $1 + y$ is a unit and defines a non-trivial element in the kernel of $K_1(R/(xy)) \to K_1(R/(y))$.

A posteriori, this is not surprising: A key point in [KST18b] was to prove Thomason's blowup formula for *derived blowups* in quasi-smooth centers together with the fact that for every closed subscheme of an affine Noetherian scheme there exists a quasi-smooth derived closed subscheme with the same underlying classical scheme. The condition that all (derived) rings and schemes appearing be Noetherian then allowed to pass from derived schemes back to classical schemes. For instance, for a Noetherian commutative ring A and elements $f_1, \ldots, f_r$ in A generating the ideal $I \subseteq A$, the pro-system of (Koszul type) derived quotients $\{A/\!\!/f_1^n, \ldots, f_r^n\}_n$ is equivalent to the pro-system of discrete rings $\{A/I^n\}_n$. This suggested that some variant of the above pro-cdh descent statement, where one takes "derived" infinitesimal thickenings everywhere, could still be true. This is precisely what we achieve in this paper. Our main K-theoretic result is the following.

Theorem A. *Let $f\colon Y \to X$ be a proper, locally almost finitely presented morphism of qcqs derived schemes which is an isomorphism outside a closed subset $Z \subseteq |X|$ whose open*

[1]We remark that it would be more appropriate to call this "pro-cdh excision" since it is not precisely a descent statement for some topology. However, together with Nisnevich excision, it does imply actual (Čech) descent for the procdh topology of [KS24], see Theorem 6.1 there. For this reason, and to be in line with the existing literature we stick to the term "pro-cdh descent."

complement is quasi-compact. Denote by $X_Z^\wedge$ and $Y_Z^\wedge$ the formal completion of X (respectively Y) along Z (respectively $f^{-1}(Z)$) viewed as ind-derived schemes. Then the square of pro-spectra

$$\begin{array}{ccc} K(X) & \longrightarrow & K(X_Z^\wedge) \\ \downarrow & & \downarrow \\ K(Y) & \longrightarrow & K(Y_Z^\wedge) \end{array}$$

is weakly cartesian.

Let us point out that this result is not specific to K-theory. It holds more generally for every localising invariant which is k-connective in the sense of [LT19] for some integer k; see Theorem 4.5 for the general statement. For instance, it also applies to THH, TC, and rational negative cyclic homology.

Remark 1.2. One may ask whether the same statement also holds for spectral schemes. As our proof makes essential use of the theory of derived blowups of derived schemes developed by Khan and Rydh [KR18], we cannot treat this more general case with our methods.

We also prove the analogous result for the cotangent complex, see Theorem 5.4. By the arguments of [EM23] one then obtains pro-cdh descent in the above form for motivic cohomology (Corollary 5.5). We thank Matthew Morrow for suggesting this result for the cotangent complex and indicating the application to motivic cohomology.

In [KST18b], pro-cdh descent of K-theory was used to derive Weibel's conjecture on the vanishing of negative K-groups of a Noetherian scheme X below $-\dim(X)$. It is an obvious question what we can say about negative K-groups if we merely assume X to be qcqs; see e.g. Morrow's Oberwolfach talk [Mor23]. In this generality, one can at least prove a vanishing result for Weibel's homotopy K-theory $KH(X)$. Let $\mathrm{vdim}(X)$ denote the valuative dimension of a scheme X introduced by Jaffard [Jaf60] (see [EHIK20] or [KS24, §7.1] for accounts). It coincides with the Krull dimension if X is Noetherian. Furthermore, write L_{cdh} for the cdh sheafification functor on presheaves of spectra and $K_{\geq 0}$ for the presheaf of connective algebraic K-theory.

Theorem 1.3. *Let X be a qcqs scheme of finite valuative dimension.*

(1) $KH_{-i}(X) = 0$ *for all* $i > \mathrm{vdim}(X)$.
(2) *The natural maps* $L_{\mathrm{cdh}}K_{\geq 0} \to L_{\mathrm{cdh}}K \to KH$ *are equivalences.*

Though we are mainly interested in (1), we include (2) for the sake of completeness. For schemes essentially of finite type over a field of characteristic 0, this was first proven in [Hae04]. For general Noetherian X, (1) was first proven in [KS17] and (2) in [KST18b]. The fact that KH is a cdh sheaf is [Cis13]. In the general case, the theorem follows easily from recent results on the cdh-topology [EHIK20] and the K-theory of valuation rings [KM21, KST21]. In fact, the proof given under Noetherian assumptions in [KM21] still works, and for the reader's convenience we reproduce this proof at the end of the paper. Alternatively, the proof of [KS17] respectively [KST18b] works mutatis mutandis, using the fact that a blowup does not increase the valuative dimension (this is not necessarily true for the Krull dimension).

The previous theorem suggests that for any qcqs scheme X, the negative K-groups vanish below $-\mathrm{vdim}(X)$. In this direction, we prove the following.

Theorem B (Theorem 6.1)**.** *Let X be a qcqs (derived) scheme. Assume that the underlying topological space of the spectrum of every local ring of every irreducible component of X is Noetherian. Then the following hold.*

(1) $K_{-i}(X) = 0$ *for all* $i > \mathrm{vdim}(X)$.
(2) *For all* $i \geq \mathrm{vdim}(X)$ *and any integer* $r \geq 0$*, the pullback map* $K_{-i}(X) \to K_{-i}(\mathbb{A}^r_X)$ *is an isomorphism. In other words,* X *is* $K_{-\mathrm{vdim}(X)}$*-regular.*

Remarks 1.4. (1) The assumption is for instance satisfied for any qcqs derived scheme of Krull dimension ≤ 1. It is also stable under passing to schemes essentially of finite type over X. In particular, the theorem holds for all schemes essentially of finite type over a valuation ring.
(2) For general qcqs $\mathbb{Q}$-schemes, (1) was announced by Elmanto and Morrow [Mor23, Thm. 4]. Moreover, they prove that the canonical map $K_{-i}(X) \to KH_{-i}(X)$ is an isomorphism for $i \geq \dim(X)$. Our result implies this only for $i \geq \mathrm{vdim}(X)$. Conversely, their isomorphism does not imply our assertion (2).
(3) We do not know if the valuative dimension is really needed in the vanishing bound. I.e., we don't know an example of a scheme X with $\mathrm{vdim}(X) > \dim(X)$ and $K_{-\mathrm{vdim}(X)}(X) \neq 0$.

Structure of the arguments and the paper. The proof of Theorems A and B follows the outline of [KST18b], but requires substantially more input from derived geometry. The proof of Theorem A is reduced to the two special cases of derived blowups and finite morphisms, respectively. This reduction is achieved by a structural result about modifications of derived schemes, Theorem 3.2, which we view as our main contribution here. In its proof, we need several preliminary results from derived algebraic geometry which we discuss in Section 2.

The case of Theorem A for derived blowups could essentially be proved as in [KST18b]. We here present a stronger result with a simplified proof due to Antieau ([Ant18]; we reproduce his proof in Proposition 4.1). For the case of finite morphisms in Theorem A, compared to [KST18b] we give a simple proof of a more general result (Proposition 4.2). This follows quite directly from [LT19].

The proof of our generalised Weibel vanishing statement is contained in Section 6. As we are working with non-Noetherian schemes, this needs some additional input.

Related work. Another approach to pro-cdh descent statements has been proposed by Clausen and Scholze. It is based on condensed mathematics [Sch] and Efimov's theory of localizing invariants of large categories [Efi24].

In forthcoming work, the first two authors introduce a "procdh topology" on qcqs derived schemes. It will follow from Theorem A (or 4.5) that every localizing invariant which is k-connective for some integer k satisfies descent for that topology.

Acknowledgement. We thank Christian Dahlhausen for discussions about pro-cdh descent in the non-Noetherian setting which led to the variant of Example 1.1 and which was one of the starting points of this project. We are grateful to Ben Antieau for allowing us to include his pro-excision result for derived blowups, which is stronger than the one obtained in [KST18b]. We thank Matthew Morrow for bringing our attention to the cotangent complex and further comments and Mauro Porta for related discussions.

2. Preliminaries on derived algebraic geometry

We freely use the language of derived algebraic geometry as developed by Toën–Vezzosi [TV08, TV05], Lurie [Lur18] and others. The following subsections mainly serve to fix some notation and recall some notions and facts that will be used later on. Readers familiar with derived algebraic geometry may safely skip this section and only come back when needed.

2.1. **Derived rings and schemes.** We write CAlg^{Δ} for the ∞-category of simplicial commutative rings [Lur18, §25.1] and refer to its objects as *derived rings.* There is a forgetful functor $\mathrm{CAlg}^{\Delta} \to \mathrm{CAlg}^{\mathrm{cn}}$, where $\mathrm{CAlg}^{\mathrm{cn}}$ denotes the ∞-category of connective $\mathbb{E}_\infty$-algebras in spectra. This functor preserves small limits and colimits and is conservative.

A *derived scheme* is a pair $X = (|X|, \mathcal{O}_X)$ consisting of a topological space $|X|$ and a sheaf $\mathcal{O}_X$ of derived rings on $|X|$ such that ${}^{\mathrm{cl}}X := (|X|, \pi_0\mathcal{O}_X)$ is a classical scheme and the higher homotopy sheaves $\pi_i\mathcal{O}_X$ are quasi-coherent $\pi_0\mathcal{O}_X$-modules. We call ${}^{\mathrm{cl}}X$ the *underlying classical scheme* or the *classical truncation* of X. Derived schemes form the objects of an ∞-category dSch, and similarly as above there is a forgetful functor from derived schemes to spectral schemes. There is also a functor of points approach to derived schemes. In other words, there is a fully faithful functor

$$\mathrm{dSch} \to \mathrm{Fun}(\mathrm{CAlg}^{\Delta}, \mathrm{Spc}),$$

which coincides with the Yoneda embedding on affine derived schemes [Lur18, §1.6].

A map $f\colon X \to Y$ of derived schemes is called *proper, a closed immersion, affine, or finite,* respectively, if the underlying map of classical schemes ${}^{\mathrm{cl}}f$ has the corresponding property. If $U \subseteq |X|$ is an open subset, then $(U, \mathcal{O}_X|_U)$ is itself a derived scheme. Such a derived scheme is called an *open subscheme of* X which we simply denote by U.

2.2. **Quasi-coherent modules.** If A is a derived ring, we write $\mathrm{Mod}(A)$ for the symmetric monoidal ∞-category of A-modules (in spectra). For a derived scheme X, we denote by $\mathrm{QCoh}(X)$ the category of quasi-coherent sheaves on X. These are stable ∞-categories with canonical t-structures, and we denote their connective part by $\mathrm{Mod}(A)^{\mathrm{cn}}$ and $\mathrm{QCoh}(X)^{\mathrm{cn}}$, respectively. Moreover, they only depend on the underlying spectrum or spectral scheme, respectively. If $X = \mathrm{Spec}(A)$ is affine, we have an equivalence $\mathrm{QCoh}(X) = \mathrm{Mod}(A)$.

If $Z \subseteq |X|$ is a closed subset, we denote by $\mathrm{QCoh}(X \,\mathrm{on}\, Z)$ the full subcategory of $\mathrm{QCoh}(X)$ spanned by those quasi-coherent sheaves which are supported on Z. If X is quasi-compact and quasi-separated (qcqs, for short) and the open complement of Z is compact, then $\mathrm{QCoh}(X \,\mathrm{on}\, Z)$ is compactly generated and its compact objects coincide with the perfect ones, [Lur18, Prop. 9.6.1.1], [CMNN, Prop. A.9]. Here a quasi-coherent sheaf on X is called *perfect* if and only its restriction to each affine open subscheme $U = \mathrm{Spec}(A) \subseteq X$ belongs to the smallest thick stable subcategory of $\mathrm{QCoh}(U) = \mathrm{Mod}(A)$ containing A. We write $\mathrm{Perf}(X)$ and $\mathrm{Perf}(X \,\mathrm{on}\, Z)$ for the corresponding subcategories. The arguments used to prove this also apply to the case of connective sheaves:

Lemma 2.1. *Let X be a qcqs derived scheme, and let $Z \subseteq |X|$ be a closed subset with compact open complement. Then $\mathrm{QCoh}(X \,\mathrm{on}\, Z)^{\mathrm{cn}}$ is compactly generated and the inclusion $\mathrm{QCoh}(X \,\mathrm{on}\, Z)^{\mathrm{cn}} \hookrightarrow \mathrm{QCoh}(X)^{\mathrm{cn}}$ preserves compact objects. An object of $\mathrm{QCoh}(X \,\mathrm{on}\, Z)^{\mathrm{cn}}$ is compact if and only if it is perfect (as an object of $\mathrm{QCoh}(X)$).*

Proof. The same argument as in the proof of [CMNN, Prop. A.9] proves the first two claims: In case X is affine, these follow from [Lur18, Prop. 7.1.1.12(e)] and the fact that $\mathrm{QCoh}(X)^{\mathrm{cn}}$

is compactly generated by [Lur18, Prop. 9.6.1.2]. The reduction of the global case to the local case is done by [Lur18, Ex. 10.3.0.2 (4), Prop. 10.3.0.3, Th. 10.3.2.1 (b)]. It then follows that an object of $\mathrm{QCoh}(X \,\mathrm{on}\, Z)^{\mathrm{cn}}$ is compact if and only if its image in $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is compact. The compact objects in the latter category coincide with the perfect, connective $\mathcal{O}_X$-modules by [Lur18, Prop. 9.6.1.2.] again. □

2.3. **Free algebras.** If A is a derived ring, we denote the category of derived A-algebras by CAlg^{Δ}_A. Its objects are derived rings B together with a map of derived rings $A \to B$. There is an obvious forgetful functor $\mathrm{CAlg}^{\Delta}_A \to \mathrm{Mod}(A)^{\mathrm{cn}}$ which admits a left adjoint, the free algebra functor, $\mathrm{LSym}^*_A \colon \mathrm{Mod}(A)^{\mathrm{cn}} \to \mathrm{CAlg}^{\Delta}_A$. If M is a connective A-module, the underlying A-module of $\mathrm{LSym}^*_A(M)$ is the direct sum $\bigoplus_{n\geq 0} \mathrm{LSym}^n_A(M)$, where $\mathrm{LSym}^n_A(M)$ is the n-th derived symmetric power of M as studied for instance by Quillen [Lur18, Constr. 25.2.2.6], whence the notation.

These constructions globalise: For a derived scheme X, write $\mathrm{CAlg}^{\Delta}_{\mathcal{O}_X}$ for the ∞-category of sheaves of derived rings on $|X|$ equipped with a map from $\mathcal{O}_X$ such that the underlying sheaf of $\mathcal{O}_X$-modules is quasi-coherent. There is an adjunction

$$\mathrm{LSym}^*_{\mathcal{O}_X} \colon \mathrm{QCoh}(X)^{\mathrm{cn}} \rightleftarrows \mathrm{CAlg}^{\Delta}_{\mathcal{O}_X} : \mathrm{forget},$$

the forgetful functor is conservative and preserves sifted colimits. Consequently, $\mathrm{LSym}^*_{\mathcal{O}_X}$ sends compact objects to compact objects.

For $\mathcal{A} \in \mathrm{CAlg}^{\Delta}_{\mathcal{O}_X}$ one can form the relative spectrum $\mathrm{Spec}(\mathcal{A})$ which comes with an affine morphism $\mathrm{Spec}(\mathcal{A}) \to X$.

2.4. **Finiteness conditions.** A map of derived rings $A \to B$ is called *locally of finite presentation* if B is a compact object of CAlg^{Δ}_A. It is called *almost of finite presentation* if B is an almost compact object CAlg^{Δ}_A, i.e. each truncation $\tau_{\leq n}B$ is a compact object of $\tau_{\leq n}\mathrm{CAlg}^{\Delta}_A$; see [Lur04, §3.1] and [Lur18, §4.1] for the analog notions for $\mathbb{E}_\infty$-algebras. It turns out that $A \to B$ is almost of finite presentation if and only if the underlying map of $\mathbb{E}_\infty$-algebras is almost of finite presentation; the analog for being locally of finite presentation is wrong.

These finiteness conditions are stable under base change: If $A \to B$ is locally or almost of finite presentation and $A \to A'$ is an arbitrary map, then also $A' \to B \otimes_A A'$ is locally or almost of finite presentation, respectively [Lur17, Rem. 7.2.4.28]. They are also stable under composition [Lur17, Rem. 7.2.4.29, Cor. 7.4.3.19].

If A is Noetherian, i.e. $\pi_0(A)$ is Noetherian in the classical sense and all higher homotopy groups are finitely generated $\pi_0(A)$-modules, then a derived A-algebra B is almost of finite presentation if and only if B is Noetherian and $\pi_0(B)$ is a classically finitely generated $\pi_0(A)$-algebra [Lur04, Prop. 3.1.5] or [Lur17, Prop. 7.2.4.31].

A map $f \colon Y \to X$ of derived schemes is *called locally of finite presentation* or *locally almost of finite presentation* (*lafp*, for short) if for all affine open subschemes $U = \mathrm{Spec}(A) \subseteq X$ and $V = \mathrm{Spec}(B) \subseteq Y$ with $f(V) \subseteq U$ the induced morphism $A \to B$ is locally of finite presentation or almost of finite presentation, respectively. As in the affine case, these notions are stable under base change and composition, and there is a characterization in the Noetherian case.

For example, if $\mathcal{F}$ is a perfect, connective $\mathcal{O}_X$-module, i.e. a compact object of $\mathrm{QCoh}(X)^{\mathrm{cn}}$, then $\mathrm{Spec}(\mathrm{LSym}^*_{\mathcal{O}_X}(\mathcal{F}))$ is locally of finite presentation over X. Using this observation, we prove the following lemma.

Lemma 2.2. *Let X be a qcqs derived scheme, and let $U \subseteq X$ be a qc open subset with complement Z. Let $i\colon Y \to X$ be closed immersion which is an isomorphism over U and such that the underlying map of classical schemes ${}^{\mathrm{cl}}Y \to {}^{\mathrm{cl}}X$ is finitely presented. Then there exists a factorization $Y \to Y' \to X$ of i such that*

1. *the morphism $Y \to Y'$ is an isomorphism on underlying classical schemes,*
2. *the morphism $Y' \to X$ is a closed immersion locally of finite presentation and an isomorphism over U.*

Proof. Let $\mathcal{I} = \mathrm{fib}(\mathcal{O}_X \to i_*\mathcal{O}_Y)$. As i is a closed immersion and an isomorphism over U, we have $\mathcal{I} \in \mathrm{QCoh}(X \,\mathrm{on}\, Z)^{\mathrm{cn}}$. As the morphism of classical schemes ${}^{\mathrm{cl}}Y \to {}^{\mathrm{cl}}X$ is classically of finite presentation, it follows that the image $\mathcal{J}$ of $\pi_0(\mathcal{I})$ in $\pi_0(\mathcal{O}_X)$ is of finite type. By Lemma 2.1, we can write $\mathcal{I}$ as a filtered colimit $\mathcal{I} = \mathrm{colim}_\alpha \mathcal{I}_\alpha$ where each $I_\alpha \in \mathrm{Perf}(X \,\mathrm{on}\, Z)^{\mathrm{cn}}$. As $\pi_0(-)$ commutes with filtered colimits, we have $\mathrm{colim}_\alpha \pi_0(\mathcal{I}_\alpha) = \pi_0(\mathcal{I})$. As $\pi_0(\mathcal{I}) \to \mathcal{J}$ is surjective and $\mathcal{J}$ is of finite type, there exists an index α such that the induced map $\pi_0(\mathcal{I}_\alpha) \to \mathcal{J}$ is surjective. By construction, we have the following commutative diagram in $\mathrm{QCoh}(X)^{\mathrm{cn}}$.

$$\begin{array}{ccc} \mathcal{I}_\alpha & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathcal{O}_X & \longrightarrow & i_*\mathcal{O}_Y \end{array}$$

By adjunction, this induces a commutative diagram

$$\begin{array}{ccc} \mathrm{LSym}^*_{\mathcal{O}_X}(\mathcal{I}_\alpha) & \longrightarrow & \mathcal{O}_X \\ \downarrow & & \downarrow \\ \mathcal{O}_X & \longrightarrow & i_*\mathcal{O}_Y \end{array}$$

in $\mathrm{CAlg}^{\Delta}_{\mathcal{O}_X}$. We define $\mathcal{A} \in \mathrm{CAlg}^{\Delta}_{\mathcal{O}_X}$ to be the tensor product $\mathcal{O}_X \otimes_{\mathrm{LSym}^*_{\mathcal{O}_X}(\mathcal{I}_\alpha)} \mathcal{O}_X$. The above diagram classifies a morphism $\mathcal{A} \to i_*\mathcal{O}_Y$ in $\mathrm{CAlg}^{\Delta}_{\mathcal{O}_X}$. We claim that it induces an isomorphism on π_0. Indeed, we compute[2]

$$\begin{aligned} \pi_0(\mathcal{A}) &\cong \pi_0(\mathcal{O}_X) \otimes^{\heartsuit}_{\pi_0(\mathrm{LSym}^*_{\mathcal{O}_X}(\mathcal{I}_\alpha))} \pi_0(\mathcal{O}_X) \\ &\cong \pi_0(\mathcal{O}_X) \otimes^{\heartsuit}_{\mathrm{Sym}_{\pi_0(\mathcal{O}_X)}(\pi_0(\mathcal{I}_\alpha))} \pi_0(\mathcal{O}_X) \\ &\cong \pi_0(\mathcal{O}_X)/\operatorname{im}(\pi_0(\mathcal{I}_\alpha) \to \pi_0(\mathcal{O}_X)) \\ &= \pi_0(\mathcal{O}_X)/\mathcal{J} \\ &\cong \pi_0(i_*\mathcal{O}_Y). \end{aligned}$$

We set $Y' = \mathrm{Spec}(\mathcal{A})$. By construction, we get the factorization $Y \to Y' \to X$ of i. The above computation shows that $Y \to Y'$ is an isomorphism on underlying classical schemes. Moreover, as I_α is supported on Z, $Y' \to X$ is an isomorphism over U. Finally, as I_α is perfect, it follows that $Y' \to X$ is locally of finite presentation, as desired. $\square$

[2]here the $\otimes^{\heartsuit}$ indicates the underived tensor product and Sym denotes the classical symmetric algebra

2.5. **Formal completion.** Let X be a qcqs derived scheme and let $Z \subseteq |X|$ be a closed subset whose open complement $|X| \setminus Z$ is quasi-compact. The *formal completion* $X^\wedge_Z$ is an ind-object of derived schemes with a map $i\colon X^\wedge_Z \to X$. It is determined by the following universal property: For any derived ring R, composition with i induces an equivalence of $\mathrm{Map}_{\mathrm{Ind(dSch)}}(\mathrm{Spec}(R), X^\wedge_Z)$ with the union of components of $\mathrm{Map}_{\mathrm{dSch}}(\mathrm{Spec}(R), X)$ consisting of those morphisms $\mathrm{Spec}(R) \to X$ that set-theoretically factor through Z; see [GR14, Prop. 6.5.5] for the existence of $X^\wedge_Z$ as an ind-derived scheme. More concretely, we can write $X^\wedge_Z$ as the ind-system

$$X^\wedge_Z = \{Z'\}_{Z' \hookrightarrow X} \tag{1}$$

of (a small cofinal subsystem of) all closed immersions of derived schemes $Z' \hookrightarrow X$ with $|Z'| = Z$.

If $X = \mathrm{Spec}(A)$ is affine, there exist finitely many elements $f_1, \dots, f_r \in \pi_0(A)$ whose zero locus is Z. In this case, $X^\wedge_Z$ can also be represented by

$$X^\wedge_Z = \{\mathrm{Spec}(A /\!\!/ f_1^\alpha, \dots, f_r^\alpha)\}_{\alpha \geq 1} \tag{2}$$

where the symbol $/\!\!/$ indicates the derived quotient, i.e. the (derived) tensor product $A \otimes_{\mathbb{Z}[t_1,\dots,t_r]} \mathbb{Z}$ where the maps send t_i to f_i^α and 0, respectively; see [Lur04, Prop. 6.1.1] or [Lur18, Lemma 8.1.2.2].

It follows immediately from the universal property that the formation of the derived completion commutes with base change: If $f\colon Y \to X$ is a quasi-compact map, then $Y^\wedge_{f^{-1}(Z)} \simeq X^\wedge_Z \times_X Y$. We therefore also write $Y^\wedge_Z$ instead of $Y^\wedge_{f^{-1}(Z)}$.

If X is a Noetherian classical scheme, then the formal completion is itself classical, equal to the classical formal completion. This follows for example from [Lur18, Lemma 17.3.5.7].

2.6. **Ample line bundles.** Ample line bundles on Noetherian derived schemes have been studied by Annala [Ann22]. We need some of the results in the more general setting of qcqs derived schemes. These are certainly well-known, the proofs are essentially the same as for classical schemes.

Let X be a qcqs derived scheme. A line bundle $\mathcal{L}$ on X is called *ample* if for any point $x \in X$ there exists an $n \geq 1$ and a global section $s \in \pi_0\Gamma(X, \mathcal{L}^{\otimes n})$ such that the non-vanishing locus X_s of s is affine and contains x. If $f\colon X \to Y$ is a morphism of derived schemes, then $\mathcal{L}$ is called *f-ample* if for every affine open subscheme $U \subseteq Y$ the restriction of $\mathcal{L}$ to $f^{-1}(U)$ is ample.

Let now $\mathcal{L}$ be any line bundle on X and $s \in \pi_0\Gamma(X, \mathcal{L})$ a global section. We view the latter as a map $s\colon \mathcal{O}_X \to \mathcal{L}$. If $\mathcal{F}$ is any quasi-coherent sheaf on X, we get a diagram

$$\mathcal{F} \xrightarrow{\otimes s} \mathcal{F} \otimes \mathcal{L} \xrightarrow{\otimes s} \mathcal{F} \otimes \mathcal{L}^{\otimes 2} \longrightarrow \dots \tag{3}$$

In the case of classical schemes, the following lemma is standard. For Noetherian derived schemes, it is [Ann22, Lemma 2.6].

Lemma 2.3. *In the above situation, there is a canonical equivalence*

$$\operatorname*{colim}_n \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \simeq \Gamma(X_s, \mathcal{F}).$$

Proof. If we take sections over X_s in (3), then all maps become equivalences. Thus the restriction maps induce a map

$$\mathrm{colim}\, \Gamma(X, (3)) \to \mathrm{colim}\, \Gamma(X_s, (3)) \simeq \Gamma(X_s, \mathcal{F}).$$

We claim that this map is an equivalence. As X is qcqs, a standard induction reduces us to the case where $X = \mathrm{Spec}(A)$ is affine and $\mathcal{L}$ is trivial. So we may assume $X = \mathrm{Spec}(A)$, $\Gamma(X, \mathcal{L}) = A$, and $\mathcal{F}$ corresponds to the A-module $M = \Gamma(X, \mathcal{F})$. In this case, the colimit in question identifies with $M[s^{-1}]$ which also identifies with $\Gamma(X_s, \mathcal{F})$ as $\mathcal{F}$ is quasi-coherent. □

Exactly as in [Ann22, Lemma 2.11], this can be used to prove the following lemma.

Lemma 2.4. *Let $f\colon X \to Y$ be a morphism of qcqs derived schemes, $\mathcal{L}$ a line bundle on X, and $(U_i)_{i\in I}$ an open covering of Y. Write f_i for the restricted morphism $f^{-1}(U_i) \to U_i$. Then the following are equivalent:*

(1) *$\mathcal{L}$ is f-ample;*
(2) *for every $i \in I$, the restriction $\mathcal{L}|_{f^{-1}(U_i)}$ of $\mathcal{L}$ is f_i-ample.* □

The existence of an ample line bundle implies the resolution property in the following form:

Lemma 2.5. *Let X be a qcqs derived scheme which carries an ample line bundle, and let $\mathcal{F}$ be a connective, quasi-coherent $\mathcal{O}_X$-module such that $\pi_0\mathcal{F}$ is of finite type as a $\pi_0\mathcal{O}_X$-module. Then there exists a vector bundle, i.e. a locally free $\mathcal{O}_X$-module of finite rank, $\mathcal{E}$ on X together with a map $\mathcal{E} \twoheadrightarrow \mathcal{F}$ which is surjective on π_0.*

Proof. Choose an ample line bundle $\mathcal{L}$. Replacing $\mathcal{L}$ by an appropriate tensor power, we may assume that there exist finitely many global sections $s_i \in \pi_0\Gamma(X, \mathcal{L})$, whose non-vanishing loci X_{s_i} form an affine covering of X. As each X_{s_i} is affine and $\mathcal{F}$ is quasi-coherent, we have isomorphisms $\pi_0\Gamma(X_{s_i}, \mathcal{F}) = \pi_0\Gamma(X_{s_i}, \pi_0\mathcal{F})$ and similarly for $\mathcal{O}_X$ in place of $\mathcal{F}$. The assumption that $\pi_0\mathcal{F}$ is a $\pi_0\mathcal{O}_X$-module of finite type then implies that each $\pi_0\Gamma(X_{s_i}, \mathcal{F})$ is a finitely generated $\pi_0\Gamma(X_{s_i}, \mathcal{O}_X)$-module. Choose finitely many generators $m_{ij} \in \pi_0\Gamma(X_{s_i}, \mathcal{F})$. By Lemma 2.3 there exists an integer N such that all the m_{ij} extend to global sections of $\mathcal{F} \otimes \mathcal{L}^{\otimes N}$. These give rise to a map $\mathcal{E} := \bigoplus_{ij} \mathcal{L}^{\otimes(-N)} \to \mathcal{F}$ which is surjective on π_0 by construction. □

2.7. **Quasi-smooth closed immersions and derived blowups.** Derived blowups were first introduced in [KST18b] for affine schemes in order to prove pro-cdh descent of algebraic K-theory on Noetherian schemes. They were then systematically studied and developed much further by Khan, Rydh, and Hekking [KR18, Hek21].

Let X be a derived scheme, and let $Z \hookrightarrow X$ be a *quasi-smooth closed immersion* [KR18, 2.3.6], i.e., Zariski locally on X, $Z \hookrightarrow X$ is the derived pullback of the map $\{0\} \hookrightarrow \mathbb{A}^r_{\mathbb{Z}}$ for some r and some morphism $X \to \mathbb{A}^r_{\mathbb{Z}}$. Equivalently, Zariski locally on X, $Z \hookrightarrow X$ is given by $\mathrm{Spec}(A/\!\!/f_1, \dots, f_r) \hookrightarrow \mathrm{Spec}(A)$ for suitable elements $f_i \in \pi_0(A)$. The number r is called the *virtual codimension* of the closed immersion. Then one can form the *derived blowup* $p\colon \mathrm{dBl}_Z(X) \to X$ *of X in Z* (or *with center Z*) which is characterized by a universal property: it classifies virtual Cartier divisors on derived X-schemes; see [KR18, §4.1] for details. The construction of the derived blowup commutes with arbitrary base change. Locally, if Z is the derived pullback of $\{0\} \hookrightarrow \mathbb{A}^r_{\mathbb{Z}}$ along a map $X \to \mathbb{A}^r_{\mathbb{Z}}$, then $\mathrm{dBl}_Z(X)$ is the derived pullback of the classical blowup $\mathrm{Bl}_{\{0\}}(\mathbb{A}^r_{\mathbb{Z}}) \to \mathbb{A}^r_{\mathbb{Z}}$. By the discussion of finiteness conditions in 2.4, as $\mathbb{A}^r_{\mathbb{Z}}$ is Noetherian, this in particular implies that the morphism $p\colon \mathrm{dBl}_Z(X) \to X$ is lafp. Clearly, p is an isomorphism outside Z and proper.

It follows from the description of the underlying classical scheme of $\mathrm{dBl}_Z(X)$ in [KR18, Thm. 4.1.5(vii)] that there is always a closed immersion $\mathrm{Bl}_{{}^{\mathrm{cl}}Z}({}^{\mathrm{cl}}X) \hookrightarrow {}^{\mathrm{cl}}\mathrm{dBl}_Z(X)$ and this is an isomorphism over ${}^{\mathrm{cl}}U$ where U is the open complement of Z.

The derived blowup $\mathrm{dBl}_Z(X)$ carries a canonical line bundle $\mathcal{O}(1)$ (the dual of the ideal sheaf defining the universal virtual Cartier divisor) and this line bundle is p-ample. Indeed, by Lemma 2.4 we may work affine-locally on X and hence assume that $p\colon \mathrm{dBl}_Z(X) \to X$ is the pullback of the classical blowup $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n_{\mathbb{Z}}) \to \mathbb{A}^n_{\mathbb{Z}}$, and the line bundle $\mathcal{O}(1)$ is the pullback of the classical canonical line bundle on $\mathrm{Bl}_{\{0\}}(\mathbb{A}^n)$ which is ample. As being relatively ample is stable under base change (the proof in [Ann22, Prop. 2.12] works in general), it follows that $\mathcal{O}(1)$ is p-ample.

In presence of an ample line bundle, every closed subset with quasi-compact complement is the support of a quasi-smooth closed subscheme. More precisely, we have the following lemma.

Lemma 2.6. *Let X be a qcqs derived scheme which carries an ample line bundle. Let $Z_0 \hookrightarrow {}^{\mathrm{cl}}X$ be a classically finitely presented closed subscheme. Then there exists a quasi-smooth closed subscheme $Z \hookrightarrow X$ whose classical truncation is Z_0.*

Proof. This is very similar to [BKRS22, Constr. A.2.2]. Let $J = \mathrm{fib}(\mathcal{O}_X \to \mathcal{O}_{Z_0})$. As $\mathcal{O}_{Z_0}$ is a discrete, quasi-coherent sheaf and $\mathcal{O}_X \to \mathcal{O}_{Z_0}$ is surjective on π_0, the sheaf J is connective, quasi-coherent, and $\pi_0 J$ is the ideal sheaf defining Z_0 in ${}^{\mathrm{cl}}X$. As $Z_0 \hookrightarrow {}^{\mathrm{cl}}X$ is classically of finite presentation, $\pi_0 J$ is a $\pi_0\mathcal{O}_X$-module of finite type. Hence Lemma 2.5 implies the existence of a vector bundle $\mathcal{E}$ on X and a map $\mathcal{E} \to \mathcal{J}$ which is surjective on π_0. Let $V(\mathcal{E})$ be the geometric vector bundle $V(\mathcal{E}) := \mathrm{Spec}(\mathrm{LSym}^*_{\mathcal{O}_X}(\mathcal{E}))$ over X. The composition $\mathcal{E} \to \mathcal{J} \to \mathcal{O}_X$ defines a section $s\colon X \to V(\mathcal{E})$, and we form the fibre product

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle 0} \\ X & \xrightarrow{\ s\ } & V(\mathcal{E}). \end{array}$$

Equivalently, we have

$$Z = \mathrm{Spec}(\mathcal{O}_X \otimes_{\mathrm{LSym}^*_{\mathcal{O}_X}(\mathcal{E})} \mathcal{O}_X).$$

By construction, $i\colon Z \hookrightarrow X$ is quasi-smooth. The same computation as in the proof of Lemma 2.2 shows that ${}^{\mathrm{cl}}Z = Z_0$, as desired. $\square$

2.8. **Pushouts of derived schemes.** We also need certain pushouts of derived schemes. These have been studied in [GR17]:

Lemma 2.7. *Let $i\colon Y_1 \to Y_1'$ be a closed immersion of derived schemes which is an isomorphism on underlying topological spaces, and let $f\colon Y_1 \to Y_2$ be an affine map in* dSch. *Then the following hold.*

(1) *The pushout square*

$$\begin{array}{ccc} Y_1 & \xrightarrow{\ i\ } & Y_1' \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f'} \\ Y_2 & \xrightarrow{\ i'\ } & Y_2 \sqcup_{Y_1} Y_1' \end{array}$$

exists in dSch. *Write $Y_2' := Y_2 \sqcup_{Y_1} Y_1'$. The map $Y_2 \to Y_2'$ is a closed immersion and an isomorphism on underlying topological spaces. In particular, if Y_2 is qcqs, then so is Y_2'.*

(2) *For an affine open subscheme* $U_2 \subseteq Y_2$ *with* $U_1 := f^{-1}(U_2) \subseteq Y_1$*, and the corresponding open subschemes* $U_i' \subseteq Y_i'$ $(i = 1, 2)$*, the map*

$$U_2 \sqcup_{U_1} U_1' \to U_2'$$

is an isomorphism.

(3) *If* f *is an open immersion, then so is* f'*.*

(4) *Assume* i *exhibits* Y_1 *as the underlying classical scheme of* Y_1'*. Then* $i' \colon Y_2 \to Y_2'$ *is an isomorphism on underlying classical schemes.*

Proof. Except for (4), this is [GR17, Ch. 1, Cor. 1.3.5]. By (2), we may assume $Y_i = \mathrm{Spec}(A_i)$ $(i = 1, 2)$ and $Y_1' = \mathrm{Spec}(A_1')$ are affine and $A_1 = \pi_0(A_1')$. By the construction of the pushout, $Y_2' = \mathrm{Spec}(A_2')$ where $A_2' = A_2 \times_{A_1} A_1'$ is the pullback in derived rings. We thus have an exact sequence of homotopy groups

$$\pi_1(A_1) \to \pi_0(A_2') \to \pi_0(A_2) \oplus \pi_0(A_1') \to \pi_0(A_1),$$

which implies (4) as $\pi_1(A_1) = 0$ and $\pi_0(A_1') \cong \pi_0(A_1)$. □

3. Modifications of derived schemes

In the following, X always denotes a qcqs derived scheme.

Definition 3.1. Let $U \subseteq X$ be a quasi-compact open subscheme. A *U-modification* of X is a proper morphism $f \colon Y \to X$ which is an isomorphism over U. A *closed U-modification* is a U-modification which is a closed immersion.

Note that we do not assume that a U-modification induces a bijection of the set of generic points. For example, if $Z \hookrightarrow X$ is a quasi-smooth closed immersion with $|Z| \cap |U| = \emptyset$, then the derived blowup $\mathrm{Bl}_Z(X) \to X$ is a U-modification which is moreover lafp (see 2.7). In fact, derived blowups and lafp closed U-modifications generate all lafp U-modifications in the following sense:

Theorem 3.2. *Assume that* X *carries an ample line bundle. Let* $U \subseteq X$ *be a quasi-compact open subscheme. Let* $f \colon Y \to X$ *be an lafp U-modification of* X*. Then there exists a commutative diagram*

$$\begin{array}{ccc} Y' & \xrightarrow{\ g\ } & Y \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \widetilde{X} & \xrightarrow{\ p\ } & X \end{array}$$

where g *is an lafp U-modification of* Y*,* h *is an lafp closed U-modification, and* p *is a derived blowup with center set-theoretically contained in* $X \setminus U$*. Moreover,* Y' *carries a* $(p \circ h)$*-ample line bundle.*

Proof. As $f \colon Y \to X$ is proper and lafp, the morphism of classical schemes ${}^{\mathrm{cl}}f \colon {}^{\mathrm{cl}}Y \to {}^{\mathrm{cl}}X$ is proper and classically locally of finite presentation, hence classically of finite presentation. It is also an isomorphism over ${}^{\mathrm{cl}}U$. By [RG71, Cor. 5.7.12] (or [Sta23, Tag 081T]), there exists a ${}^{\mathrm{cl}}U$-admissible blowup[3] $Y_0 \to {}^{\mathrm{cl}}X$ such that the morphism $Y_0 \to {}^{\mathrm{cl}}X$ factors through ${}^{\mathrm{cl}}Y \to {}^{\mathrm{cl}}X$. For the constructions to come, we need the open immersion ${}^{\mathrm{cl}}U \to Y_0$ to be affine. As this is not necessarily the case, we make a further blowup: As ${}^{\mathrm{cl}}U$ is quasi-compact, there

[3] i.e., a blowup in a finitely presented closed subscheme of ${}^{\mathrm{cl}}X$ which is set-theoretically contained in the complement of ${}^{\mathrm{cl}}U \subseteq {}^{\mathrm{cl}}X$

exists a finitely presented closed subscheme $T_0 \hookrightarrow Y_0$ whose underlying topological space is $Y_0 \backslash {}^{\mathrm{cl}}U$.[4] Let $Y_1 \to Y_0$ be the blowup of Y_0 in T_0. Then the canonical open immersion ${}^{\mathrm{cl}}U \to Y_1$ is affine as its complement is a Cartier divisor. As the composite of two ${}^{\mathrm{cl}}U$-admissible blowups is a ${}^{\mathrm{cl}}U$-admissible blowup [Sta23, Tag 080L], the composition $Y_1 \to Y_0 \to {}^{\mathrm{cl}}X$ is a ${}^{\mathrm{cl}}U$-admissible blowup, say $Y_1 = \mathrm{Bl}_{S_0}({}^{\mathrm{cl}}X)$ for some classically finitely presented closed subscheme $S_0 \hookrightarrow {}^{\mathrm{cl}}X$.

As X carries an ample line bundle, Lemma 2.6 implies the existence of a quasi-smooth closed immersion of derived schemes $S \hookrightarrow X$ whose classical truncation is $S_0 \hookrightarrow {}^{\mathrm{cl}}X$. Let $p\colon \widetilde{X} \to X$ be the derived blowup of X in S. Then there is a canonical closed immersion $Y_1 \hookrightarrow {}^{\mathrm{cl}}\widetilde{X}$, which is an isomorphism over ${}^{\mathrm{cl}}U$. Note that Y_1 need not be classically of finite presentation over X. However, by [GD71, Cor. 6.9.15] we can write the ideal sheaf defining Y_1 in ${}^{\mathrm{cl}}\widetilde{X}$ as a filtered colimit of sub-ideal sheaves of finite type. Passing to relative spectra, we deduce that the closed immersion $Y_1 \hookrightarrow {}^{\mathrm{cl}}\widetilde{X}$ can be written as a cofiltered limit of classically finitely presented closed immersions $Y_\alpha \to {}^{\mathrm{cl}}\widetilde{X}$ all of which are isomorphisms over ${}^{\mathrm{cl}}U$. As ${}^{\mathrm{cl}}Y \to {}^{\mathrm{cl}}X$ is classically of finite presentation, there exists an α such that the ${}^{\mathrm{cl}}X$-morphism $Y_1 \to {}^{\mathrm{cl}}Y$ factors through a morphism $Y_\alpha \to {}^{\mathrm{cl}}Y$, see [Gro66, Prop. 8.14.2]. Thus, so far, we have constructed a commutative diagram of classical schemes finitely presented over ${}^{\mathrm{cl}}X$

$$\begin{array}{ccc} Y_\alpha & \longrightarrow & {}^{\mathrm{cl}}Y \\ \big\downarrow & & \big\downarrow \\ {}^{\mathrm{cl}}\widetilde{X} & \longrightarrow & {}^{\mathrm{cl}}X \end{array}$$

in which all morphisms are isomorphisms over ${}^{\mathrm{cl}}U$. The lower right corner is the classical truncation of the cospan $\widetilde{X} \xrightarrow{p} X \xleftarrow{f} Y$.

As the composite of the affine open immersion ${}^{\mathrm{cl}}U \to Y_1$ with the closed immersion $Y_1 \hookrightarrow Y_\alpha$, the open immersion ${}^{\mathrm{cl}}U \to Y_\alpha$ is affine, too. By Lemma 2.7 we may hence form the pushout $Y_2 = Y_\alpha \sqcup_{{}^{\mathrm{cl}}U} U$ of derived schemes, for which we have ${}^{\mathrm{cl}}Y_2 = Y_\alpha$. As p and f are isomorphisms over U, we get induced morphisms $h_2\colon Y_2 \to \widetilde{X}$, $g_2\colon Y_2 \to Y$, and a commutative diagram

$$\begin{array}{ccc} Y_2 & \xrightarrow{g_2} & Y \\ {\scriptstyle h_2}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \widetilde{X} & \xrightarrow{p} & X. \end{array}$$

Note that h_2 is a closed immersion, as this only depends on the underlying map of classical schemes. Moreover, all morphisms in the above diagram are U-modifications. However, h_2 and g_2 need not be locally of almost finite presentation. In order to remedy this, we consider the induced morphism $k_2 = (h_2, g_2)\colon Y_2 \to \widetilde{X} \times_X Y$. As ${}^{\mathrm{cl}}Y \to {}^{\mathrm{cl}}X$ is separated, and as h_2 is a closed immersion, k_2 is a closed immersion, too. It is also an isomorphism over U. By construction, the map of underlying classical schemes ${}^{\mathrm{cl}}k_2\colon Y_\alpha = {}^{\mathrm{cl}}Y_2 \to {}^{\mathrm{cl}}(\widetilde{X} \times_X Y)$ is classically of finite presentation. We can hence apply Lemma 2.2 to obtain a factorization of k_2 through a closed derived subscheme $k'\colon Y' \hookrightarrow \widetilde{X} \times_X Y$ such that k' is locally of finite presentation and an isomorphism over U, and ${}^{\mathrm{cl}}Y' \cong {}^{\mathrm{cl}}Y_2$. The composite $h\colon Y' \hookrightarrow \widetilde{X} \times_X Y \to$

[4] For example, one can use absolute Noetherian approximation [TT90, Thm. C.9, C.2] to see this.

$\widetilde{X}$ is then lafp and an isomorphism over U. As its underlying map of classical schemes identifies with ${}^{\mathrm{cl}}h_2\colon {}^{\mathrm{cl}}Y_2 \hookrightarrow {}^{\mathrm{cl}}\widetilde{X}$, it is a closed immersion. Similarly, the composite $g\colon Y' \to \widetilde{X} \times_X Y \to Y$ is lafp and an isomorphism over U. This finishes the construction of the asserted commutative diagram.

It remains to prove the claim about ample line bundles. As discussed in 2.7, the canonical line bundle $\mathcal{O}(1)$ on the derived blowup $\widetilde{X}$ is p-ample. As h is a closed immersion and thus in particular affine, the pullback $h^*\mathcal{O}(1)$ is then $(p \circ h)$-ample. □

4. Pro-cdh descent for connective localizing invariants

In this section, we prove our main results on pro-descent for localizing invariants. The strategy is the same as in [KST18b]: We first prove the result for the special cases of derived blowups and finite modifications and then use the geometric input from Theorem 3.2 to handle the general case.

We begin by fixing some notation. Let k be a fixed commutative base ring (e.g. $k = \mathbb{Z}$). If E is an additive invariant of small k-linear ∞-categories with values in a stable presentable ∞-category $\mathcal{C}$, e.g. the ∞-category of spectra, and X is a qcqs derived k-scheme, we write $E(X)$ for $E(\mathrm{Perf}(X))$. Let $Z \subseteq |X|$ be a closed subset with quasi-compact open complement. Recall from 2.5 that the formal completion $X^\wedge_Z$ is an ind-derived scheme. Applying E we thus obtain a pro-object $E(X^\wedge_Z)$. We write $E(X, X^\wedge_Z)$ for the relative term $\mathrm{fib}(E(X) \to E(X^\wedge_Z))$ in $\mathrm{Pro}(\mathcal{C})$.

A version of the following Proposition was first proven in [KST18b]. There all schemes were assumed to be classical Noetherian schemes, E was K-theory, and the result only gave a weakly cartesian square of pro-spectra, see below for this notion. In a letter to Kerz, Antieau [Ant18] described a simplification of the proof which at the same time gives a cartesian square. We thank Ben Antieau for allowing us to include his argument in our paper.

Proposition 4.1. *Let X, Z, and E be as above. Let $\widetilde{X} \to X$ be a derived blowup in some quasi-smooth closed immersion $S \hookrightarrow X$ with S set-theoretically contained in Z. Then the square*

$$\begin{array}{ccc} E(X) & \longrightarrow & E(X^\wedge_Z) \\ \downarrow & & \downarrow \\ E(\widetilde{X}) & \longrightarrow & E(\widetilde{X}^\wedge_Z) \end{array}$$

is cartesian in $\mathrm{Pro}(\mathcal{C})$.

Proof. Let $r \geq 1$ be the virtual codimension of the derived blowup, and let D be its exceptional divisor, i.e., the universal virtual Cartier divisor on the derived blowup, so that there is a commutative diagram

$$\begin{array}{ccc} D & \xhookrightarrow{j} & \widetilde{X} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ S & \xhookrightarrow{i} & X. \end{array}$$

Recall from [Kha20, Thm. C] that $\mathrm{Perf}(\widetilde{X})$ has a semi-orthogonal decomposition as follows. The functor $p^*\colon \mathrm{Perf}(X) \to \mathrm{Perf}(\widetilde{X})$ is fully faithful, denote its essential image by $\mathcal{B}(0)$. For $1 \leq k \leq r-1$, the composed functor $j_*(q^*(-) \otimes_{\mathcal{O}_D} \mathcal{O}_D(-k))\colon \mathrm{Perf}(S) \to \mathrm{Perf}(\widetilde{X})$ is fully faithful, denote its essential image by $\mathcal{B}(-k)$. Then the sequence of full subcategories $(\mathcal{B}(0), \mathcal{B}(-1), \ldots, \mathcal{B}(-r+1))$ forms a semi-orthogonal decomposition of $\mathrm{Perf}(\widetilde{X})$. In particular, there is a decomposition

$$E(\widetilde{X}) \simeq E(\mathcal{B}(0)) \oplus \bigoplus_{k=1}^{r-1} E(\mathcal{B}(-k)) \simeq E(X) \oplus \bigoplus_{k=1}^{r-1} E(S). \tag{4}$$

Now let $Z' \hookrightarrow X$ be any closed derived subscheme with $|Z'| = Z$. Note that all ∞-categories appearing above are in fact $\mathrm{Perf}(X)$-linear, as are the functors between them. In particular, we can base change the semi-orthogonal decomposition of $\mathrm{Perf}(\widetilde{X})$ along $\mathrm{Perf}(X) \to \mathrm{Perf}(Z')$. As there are canonical equivalences (as follows from [Lur18, Cor. 9.4.3.8] by passing to compact objects; see also [BZFN10, Thm. 4.7] with slightly different hypotheses)

$$\mathrm{Perf}(\widetilde{X}) \otimes_{\mathrm{Perf}(X)} \mathrm{Perf}(Z') \simeq \mathrm{Perf}(\widetilde{X} \times_X Z'),$$
$$\mathrm{Perf}(S) \otimes_{\mathrm{Perf}(X)} \mathrm{Perf}(Z') \simeq \mathrm{Perf}(S \times_X Z'),$$

we conclude that $\mathrm{Perf}(\widetilde{X} \times_X Z')$ admits a semi-orthogonal decomposition

$$(\mathcal{B}(0)_{Z'}, \mathcal{B}(-1)_{Z'}, \ldots, \mathcal{B}(-r+1)_{Z'})$$

with $\mathcal{B}(0)_{Z'} \simeq \mathrm{Perf}(Z')$ and $\mathcal{B}(-k)_{Z'} \simeq \mathrm{Perf}(S \times_X Z')$ for $1 \le k \le r-1$. In particular,

$$E(\widetilde{X} \times_X Z') \simeq E(Z') \oplus \bigoplus_{k=1}^{r-1} E(S \times_X Z'). \tag{5}$$

Recall from (1) that $E(X^\wedge_Z) \in \mathrm{Pro}(\mathcal{C})$ is given concretely as the pro-object $\{E(Z')\}_{Z' \hookrightarrow X, |Z'|=Z}$ where Z' runs through all closed derived subschemes of X with $|Z'| = Z$. As formal completion is compatible with base change (see 2.5), we similarly have $E(\widetilde{X}^\wedge_Z) = \{E(\widetilde{X} \times_X Z')\}_{Z' \hookrightarrow X, |Z'|=Z}$. Comparing the decompositions (4) and (5) it thus suffices to prove that the functor induced by pullback

$$\mathrm{Perf}(S) \to \{\mathrm{Perf}(S \times_X Z')\}_{Z' \hookrightarrow X, |Z'|=Z}$$

is an equivalence of pro-∞-categories. For this, it suffices to check that the map of ind-derived schemes $\{S \times_X Z'\}_{Z' \hookrightarrow X, |Z'|=Z} \to S$ is an equivalence. But this is clear: By 2.5 again, the source represents the Z-completion $S^\wedge_Z$ of the target. As S is set-theoretically contained in Z, we clearly have $S^\wedge_Z = S$. □

Let ℓ be an integer. Recall from [LT19, Def. 2.5] that a spectra valued localizing invariant E is called *ℓ-connective* if, for any n-connective map ($n \ge 1$) of connective $\mathbb{E}_1$-ring spectra $A \to B$, the induced map $E(A) \to E(B)$ is $(n+\ell)$-connective. For example, K-theory, topological cyclic homology TC and rational negative cyclic homology $\mathrm{HN}(- \otimes \mathbb{Q}/\mathbb{Q})$ are 1-connective, THH is 0-connective; see [LT19, Ex. 2.6].

Recall also that a map of pro-spectra $\{C_\alpha\}_\alpha \to \{D_\alpha\}_\alpha$ is called a *weak equivalence* if each truncation $\{\tau_{\le n} C_\alpha\}_\alpha \to \{\tau_{\le n} D_\alpha\}_\alpha$ is an equivalence in Pro(Sp) and there are similar notions of being weakly cartesian, weakly contractible, and so on; see [LT19, Def. 2.27].

Proposition 4.2. *Let E be a localizing invariant of small k-linear ∞-categories that is ℓ-connective for some integer ℓ. Let $f \colon Y \to X$ be a finite, lafp morphism of qcqs derived*

k-schemes which is an isomorphism outside the closed subset Z with $|X| \setminus Z$ quasi-compact. Then the commutative square of pro-spectra

$$\begin{array}{ccc} E(X) & \longrightarrow & E(X^\wedge_Z) \\ \downarrow & & \downarrow \\ E(Y) & \longrightarrow & E(Y^\wedge_Z). \end{array} \tag{6}$$

is weakly cartesian.

Proof. We first reduce to the case that X is affine: As X is qcqs, we can write X as the colimit of a finite diagram of open affine subschemes $V_i \hookrightarrow X$. As any localizing invariant satisfies Zariski descent , we get $E(X) = \lim_i E(V_i)$ and $E(Y) = \lim_i E(Y \times_X V_i)$. If $Z' \hookrightarrow X$ is a closed subscheme with $|Z'| = Z$, we also have $E(Z') = \lim_i E(V_i \times_X Z')$. As finite limits in $\mathrm{Pro}(\mathcal{C})$ are computed level-wise and formal completion is compatible with base change, this implies $E(X^\wedge_Z) = \lim_i E((V_i)^\wedge_Z)$ and similarly for $E(Y^\wedge_Z)$. Replacing X by V_i and Z by $V_i \cap Z$ we thus reduce to the case that X is affine.

So assume now that X is affine, say $X = \mathrm{Spec}(A)$. As f is finite, also Y is affine, say $Y = \mathrm{Spec}(B)$. Let $\phi\colon A \to B$ be the corresponding morphism of derived rings and write $J = \mathrm{fib}(A \to B)$. As A and B are connective, J is (-1)-connective. By [Lur18, Cor. 5.2.2.2] the A-algebra B is almost perfect as an A-module, hence also J is almost perfect as an A-module, i.e. $\tau_{\leq n} J$ is a compact object in $\tau_{\leq n+1} \mathrm{Mod}(A)_{\geq -1} = \mathrm{Mod}(A)_{[-1,n]}$ for every n.[5]

Choose $f_1, \dots, f_r \in \pi_0(A)$ whose zero set is Z. Recall from (2) that $X^\wedge_Z$ is then represented by the ind-derived scheme $\{\mathrm{Spec}(A/\!\!/ f_1^\alpha, \dots, f_r^\alpha)\}_{\alpha \geq 1}$. Consider the commutative diagram of derived rings

$$\begin{array}{ccc} A & \longrightarrow & A/\!\!/ f_1^\alpha, \dots, f_r^\alpha \\ \downarrow & & \downarrow \\ B & \longrightarrow & B/\!\!/ f_1^\alpha, \dots, f_r^\alpha. \end{array} \tag{7}$$

We claim that as pro-system in α, this square is weakly cartesian. The map of vertical fibres (in A-modules) is the canonical map

$$J \longrightarrow J/\!\!/ f_1^\alpha, \dots, f_r^\alpha,$$

so we have to prove that this map is a weak equivalence as a pro-system in α. For any $i = 1, \dots, r$, the fibre of the map of pro-systems $J \to \{J/\!\!/ f_i^\alpha\}_\alpha$ is the pro-system

$$\{\ J \xleftarrow{f_i} J \xleftarrow{f_i} \dots\ \}. \tag{8}$$

We show below that this system is weakly contractible. We then get weak equivalences $J \xrightarrow{\simeq} \{J/\!\!/ f_1^\alpha\}_\alpha$ and $J \xrightarrow{\simeq} \{J/\!\!/ f_2^\alpha\}_\alpha \xrightarrow{\simeq} \{J/\!\!/ f_1^\alpha, f_2^\alpha\}_\alpha$, and so on, so $\{(7)\}_\alpha$ is indeed weakly cartesian.

The assumption that f be an isomorphism outside Z implies that $J[f_i^{-1}] = 0$ for $i = 1, \dots, r$. Note that

$$J[f_i^{-1}] = \mathrm{colim}(J \xrightarrow{f_i} J \xrightarrow{f_i} J \xrightarrow{f_i} \dots).$$

[5] The $(n+1)$-truncated objects in $\mathrm{Mod}(A)_{\geq -1}$ are precisely the n-truncated, (-1)-connective objects in $\mathrm{Mod}(A)$ with respect to the standard t-structure. Hence the usual truncation $\tau_{\leq n} J$ coming from the t-structure is the categorical $(n+1)$-truncation $\mathrm{Mod}(A)_{\geq -1} \to \tau_{\leq n+1} \mathrm{Mod}(A)_{\geq -1}$.

As the standard t-structure on $\mathrm{Mod}(A)$ is compatible with filtered colimits, we have

$$0 = \tau_{\leq n} J[f_i^{-1}] = \mathrm{colim}(\tau_{\leq n} J \xrightarrow{f_i} \tau_{\leq n} J \xrightarrow{f_i} \tau_{\leq n} J \xrightarrow{f_i} \dots)$$

As $\tau_{\leq n} J$ is compact in $\mathrm{Mod}(A)_{[-1,n]}$, we have

$$0 = \pi_0(\mathrm{map}(\tau_{\leq n} J, \tau_{\leq n} J[f_i^{-1}])) \cong \mathrm{colim}\, \pi_0(\mathrm{map}(\tau_{\leq n} J, \tau_{\leq n} J))$$

which means that there is an N such that the power f_i^N acts nullhomotopically on $\tau_{\leq n} J$. It follows that (8) is weakly contractible, and hence the pro-system of squares $\{(7)\}_\alpha$ is indeed weakly cartesian.

Note that for every α the canonical map $B \otimes_A (A /\!\!/ f_1^\alpha, \dots, f_r^\alpha) \to B /\!\!/ f_1^\alpha, \dots, f_r^\alpha$ is an equivalence. Thus we may apply the variant of [LT19, Thm. 2.32] for ℓ-connective localizing invariants to deduce that $\{(7)\}_\alpha$ induces a weakly cartesian square of E-theory pro-spectra.

□

Remark 4.3. In Proposition 4.2 one can actually relax the finiteness assumption if one adds other hypotheses: As the proof shows, we only need that the pro-systems (8) are weakly contractible for each i (using notation of the proof). This is satisfied if, on each truncation $\tau_{\leq n} J$, some power of each f_i acts null-homotopically.

If X and Y are n-truncated, then also J is n-truncated. It is then enough to assume that J is perfect to order n in the sense of [Lur18, Def. 2.7.0.1] in order to conclude that the pro-systems (8) are weakly contractible.

As a special case, if the map f in Proposition 4.2 is a closed immersion of classical qcqs schemes which is classically finitely presented, then the conclusion of the proposition holds, i.e. (6) is weakly cartesian.

Remark 4.4. There is a version of the above Proposition for stacks: Let X be a qcqs ANS derived algebraic stack [BKRS22, A.1], $Y \to X$ a finite, locally almost finitely presented morphism of derived algebraic stacks, and $Z \hookrightarrow X$ a closed immersion with quasi-compact open complement. Let E be any connected localizing invariant in the sense of [BKRS22, Def. C.1.3] (e.g. a 2-connective or a finitary 1-connective localizing [BKRS22, Rem. C.1.5]). Then the square (6) of pro-spectra is weakly cartesian.

Indeed, the proof of [BKRS22, Thm. 4.2.1] works with the following changes: As in the proof of Lemma 2.3.2 in *op. cit.*, the proof of our Proposition 4.2 shows that the formally completed square

$$\begin{array}{ccc} Y_Z^\wedge & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X_Z^\wedge & \longrightarrow & X \end{array} \tag{9}$$

is weakly cocartesian. As in the proof of Theorem 2.4.1 of *op. cit.* this implies that the square of derived (pro-)categories induced by (9) is weak pro-Milnor and satisfies weak pro-base change in the sense of Definitions C.2.4 and C.2.6 there. Hence by Theorem C.3.1 there the square (6) is weakly cartesian.

We now come to our main descent result, which in particular includes Theorem A.

Theorem 4.5. *Let E be a localizing invariant of small k-linear ∞-categories which is ℓ-connective for some integer ℓ. Let X be a qcqs derived k-scheme, $U \subseteq X$ a quasi-compact* *open subscheme, and denote by Z the closed subset $X \setminus U$. Let $f\colon Y \to X$ be a locally almost finitely presented U-modification of X. Then the square of pro-spectra*

$$\begin{array}{ccc} E(X) & \longrightarrow & E(X^\wedge_Z) \\ \downarrow & & \downarrow \\ E(Y) & \longrightarrow & E(Y^\wedge_Z) \end{array} \tag{10}$$

is weakly cartesian.

If X and Y are Noetherian classical schemes, and f is classically of finite type, then f is lafp (see 2.4). Moreover, the formal derived completions are equivalent to the classical formal completions (see 2.5). We thus recover the classical pro-cdh descent statement as formulated (for K-theory) for example in [KST18b, Thm. A].

Proof of Theorem 4.5. We first prove the theorem under the additional assumption that Y carries an f-ample line bundle. Exactly as in the proof of Proposition 4.2 we reduce to the case that X is affine. Then Y carries an ample line bundle.

By Theorem 3.2, we find an lafp U-modification $g\colon Y' \to Y$ such that $f \circ g$ factors as an lafp, closed U-modification $h\colon Y' \hookrightarrow \widetilde{X}$ followed by a derived blowup $p\colon \widetilde{X} \to X$ with center set-theoretically contained in Z. Both of these are isomorphisms over U. By Proposition 4.1 and Proposition 4.2, the maps of relative E-theory pro-spectra

$$E(X, X^\wedge_Z) \to E(\widetilde{X}, \widetilde{X}^\wedge_Z) \to E(Y', Y'^\wedge_Z)$$

are weak equivalences. Hence also the composite

$$E(X, X^\wedge_Z) \to E(Y, Y^\wedge_Z) \to E(Y', Y'^\wedge_Z) \tag{11}$$

is a weak equivalence. It follows that $E(X, X^\wedge_Z) \to E(Y, Y^\wedge_Z)$ is the inclusion of a direct summand (in the weak sense). As Y carries an ample line bundle, we can repeat this argument for the U-modification $Y' \to Y$. So also the second map in (11) is the inclusion of a direct summand. It now follows that both maps are in fact weak equivalences.

We now prove the theorem for general Y. As before, we may assume that X is affine. By Theorem 3.2 again, there exists an lafp U-modification $g\colon Y' \to Y$ such that Y' carries an ample line bundle relative to X. As X is affine, this line bundle is in fact ample and it is also ample relative to Y. Hence, by Step 1, the maps of pro-spectra $E(X, X^\wedge_Z) \to E(Y', Y'^\wedge_Z)$ and $E(Y, Y^\wedge_Z) \to E(Y', Y'^\wedge_Z)$ both are weak equivalences. By 3-for-2, also $E(X, X^\wedge_Z) \to E(Y, Y^\wedge_Z)$ is a weak equivalence, as desired. □

5. Pro-cdh descent for the cotangent complex and motivic cohomology

We fix some base ring k (e.g., $k = \mathbb{Z}$). For X a qcqs derived k-scheme, we denote by $L_X \in \mathrm{QCoh}(X)$ its (algebraic) cotangent complex relative to k, and by L^i_X its i-th derived exterior power ($i \geq 0$). If $X = \mathrm{Spec}(A)$ is affine, we also write L^i_A for the A-module $\Gamma(X, L^i_X)$ corresponding to L^i_X. If $Y \to X$ is a morphism of qcqs derived k-schemes, we denote by $L_{Y/X}$ its relative cotangent complex and by $L^i_{Y/X}$ its derived exterior powers. Similarly for a morphism of derived k-algebras $A \to B$.

If $Z \subseteq |X|$ is a closed subset with quasi-compact open complement, we obtain a *pro-completion along Z* functor

$$(-)^\wedge_Z\colon \mathrm{QCoh}(X) \to \mathrm{Pro}(\mathrm{QCoh}(X))$$

by pulling back to the ind-derived scheme $X^\wedge_Z$ and pushing forward. The usual formal completion along Z is given by composing the above functor with $\lim\colon \operatorname{Pro}(\operatorname{QCoh}(X)) \to \operatorname{QCoh}(X)$.

Lemma 5.1. *Let $X = \operatorname{Spec}(A)$ be an affine derived k-scheme, $Z \subseteq |X|$ a closed subset with quasi-compact open complement. Let $M \in \operatorname{Mod}(A \operatorname{on} Z)^{\mathrm{aperf}}$ be an almost perfect A-module supported on Z. Then the canonical map*

$$M \to M^\wedge_Z$$

is a weak equivalence in $\operatorname{Pro}(\operatorname{Mod}(A))$.

Proof. This was proven in the proof of Proposition 4.2 (replace J there by M). □

For a pro-system of derived rings $\{A(\alpha)\}_\alpha$ we denote by $\operatorname{Pro}(\operatorname{Mod})(\{A(\alpha)\})$ the ∞-category of pro-systems of modules over the pro-ring $\{A(\alpha)\}_\alpha$ (see [LT19, §2.4] for a precise definition).

Lemma 5.2. *In the situation of the previous lemma, choose $f_1, \dots, f_r \in \pi_0(A)$ defining Z, and write $A(\alpha) = A/\!\!/ f_1^\alpha, \dots, f_r^\alpha$. Then the pro-system*

$$\{L_{A(\alpha)/A}\}_\alpha \in \operatorname{Pro}(\operatorname{Mod})(\{A(\alpha)\})$$

vanishes. In fact, all transition maps in this pro-system are null-homotopic.

Proof. This follows by base change from the universal case: Let $R = k[T_1, \dots, T_r]$ be the polynomial ring over k, let $R \to k$ be the map sending all T_i to 0, and let $g_\alpha\colon R \to A$ be the map sending T_i to f_i^α. Then $A(\alpha) = k \otimes_{R,g_\alpha} A$ and consequently

$$L_{A(\alpha)/A} \simeq L_{k/R} \otimes_{R,g_\alpha} A \simeq L_{k/R} \otimes_k A(\alpha).$$

Consider the pro-system $\{L_{k/R}\}_\alpha \in \operatorname{Pro}(\operatorname{Mod}(k))$ whose transition maps are induced by the maps $R \to R$ sending the T_i to T_i^β. Then

$$\{L_{A(\alpha)/A}\}_\alpha \simeq \{L_{k/R}\}_\alpha \otimes_k \{A(\alpha)\}_\alpha.$$

Hence it suffices to prove that all transition maps $L_{k/R} \to L_{k/R}$ are null-homotopic in $\operatorname{Mod}(k)$. As $R \to k$ has a section, the transitivity triangle yields an equivalence $L_{k/R} \simeq \Sigma L_{R/k} \otimes_R k$. As R is a polynomial ring, $L_{R/k}$ is discrete, given by the module of Kähler differentials $\Omega^1_{R/k} = \bigoplus_i R dT_i$. The map $T_i \mapsto T_i^\beta$ induces $dT_i \mapsto \beta T_i^{\beta-1} dT_i$ in $\Omega^1_{R/k}$, and hence the zero map in $\Omega^1_{R/k} \otimes_R k \simeq \Sigma^{-1} L_{k/R}$ for $\beta > 1$. This proves our claim. □

Lemma 5.3. *In the situation of Lemma 5.2, the canonical map*

$$\{L^i_A \otimes_A A(\alpha)\}_\alpha \to \{L^i_{A(\alpha)}\}_\alpha$$

is an equivalence in $\operatorname{Pro}(\operatorname{Mod})(\{A(\alpha)\})$.

Proof. The transitivity triangle for the maps $A \to A(\alpha)$ and Lemma 5.2 imply the case $i = 1$. Passing to derived exterior powers over $\{A(\alpha)\}$ implies the general case. □

The following theorem was suggested by Matthew Morrow. It generalizes [Mor16a, Thm. 2.4] (see also [EM23, Lemma 8.5]).

Theorem 5.4. *Let X be a qcqs derived k-scheme, $U \subseteq X$ a quasi-compact open subscheme, and denote by Z the closed subset $X \setminus U$. Let $f\colon Y \to X$ be a locally almost finitely presented U-modification of X. Then for every $i \geq 0$, the square of pro-spectra (or pro-complexes)*

$$\begin{array}{ccc} \Gamma(X, L^i_X) & \longrightarrow & \Gamma(X^\wedge_Z, L^i_{X^\wedge_Z}) \\ \downarrow & & \downarrow \\ \Gamma(Y, L^i_Y) & \longrightarrow & \Gamma(Y^\wedge_Z, L^i_{Y^\wedge_Z}) \end{array} \tag{12}$$

is weakly cartesian.

Here, similarly as in the previous section, $\Gamma(X^\wedge_Z, L^i_{X^\wedge_Z})$ denotes the pro-object $\{\Gamma(Z', L^i_{Z'})\}_{Z' \hookrightarrow X}$ where $Z' \hookrightarrow X$ runs through all closed immersions of derived schemes with $|Z'| = Z$.

Proof. By Zariski descent we may assume that $X = \operatorname{Spec}(A)$ is affine. Choose $f_1, \dots, f_r \in \pi_0(A)$ defining Z, and write $A(\alpha) = A /\!\!/ f_1^\alpha, \dots, f_r^\alpha$ so that $X^\wedge_Z = \{\operatorname{Spec}(A(\alpha))\}_{\alpha \geq 1}$. Abusing notation slightly, we also write $f_*L^i_Y \in \operatorname{Mod}(A)$ for the module $\Gamma(X, f_*L^i_Y) = \Gamma(Y, L^i_Y)$ corresponding to $f_*L^i_Y \in \operatorname{QCoh}(X)$.

By Lemma 5.3 we have

$$\Gamma(X^\wedge_Z, L^i_{X^\wedge_Z}) \simeq L^i_A \otimes_A \{A(\alpha)\}.$$

Similarly, using affine coverings of Y and the usual induction we get

$$\Gamma(Y^\wedge_Z, L^i_{Y^\wedge_Z}) \simeq (f_*L^i_Y) \otimes_A \{A(\alpha)\}.$$

So we have to prove that

$$\begin{array}{ccc} L^i_A & \longrightarrow & L^i_A \otimes_A \{A(\alpha)\} \\ \downarrow & & \downarrow \\ f_*L^i_Y & \longrightarrow & (f_*L^i_Y) \otimes_A \{A(\alpha)\} \end{array} \tag{13}$$

is weakly cartesian. The map $L^i_A \to f_*L^i_Y$ factors as $L^i_A \to f_*f^*L^i_A \to f_*L^i_Y$, so it suffices to show that the two squares

$$\begin{array}{ccc} L^i_A & \longrightarrow & L^i_A \otimes_A \{A(\alpha)\} \\ \downarrow & & \downarrow \\ f_*f^*L^i_A & \longrightarrow & (f_*f^*L^i_A) \otimes_A \{A(\alpha)\} \end{array} \quad \text{and} \quad \begin{array}{ccc} f_*f^*L^i_A & \longrightarrow & (f_*f^*L^i_A) \otimes_A \{A(\alpha)\} \\ \downarrow & & \downarrow \\ f_*L^i_Y & \longrightarrow & (f_*L^i_Y) \otimes_A \{A(\alpha)\} \end{array} \tag{14}$$

are weakly cartesian.

We first treat the left one. Using the projection formula, we rewrite that square as the tensor product of L^i_A with the square

$$\begin{array}{ccc} A & \longrightarrow & \{A(\alpha)\} \\ \downarrow & & \downarrow \\ f_*\mathcal{O}_Y & \longrightarrow & f_*\mathcal{O}_Y \otimes_A \{A(\alpha)\}. \end{array} \tag{15}$$

As f is lafp, $f_*\mathcal{O}_Y$ is almost perfect [Lur18, Thm. 5.6.0.2]. On the other hand, by assumption $A \to f_*\mathcal{O}_Y$ is an isomorphism outside Z. Hence the left vertical fibre in (15) lies in

$\mathrm{Mod}(A\,\mathrm{on}\,Z)^{\mathrm{aperf}}$. It now follows from Lemma 5.1 that the map on vertical fibres is a weak equivalence, and hence that (15) is weakly cartesian. As L^i_A is connective, tensoring with L^i_A preserves weak equivalences and weakly cartesian squares (see e.g. [LT19, Lemma 2.29]). Thus the left-hand square in (14) is weakly cartesian.

We now consider the right square in (14). The transitivity triangle for f gives rise to a finite filtration on L^i_Y whose graded pieces are given by

$$f^*L^k_X \otimes_{\mathcal{O}_Y} L^{i-k}_{Y/X}, \quad k = 0, \dots, i.$$

It follows that the left vertical cofibre in our square has a finite filtration whose graded pieces are given by

$$f_*(f^*L^k_X \otimes_{\mathcal{O}_Y} L^{i-k}_{Y/X}) \simeq L^k_X \otimes_A f_*L^{i-k}_{Y/X}, \quad k = 0, \dots, i-1.$$

As f is lafp, the relative cotangent complex $L_{Y/X}$ is almost perfect [Lur04, Prop. 3.2.14] and so are its wedge powers $L^{i-k}_{Y/X}$. Again because f is lafp, the direct images $f_*L^{i-k}_{Y/X}$ are almost perfect [Lur18, Thm. 5.6.0.2]. As f is an isomorphism outside Z, the sheaf $f_*L^{i-k}_{Y/X}$ is supported on Z for $k < i$. So we may again apply Lemma 5.1 to deduce that the map $f_*L^{i-k}_{Y/X} \to (f_*L^{i-k}_{Y/X}) \otimes_A \{A(\alpha)\}_\alpha$ is a weak equivalence for $k = 0, \dots, i-1$ and hence so is the map

$$L^k_X \otimes_A f_*L^{i-k}_{Y/X} \to (L^k_X \otimes_A f_*L^{i-k}_{Y/X}) \otimes_A \{A(\alpha)\}_\alpha.$$

Using the above filtration, it follows that the map on vertical fibres in the right-hand square in (15) is a weak equivalence, and hence that square is also weakly cartesian. This finishes the proof of the theorem. $\square$

Combining Theorem 5.4 with the arguments of [EM23, §8.1] we obtain pro-cdh descent for Elmanto–Morrow's motivic cohomology denoted by $\mathbb{Z}(j)^{\mathrm{mot}}(-)$.

Corollary 5.5. *Let $\mathbb{F}$ be a prime field. Let X be a qcqs derived $\mathbb{F}$-scheme, $U \subseteq X$ a quasi-compact open subscheme, and denote by Z the closed subset $X \setminus U$. Let $f\colon Y \to X$ be a locally almost finitely presented U-modification of X. Then the square of pro-complexes*

$$\begin{array}{ccc} \mathbb{Z}(j)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}(j)^{\mathrm{mot}}(X^\wedge_Z) \\ \downarrow & & \downarrow \\ \mathbb{Z}(j)^{\mathrm{mot}}(Y) & \longrightarrow & \mathbb{Z}(j)^{\mathrm{mot}}(Y^\wedge_Z) \end{array}$$

is weakly cartesian.

Proof. The proof of Elmanto and Morrow goes through verbatim once we replace their Lemma 8.5 by the above Theorem 5.4. $\square$

6. Generalised Weibel vanishing

In this section we prove our generalised Weibel vanishing result. As its formulation involves the valuative dimension, we start by recalling the latter. The valuative dimension of a commutative ring was introduced and studied by Jaffard [Jaf60]; we refer to [EHIK20, §2.3] for an account. For an integral domain A, it is defined as

$$\mathrm{vdim}(A) = \sup\{n \mid \exists A \subseteq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n \subseteq \mathrm{Frac}(A),\ V_i \text{ valuation ring}\},$$

and in general as $\mathrm{vdim}(A) = \sup\{\mathrm{vdim}(A/\mathfrak{p}) \,|\, \mathfrak{p} \in \mathrm{Spec}(A)\}$. For a classical scheme X, one then sets

$$\mathrm{vdim}(X) = \sup\{\mathrm{vdim}(\mathcal{O}_X(U)) \,|\, U \subseteq X \text{ affine open}\}.$$

We always have $\dim(X) \leq \mathrm{vdim}(X)$ and equality holds if X is Noetherian [Jaf60, Ch. IV, Thm. 1, Cor. 2 of Thm. 5]. Note that the valuative dimension only depends on the underlying reduced scheme of a scheme. In particular, it makes sense to talk about the valuative dimension of a closed subset of a scheme, by equipping it with any closed subscheme structure.

Let now X be a derived scheme. The Krull dimension $\dim(X)$ of X is defined to be the Krull dimension of the underlying topological space of X. The valuative dimension $\mathrm{vdim}(X)$ is defined to be the valuative dimension of the underlying classical scheme ${}^{\mathrm{cl}}X$.

The following theorem generalizes [KST18b, Thm. B], which treats the case of Noetherian classical schemes.

Theorem 6.1. *Let X be a qcqs derived scheme. Assume that the underlying topological space of the spectrum of every local ring of every irreducible component of ${}^{\mathrm{cl}}X$ is Noetherian. Then the following hold.*

1. $K_{-i}(X) = 0$ *for all* $i > \mathrm{vdim}(X)$.
2. *For all* $i \geq \mathrm{vdim}(X)$ *and any integer* $r \geq 0$, *the pullback map* $K_{-i}(X) \to K_{-i}(\mathbb{A}^r_X)$ *is an isomorphism.*

Remarks 6.2.
1. We point out that Lemma 6.6 on the annihilation of negative K-theory classes is the only ingredient in our proof of Theorem 6.1 where we need a Noetherian assumption on the underlying topological space.
2. Any qcqs scheme of Krull dimension ≤ 1 satisfies the assumptions of the theorem.
3. The assumptions in the theorem are stable under maps essentially of finite type: Assume that X satisfies the hypothesis of Theorem 6.1. If $f\colon Y \to X$ is a morphism of qcqs derived schemes such that the morphism ${}^{\mathrm{cl}}f$ of underlying classical schemes is essentially of finite type, then also Y satisfies the hypothesis of the theorem.

   Indeed, we may assume that Y and X are classical. Let Y_0 be the spectrum of a local ring of an irreducible component of Y. Then the restriction of f to Y_0 factors through the spectrum X_0 of a local ring of an irreducible component of X and the induced morphism $Y_0 \to X_0$ is essentially of finite type. Say $X_0 = \mathrm{Spec}(A)$ and $Y_0 = \mathrm{Spec}(S^{-1}B)$ where B is a finitely generated A-algebra and $S \subseteq B$ is a multiplicative subset. As the topological space $|X_0|$ is Noetherian by assumption, so is $|\mathrm{Spec}(B)|$ by [OP68, Cor. 2.6]. As $\mathrm{Spec}(S^{-1}B)$ is homeomorphic to a subspace of $\mathrm{Spec}(B)$, it is Noetherian, too.

For proving Theorem 6.1, we follow the strategy of the proof of [KST18b, Thm. B] though we need some additional inputs. The following lemma replaces [KS17, Lemma 4] and is essentially due to Scheiderer [Sch92].

Lemma 6.3. *Let X be a spectral space, and $\mathcal{F}$ a sheaf of abelian groups on X. Let $r \geq 0$ be an integer. Assume that $\mathcal{F}_y = 0$ for all points $y \in X$ with $\dim(\overline{\{y\}}) > r$. Then $H^n(X, \mathcal{F}) = 0$ for all integers $n > r$.*

Proof. This is a direct consequence of results of Scheiderer [Sch92], see the proof of Proposition 4.7 there. Let $\mathrm{sp}_\bullet(X) \to X$ be the quasi-augmented simplicial topological space defined in [Sch92, §2]: Its set of n-simplices is given by chains of specialisations

$$x_0 \succ \cdots \succ x_n$$

of points in X with coincidences between the x_i allowed. The topology is induced by the constructible topology on X^{n+1}. The quasi-augmentation is given by the canonical map $\mathrm{sp}_0(X) \to X$. Let $\gamma_n : \mathrm{sp}_n(X) \to X$ be the map sending a chain as above to x_0. By [Sch92, Rem. 2.5 and Thm. 4.1], the cohomology groups $H^*(X, \mathcal{F})$ are computed by the complex

$$\Gamma(\mathrm{sp}_0(X), \gamma_0^*\mathcal{F}) \to \Gamma(\mathrm{sp}_1(X), \gamma_1^*\mathcal{F}) \to \Gamma(\mathrm{sp}_2(X), \gamma_2^*\mathcal{F}) \to \cdots$$

which arises from the cosimplicial abelian group $[n] \to \Gamma(\mathrm{sp}_n(X), \gamma_n^*\mathcal{F})$. Let

$$B_0 \to B_1 \to B_2 \to \cdots$$

be the associated normalized subcomplex, which also computes $H^*(X, \mathcal{F})$. As in the proof of [Sch92, Prop. 4.7], B_n coincides with the group of those sections $a \in \Gamma(\mathrm{sp}_n(X), \gamma_n^*\mathcal{F})$ whose supports consist only of non-degenerate simplices $x \in \mathrm{sp}_n(X)$. Hence, it suffices to show $(\gamma_n^*\mathcal{F})_x = 0$ for such x if $n > r$. So let now $x \in \mathrm{sp}_n(X)$ be given by a chain of specialisations $x_0 \succ \cdots \succ x_n$ with pairwise different x_i's. Then, we must have $\dim(\overline{\{x_0\}}) \geq n > r$. Thus the assumption implies $(\gamma_n^*\mathcal{F})_x = \mathcal{F}_{x_0} = 0$ as wanted. This completes the proof of the lemma. □

Lemma 6.4. *For a scheme X and a point $x \in X$ with closure $\overline{\{x\}}$ in X,*

$$\mathrm{vdim}(\mathcal{O}_{X,x}) + \mathrm{vdim}(\overline{\{x\}}) \leq \mathrm{vdim}(X).$$

In particular,

$$\mathrm{vdim}(\mathcal{O}_{X,x}) + \dim(\overline{\{x\}}) \leq \mathrm{vdim}(X).$$

Proof. By definition it suffices to prove the lemma in case $X = \mathrm{Spec}(A)$ with A integral. In this case, the first assertion is [Jaf60, Ch. IV, Prop. 2], the second assertion follows from the fact that $\dim(Y) \leq \mathrm{vdim}(Y)$ for every scheme Y. □

Lemma 6.5. *Let X be a qcqs derived scheme of finite valuative dimension. Let F be a sheaf of spectra on X for the Zariski topology. Assume that for every $x \in X$, the homotopy groups of the stalks $\pi_{-i}(F_x)$ vanish for $i > \mathrm{vdim}(\mathcal{O}_{X,x})$. Then $\pi_{-i}(F(X)) = 0$ for all $i > \mathrm{vdim}(X)$.*

Note that if F is K-theory, then $\pi_{-i}(K_x) \cong K_{-i}(\mathcal{O}_{X,x})$ as K-theory commutes with filtered colimits.

Proof. This is deduced from Lemma 6.3 by the same argument as the proof of [KS17, Prop. 3]. From [CM21, Thm. 3.12] we know that the homotopy dimension of X_{Zar} is bounded by the Krull dimension $\dim(X)$, and in particular the ∞-topos of sheaves of spaces on X_{Zar} is hypercomplete. Hence there is a convergent Zariski descent spectral sequence

$$E_2^{p,q} = H^p(X, \widetilde{F}_{-q}) \Longrightarrow \pi_{-p-q}(F(X)),$$

where $\widetilde{F}_{-q}$ is the Zariski sheaf of abelian groups on X associated to the presheaf $U \mapsto \pi_{-q}(F(U))$, and $E_2^{p,*} = 0$ unless $0 \leq p \leq \dim(X)$. It suffices to show $E_2^{p,q} = 0$ for $p + q > \mathrm{vdim}(X)$. We may assume $p \leq \dim(X)$ so that $q > 0$. Note that $\widetilde{F}_{-q,x} = \pi_{-q}(F_x)$. By Lemma 6.3 it now suffices to check that $\pi_{-q}(F_x) = 0$ for all points x with $\dim(\overline{\{x\}}) > \mathrm{vdim}(X) - q$. But for such points x we have

$$\mathrm{vdim}(\mathcal{O}_{X,x}) \leq \mathrm{vdim}(X) - \dim(\overline{\{x\}}) < q$$

by Lemma 6.4, and hence $\pi_{-q}(F_x) = 0$ by assumption. □

In the following lemma, given a morphism $f\colon Y \to X$, we abusively denote the induced morphism $\mathbb{A}^r_Y \to \mathbb{A}^r_X$ by the same letter.

Lemma 6.6. *Let $X = \operatorname{Spec}(A)$ be a classical integral affine scheme whose underlying topological space is Noetherian. Let $i > 0$, $r \geq 0$, and let γ be an element in $K_{-i}(\mathbb{A}^r_X)$. Then there exists an irreducible derived scheme Y and a proper lafp morphism $f\colon Y \to X$ which is an isomorphism over a non-empty open subset $U \subseteq X$ with the property that $f^*(\gamma) = 0$ in $K_{-i}(\mathbb{A}^r_Y)$.*

Proof. As K-theory commutes with filtered colimits of rings, there exists a subring $A_0 \subseteq A$, finitely generated as a $\mathbb{Z}$-algebra, and an element $\gamma_0 \in K_{-i}(\mathbb{A}^r_{X_0})$ with $a^*(\gamma_0) = \gamma$. Here $X_0 = \operatorname{Spec}(A_0)$ and $a\colon X \to X_0$ is the induced morphism.

By [KS17, Prop. 5] there exists a proper birational morphism of schemes $f_0\colon Y_0 \to X_0$ such that $f_0^*(\gamma_0) = 0$ in $K_{-i}(\mathbb{A}^r_{Y_0})$. As X_0 is Noetherian, f_0 is lafp (see 2.4). Let $f_1\colon Y_1 \to X$ be the derived pullback of f_0 along $a\colon X \to X_0$. Then f_1 is proper, lafp, and an isomorphism over a non-empty open subscheme $U \subseteq X$. From the commutative diagram

$$\begin{array}{ccc} Y_1 & \longrightarrow & Y_0 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ X & \xrightarrow{\ a\ } & X_0 \end{array}$$

it is clear that $f_1^*(\gamma) = 0$ in $K_{-i}(\mathbb{A}^r_{Y_1})$.

As ${}^{\mathrm{cl}}f_1$ is of finite type, the topological space $|Y_1|$ is Noetherian [OP68, Cor. 2.6]. In particular, the open complement of the closure of $|U|$ in $|Y_1|$ is quasi-compact, and hence there exists a classically finitely presented closed subscheme $Y_2 \hookrightarrow {}^{\mathrm{cl}}Y_1$ whose underlying topological space $|Y_2|$ is the closure of $|U|$ in $|Y_1|$ (see footnote 4 in the proof of Theorem 3.2 or [KS24, Lemma 4.14(1)] for an alternative argument). Note that $Y_2 \hookrightarrow Y_1$ is an isomorphism over U. By Lemma 2.2 there exists a finitely presented closed subscheme $Y \hookrightarrow Y_1$ with ${}^{\mathrm{cl}}Y = Y_2$. By construction, Y is irreducible, the composite $f\colon Y \hookrightarrow Y_1 \to X$ is proper, lafp, and an isomorphism over U, and $f^*(\gamma) = 0$, as desired. ☐

In the proof, we use the following well known facts about non-positive K-theory on affine (derived) schemes.

Lemma 6.7.
(1) *Let A be a derived ring. Then the canonical map $K(A) \to K(\pi_0(A))$ is 2-connective, i.e. it induces an isomorphism on π_i for $i \leq 1$ and a surjection on π_2.*
(2) *Let A be a discrete commutative ring, and let $I \subseteq A$ be a locally nilpotent ideal (i.e., every element of I is nilpotent). Then the map $K(A) \to K(A/I)$ is 1-connective.*
(3) *Let A be a discrete, commutative ring, $X = \operatorname{Spec}(A)$, and let $X = X_1 \cup X_2$ be a closed covering of X. Write $X_{12} = X_1 \cap X_2$. Then there is a long exact sequence*

$$K_1(X) \to K_1(X_1) \oplus K_1(X_2) \to K_1(X_{12}) \to K_0(X) \to \dots$$

Proof. (1) For connective K-theory, this is due to Waldhausen [Wal78, Prop. 1.1]. The general case follows from this together with [BGT13, Thm. 9.53] (or [KST18b, Thm. 2.16]). Alternatively, see [LT19, Lemma 2.4] for a slightly more general statement.

(2) Writing I as a filtered colimit of nilpotent ideals, we may assume that I itself is nilpotent. Then $K_0(I) = 0$ by [Wei13, Exc. II.2.5] and hence $K_1(A) \to K_1(A/I)$ is surjective by Prop. III.2.3 there. Moreover, $K_0(A) \to K_0(A/I)$ is an isomorphism by [Wei13, Lemma II.2.2]. As the ideal generated by I in any A-algebra is still nilpotent, it then follows from the definition of negative K-groups [Wei13, Def. III.4.1] that $K_i(A) \to K_i(A/I)$ is an isomorphism for all $i \leq 0$.

(3) Let I and J be the ideals defining X_1 and X_2, respectively. Hence $X_{12} = \mathrm{Spec}(A/I+J)$. Let $A' = A/I \cap J$. Then A' sits in a Milnor square

$$\begin{array}{ccc} A' & \longrightarrow & A/I \\ \big\downarrow & & \big\downarrow \\ A/J & \longrightarrow & A/I+J. \end{array}$$

Hence, by [Wei13, Thm. III.4.3], there is a long exact sequence

$$K_1(A') \to K_1(A/I) \oplus K_1(A/J) \to K_1(A/I+J) \to K_0(A') \to \dots$$

As $X_1 \cup X_2 = X$, the ideal $I \cap J$ is contained in the nilradical of A and so is locally nilpotent. Using (2), we may thus replace A' by A in the above sequence to get a long exact sequence of the statement. □

Proof of Theorem 6.1. We write $N^{(r)}K(X)$ for the cofiber of the canonical split inclusion $K(X) \to K(\mathbb{A}^r_X)$ so that we have $K(\mathbb{A}^r_X) \cong K(X) \oplus N^{(r)}K(X)$. Then assertion (2) of the theorem is equivalent to the statement that $N^{(r)}K_{-i}(X) = 0$ for all $i \geq \mathrm{vdim}(X)$.

We prove the theorem by induction on $d = \mathrm{vdim}(X)$. By Lemma 6.5, applied to K and $\Sigma N^{(r)}K$ respectively, we may assume that X is affine and local. In this case, $K_{-i}(\mathbb{A}^r_X) \cong K_{-i}(\mathbb{A}^r_{\mathrm{cl}\,X_{\mathrm{red}}})$ for all $i \geq 0$ and all $r \geq 0$ by Lemma 6.7 (1), (2). So we can assume that X is a classical reduced affine local scheme.

If $d = 0$, then also the Krull dimension $\dim(X)$ is 0. As X is local and reduced, it is the spectrum of a field. Hence $K_{-i}(X) = 0$ and for all $i > 0$ and $N^{(r)}K(X) = 0$ (by [Wei13, Thm. II.7.8] and the definition of negative K-groups), as X is regular Noetherian.

Now assume that d is positive and the assertion is proven for all derived schemes satisfying the assumptions of Theorem 6.1 that are of valuative dimension $< d$. We first prove the result for X under the additional assumption that X is irreducible, and hence has Noetherian topological space.

Let γ be an element in $K_{-i}(X)$ with $i > \mathrm{vdim}(X)$ or in $N^{(r)}K_{-i}(X)$ with $i \geq \mathrm{vdim}(X)$. We have to show that $\gamma = 0$. Let $f\colon Y \to X$ be a proper, lafp morphism as provided by Lemma 6.6. In particular $f^*(\gamma) = 0$. Let Z be the closed complement of U in X. By Theorem 4.5 we obtain an exact sequence of pro-abelian groups

$$K_{-i+1}(Y^\wedge_Z) \to K_{-i}(X) \to K_{-i}(Y) \oplus K_{-i}(X^\wedge_Z)$$

and similarly for $N^{(r)}K$ in place of K. Recall from (1) in 2.5 that $K_{-i}(X^\wedge_Z)$ is the pro-abelian group $\{K_{-i}(Z')\}_{Z'}$ where Z' runs through all closed (derived) subschemes $Z' \hookrightarrow X$ with $|Z'| = Z$ and similarly $K_{-i+1}(Y^\wedge_Z) = \{K_{-i+1}(Y \times_X Z')\}_{Z'}$.

As Z does not contain the generic point of X, we have $\mathrm{vdim}(Z') < \mathrm{vdim}(X)$ for every Z' as above by [EHIK20, Prop. 2.3.2(4)]. Hence $K_{-i}(X^\wedge_Z)$ vanishes by induction. By [EHIK20, Prop. 2.3.2(6)] we have $\mathrm{vdim}(Y) = \mathrm{vdim}(X)$. As $f^{-1}(Z)$ does not contain the generic point of Y, the same argument as before implies that $\mathrm{vdim}(Y \times_X Z') < \mathrm{vdim}(Y)$ for all Z' as above. Hence also $K_{-i+1}(Y^\wedge_Z)$ vanishes by induction. It follows that $f^*\colon K_{-i}(X) \to K_{-i}(Y)$ is injective. Similarly, $f^*\colon N^{(r)}K_{-i}(X) \to N^{(r)}K_{-i}(Y)$ is injective. As $f^*(\gamma) = 0$, we must have $\gamma = 0$, as desired.

We now consider the general case. So X is now a classical, reduced, affine, local scheme of valuative dimension $d > 0$ and every irreducible component of X has Noetherian topological space. For ease of notation, we only consider assertion (1) of the theorem. The proof of (2) is completely parallel. To reduce to the integral case treated above, we apply an argument from [EHIK20, Thm. 2.4.15] as follows: Take an integer $i > d = \mathrm{vdim}(X)$ and an element $\gamma \in K_{-i}(X)$. We wish to show that $\gamma = 0$. Let

$$\mathcal{E} = \{Z \hookrightarrow X \text{ reduced, closed} \mid \gamma_{|Z} \neq 0 \in K_{-i}(Z)\}.$$

We need to prove that $\mathcal{E} = \emptyset$. Note that every $Z \in \mathcal{E}$ is itself a reduced local affine scheme and that $\mathcal{E}$ is ordered by inclusion. Let $(Z_\lambda)_{\lambda\in\Lambda}$ be a descending chain in $\mathcal{E}$ and put $Z = \lim_{\lambda\in\Lambda} Z_\lambda$. As K-theory commutes with filtered colimits of rings, we have $K_{-i}(Z) = \mathrm{colim}_{\lambda\in\Lambda} K_{-i}(Z_\lambda)$. So if $\gamma_{|Z} = 0$, then there exists a λ such that $\gamma_{|Z_\lambda} = 0$, which is a contradiction. Hence $\gamma_{|Z} \neq 0$ so that $Z \in \mathcal{E}$ and Z is a lower bound of $(Z_\lambda)_{\lambda\in\Lambda}$. If $\mathcal{E} \neq \emptyset$, we may apply Zorn's lemma to conclude that $\mathcal{E}$ has a minimal element Z. As $K_{-i}(Z) \neq 0$, Z must be reducible. Let Z^{gen} be the set of the generic points of Z equipped with the induced topology from the underlying topological space of Z. By [HJ65, Cor. 2.4] (see also [EHIK20, Lemma 2.4.14]), there exists a decomposition $Z^{\mathrm{gen}} = S_1 \sqcup S_2$ with S_i closed and non-empty for $i = 1, 2$. Letting Z_i be the closure of S_i in Z with reduced scheme structure, we have $Z = Z_1 \cup Z_2$ and $Z_i \cap Z^{\mathrm{gen}} = S_i$ for $i = 1, 2$. In particular, $Z_1 \cap Z_2 \cap Z^{\mathrm{gen}} = \emptyset$ so that $\mathrm{vdim}(Z_1 \cap Z_2) < d$ by [EHIK20, Prop. 2.3.2(4)]. By excision in non-positive K-theory for closed coverings of affine schemes (Lemma 6.7(3)), we have an exact sequence

$$K_{-i+1}(Z_1 \cap Z_2) \to K_{-i}(Z) \to K_{-i}(Z_1) \oplus K_{-i}(Z_2).$$

The group on the left-hand side vanishes by induction. As Z was minimal in $\mathcal{E}$, we must have $\gamma_{|Z_i} = 0$ for $i = 1, 2$. Thus we get $\gamma_{|Z} = 0$, which is a contradiction. Thus we must have $\mathcal{E} = \emptyset$, which completes the proof of Theorem 6.1. □

We now give a proof of Theorem 1.3, reproduced from [KM21, Rem. 3.5]. We refer to [EHIK20, §2.1] for a discussion of the cdh topology in this generality but note that the definition of the cdh topology used in [EHIK20] is the one used by Suslin and Voevodsky, [SV00, Def. 5.7], and the proof that cdh Čech descent is equivalent to cdh excision is Voevodsky's proof from [Voe10], rewritten in modern language in [AHW17, Thm. 3.2.5]. Voevodsky's proof that cdh Čech descent is equivalent to cdh hyperdescent requires Noetherian hypotheses that were lifted in [EHIK20].

Proof. We note that homotopy K-theory is a cdh sheaf. This was first proven by Cisinski [Cis13] for Noetherian schemes of finite dimension, which implies the general statement by absolute Noetherian approximation; alternatively it follows from the fact that KH is truncating, see [LT19, Cor. A.5]. In particular, we get the maps

$$L_{\mathrm{cdh}} K_{\geq 0} \to L_{\mathrm{cdh}} K \to KH \tag{16}$$

which we want to show are equivalences. By [EHIK20, Thm. 2.4.15, Cor. 2.3.3] the ∞-topos of cdh sheaves of spaces on finitely presented X-schemes is locally of finite homotopy dimension and of homotopy dimension $\leq \mathrm{vdim}(X)$. This implies that we get a convergent spectral sequence

$$E_2^{p,q} = H^p(X_{\mathrm{cdh}}, \tilde{\pi}_{-q} KH) \Longrightarrow KH_{-p-q}(X). \tag{17}$$

where $\tilde{\pi}_q KH$ denotes the cdh sheafified homotopy groups of KH. A conservative family of points for the cdh topology is given by the spectra of henselian valuation rings [GL01], [GK15, Thm. 2.3, Thm. 2.6], [EHIK20, Cor. 2.4.19]. As K-theory of valuation rings is connective and agrees with its homotopy K-theory [KM21, Thm. 3.4], [KST21, Lem. 4.3], and K and

KH commute with filtered colimits, the maps (16) are equivalences so we get part (2) of Theorem 1.3. The vanishing

$$KH_{-i}(X) = 0, \qquad \text{for all } i > \mathrm{vdim}(X)$$

claimed in part (1) follows from the spectral sequence (17). □

References

[AHW17] Aravind Asok, Marc Hoyois, and Matthias Wendt, *Affine representability results in $\mathbb{A}^1$-homotopy theory, I: Vector bundles*, Duke Mathematical Journal **166** (2017), no. 10, 1923 – 1953.

[Ann22] Toni Annala, *Base independent algebraic cobordism*, J. Pure Appl. Algebra **226** (2022), no. 6, Paper No. 106977, 44. MR 4346005

[Ant18] Ben Antieau, *Various remarks on Weibel's conjecture*, Letter to Kerz, June 2018.

[BGT13] A. J. Blumberg, D. Gepner, and G. Tabuada, *A universal characterization of higher algebraic K-theory*, Geom. Topol. **17** (2013), no. 2, 733–838.

[BKRS22] Tom Bachmann, Adeel A. Khan, Charanya Ravi, and Vladimir Sosnilo, *Categorical Milnor squares and K-theory of algebraic stacks*, Selecta Math. (N.S.) **28** (2022), no. 5, Paper No. 85, 72. MR 4487748

[BZFN10] David Ben-Zvi, John Francis, and David Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry*, J. Amer. Math. Soc. **23** (2010), no. 4, 909–966. MR 2669705

[CHSW08] G. Cortiñas, C. Haesemeyer, M. Schlichting, and C. Weibel, *Cyclic homology, cdh-cohomology and negative K-theory*, Ann. of Math. (2) **167** (2008), no. 2, 549–573.

[Cis13] D.-C. Cisinski, *Descente par éclatements en K-théorie invariante par homotopie*, Ann. of Math. (2) **177** (2013), no. 2, 425–448. MR 3010804

[CM21] Dustin Clausen and Akhil Mathew, *Hyperdescent and étale K-theory*, Invent. Math. **225** (2021), no. 3, 981–1076. MR 4296353

[CMNN] D. Clausen, A. Mathew, N. Naumann, and J. Noel, *Descent in algebraic K-theory and a conjecture of Ausoni-Rognes*, J. Eur. Math. Soc. (JEMS) (to appear).

[Cor06] G. Cortiñas, *The obstruction to excision in K-theory and in cyclic homology*, Invent. Math. **164** (2006), no. 1, 143–173.

[DT22] Christian Dahlhausen and Georg Tamme, *A counterexample to pro-cdh descent for non-noetherian schemes*, `https://cdahlhausen.eu/Notes/pro-cdh-counterex.pdf`, 2022.

[Efi24] Alexander I. Efimov, *K-theory and lozalizing invariants of large categories*, arXiv:2405.12169, 2024.

[EHIK20] Elden Elmanto, Marc Hoyois, Ryomei Iwasa, and Shane Kelly, *Cdh descent, cdarc descent, and Milnor excision*, Mathematische Annalen **379** (2020), no. 3–4, 1011–1045.

[EM23] Elden Elmanto and Matthew Morrow, *Motivic cohomology of equicharacteristic schemes*, arXiv:2309.08463, 2023.

[GD71] A. Grothendieck and J. A. Dieudonné, *Éléments de géométrie algébrique. I*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 166, Springer-Verlag, Berlin, 1971. MR 3075000

[GH06] T. Geisser and L. Hesselholt, *Bi-relative algebraic K-theory and topological cyclic homology*, Invent. Math. **166** (2006), no. 2, 359–395.

[GH11] ———, *On relative and bi-relative algebraic K-theory of rings of finite characteristic*, J. Amer. Math. Soc. **24** (2011), no. 1, 29–49.

[GK15] Ofer Gabber and Shane Kelly, *Points in algebraic geometry*, J. Pure Appl. Algebra **219** (2015), no. 10, 4667–4680. MR 3346512

[GL01] T. G. Goodwillie and S. Lichtenbaum, *A cohomological bound for the h-topology*, Amer. J. Math. **123** (2001), no. 3, 425–443. MR 1833147

[GR14] Dennis Gaitsgory and Nick Rozenblyum, *DG indschemes*, Perspectives in representation theory, Contemp. Math., vol. 610, Amer. Math. Soc., Providence, RI, 2014, pp. 139–251. MR 3220628

[GR17] ———, *A study in derived algebraic geometry. Vol. II. Deformations, Lie theory and formal geometry*, Mathematical Surveys and Monographs, vol. 221, American Mathematical Society, Providence, RI, 2017. MR 3701353

[Gro66] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 28, 255. MR 217086

[Hae04] C. Haesemeyer, *Descent properties of homotopy K-theory*, Duke Math. J. **125** (2004), no. 3, 589–620.

[Hek21] Jeroen Hekking, *Graded algebras, projective spectra and blow-ups in derived algebraic geometry*, arXiv:2106.01270, 2021.

[HJ65] M. Henriksen and M. Jerison, *The space of minimal prime ideals of a commutative ring*, Trans. Amer. Math. Soc. **115** (1965), 110–130. MR 194880

[Jaf60] Paul Jaffard, *Théorie de la dimension dans les anneaux de polynomes*, Mémor. Sci. Math., Fasc. 146, Gauthier-Villars, Paris, 1960. MR 117256

[Kel24] Shane Kelly, *Non-reduced valuation rings and descent for smooth blowup squares*, arXiv:2401.02706, 2024.

[Kha20] Adeel A. Khan, *Algebraic K-theory of quasi-smooth blow-ups and CDH descent*, Ann. H. Lebesgue **3** (2020), 1091–1116. MR 4149835

[KM21] Shane Kelly and Matthew Morrow, *K-theory of valuation rings*, Compos. Math. **157** (2021), no. 6, 1121–1142. MR 4264079

[KR18] Adeel A. Khan and David Rydh, *Virtual cartier divisors and blow-ups*, arXiv:1802.05702, 02 2018.

[Kri10] Amalendu Krishna, *An Artin-Rees theorem in K-theory and applications to zero cycles*, J. Algebraic Geom. **19** (2010), no. 3, 555–598. MR 2629600

[KS02] Amalendu Krishna and V. Srinivas, *Zero-cycles and K-theory on normal surfaces*, Ann. of Math. (2) **156** (2002), no. 1, 155–195. MR 1935844

[KS17] Moritz Kerz and Florian Strunk, *On the vanishing of negative homotopy K-theory*, J. Pure Appl. Algebra **221** (2017), no. 7, 1641–1644. MR 3614971

[KS24] Shane Kelly and Shuji Saito, *A procdh topology*, arXiv:2401.02699, 2024.

[KST18a] M. Kerz, S. Saito, and G. Tamme, *K-theory of non-archimedean rings I*, arXiv:1802.09819, 2018.

[KST18b] M. Kerz, F. Strunk, and G. Tamme, *Algebraic K-theory and descent for blow-ups*, Invent. Math. **211** (2018), no. 2, 523–577.

[KST21] Moritz Kerz, Florian Strunk, and Georg Tamme, *Towards Vorst's conjecture in positive characteristic*, Compos. Math. **157** (2021), no. 6, 1143–1171. MR 4270122

[KST23] Moritz Kerz, Shuji Saito, and Georg Tamme, *K-theory of non-Archimedean rings II*, Nagoya Math. J. **251** (2023), 669–685. MR 4637146

[LT19] M. Land and G. Tamme, *On the K-theory of pullbacks*, Ann. of Math. (2) **190** (2019), no. 3, 877–930.

[Lur04] Jacob Lurie, *Derived algebraic geometry*, Ph.D. thesis, Massachusetts Institute of Technology. Dept. of Mathematics., 2004.

[Lur17] J. Lurie, *Higher Algebra*, Available at the author's homepage `http://www.math.ias.edu/~lurie`, 2017.

[Lur18] ———, *Spectral Algebraic Geometry*, Available at the author's homepage `http://www.math.ias.edu/~lurie`, 2018.

[Mor16a] M. Morrow, *Pro CDH-descent for cyclic homology and K-theory*, J. Inst. Math. Jussieu **15** (2016), no. 3, 539–567.

[Mor16b] Matthew Morrow, *A historical overview of pro cdh descent in algebraic k-theory and its relation to rigid analytic varieties*, arXiv:1612.00418, 2016.

[Mor18] M. Morrow, *Pro unitality and pro excision in algebraic K-theory and cyclic homology*, J. Reine Angew. Math. **736** (2018), 95–139.

[Mor23] Matthew Morrow, *Thoughts on Weibel's conjecture* in *Algebraic K-Theory*, Oberwolfach Reports **19** (2023), no. 2, 1354–1356.

[OP68] Jack Ohm and R. L. Pendleton, *Rings with noetherian spectrum*, Duke Math. J. **35** (1968), 631–639. MR 229627

[RG71] Michel Raynaud and Laurent Gruson, *Critères de platitude et de projectivité. Techniques de "platification" d'un module*, Invent. Math. **13** (1971), 1–89. MR 308104

[Sch] Peter Scholze, *Lectures on condensed mathematics*, `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`.

[Sch92] Claus Scheiderer, *Quasi-augmented simplicial spaces, with an application to cohomological dimension*, J. Pure Appl. Algebra **81** (1992), no. 3, 293–311. MR 1179103

[Sta23] The Stacks project authors, *The stacks project*, `https://stacks.math.columbia.edu`, 2023.

[SV00] Andrei Suslin and Vladimir Voevodsky, *Bloch-Kato conjecture and motivic cohomology with finite coefficients*, The arithmetic and geometry of algebraic cycles, Springer, 2000, pp. 117–189.

[Tho93] R. W. Thomason, *Les K-groupes d'un schéma éclaté et une formule d'intersection excédentaire*, Invent. Math. **112** (1993), no. 1, 195–215. MR 1207482

[TT90] R. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435.

[TV05] Bertrand Toën and Gabriele Vezzosi, *Homotopical algebraic geometry. I. Topos theory*, Adv. Math. **193** (2005), no. 2, 257–372. MR 2137288

[TV08] ———, *Homotopical algebraic geometry. II. Geometric stacks and applications*, Mem. Amer. Math. Soc. **193** (2008), no. 902, x+224. MR 2394633

[Voe10] Vladimir Voevodsky, *Homotopy theory of simplicial sheaves in completely decomposable topologies*, Journal of pure and applied algebra **214** (2010), no. 8, 1384–1398.

[Wal78] F. Waldhausen, *Algebraic K-theory of topological spaces. I*, Proceedings of Symposia in Pure Mathematics **32** (1978).

[Wei13] C. Weibel, *The K-book*, Graduate Studies in Mathematics, vol. 145, American Mathematical Society, Providence, RI, 2013, An introduction to algebraic K-theory.

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba Meguro-ku, Tokyo 153-8914, Japan
Email address: shanekelly@g.ecc.u-tokyo.ac.jp

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba Meguro-ku, Tokyo 153-8914, Japan
Email address: sshuji.goo@gmail.com

Institut für Mathematik, Fachbereich 08, Johannes Gutenberg-Universität Mainz, D-55099 Mainz, Germany
Email address: georg.tamme@uni-mainz.de