1 Review on Hodge conjecture and Noether-Lefschetz problem

$X \subset \mathbb{P}^n$: a smooth projective variety over $\mathbb{C}$.

The space of Hodge cycles of codimension q on X:

$$F^0H^{2q}(X, \mathbb{Q}(q)) := H^{2q}(X, \mathbb{Q}(q)) \cap F^qH^{2q}(X, \mathbb{C}).$$

Hodge Conjecture : *$F^0H^{2q}(X, \mathbb{Q}(q))$ is generated by the cycle classes of algebraic subvarieties of X, namely the cycle class map :*

$$\rho_X^q \ : \ CH^q(X) \otimes \mathbb{Q} \to H^{2q}(X, \mathbb{Q}(q)) \cap F^qH^{2q}(X, \mathbb{C})$$

is surjective. Here:

$CH^q(X)$: the Chow group of algebraic cycles of codimension q on X modulo rational equivalence,

$\mathbb{Q}(q) = (2\pi\sqrt{-1})^q\mathbb{Q} \subset \mathbb{C}$,

$F^qH^{2q}(X, \mathbb{C}) \subset H^{2q}(X, \mathbb{C})$ *is the Hodge filtration.*

The space of trivial cycles on $X \subset \mathbb{P}^n$:

$$H^{2q}(X, \mathbb{Q}(q))_{triv} := \mathbb{Q} \cdot [X \cap L] \ \subset \mathrm{Image}(\rho_X^q)$$

the subspace generated by the class of the section on X of a linear subspace $L \subset \mathbb{P}^n$ of codimension q.

S : a non-singular quasi-projective variety over $\mathbb{C}$,

$\mathcal{X} \hookrightarrow \mathbb{P}^n_S = \mathbb{P}^n \times S$: an algebraic family over S of smooth projective varieties,

X_t : the fiber of $\mathcal{X}$ over $t \in S$,

The Noether-Lefschetz locus for Hodge cycles in codimension q on $\mathcal{X}/S$:

$$S^q_{NL} = \{t \in S \mid F^0H^{2q}(X_t, \mathbb{Q}(q)) \neq H^{2q}(X_t, \mathbb{Q}(q))_{triv}\}.$$

$$F^0H^{2q}(X_t, \mathbb{Q}(q)) = F^qH^{2q}(X_t, \mathbb{C}) \cap H^{2q}(X_t, \mathbb{Q}(q)).$$

It is the locus of such $t \in S$ that there exist non-trivial Hodge cycles in codimension q on X_t and hence that the Hodge conjecture is non-trivial for X_t.

Lemma *S^q_{NL} is the union of countable number of (not necessarily proper) closed analytic subsets of S.*

S : the moduli space of smooth hypersurfaces of degree d in $\mathbb{P}^3$,

$\mathcal{X}/S$: the universal family of hypersurfaces in $\mathbb{P}^3$.

$S_{NL} = S^q_{NL}$ for $q = 1$,

It is then the locus of those surfaces that possess curves which are not complete intersections of the given surface with another surface.

The celebrated theorem of Noether-Lefschetz affirms that every component of S_{NL} has positive codimension in S when $d \geq 4$.

Theorem 1.1 *(M. Green and C. Voisin)*

Let T be an irreducible component of S_{NL}.

(1) $\operatorname{codim}(T) \geq d - 3$.

(2) Assume $d \geq 5$. If $\operatorname{codim}(T) = d - 3$, *then*

$$T = \{t \in S \mid \ X_t \textit{ contains a line}\}.$$

2 Beilinson's Hodge conjecture

U : a (non-complete) quasi-projective smooth variety over $\mathbb{C}$.

The space of Beilinson-Hodge cycles on U:

$$F^0 H^q(U, \mathbb{Q}(q)) := H^q(U, \mathbb{Q}(q)) \cap F^q H^q(U, \mathbb{C}).$$

If U is projecitve and $q \neq 0$, $F^0 H^q(U, \mathbb{Q}(q)) = 0$ by the Hodge symmetry.

The analogue of the cycle map ρ_X^q is the regulator maps:

$$reg_U^q : CH^q(U, q) \otimes \mathbb{Q} \to H^q(U, \mathbb{Q}(q)) \cap F^q H^q(U, \mathbb{C}),$$

Here:

$CH^q(U, q)$: Bloch's higher Chow group,

$F^q H^q(U, \mathbb{C}) \subset H^q(X, \mathbb{C})$ is the Hodge filtration defined by Deligne.

Beilinson-Hodge Conjecture : *reg_U^q is surjective.*

(1) $CH^1(U,1) = \Gamma(U, \mathcal{O}^*_{Zar})$
(invertible (algebraic) functions on U)

(2) There is a product map

$$\overbrace{CH^1(U,1) \otimes \cdots \otimes CH^1(U,1)}^{q \text{ times}} \to CH^q(U,q)$$

$$g_1 \otimes \cdots \otimes g_q \to \{g_1, \ldots, g_q\}$$

The subspace generated by the *decomposable elements :*

$$CH^q(U,q)_{dec} =< \{g_1, \ldots, g_q\} \mid g_j \in CH^1(U,1) >$$

(3) The formula for the value of reg^q_U on decomposable elements:

$$reg^q_U(\{g_1, \ldots, g_q\}) = \frac{dg_1}{g_1} \wedge \cdots \wedge \frac{dg_q}{g_q} \in H^0(X, \Omega^q_X(\log Z)),$$

$$H^0(X, \Omega^q_X(\log Z)) = F^q H^q(U, \mathbb{C}) \quad \text{(Deligne)}$$

Here, $U \subset X$ is a smooth compactification with $Z = X \setminus U$, a simple normal crossing divisor on X.

(4) There is a map

$$\phi \ : \ CH^q(U,q) \to \mathrm{Ker}\Big(K_q^M(\mathbb{C}(U)) \xrightarrow{\partial} \bigoplus_{\substack{Z\subset U \\ \text{prime divisor}}} K_{q-1}^M(\mathbb{C}(Z))\Big),$$

∂ : *tame symbols.*

Here, for a field L, $K_q^M(L)$ is the Milnor K-group of L:

$$K_q^M(L) = \overbrace{L^\times \otimes \cdots \otimes L^\times}^{q \text{ times}} / < \text{Steinberg relation} >$$

If $q = 2$, then ϕ is surjective and reg_U^2 factors through it.

Noether-Lefschetz problem for Beilinson-Hodge cycles

The space of trivial Beilinson-Hodge cycles:

$$H^q(U, \mathbb{Q}(q))_{triv} := reg_U^q(CH^q(U, q)_{dec}) \subset F^0H^q(U, \mathbb{Q}(q))$$

$$F^0H^q(U, \mathbb{Q}(q)) := H^q(U, \mathbb{Q}(q)) \cap F^qH^q(U, \mathbb{C})$$

S : a non-singular quasi-projective variety over $\mathbb{C}$,

$\mathcal{U} \to S$; an algebraic family over S of non-complete smooth varieties.

U_t : the fiber of $\mathcal{U}$ over $t \in S$.

The Noether-Lefschetz locus for Beilinson-Hodge cycles on $\mathcal{U}/S$:

$$S_{NL}^q = \{t \in S | \; F^0H^q(U_t, \mathbb{Q}(q)) \neq H^q(U_t, \mathbb{Q}(q))_{triv}\}.$$

In this lecture we explain some results on Noether-Lefschetz locus for Beilinson-Hodge cycles that are analogous to the theorem of Green-Voisin.