Bertini theorems and Lefschetz pencils over discrete valuation rings, with applications to higher class field theory

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Abstract

We show the existence of good hyperplane sections for schemes over discrete valuation rings with good or (quasi-)semistable reduction, and the existence of good Lefschetz pencils for schemes with good reduction or ordinary quadratic reduction. As an application we prove that the reciprocity map introduced for smooth projective varieties over local fields by Bloch, Kato and Saito is an isomorphism after profinite completion, if the variety has good reduction or ‘almost good’ reduction.

Good hyperplane sections, whose existence is assured by Bertini’s theorem, and good families of hyperplane sections, so-called Lefschetz pencils, are well-known constructions and powerful tools in classical geometry, i.e., for varieties over a field. But for arithmetic questions one is naturally led to the consideration of models over Dedekind rings and, for local questions, to schemes over discrete valuation rings. It is the aim of this note to provide extensions of the mentioned constructions to the latter situation. We point out some new phenomena, and give an application to the class field theory of varieties over local fields with good reduction or, more generally, ordinary quadratic reduction. For this we also discuss the resolution of the latter to semi-stable models. For more arithmetic applications see [JS].

Let A be a discrete valuation ring with fraction field K, maximal ideal $\mathfrak{m}$ and residue field $F = A/\mathfrak{m}$. Let $\eta = \mathrm{Spec}(K)$ and $s = \mathrm{Spec}(F)$ be the generic and closed point of $\mathrm{Spec}(A)$, respectively. For any scheme X over A we let $X_\eta = X \times_A K$ and $X_s = X \times_A F$ be its generic and special fibre, respectively.

0. Good hyperplane sections for good reduction schemes

As a ‘warm-up’, we recall the classical Bertini theorem and extend it to varieties over K with good reduction. Let $X \subset \mathbb{P}^N_L$ be a smooth quasi-projective variety over a field L. Recall that another irreducible smooth subscheme $Z \subset \mathbb{P}^N_L$ is said to intersect X transversally, if the scheme-theoretic intersection $X \cdot Z = X \times_{\mathbb{P}^N} Z$ (which is just defined by the ideal generated by the equations of X and Z) is smooth and of pure codimension $\mathrm{codim}_{\mathbb{P}^N}(Z)$ in X. Then the Bertini theorem asserts that for infinite L, there exists an L-rational hyperplane $H \subset \mathbb{P}^N_L$ intersecting X transversally (cf. [Jou, 6.11, 2], and also Theorem 3 below). In this case, one calls $Y = X \cdot H$ a smooth (or good) hyperplane section of X.

More precisely, the following holds. One has the dual projective space $(\mathbb{P}^N_L)^\vee$ parameterizing the hyperplanes in $\mathbb{P}^N_L$ (a point $a = (a_0 : \ldots : a_N)$ corresponds to the hyperplane with the equation $a_0x_0 + \ldots + a_Nx_N = 0$ for the homogeneous coordinates x_i of $\mathbb{P}^N_L$). Then, for an arbitrary field L, there is a dense Zariski open $V_X \subset (\mathbb{P}^N_L)^\vee$ parameterizing those hyperplanes which intersect X transversally. Moreover, if L is infinite, then the set $V_X(L)$ of L-rational points is non-empty, since $\mathbb{P}^N_L(L)$ is Zariski dense in $\mathbb{P}^N_L$. This shows that, for an infinite field L, and finitely many smooth varieties $X_1, \ldots, X_n$ in $\mathbb{P}^N_L$, there also exists an L-rational hyperplane H intersecting all X_i transversally, because $V_{X_1} \cap \ldots \cap V_{X_n}$ is non-empty.

If L is finite, it may happen that V_X does not have any L-rational point. But, by sieve methods, Poonen [Po] showed that in this case there always exists an L-rational point after replacing the projective embedding by the d-fold embedding for some $d > 0$, i.e., there always exists a smooth L-rational *hypersurface* section of X.

Now consider a quasi-projective A-scheme (A a discrete valuation ring as above), i.e., a subscheme X of the projective space $\mathbb{P}^N_A$ over A.

By a hyperplane $H \subseteq \mathbb{P}^N_A$ over A we mean a closed subscheme which corresponds to an A-rational point of the dual projective space $(\mathbb{P}^N_A)^\vee$ (= Grassmannian of linear subspaces of codimension 1). Since every invertible module over A is free, H is given by a surjection $\varphi : A^{N+1} \twoheadrightarrow A^N$; or, equivalently, by an equation $\sum_{i=0}^N a_ix_i = 0$, $a_i \in A$ $(i = 0, \ldots, N)$, not all in the maximal ideal $\mathfrak{m}$, for the coordinates x_i on $\mathbb{P}^n_A$. The correspondence is given by

$$\ker\varphi = A \cdot \sum_{i=0}^N a_ie_i \qquad ,$$

where $e_0, \ldots, e_N$ is the basis of A^{N+1}.

Theorem 0 *Let $X \subset \mathbb{P}^N_A$ be a smooth quasi-projective A-scheme. If F is infinite, then there exists a hyperplane $H \subset \mathbb{P}^N_A$ over A such that the scheme-theoretic intersection $X \cdot H = X \times_{\mathbb{P}^N_A} H$ is smooth over A and of pure codimension 1 in X. If F is finite and A is Henselian, then, for every given prime number ℓ, such a hyperplane exists after replacing A by a finite étale covering A'/A of ℓ-power-degree.*

Proof Let $H \subset \mathbb{P}^N_A$ be a hyperplane over A. Then H_η and H_s are hyperplanes in $\mathbb{P}^N_K$ and $\mathbb{P}^N_F$, respectively. With the notations as above, the condition on the hyperplane is that (the K-rational point corresponding to) H_η lies in the good locus $V_{X_\eta} \subset (\mathbb{P}^N_K)^\vee$, and that H_s lies in V_{X_s}. Since H is completely determined by H_η, this means that $H_\eta \in V_{X_\eta}(K) \cap sp^{-1}(V_{X_s}(F))$, where $sp : (\mathbb{P}^N)^\vee(K) \to (\mathbb{P}^N(F)$ is the specialization map, which sends H_η to H_s.

It remains to see when this intersection is non-empty. But for any proper scheme P over A and any open subschemes $V_1 \subset P_\eta$ and $V_2 \subset P_s$, with closed complements $Z_1 = P_\eta \setminus V_1$ and $Z_2 = P_s \setminus V_2$, respectively, one has $Z_1(K) \subset sp^{-1}(sp(Z_1)(F))$ where $sp(Z_1) = \overline{Z}_1 \cap P_s$ for the Zariski closure $\overline{Z}_1$ of Z_1 in $\mathbb{P}^N_A$. Therefore $V_1(K) \cap sp^{-1}(V_2(F))$ contains $sp^{-1}((V_2 \setminus sp(Z_1))(F))$. The latter set has K-rational points, if $sp : P(K) \to P(F)$ is surjective and $V_2 \setminus sp(Z_1)$ has F-rational points. The latter set is open and dense in P_s, if P/S has irreducible fibres, and V_1 and V_2 are dense in their fibres.

Applying this to $P = (\mathbb{P}^N_A)^\vee$, $V_1 = V_{X_\eta}$ and $V_2 = V_{X_s}$, where all conditions are fulfilled, we see it suffices that the non-empty open subset $W = V_2 \setminus sp(Z_1)$ has F-rational points. As explained above, this is the case if F is infinite. Hence, if F is finite, it is the case over the maximal pro-ℓ-extension of F, hence over some extension F'/F of ℓ-power degree. If A is Henselian, and A'/A is the unramified extension corresponding to F'/F, then the F'-rational point lifts to an A'-rational point of P. Since the formation of the sets V_1 and V_2 is compatible with étale base change in the base, this means there is a good hyperplane section for X over A'.

Remarks 0 (i) In contrast to the classical situation, the good hyperplanes over A are not parametrized by a Zariski open in $\mathbb{P}^N_K$, but by a subset of the type $V_1(K) \cap sp^{-1}((V_2)(F))$ for Zariski opens $V_1 \subset (\mathbb{P}^N_K)^\vee$ and $V_2 \subset (\mathbb{P}^N_F)^\vee$.

(ii) If, with the notations as in the proof, H_s intersects the smooth variety X_s transversally, and if $X_\eta \cap H_\eta$ is non-empty, then $X \cdot H$ is a flat A-scheme of finite type whose special fibre $(X \cdot H)_s = X_s \cdot H_s$ is smooth. Since the smooth locus of $X \cdot H$ is open, $X \cdot H$ must be smooth, if X and hence $X \cdot H$ is proper. This shows that, for smooth and proper X, one has $sp^{-1}(V_{X_s}) \subset V_{X_\eta}$, and the locus in $(\mathbb{P}^N_K)^\vee(K)$ of good hyperplanes for X/A is just $sp^{-1}(V_{X_s}(F))$. Moreover, by applying the mentioned result of Poonen, this has a K-rational point after passing to some multiple embedding.

Recall that a smooth proper variety V over K is said to have good reduction (over A) if there is a smooth proper A-scheme X with generic fiber $X_\eta = X \times_A K \cong V$.

Corollary 0 *If F is finite and V/K is a smooth projective variety with good reduction, there exists a smooth hypersurface section which again has good reduction.*

1. Good hyperplane sections for quasi-semi-stable schemes

For the applications, the case of good reduction is too restrictive. We now introduce the following more general objects:

Definition 1 Let $\mathcal{C}$ be the category of quasi-projective schemes over A. An object $X \in \mathcal{C}$ is *quasi-semistable* if the following conditions holds:

(1) X is regular flat over $\mathrm{Spec}(A)$,

(2) for each closed point $x \in X_s$ the completion of $\mathcal{O}_{X,x}$ is isomorphic to

$$B = A[[x_1, \dots, x_r, y_1, \dots, y_n]]/\langle \pi - u x_1^{e_1} \dots x_r^{e_r} \rangle$$

where $e_1, \dots, e_r \geq 1$ are suitable integers and u is a unit in the ring of the formal power series $A[[x_1, \dots, x_r, y_1, \dots, y_n]]$.

Recall that one calls X *semi-stable* if in addition X_s is reduced. Then one has $e_1 = \cdots e_r = 1$ in condition (2), so that one can also choose $u = 1$. (One also says that X has semi-stable reduction.) Let $\mathcal{QS} \subset \mathcal{C}$ be the subcategory of all $X \in Ob(\mathcal{C})$ which are quasi-semistable and let $\mathcal{S} \subset \mathcal{QS}$ be the subcategory of semi-stable objects. Finally let $s\mathcal{QS} \subset \mathcal{QS}$ and $s\mathcal{S} \subset \mathcal{S}$ be the subcategories of strictly quasi-semistable and strictly semi-stable objects, i.e., those objects for which $X_{s,red}$ has regular irreducible components.

The aim of this section is to prove:

Theorem 1 *Let X be an object of $s\mathcal{QS}$ (resp. $s\mathcal{S}$). If F is infinite, then there exists a hyperplane $H \subset \mathbb{P}^N_A$ over A such that X and H intersect transversally in $\mathbb{P}^N_A$, $X \cdot H := X \times_{\mathbb{P}^N_A} H$ is in $s\mathcal{QS}$ (resp. $s\mathcal{S}$), and $(X \cdot H) \cup X_{s,red}$ is a simple normal crossing divisor on X. If F is finite and A is Henselian, then for every prime ℓ, there is a finite unramified extension A' of A of ℓ-power degree such that the same conclusion holds after base change with A'.*

For the proof we need the following lemma.

Lemma 1 *Let X be an object of $s\mathcal{QS}$ (resp. $s\mathcal{S}$). Let $H \subset \mathbb{P}^N_A$ be a hyperplane over A, with special fibre $H_s \subset \mathbb{P}^N_F$ and generic fibre $H_\eta \subset \mathbb{P}^N_K$. Let $Y_1, \ldots, Y_M$ be the irreducible components of $X_{s,red}$, which are by definition smooth varieties intersecting transversally in $\mathbb{P}^N_F$. Assume that*

- *(i) H_s and $Y_{i_1,\ldots,i_p}$ for any $i_1, \ldots, i_a$ intersect transversally in $\mathbb{P}^N_F$, where $Y_{i_1,\ldots,i_p} := Y_{i_1} \cap \cdots \cap Y_{i_p}$.*
- *(ii) H_η and X_η intersect transversally in $\mathbb{P}^N_K$.*

Then X and H intersect transversally in $\mathbb{P}^N_A$ and $X \cdot H := X \times_{\mathbb{P}^N_A} H$ is an object of $s\mathcal{QS}$ (resp. $s\mathcal{S}$) and $(X \cdot H) \cup X_{s,red}$ is a simple normal crossing divisor on X. If X is proper over A, condition (ii) is implied by condition (i).

Proof of Lemma 1 Noting

$$(X \cdot H) \times_X Y_{i_1,\ldots,i_p} = (H \times_{\mathbb{P}^N_A} X) \times_X Y_{i_1,\ldots,i_p} = H \times_{\mathbb{P}^N_A} Y_{i_1,\ldots,i_p} = H_s \times_{\mathbb{P}^N_F} Y_{i_1,\ldots,i_p},$$

it suffices to show that $X \cdot H$ is an object of $s\mathcal{QS}$ (resp. $s\mathcal{S}$). We may assume that the residue field F of A is algebraically closed. Choose a closed point $x \in X_s$ and assume that the completion of $\mathcal{O}_{X,x}$ is isomorphic to

$$B = A[[x_1, \ldots, x_r, y_1, \ldots, y_n]]/\langle \pi - u x_1^{e_1} \ldots x_r^{e_r} \rangle$$

as in condition (2) above. Let $f \in B$ be the image of the local equation for H at x, and let $\mathfrak{n} \subseteq B$ be the maximal ideal. Since $B/\langle f \rangle$ is the completion of the local ring of $X \cdot H$ at x if $x \in X \cdot H$, and since the irreducible components of $(X \cdot H)_{s,red} = X_{s,red} \cap H_s$ are the connected components of the smooth varieties

$Y_i \cap H_s$, the lemma follows from the following two claims. In fact, Claim 2 shows that every $x \in (X \cdot H)_{s,red}$ has an open neighborhood in $X \cdot H$ which is an object $s\mathcal{QS}$ (resp. $s\mathcal{S}$). If X/A is proper, these neighborhoods cover $X \cdot H$.

Claim 1 Assumption (i) implies that

either (a) f is a unit in B,
or (b) $s \geq 1$, f is in $\mathfrak{n}$, and has non-zero image in $\mathfrak{n}/(\mathfrak{n}^2 + \langle x_1, \ldots, x_r \rangle)$.

Claim 2 Assume condition (b) holds. Then

$$B/\langle f \rangle \cong A[[x_1, \ldots, x_r, z_1, \ldots, z_{s-1}]]/\langle \pi - \overline{u} \cdot x_1^{e_1} \ldots x_r^{e_r} \rangle,$$

where $\overline{u}$ is a unit of $A[[x_1, \ldots, x_r, z_1, \ldots, z_{s-1}]]$.

Proof of claim 2 The elements x_i and $y_j \mod \mathfrak{n}^2$ form an F-basis of $\mathfrak{n}/\mathfrak{n}^2$ $(1 \leq i \leq r,\ 1 \leq j \leq n)$. Hence we have

$$f = \sum_{i=1}^{r} a_i x_i + \sum_{j=1}^{s} a_{j+r} y_j \qquad \mod \mathfrak{n}^2$$

with elements $a_i,\ a_{j+r} \in A$ which are determined modulo $\langle \pi \rangle$. If (b) holds, then $a_{j+r} \in A^\times$ for some j, and by possibly renumbering and multiplying f by a unit we may assume $j = s$ and $a_{r+s} = 1$. But then

$$B/\langle f \rangle \cong A[[x_1, \ldots, x_r, y_1, \ldots, y_{s-1}]]/\langle \pi - \overline{u} \cdot x_1^{e_1} \ldots x_r^{e_r} \rangle.$$

Proof of claim 1 The elements $x_1, \ldots, x_r$ are the images of the local equations for $Y_{i_1}, \ldots, Y_{i_r}$ for suitable $1 \leq i_1 < \cdots < i_r \leq M$. Thus the trace of $Y_{i_1} \cap \cdots \cap Y_{i_r}$ in $\hat{\mathcal{O}}_{X,x} \cong B$ is given by the ideal $\langle x_1, \ldots, x_r \rangle$, i.e., by the quotient

$$B' = B/\langle x_1, \ldots, x_r \rangle \cong F[[y_1, \ldots, y_s]] \quad .$$

This is zero-dimensional if and only if $s = 0$, and in this case $Y_{i_1} \cap \cdots \cap Y_{i_r}$ is zero-dimensional as well. Then, by assumption on H, H does not intersect $Y_{i_1} \cap \cdots \cap Y_{i_r}$, and so f is a unit in $B/\langle x_1, \ldots, x_r \rangle$ and hence so is in B.

If $s \geq 1$, then H intersects $Y_{i_1} \cap \cdots \cap Y_{i_r}$ transversally at x if and only if the image of f in B' lies in $\mathfrak{n}' - (\mathfrak{n}')^2$, where $\mathfrak{n}'$ is the maximal ideal of B'. Now Claim 1 follows from the isomorphism

$$\mathfrak{n}'/(\mathfrak{n}')^2 \cong \mathfrak{n}/(\mathfrak{n}^2 + \langle x_1, \ldots, x_r \rangle).$$

Proof of Theorem 1 It suffices to find a hyperplane satisfying the assumption of Lemma 1, i.e., to show that, with the notations introduced earlier, the set $V_{X_\eta}(K) \cap sp^{-1}(V_2(F))$ is non-empty, where V_2 is the intersection of the sets $V_{Y_{i_1,\ldots,i_p}}$, and hence open and dense in $(\mathbb{P}_F^N)^\vee$. This holds under the conditions of Theorem 1, by the arguments used in the proof of Theorem 0. □

If X/A is proper, we noted that $sp^{-1}(V_2(F))$ is contained in $V_{X_\eta}(K)$. Combining this with the mentioned results of Poonen [Po] we get:

Corollary 1 *If F is finite and V/K is a smooth projective variety with strictly semi-stable reduction, there is a smooth hypersurface section which again has strictly semi-stable reduction.*

2. Lefschetz pencils for schemes with ordinary quadratic reduction

Even if one starts with a variety V over K with good reduction, in general infinitely many fibres in a Lefschetz pencil (cf. below) for V will not have good reduction, because infinitely many hyperplanes specialize to the same hyperplane in the reduction, and usually the induced pencil for the reduction of V has a bad member. But one can arrange very mild singularities:

Definition 2 We say that a smooth projective variety V over K has ordinary quadratic reduction, if there is a projective A-scheme X such that $X_\eta \cong V$, and X_s is smooth over F except for a finite number of singular points which are ordinary quadratic (cf. [SGA 7 XV, 1.2.1, XII 1.1] and below).

We will show that one can even start with such singularities, and still get singularities which are not worse - which is useful for induction on dimension:

Theorem 2 *Let V be projective K-variety with ordinary quadratic reduction, and let $X \subset \mathbb{P}^N_A$ be a model of V as in Definition 1. Let $d \geq 2$ be an integer and suppose F is infinite. Then, after possibly passing to the d-fold embedding of X, there exists a Lefschetz pencil $\{V_t\}_{t\in D}$, where D is a line in the dual projective space $(\mathbb{P}^N_K)^\vee$, satisfying the following conditions:*

(1) The axis of the pencil has good reduction over A.

(2) There exists a finite subset $\Sigma \subset \mathbb{P}^1_K$ of closed points such that for any $t \notin \Sigma$, V_t has ordinary quadratic reduction over $A_{K(t)}$, the integral closure of A in the residue field $K(t)$ of t.

Suppose F is finite, A is Henselian, and ℓ is a fixed prime. Then the same result holds after possibly passing to a finite unramified extension K'/K of ℓ-power degree.

The proof will be achieved in four steps, numbered (2.1) ∼ (2.4).

(2.1) Let $(\mathbb{P}^N_A)^\vee$ be the dual projective space over A. Its fibres over K and F coincide with the dual projective spaces $(\mathbb{P}^N_K)^\vee$ and $(\mathbb{P}^N_F)^\vee$, respectively. Furthermore, let $\mathcal{G} = Gr(1, (\mathbb{P}^N_A)^\vee)$ be the Grassmannian of lines in $(\mathbb{P}^N_A)^\vee$; again its fibres over K and F are the corresponding Grassmannians for $(\mathbb{P}^N_K)^\vee$ and $(\mathbb{P}^N_F)^\vee$, respectively. According to [SGA 7, XVII, 2.5], after possibly passing to the d-fold projective embedding, there is a dense open subscheme $W_{X_\eta} \subset \mathcal{G}_K$ such that the lines in W_{X_η} give Lefschetz pencils for $X_\eta \subset \mathbb{P}^N_K$.

(2.2) Since X_F is possibly singular, we need a slight extension of the results in [SGA 7, XVII]. First we extend the results to smooth, but only quasi-projective varieties.

Theorem 3 *Let L be any field, let $U \subset \mathbb{P}^N_L$ be a smooth irreducible quasi-projective variety, and let $d \geq 2$ be an integer. After possibly passing to the d-fold embedding, there is a non-empty open subscheme W_U in the Grassmannian $Gr(1, (\mathbb{P}^N_L)^\vee)$ of lines in the dual projective space, such that the lines D in W_U satisfy all properties of Lefschetz pencils with respect to U, i.e.:*

1. *The axis of D (i.e., the intersection of any two different and hence all hyperplanes parametrized by D) intersects U transversally.*
2. *There is a finite subscheme $\Sigma \subset D$ such that for $t \in D \setminus \Sigma$ the hyperplane H_t corresponding to t intersects U transversally.*
3. *For $t \in \Sigma$ the scheme-theoretic intersection $U \cdot H_t = U \times_{\mathbb{P}^N_L} H_t$ is smooth except for one singular point which is ordinary quadratic.*

Proof Let X be the closure of U in $\mathbb{P} = \mathbb{P}^N_L$, and let $Z = X \setminus U$ (both endowed with the reduced subscheme structure). For $Q = \mathbb{P}^N_L \setminus Z$, let $\mathcal{J}$ be the ideal sheaf of the closed immersion $U \subset Q$, and denote by $\mathcal{N} = \mathcal{J}/\mathcal{J}^2$ the conormal sheaf, regarded as a locally free sheaf on U, and by $\mathcal{N}^\vee$ its dual. As in [SGA 7, XVII] consider the closed immersion of projective bundles on U

$$\mathbb{P}_U(\mathcal{N}^\vee) \hookrightarrow \mathbb{P}_U(\mathcal{O}_U(1) \otimes_L \Gamma(\mathbb{P}, \mathcal{O}_\mathbb{P}(1))^\vee) \cong U \times (\mathbb{P}^N_L)^\vee$$

induced by the canonical monomorphism of bundles

$$\mathcal{J}/\mathcal{J}^2 \hookrightarrow \Omega^1_{Q|U} \hookrightarrow \mathcal{O}_U(-1)^{N+1} = \mathcal{O}_U(-1) \otimes_L \Gamma(\mathbb{P}, \mathcal{O}_\mathbb{P}(1)).$$

(Here we adopt the convention that, for a vector bundle $\mathcal{F}$ on U, the projective bundle $\mathbb{P}(\mathcal{F}) = \mathrm{Proj}(\mathrm{Sym}(\mathcal{F}))$ parametrizes line bundle quotients of $\mathcal{F}$.) The above immersion identifies $\mathbb{P}_U(\mathcal{N}^\vee)$ with the subvariety of points (x, H) in $U \times (\mathbb{P}^N_L)^\vee$ for which H touches U in x. Let $U^\vee$ be the closure of the image of $\mathbb{P}_U(\mathcal{N}^\vee)$ in $(\mathbb{P}^N_L)^\vee$. It is the dual variety to U and contains all hyperplanes in $\mathbb{P}^N_L$ which touch U in some point. One has $\dim U^\vee \leq \dim \mathbb{P}_U(\mathcal{N}^\vee) = N - 1$. Hence $(\mathbb{P}^N_L)^\vee \smallsetminus U^\vee$ is non-empty, and the set $M''_U \subseteq \mathcal{G}_L = Gr(1, (\mathbb{P}^N_L)^\vee)$ of lines in $(\mathbb{P}^N_L)^\vee$ contained in $U^\vee$ is closed and different from $\mathcal{G}_L$.

Moreover, let $(U^\vee)^0$ be the set of hyperplanes which touch U in exactly one point which is an ordinary quadratic singularity. Then $(U^\vee)^0$ is open in $U^\vee$ by results of Elkik and Deligne ([SGA 7, XVII, 3.2], [SGA 7, XV, 1.3.4]). (If char $L \neq 2$ or if $n = \dim U$ is even, then it is the locus where $\mathbb{P}_U(\mathcal{N}^\vee) \to (\mathbb{P}^N_L)^\vee$ is unramified.) It is non-empty after replacing the given embedding by its d-multiple ($d \geq 2$), by the argument in [SGA 7, XVII, 3.7, 4.2]. Since $U^\vee$ is irreducible (by irreducibility of $\mathbb{P}_U(\mathcal{N}^\vee)$), the closed subscheme $F''' = U^\vee \smallsetminus (U^\vee)^0$ has codimension ≥ 2 in $(\mathbb{P}^N_L)^\vee$

in this case. Then the set $M_U''' \subset \mathcal{G}_L$ of lines in $\mathbb{P}_L^N$ which meet F''' is closed and different from $\mathcal{G}_L$.

Finally, the set $W_U' \subseteq Gr(N-2, \mathbb{P}_L^N)$ of codimension 2 linear subspaces in $\mathbb{P}_L^N$ which intersect U transversally is open [Jou, 6.11, 2)]. It is also non-empty: Since $U^\vee \neq (\mathbb{P}_L^N)^\vee$, over the algebraic closure there is a hyperplane H_1 intersecting U transversally, and similarly, there is a hyperplane H_2 intersecting $U \cdot H_1$ transversally. This means that the codimension 2 linear subspace $H_1 \cdot H_2$ intersect U transversally. Recall the isomorphism

$$\mathcal{G}_L = Gr(1, (\mathbb{P}_L^N)^\vee) \xrightarrow{\sim} Gr(N-2, \mathbb{P}_L^N)$$

sending a pencil to its axis. We denote the preimage of W_U' in $\mathcal{G}_L$ by W_U' again.

The conclusion is that there is a non-empty open subscheme $W_U = W_U' \cap (\mathcal{G}_L \smallsetminus M_U'') \cap (\mathcal{G}_L \smallsetminus M_U''') \subseteq \mathcal{G}_L$ such that the lines in W_U satisfy all properties of Lefschetz pencils with respect to U, and thus Theorem 3 is proved.

(2.3) Now we deal with the singular points of the special fibre X_F of X in Theorem 2.

Theorem 4 *Let L be any field, and let $X \subset \mathbb{P}_L^N$ be a projective variety which is smooth except for finitely many singular points $x_1, \ldots, x_r$ which are ordinary quadratic. After possibly passing to the d-fold embedding (any $d \geq 2$), there is a non-empty open subscheme $W_X \subset Gr(1, (\mathbb{P}_L^N)^\vee)$ such that for the lines D in W_X the following holds:*

- *(i) The axis of D does not meet the singular points of X and intersects the regular locus X^{reg} transversally.*
- *(ii) There is a finite subscheme $\Sigma \subseteq D$ such that for $t \in D \smallsetminus \Sigma$ the hyperplane H_t does not meet the singular points of X and intersects X^{reg} transversally.*
- *(iii) For $t \in \Sigma$ the scheme-theoretic intersection $X^{reg} \cdot H_t$ is smooth except for possibly one singular point which is an ordinary quadratic singularity.*
- *(iv) If $t \in \Sigma$ and $x_i \in H_t$, then x_i is an ordinary quadratic singularity of $X \cdot H_t$.*

Proof Applying Theorem 3 to $X^{reg} = X \smallsetminus \{x_1, \ldots, x_r\}$ we find a non-empty open subset $W' = W_{X^{reg}} \subseteq \mathcal{G}_L$ such that the lines D in V satisfies the properties (i) to (iii) for X^{reg} instead of X.

It remains to consider the singular points $x_1, \ldots, x_r$. For each x_i, the hyperplanes in $\mathbb{P}_L^N$ which pass through x_i form a hyperplane $\widetilde{H}_i \subseteq (\mathbb{P}_L^N)^\vee$. By the following lemma there is a non-empty open subset $U_i \subseteq \widetilde{H}_i$ such that for any hyperplane H in U_i the intersection $Y \cdot H$ has an ordinary quadratic singularity at x_i. Then $F_i = \widetilde{H}_i \smallsetminus U_i$ is closed and of codimension ≥ 2 in $(\mathbb{P}_L^N)^\vee$, and so is $F = \cup_{i=1}^r F_i$. The set $W'' \subseteq \mathcal{G}_L$ of lines in $(\mathbb{P}_L^N)^\vee$ which do not meet F and are not contained in any $\widetilde{H}_i$

is thus open and non-empty, and the properties (i) to (iv) above hold for the lines in $W_X = W' \cap W'' \subseteq \mathcal{G}_L$.

Lemma 2 *Let L be any field, let $X \subset \mathbb{P}^N_L$ be a projective variety of positive dimension, and let x be an isolated singularity which is an ordinary quadratic point. If $\widetilde{H}_x \subset (\mathbb{P}^N_L)^\vee$ denotes the locus of hyperplanes passing through x, then there is an open dense subset $U \subset \widetilde{H}_x$ such that for all hyperplanes H in U the point x is an ordinary quadratic singularity of $X \cdot H$.*

Proof We may assume that L is algebraically closed. Let $A = \hat{\mathcal{O}}_{X,x}$ be the completion of the local ring at x. Then x is called an ordinary quadratic singularity, if A is isomorphic to the quotient

$$L\big[[x_1, \dots, x_{n+1}]\big]/\langle f \rangle \ ,$$

where f starts in degree 2, and where f_2, the homogeneous part of degree 2 of f, is non-zero, and defines a non-singular quadric in $\mathbb{P}^n_L$ (where $n \geq 1$ by assumption). We shall call A the ring of an ordinary quadratic singularity in this case.

Lemma 3 *Let $\mathfrak{m} \subset A$ be the maximal ideal, and let $g \in \mathfrak{m} \smallsetminus \{0\}$ be an element. Then $A' := A/\langle g \rangle$ is the ring of an ordinary quadratic singularity if the following two conditions hold*

(i) The image $\bar{g}$ of g in $\mathfrak{m}/\mathfrak{m}^2$ is non-zero.

(ii) The non-singular projective quadric $\operatorname{Proj}(\operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2)/\langle Q \rangle)$ and the hypersurface $\operatorname{Proj}(\operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2)/\langle \bar{g} \rangle)$ intersect transversally in $\operatorname{Proj}(\operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2)) \cong \mathbb{P}^n_L$. Here Q corresponds to f_2 under the isomorphism

$$L[[x_1, \dots, x_{n+1}]] \xrightarrow{\sim} \operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2) \ .$$

(More intrinsically, Q is determined up to a scalar factor as the generator of the 1-dimensional kernel of the surjection $\operatorname{Sym}^2(\mathfrak{m}/\mathfrak{m}^2) \twoheadrightarrow \mathfrak{m}^2/\mathfrak{m}^3$).

Proof Lift g to an element $\bar{g} \in B := L\big[[x_1, \dots, x_{n+1}]\big]$, and let $\mathfrak{n}$ be the maximal ideal of B. By (i) and a substitution we may assume $\bar{g} = x_{n+1}$. Then $B' := B/\langle \bar{g} \rangle \cong L[[x_1, \dots, x_n]]$, and

$$A' = B'/\langle f' \rangle$$

where f' is the image of f in B'. Then f' starts in degree 2 as well, and f'_2, its degree 2 part with respect to the variables $x_1, \dots, x_n$, is just the image of f_2. If f'_2 is zero, then $\langle Q \rangle \subseteq \langle \bar{g} \rangle$ in $\operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2)$, in contradiction to (ii). Hence $f'_2 \neq 0$, and by (ii) it gives rise to a non-singular quadric in

$$\operatorname{Proj}(\operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2)/\langle \bar{g} \rangle) \cong \operatorname{Proj}(\operatorname{Sym}(\mathfrak{m}/(\mathfrak{m}^2 + \langle g \rangle))) \cong \mathbb{P}^{n-1}_L$$

by (ii) for $n \geq 2$, i.e., A' is the ring of an ordinary quadric singularity.

We proceed with the proof of Lemma 2. Choose coordinates $X_0, \dots, X_N$ on $\mathbb{P}^N_L$ such that $x = (1 : 0 : \dots : 0)$. The hyperplanes in $\mathbb{P}^N_L$ are given by points $b = (b_0 : \dots : b_N)$ in the dual projective space $(\mathbb{P}^N_L)^\vee$, corresponding to the hyperplanes

$$\sum_{i=0}^{N} b_i X_i = 0 \qquad .$$

The hyperplanes through x are given by those b with $b_0 = 0$ and are parametrized by $(b_1 : \dots : b_N) \subseteq (\mathbb{P}^{N-1}_L)^\vee$. If $x_i = \frac{X_i}{X_0}$, $i = 1, \dots, N$, are the affine coordinates on the open affine neighborhood $\{x_0 \neq 0\} \cong \mathbb{A}^N_L$, x corresponds to the zero point, and the hyperplane associated to $(b_1 : \dots : b_N)$ is determined by the element $\sum\limits_{i=1}^{N} b_i x_i \in L[x_1, \dots, x_N]$.

Let $\mathfrak{n}$ be the maximal ideal $\langle x_1, \dots, x_N \rangle$. Then one has an isomorphism

$$\begin{array}{rcl} L^N & \xrightarrow{\sim} & \mathfrak{n}/\mathfrak{n}^2 \\ (b_1, \dots, b_N) & \longmapsto & \sum\limits_{i=1}^{N} b_i x_i \quad \mod \mathfrak{n}^2 \quad . \end{array}$$

Now let $\mathfrak{m} \subset \mathcal{O}_{X,x}$ be the maximal ideal. Then we get a surjection

$$\varphi : \mathfrak{n}/\mathfrak{n}^2 \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2 \qquad ,$$

and for a point $b = (b_1, \dots, b_N) \in L^N$ and the associated hyperplane H_b, the local ring of x in $X \cdot H_b$ is

$$\mathcal{O}_{X,x}/\langle \sum_{i=1}^{N} b_i x_i \rangle .$$

By the above lemma, x is an ordinary quadratic singularity if the image $\overline{\sum b_i x_i}$ of $\sum b_i x_i$ in $\mathfrak{m}/\mathfrak{m}^2$ is non-zero, and if the associated hyperplane in $\mathbb{P}_L(\mathfrak{m}/\mathfrak{m}^2)$ intersects the quadric in $\mathbb{P}_L(\mathfrak{m}/\mathfrak{m}^2)$ associated to the singularity transversally. The latter condition defines an open subset U' in the dual projective space $\mathbb{P}_L((\mathfrak{m}/\mathfrak{m}^2)^\vee)$ parametrizing the hyperplanes in $\mathbb{P}_L(\mathfrak{m}/\mathfrak{m}^2)$. Consider the non-empty open subset $U'' \subseteq \mathbb{P}_L((\mathfrak{n}/\mathfrak{n}^2)^\vee)$ on which the projection

$$p : \mathbb{P}_L((\mathfrak{n}/\mathfrak{n}^2)^\vee) \dashrightarrow \mathbb{P}_L((\mathfrak{m}/\mathfrak{m}^2)^\vee)$$

associated to $\varphi^\vee$ is defined. (To wit: U'' is the complement of $\mathbb{P}_L((\ker \varphi)^\vee) \subseteq \mathbb{P}_L((\mathfrak{n}/\mathfrak{n}^2)^\vee)$. Letting $U = p^{-1}(U')$, we see that for the hyperplanes H in U the intersection $X \cdot H$ has an ordinary quadratic singularity at x.

(2.4) We can now finish the proof of Theorem 2. Applying Theorem 4 to X_F and combining it with the result on X_K, we obtain the wanted Lefschetz pencil over $\mathrm{Spec}(A)$ provided there is an A-rational point in $\mathcal{G}$, corresponding to a line

L over A, such that L_η lies in the open $W_{X_\eta} \subset \mathcal{G}_\eta$ (constructed in (2.1)) and L_s lies in the open $U_{X_s} \subset \mathcal{G}_s$ (constructed in Theorem 4). This existence, under the conditions of Theorem 2, follows now by applying the arguments in the proof of Theorem 0 to $P = \mathcal{G}$, $V_1 = W_{X_\eta}$ and $V_2 = U_{X_s}$. Note that the specialization map $\mathcal{G}(K) = \mathcal{G}(A) \to \mathcal{G}(F)$ is surjective, and that $\mathcal{G}_L$, over a field L, has a cellular decomposition, so that $\mathcal{G}(L)$ is dense in $\mathcal{G}$ for infinite L.

3. Desingularization of ordinary quadratic singularities

For the applications, it is important to have a good description of varieties with ordinary quadratic reduction, and also a description of their desingularization, because such schemes may be non-regular. We recall the following description of local rings around an ordinary quadratic singularity [SGA 7, XV, 1.3.2].

Lemma 4 *Let X be a flat scheme of finite type over A, and assume that X is smooth over A except for one singular point $x \in X_s$ which is an ordinary quadratic singularity (in X_s). Assume that X_s is of dimension n at x. Then $\mathcal{O}_{X,x}$ is étale locally isomorphic to the localization of the following ring B at the maximal ideal $\mathfrak{n} = \langle x_1, \ldots, x_{n+1}, \pi\rangle$.*

(i) If x is non-degenerate:

$$B = A[x_1, \ldots, x_{n+1}]/\langle Q(x_1, \ldots, x_{n+1}) - c\rangle,$$

where Q is a non-degenerate quadratic form over A and $c \in \mathfrak{m} \setminus \{0\}$.

(ii) If x is degenerate (which can only happen if char$(F) = 2$ and $n = 2m$ is even):

$$B = A[x_1, \ldots, x_{n+1}]/\langle P(x_1, \ldots, x_{2m}) + x_{n+1}^2 + bx_{n+1} + c\rangle,$$

where P is a non-degenerate quadratic form over A and $b, c \in A$ with $b^2 - 4c \in \mathfrak{m} \setminus \{0\}$.

Proof In loc. cit. it is shown that the henselianizations of $\mathcal{O}_{X,x}$ and $B_\mathfrak{n}$ are isomorphic in the case that A is henselian and $k(x)/k(s)$ is purely inseparable. After passing to the strict henselization $\tilde{A}$ of A we achieve the last two conditions, and obtain the mentioned result for the base change $\tilde{X} = X \times_A \tilde{A}$ and any singular point $\tilde{x} \in \tilde{X}$ above x. Thus $R_1 = \mathcal{O}_{\tilde{X},\tilde{x}}$ and the localization R_2 of $B \otimes_A \tilde{A}$ at the ideal $\langle x_1, \ldots, x_{n+1}, \pi\rangle$ are étale locally isomorphic, i.e., there is a local ring R with two étale local morphisms $R_1 \leftarrow R \to R_2$. But $\tilde{A}$ is the inductive limit of étale local morphisms $A \to A_i$, so by the usual limit theorems we get a similar diagram already with $A' = A_{i_0}$, for some i_0, instead of $\tilde{A}$ and hence the claim, because $B \to B \otimes_A A'$ is étale.

In the situation of Lemma 4 (i), let $r = v(c)$, where v is the normalized valuation of K, so that $c = \eta\pi^r$, where π is a prime element in A and η is a unit in A. Then we say that X has an ordinary quadratic singularity of order r. By possibly passing to a ramified extension of degree 2 (extracting a square root of π), we may assume that r is even.

In the situation of Lemma 4 (ii), by possibly passing to a ramified extension of degree 2 (the splitting field of $x^2 + bx + c$), and by a coordinate transformation, we may assume that $c = 0$. In this case we let $q = v(b)$, so that $b = \epsilon\pi^q$ with a unit ϵ, and say that X has an ordinary quadratic singularity of order q.

Theorem 5 *Let X be as in Lemma 4, and let $\varphi : \tilde{X} \to X$ be the blowing up of X at the singular point x. Assume r is even in case (i), and $c = 0$ in case (ii). Then the strict transform $\tilde{Y}$ of $Y = X_s$ is smooth, and the exceptional fibre $F_x = \varphi^{-1}(x)$ contains a point $\tilde{x} \notin \tilde{Y}$ such that the following holds:*

(a) $F_x \setminus \{\tilde{x}\}$ is smooth, and $\tilde{Y}$ and F_x intersect transversally, i.e., the scheme-theoretic intersection of these inside $\tilde{X}$ is smooth.

(b) $\tilde{X} \setminus \{\tilde{x}\}$ is regular and strictly semi-stable.

(c) In case (i), if x is of order r, then the behavior of $\tilde{X}$ at $\tilde{x}$ is as follows: If $r > 2$, then $\tilde{x}$ is ordinary quadratic of order $r - 2$. If $r = 2$, then $\tilde{X}$ is also smooth at $\tilde{x}$, and hence strictly semi-stable.

(d) In case (ii), if x is of order q, then the behavior of $\tilde{X}$ at $\tilde{x}$ is as follows: If $q > 1$, then $\tilde{x}$ is ordinary quadratic of order $q - 1$. If $q = 1$, then $\tilde{X}$ is also smooth at $\tilde{x}$, and hence strictly semi-stable.

Proof Since blowing-ups are compatible with flat base change, and since smoothness, semi-stability and type of the quadratic singularity just depend on the Henselization of the local ring, we may consider the rings B in Lemma 4.

Case (i): 1) Here the blowing-up of B at the ideal $\mathfrak{n} = \langle x_1, \ldots, x_{n+1}, \pi\rangle$ is $\mathrm{Proj}(C)$, for the B-algebra

$$C = B[U_1, \ldots, U_{n+1}, T]/I$$
$$I = \langle\ x_iU_j - x_jU_i,\ \ x_iT - \pi U_i,\ \ Q(U_1, \ldots, U_{n+1}) - \eta\pi^{r-2}T^2\ \rangle,$$

which is graded as quotient of the polynomial ring over B. In fact, the coordinate ring of the affine chart $\{U_{n+1} \neq 0\}$ is

$$A[u_1, \ldots, u_n, x_{n+1}, t]/\langle\ Q(u_1, \ldots, u_n, 1) - \eta\pi^{r-2}t^2,\ \ x_{n+1}t - \pi\ \rangle,$$

with $x_{n+1}u_i = x_i \quad (i = 1, \ldots, n)$. A similar description holds for the other charts $\{U_i \neq 0\}$. The coordinate ring for the chart $\{T \neq 0\}$ is

$$A[u_1, \ldots, u_{n+1}]/\langle\ Q(u_1, \ldots, u_{n+1}) - \eta\pi^{r-2}\ \rangle,$$

with $\pi u_i = x_i \quad (i = 1, \ldots, n + 1)$. This shows that the inverse image of $\mathfrak{n}$ is an invertible ideal: it is generated by one element (by x_{n+1}, x_i, and π, respectively), which is not a zero divisor. Moreover, the morphism $\mathrm{Proj}(C) \to \mathrm{Spec}(B)$ becomes an isomorphism after inverting any of the elements $x_1, \ldots, x_{n+1}, \pi$. Finally there is a surjection of graded B-algebras

$$C \longrightarrow \bigoplus_{n\geq 0} \mathfrak{n}^n,$$

by sending U_i and T to x_i and π in $\mathfrak{n}$, respectively. Thus, by lemma 5 below, $\mathrm{Proj}(C)$ is isomorphic to $\mathrm{Proj}(\oplus_{n\geq 0}\ \mathfrak{n}^n)$, the blowing-up of B in $\mathfrak{n}$.

2) Assume $r > 2$. We consider the special fiber of the blowing-up, obtained by setting $\pi = 0$. Thus its chart $\{U_{n+1} \neq 0\}$ is

$$\begin{aligned} &\mathrm{Spec}(\ k[u_1,\dots,u_n,x_{n+1},t]/\langle\ Q(u_1,\dots,u_n,1),\quad x_{n+1}t\ \rangle\ ) \\ = \ &\mathrm{Spec}(\ R[x_{n+1},t]/\langle\ x_{n+1}t\ \rangle\ ) \end{aligned}$$

where $R = k[u_1,\dots,u_n]/\langle\ Q(u_1,\dots,u_n,1)\ \rangle$. It is reduced, with two smooth irreducible components intersecting transversally - the first one being the locus $\{t = 0\}$, the second one being the locus $\{x_{n+1} = 0\}$. A similar result holds for the other charts $\{U_i \neq 0\}$. The chart $\{T \neq 0\}$ is

$$\mathrm{Spec}(\ k[u_1,\dots,u_{n+1}]/\langle\ Q(u_1,\dots,u_{n+1})\ \rangle\ ),$$

which is smooth except for one ordinary quadratic singularity at $u = (0,\dots,0)$.

We may identify the irreducible components as follows. The strict transform $\tilde{Y}$ of the special fiber of $Spec(B)$ is obtained by blowing up

$$\bar{B} = B/\langle\pi\rangle = F[x_1,\dots,x_{n+1}]/\langle Q(x_1,\dots,x_{n+1})\rangle$$

in the ideal $\bar{\mathfrak{n}} = \langle x_1,\dots,x_{n+1}\rangle$. This is $\mathrm{Proj}(\bar{C})$, for

$$\bar{C} = \bar{B}[U_1,\dots,U_{n+1}]/\langle\ x_iU_j - x_jU_i,\ Q(U_1,\dots,U_{n+1})\ \rangle.$$

The affine ring of the chart $\{U_i \neq 0\}$ is

$$F[x_i,u_1,\dots,\check{u}_i,\dots,u_{n+1}]/\langle\ Q(u_1,\dots,1,\dots,u_{n+1})\ \rangle,$$

where $x_iu_j = x_j \quad (j \neq i)$, $\check{u}_i$ means omission of u_i, and the 1 in $Q(u_1,\dots,1,\dots,u_{n+1})$ is at position i. This is smooth over F, and corresponds to the locus $T = 0$ in $\tilde{X}$.

The exceptional fibre F_x is obtained by letting $x_1 = \dots = x_{n+1} = 0 = \pi$ in C. For $r > 2$ we get

$$\mathrm{Proj}(F[U_1\dots,U_{n+1},T]/\langle\ Q(U_1,\dots,U_{n+1})\ \rangle).$$

In the chart $\{U_i \neq 0\}$ this corresponds to the locus $x_i = 0 = \pi$ which is

$$\mathrm{Spec}(F[u_1,\dots,\check{u}_i,\dots,u_{n+1},t]/\langle\ Q(u_1,\dots,1,\dots,u_{n+1})\ \rangle)$$

and thus smooth. In the chart $\{T \neq 0\}$ we get

$$\mathrm{Spec}(F[u_1,\dots,u_{n+1}]/\langle Q(u_1,\dots,u_{n+1}\rangle).$$

This shows that the exceptional fiber has one ordinary quadratic singular point which does not lie on $\tilde{Y}$. From the previous description of the chart $\{T \neq 0\}$ for the whole blowing-up we see that the order of the quadratic singularity is $r - 2$.

3) Now let $r = 2$. Then the chart $\{U_{n+1} \neq 0\}$ of the whole blowing-up is

$$\mathrm{Spec}(\ S/\langle\ x_{n+1}\ t - \pi\ \rangle\ ),$$

where $S = A[u_1, \ldots, u_n, x_{n+1}, t]/\langle\ Q(u_1, \ldots, u_n, 1) - \eta t^2\ \rangle$ is smooth over A and x_{n+1}, t are part of a local parameter system where they vanish. Thus we get a regular scheme with semi-stable reduction over A. The same holds in the other charts $\{U_i \neq 0\}$. In the chart $\{T \neq 0\}$ we get the smooth A-scheme

$$\mathrm{Spec}(\ A[u_1, \ldots, u_{n+1}]/\langle\ Q(u_1, \ldots, u_{n+1}) - \eta\ \rangle\ ).$$

The strict transform $\tilde{Y}$ of Y has exactly the same description as before; it is smooth, and it is again the locus where $T = 0$. The exceptional fiber is

$$\mathrm{Proj}(\ F[U_1, \ldots, U_{n+1}, T]/\langle\ Q(U_1, \ldots, U_{n+1}) - \eta T^2\ \rangle\ )$$

which is smooth as well. Therefore $\tilde{X}$ has strict semistable reduction.

Case (ii): Here the blowing-up of

$$B = A[x_1, \ldots, x_{n+1}]/\langle\ P(x_1, \ldots, x_{2m}) + x_{n+1}^2 + bx_{n+1}\ \rangle$$

($b \in \mathfrak{m} \setminus \{0\}$) in the ideal $\mathfrak{n} = \langle\ x_1, \ldots, x_{n+1}, \pi\ \rangle$ is $\mathrm{Proj}(C)$, for

$$C = B[U_1, \ldots, U_{n+1}, T]/I,$$

where the ideal I is generated by the elements

$$\begin{array}{l} x_iU_j - x_jU_i \quad \text{for} \quad i, j \in \{1, \ldots, n+1\} \\ x_iT = \pi U_i \quad \text{for} \quad i \in \{1, \ldots, n+1\} \\ P(U_1, \ldots, U_n) + U_{n+1}^2 + \epsilon\pi^{q-1}TU_{n+1}. \end{array}$$

The coordinate ring of the chart $\{U_{n+1} \neq 0\}$ is

$$A[u_1, \ldots, u_n, x_{n+1}, t]/\langle\ x_{n+1}t - \pi, \quad P(u_1, \ldots, u_n) + 1 + \epsilon\pi^{q-1}t\ \ \rangle\ ).$$

For $i \in \{1, \ldots, n\}$, the chart $\{U_i \neq 0\}$ is

$$\mathrm{Spec}(\ A[u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_{n+1}, t]/J\ ),$$

where

$$J = \langle\ x_it - \pi, \quad P(u_1, \ldots, u_{i-1}, 1, u_{i+1}, \ldots, u_n) + u_{n+1}^2 + \epsilon\pi^{q-1}tu_{n+1}\ \rangle,$$

The affine ring for the chart $\{T \neq 0\}$ is

$$A[u_1, \ldots, u_{n+1}]/\langle\ P(u_1, \ldots, u_n) + u_{n+1}^2 + \epsilon\pi^{q-1}u_{n+1}\ \rangle.$$

The strict transform of the special fiber X_s is the locus $T = 0$, and it has exactly the same description as in case (i)2), except that the quadratic form is now $Q(U_1, \ldots, U_{n+1}) = P(U_1, \ldots, U_n) + U_{n+1}^2$. Thus it is smooth.

The exceptional fibre corresponds to the locus $\{x_i = 0 = \pi\}$. In the chart $\{U_i \neq 0\}$, for $i \neq n+1$, we get the subscheme

$$\mathrm{Spec}(\ F[u_1, \ldots, \check{u}_i, \ldots, u_{n+1}, t]/\langle\ f_i\ \rangle\ )$$

with

$$f_i = P(u_1, \ldots, 1, \ldots, u_n) + u_{n+1}^2 + \overline{\epsilon\pi^{q-1}} t u_{n+1}$$

where $\overline{a} = a \bmod \pi$. In the chart $\{U_{n+1} \neq 0\}$ we get the subscheme

$$\mathrm{Spec}(\ F[u_1, \ldots, u_n, t]/\langle\ f_{n+1}\ \rangle\ ),$$

$$f_{n+1} = P(u_1, \ldots, u_n) + 1 + \overline{\epsilon\pi^{q-1}} t.$$

These are smooth. In the chart $\{T \neq 0\}$ we get the scheme

$$\mathrm{Spec}(\ F[u_1, \ldots, u_{n+1}]/\langle\ g\ \rangle\ ),$$

$$g = P(u_1, \ldots, u_n) + u_{n+1}^2 + \overline{\epsilon\pi^{q-1}} u_{n+1}.$$

If $q > 1$, this has one quadratic singularity of order $q-1$. If $q = 1$, the scheme is smooth, since $\partial g/\partial u_{n+1} = \overline{\epsilon} \neq 0$. It is also clear that the strict transform of the special fibre and the exceptional fibre intersect transversally (in their smooth loci). Hence the claim follows.

Lemma 5 *Let B be a noetherian ring, let $I \subset B$ be an ideal, and let $\tilde{X} = \mathrm{Proj}(\oplus\, I^n)$ be the blowing-up of $X = \mathrm{Spec}(B)$ in the closed subscheme $Y = \mathrm{Spec}(B/I)$ corresponding to I. Let*

$$\varphi:\ C \longrightarrow \bigoplus_{n\geq 0} I^n$$

be a surjection of graded B-algebras. Then the X-morphism

$$f = \varphi^*:\ \ \mathrm{Proj}(\oplus\, I^n) = \tilde{X}\ \ \longrightarrow\ \ Z = \mathrm{Proj}(C)$$

induced by φ is an isomorphism if and only if the following two conditions hold.

(i) I generates an invertible ideal in $Z = \mathrm{Proj}(C)$.

(ii) g induces an isomorphism $g^{-1}(X \setminus Y) \xrightarrow{\sim} X \setminus Y$.

Proof The two conditions are known to hold for $Z = \tilde{X}$, and by the surjectivity of φ, the morphism f is a closed immersion. In particular, f is affine. By (i), each point in Z has an open affine neighborhood $V = \mathrm{Spec}(R) \subset Z$ over which the image of I is generated by one element $a \in R$ which is not a zero divisor. Hence $f^{-1}(V) \to V$ corresponds to a surjection of rings $R \to R/J$, which induces an isomorphism after inverting a, by condition (ii) (for Z and $\tilde{X}$). This means that $J_a = 0$ for the localization of the ideal J with respect to a. It follows that $J = 0$, because a is not a zero divisor. Therefore f is an isomorphism.

4. Application to class field theory of varieties over local fields with (almost) good reduction

Now assume that A is a Henselian discrete valuation ring with finite residue field $F = A/\mathfrak{m}$. Thus the fraction field K of A is a non-archimedean local field (in the usual sense if A is complete). Let V be a proper variety over K. Then we have the *reciprocity map for V*

$$\rho_V \;:\; SK_1(V) \to \pi_1^{ab}(V)$$

introduced in [Bl], [Sa1] and [KS1]. Here $\pi_1^{ab}(V)$ is the abelianized algebraic fundamental group of V and

$$SK_1(V) = \mathrm{Coker}(\bigoplus_{x\in V_1} K_2(k(y)) \xrightarrow{\partial} \bigoplus_{x\in V_0} K_1(k(x)))$$

where V_i denotes the set of points $x \in V$ of dimension i, $K_q(k(x))$ denotes the q-th algebraic K-group of the residue field $k(x)$ of x, and ∂ is induced by tame symbols. For an integer $n > 0$ prime to $\mathrm{ch}(K)$ let

$$\rho_{V,n} \;:\; SK_1(V)/n \to \pi_1^{ab}(V)/n$$

denote the induced map. Finally, for a field L and a prime ℓ invertible in L recall the following

Conjecture $\mathbf{B}K_q(L,\ell)$ The Galois cohomology group $H^q(L, \mathbb{Q}_\ell/\mathbb{Z}_\ell(q))$ is divisible.

Here and below (q) denotes the usual q-fold Tate twist. This conjecture is a consequence of a conjecture of Bloch and Kato asserting the surjectivity of the symbol map $K_q^M(L) \to H^q(L, \mathbb{Z}/\ell\mathbb{Z}(q))$ from Milnor K-theory to Galois cohomology. The above form is weaker if restricted to particular fields L, but known to be equivalent if stated for all fields. By Kummer theory, $BK_1(L,\ell)$ holds for any L and any ℓ. The celebrated work of Merkuriev and Suslin [MS] shows that $BK_2(L,\ell)$ holds for any L and any ℓ. Voevodsky [V1] proved $BK_q(L,2)$ for any L and any q. It has been announced by Rost [Ro] and Voevodsky [V2] (see also [SJ] and [Weib]) that it it holds in general.

Theorem 6 *Let V be a connected smooth projective variety over K of dimension ≥ 2 with good or ordinary quadratic reduction, and let ℓ be a prime invertible in K. Assume $BK_3(k(Z),\ell)$ for any proper smooth surface Z lying on V which has almost good reduction over* $\mathrm{Spec}(\mathcal{O}_{K'})$ *for some finite extension K' of K. Then ρ_{V,ℓ^ν} is an isomorphism for all $\nu > 0$.*

Proof Let $n = \ell^\nu$, and let X be as in Definition 2. By [JS (1-6) and (7-1)] there exists a fundamental exact sequence

$$(4.1) \qquad H_2^K(V, \mathbb{Z}/n\mathbb{Z}) \to SK_1(V)/n \xrightarrow{\rho_{V,n}} \pi_1^{ab}(V)/n \to H_1^K(V, \mathbb{Z}/n\mathbb{Z}) \to 0$$

where $H_i^K(V, \mathbb{Z}/n\mathbb{Z})$ denotes the i-th Kato homology of V with $\mathbb{Z}/n\mathbb{Z}$-coefficients introduced in [JS Def. 1.2]. In order to show the surjectivity of $\rho_{V,n}$, it thus suffices to show $H_1^K(V, \mathbb{Z}/n\mathbb{Z}) = 0$. According to Theorem 5, by blowing up X at the points where X_s is not smooth, and repeating this finitely many times, we obtain a scheme $\widetilde{X}$ which is regular with strict semi-stable reduction, and has the same generic fibre V, by construction.

Then, by the isomorphism $H_1^K(V, \mathbb{Z}/n\mathbb{Z}) \cong H_1^K(\widetilde{X}_s, \mathbb{Z}/n\mathbb{Z})$ proved in [JS Thm. 1.5 (2)] it suffices to show the vanishing of the latter group, which is the Kato homology of the special fibre $\widetilde{X}_s$ as defined in [JS Def. 1.2]. Moreover it is easy to see that the configuration complex $\Gamma_{\widetilde{X}_s}$ of $\widetilde{X}_s$ introduced in [JS Rem. 3.8] is contractible. Thus the isomorphism $H_1^K(\widetilde{X}_s, \mathbb{Z}/n\mathbb{Z}) \cong H_1(\Gamma_{\widetilde{X}_s}, \mathbb{Z}/n\mathbb{Z})$ proved in [JS Thm. 1.4] gives the desired vanishing.

Now we show the injectivity of $\rho_{V,n}$. Let $X \subset \mathbb{P}_A^N$ be an embedding of X into the projective space over A, and fix a prime q. By Theorem 2, after possibly taking the base change with a finite unramified covering of A of q-power degree, there exists a Lefschetz pencil $\{V_t\}_{t \in D}$, where $D \cong \mathbb{P}^1$ is a K-line in the dual projective space of $\mathbb{P}_K^N$, satisfying the following conditions:

(1) The axis of the pencil has good reduction over $A = \mathcal{O}_K$.

(2) There exists a finite subset $\Sigma \subset \mathbb{P}_K^1$ of closed points such that for every $t \notin \Sigma$, V_t has ordinary quadratic reduction over $\mathrm{Spec}(\mathcal{O}_{k(t)})$.

We write $V_\Sigma = \cup_{t \in \Sigma} V_t$. Fix an integer $n > 0$.

Claim 1 *Any element $\alpha \in SK_1(V)/n$ is represented by*

$(*)$ $\sum_{i=1}^r f_i \otimes [x_i]$ *with* $f_i \in k(x_i)^*$,

where x_i is a closed point of $V \backslash V_\Sigma$ for $1 \leq i \leq r$.

Proof By definition α is represented by a sum of the form $(*)$, where however the x_i may lie on V_Σ. By a standard Bertini argument there exists a proper smooth curve C over k lying on V such that

(1) C is not contained in any fiber V_t.

(2) $\{x_1, \ldots, x_n\} \subset C$.

By (1) $C_\Sigma := C \cap V_\Sigma$ is finite and we put $U = C \backslash C_\Sigma$. By (2) α lies in the image of $SK_1(C) \to SK_1(V)$. The claim follows from the surjectivity of the natural map

$$\bigoplus_{x \in U_0} k(x)^* \to SK_1(C)/n \tag{7-4}$$

which is a consequence of the class field theory of curves over local fields. Indeed, by [Sa1] one knows that the reciprocity map $SK_1(C)/n \to \pi_1^{ab}(C)/n$ is injective and that every character of $\pi_1^{ab}(C)/n$ which is trivial on the image of (7-4) is trivial on $SK_1(C)/n$. Namely, if every closed point of U splits completely in a given abelian covering of C, each point of C_Σ splits completely as well.

Now, fixing $n = \ell^\nu$, we show the injectivity of $\rho_{V,n}$ by induction on $d := \dim(V)$. The case $d = 2$ follows from [JS]. In fact, by (4.1) it suffices to show $H_2^K(V, \mathbb{Z}/n\mathbb{Z}) = 0$. But for $d = 2$ we have $H_3^K(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ by definition, and [JS Lem.7.3] gives an isomorphism $H_2^K(V, \mathbb{Z}/n) \cong H_2^K(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell)[n]$, where $B[n] = \{b \in B \mid nb = 0\}$ for an abelian group B. Moreover, by [JS Thm.1.6] we have an isomorphism $H_2^K(V, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong H_2^K(\widetilde{X}_s, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$, and by [JS Thm.1.4] we have an isomorphism $H_2^K(\widetilde{X}_s, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong H_2(\Gamma_{\widetilde{X}_s}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. Since the latter group vanishes by contractibility of $\Gamma_{\widetilde{X}_s}$, the claim follows.

Now assume $d > 2$. Let $\alpha \in SK_1(V)/n$ and assume $\rho_{V,n}(\alpha) = 0$. We want to show $\alpha = 0$. Take a Lefschetz pencil as in Theorem 2. By Claim (1) there exist $t_1, \dots, t_m \in \mathbb{P}^1_k$ such that $Y_i := V_{t_i}$ has ordinary quadratic reduction over k and α lies in the image of

$$SK_1(Y)/n \to SK_1(V)/n \quad \text{with } Y := \cup_{i=1}^m Y_i.$$

We have the commutative diagram

$$\begin{tikzcd}
SK_1(Y)/n \arrow[r, "\rho_{Y,n}"] \arrow[d] & \pi_1^{ab}(Y)/n \arrow[d] \\
SK_1(V)/n \arrow[r, "\rho_{V,n}"] & \pi_1^{ab}(V)/n
\end{tikzcd}$$

Claim 2 *The right vertical map is injective.*

Proof By Poincaré duality, we have isomorphisms

$$\pi_1^{ab}(V)/n \xrightarrow{\cong} H^{2d+1}(V, \mathbb{Z}/n\mathbb{Z}(d+1)), \quad \pi_1^{ab}(Y)/n \xrightarrow{\cong} H_Y^{2d+1}(V, \mathbb{Z}/n\mathbb{Z}(d+1)).$$

But in the localization sequence

$$H^{2d}(V \backslash Y, \mathbb{Z}/n\mathbb{Z}(d+1)) \to H_Y^{2d+1}(V, \mathbb{Z}/n\mathbb{Z}(d+1)) \to H^{2d+1}(V, \mathbb{Z}/n\mathbb{Z}(d+1))$$

we have $H^{2d}(V \backslash Y, \mathbb{Z}/n\mathbb{Z}(d+1)) = 0$ since $V \backslash Y$ is affine and $2d > \dim(V) + 2 = d + 2$ by the assumption that $d > 2$. This proves the claim.

By Claim (2) the desired assertion follows if we show that $\rho_{Y,n}$ is an isomorphism. Let A be the axis of the pencil. We have a commutative diagram

$$\begin{tikzcd}
\bigoplus_{i=1}^{m-1} SK_1(A)/n \arrow[r, "\cong"] \arrow[d] & \bigoplus_{1=1}^{m-1} \pi_1^{ab}(A)/n \arrow[d] \\
\bigoplus_{i=1}^{m} SK_1(Y_i)/n \arrow[r, "\cong"] \arrow[d] & \bigoplus_{i=1}^{m} \pi_1^{ab}(Y_i)/n \arrow[d] \\
SK_1(Y)/n \arrow[r, "\rho_{Y,n}"] \arrow[d] & \pi_1^{ab}(Y)/n \\
0 &
\end{tikzcd}$$

Here the vertical maps from the second to the third line are simply induced by the inclusions $\rho_i : Y_i \hookrightarrow Y$. The vertical maps above are obtained by noting that $A = Y_1 \cap Y_2 = Y_2 \cap Y_3 = \ldots = Y_{m-1} \cap Y_m$, and using the inclusions $Y_i \cap Y_{i+1} \hookrightarrow Y_i$ and $Y_i \cap Y_{i+1} \hookrightarrow Y_{i+1}$. Then the exactness of the left vertical sequence is easily seen. Similarly, one can show that the right vertical sequence is exact, by using the Poincaré duality and the localization sequence of étale cohomology. By induction hypothesis, the two upper horizontal maps are isomorphisms. This shows that $\rho_{Y,n}$ is an isomorphism and completes the proof of Theorem 6.

Remark 1 By [JS] Lemma 7.7, Theorem 6 implies that $\mathrm{Ker}(\rho_V)$ coincides with the maximal divisible subgroup $SK_1(V)_{Div}$ of $SK_1(V)$. When V does not have (almost) good reduction, this is not the case in general. In fact, K. Sato [Sat] constructed smooth projective surfaces X_1 and X_2 over a p-adic field with the following properties (provided the Bloch-Kato conjecture in degree 3 holds).

(a) $SK_1(X_1)_{Div} \subsetneqq \mathrm{Ker}(\rho_{X_1})$,

(b) $SK_1(X_2)_{Div} = \mathrm{Ker}(\rho_{X_2})$ but $\rho_{X_2,n}$ is not injective for some $n > 1$,

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