WEIGHT HOMOLOGY OF MOTIVES

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Abstract. In the first half of this article we define a new weight homology functor on Voevodsky's category of effective motives, and investigate some of its properties. In special cases we recover Gillet-Soulé's weight homology, and Geisser's Kato-Suslin homology. In the second half, we consider the notions of "co-étale" and "reduced" motives, and use the notions to a prove a theorem comparing motivic homology to étale motivic homology.

Due to [Ke1] we do not have to restrict to smooth schemes.

Contents

1. Introduction

Weight homology. In their highly cited work [GS1, §2], Gillet and Soulé associate to every variety X over a perfect field[1] k, a complex $W(X)$ of (effective) Chow motives, and show it is well-defined up to chain homotopy. Using this, in [GS1, §3] they define the *weight homology* (with compact support) of X with (Γ-coefficients) as

$$H_n^{GS}(X,\Gamma) = H_n(\Gamma(W(X)))$$

where $\Gamma : Chow^{\mathrm{eff}} \to \mathcal{A}$ is an additive covariant functor from effective Chow motives $Chow^{\mathrm{eff}}$ to an abelian category $\mathcal{A}$. A distinguished property of Gillet and Soulé's weight homology is : For X projective smooth over k, we have

$$H_n^{GS}(X,\Gamma) = \begin{cases} \Gamma(X) & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases}$$

On the other hand, for a variety X over a finite field k, Geisser defines in [Ge2, §8] a homology theory he calls the Kato-Suslin homology with coefficients in an abelian group A as

$$H_n^{KS}(X,A) = H_n((\underline{C}_*(X)(\overline{k}) \otimes A)_G)$$

where $\underline{C}_*(X)(\overline{k}) = \hom_{Cor}(\Delta^*_{\overline{k}}, \overline{k} \times_k X)$ and G is the Weil group of k (i.e., the subgroup of $Gal(\overline{k}/k)$ generated by the Frobenius). In [Ge2] Geisser shows the following

[1] In their article they state their result for fields of characteristic zero since they use resolution of singularities, however if one uses $\mathbb{Z}[\frac{1}{p}]$-coefficients (resp. $\mathbb{Q}$-coefficients) one can replace this with the appropriate theorem of Gabber (resp. de Jong). Alternatively, one can work under the assumption that the field admits resolution of singularities.

(see Prop. 3.4): Assume either that A is torsion prime to ch(k), or that Parshin's conjecture P_0 stated in Prop. 3.4(2) holds. Then, for X smooth over k, we have

$$H_n^{KS}(X, A) = \begin{cases} A^{c(X)} & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

where $c(X)$ is the set of connected components of X.

The first main purpose of this article is to introduce a general homology theory for motives from which the above two can be recovered as special cases.

Recall that in [Bo2] Bondarko extends Gillet-Soulé's weight complex to a triangulated functor defined on all geometric effective motives (cf. Theorem 2.1). We observe that a consequence of this is an induced equivalence of categories (see §2.1 for the notations):

Theorem (Thm. 2.4)**.** *Let k be a perfect field of exponential characteristic p. The functor*

$$\left\{ \begin{array}{c} \textit{homological functors}^2 \\ \textit{on } DM_{gm}^{\mathrm{eff}}(k, \mathbb{Z}[\tfrac{1}{p}]) \end{array} \right\} \to \left\{ \begin{array}{c} \textit{additive functors} \\ \textit{on } Chow^{\mathrm{eff}}(k, \mathbb{Z}[\tfrac{1}{p}]) \end{array} \right\} \tag{1.1}$$

induced by the inclusion $Chow^{\mathrm{eff}}(k, \mathbb{Z}[\frac{1}{p}]) \to DM_{gm}^{\mathrm{eff}}(k, \mathbb{Z}[\frac{1}{p}])$ induces an equivalence when restricted to the full subcategory of homological functors $\mathcal{H}$ satisfying:

(SPA) $\mathcal{H}(M(X)[n]) = 0$ *when X is smooth and projective and $n \neq 0$.*

For any additive covariant functor $\Gamma : \mathrm{Chow}^{\mathrm{eff}}(k, \mathbb{Z}[\frac{1}{p}]) \to \mathcal{A}$ we write

$$H_0^W(-, \Gamma) : DM_{gm}^{\mathrm{eff}}(k, \mathbb{Z}[\tfrac{1}{p}]) \to \mathcal{A} \tag{1.2}$$

for its associated homological functor (Def. 2.6) and write $H_n^W(-, \Gamma) = H_0^W(-[n], \Gamma)$ for $n \in \mathbb{Z}$. For an object A of the category $\mathcal{AB}$ of abelian groups, we put $H_n^W(-, A) = H_n^W(-, \Gamma_A)$ where $\Gamma_A : \mathrm{Chow}^{\mathrm{eff}}(k, \mathbb{Z}[\frac{1}{p}]) \to \mathcal{AB}$ is the additive functor such that $\Gamma_A(X) = A^{c(X)}$ for X smooth projective, where $c(X)$ is the set of connected components of X.

In this way, we obtain a general weight homology theory. Our theory recovers the other two:

Theorem (Thm. 3.1, Thm. 3.6)**.** *Let k be a perfect field and $\mathcal{S}ch/k$ the category of finite type separated k-schemes.*

(1) *For an additive covariant functor $\Gamma : Chow^{\mathrm{eff}}(k, \mathbb{Z}[\frac{1}{p}]) \to \mathcal{A}$, there are canonical isomorphisms for $X \in \mathcal{S}ch/k$*

$$H_n^{GS}(X, \Gamma) \cong H_n^W(M^c(X), \Gamma)$$

which are covariantly functorial for proper morphisms in $\mathcal{S}ch/k$.

(2) *Assume k is a finite field. If A is a torsion abelian group of torsion prime to the characteristic, there are canonical isomorphisms for $X \in \mathcal{S}ch/k$*

$$H_n^{KS}(X, A) \cong H_n^W(M(X), A) \tag{1.3}$$

which are covariantly functorial for morphisms in $\mathcal{S}ch/k$. Moreover, for a general $\mathbb{Z}[1/p]$-module A, the existence of isomorphisms such as (1.3) *follows from Parshin's conjecture P_0.*

See §3.1 and §3.2 for a number of properties that our weight homology (and consequently, Gillet-Soulé weight homology and Geisser's Kato-Suslin homology) satisfies. In particular, we notice that $H_n^W(M^c(-), \Gamma)$ is contravariant for quasi-finite flat morphisms in $\mathcal{S}ch/k$, and for arbitrary morphisms in $\mathcal{S}m/k$ between schemes of the same dimension (our properties (Wc3) and (Wc4)). The corresponding contravariance of

Gillet-Soulé's theory $H_n^{GS}(-,\Gamma)$ seems to be new, and not at all obvious from its construction.

For functors Γ on $\mathrm{Chow}^{\mathrm{eff}}(k,R)$ such as Γ_A which are $\mathbb{P}^1$-invariant, the associated weight homology functor satisfies a number of additional interesting properties: it vanishes on all twisted motives, it is birationally invariant on smooth (not necessarily projective) schemes, and the weight homology of (not necessarily projective) smooth schemes is concentrated in degree zero (cf. Lem. 2.7 and Thm. 2.8).

Cycle maps. The canonical functor

$$\alpha^* : DM^{\mathrm{eff}}(k,R) \to DM^{\mathrm{eff}}_{\text{ét}}(k,R)$$

connecting Voevodsky's category of motives and its étale version admits a right adjoint α_*. The motivic homology and étale motivic homology with R-coefficients of an object C in $DM^{\mathrm{eff}}(k,R)$ are defined as

$$H_i^M(C,R) = \hom_{DM^{\mathrm{eff}}(k,R)}(R[i],C), \qquad \text{and}$$
$$H_i^{M,\text{ét}}(C,R) = \hom_{DM^{\mathrm{eff}}(k,R)}(R[i],\alpha_*\alpha^*C)$$

where $R[i]=M(k)[i]$ is a shift of the unit for the tensor structures. The second half of the article concerns the following maps functorial in $C \in DM^{\mathrm{eff}}(k,R)$:

$$\alpha^* : H_i^M(C,R) \to H_i^{M,\text{ét}}(C,R)$$

induced by the unit $\mathrm{id} \to \alpha_*\alpha^*$ of the adjunction (α^*,α_*).

Recall that motivic homology is related to fundamental invariants of schemes. For example, if p is invertible in R, or if we assume strong resolution of singularities, for $X \in \mathcal{S}ch/k$, $r,i\in\mathbb{Z}$, there are canonical isomorphisms ([V, §2.2] and [Ke1, Chap. 5])

$$H_i^M(M(X),R) \cong H_i^S(X,R) \tag{1.4}$$

$$H_i^M(M^c(X)(r),R) \cong CH_{-r}(X,i+2r;R) \tag{1.5}$$

where $H_i^S(X,R)$ is the Suslin homology with R-coefficients ([SV1]), and $CH_*(X,\bullet;R)$ are the higher Chow groups with R-coefficients ([B]).

On the ther hand, for $R=\mathbb{Z}/n\mathbb{Z}$ with n coprime to p, the étale motivic homology $H_i^{M,\text{ét}}(C,\mathbb{Z}/n\mathbb{Z})$ for $C=M(X)$ (resp. $M^c(X)$) are identified with the dual of étale cohomology (resp. étale cohomology with compact support) (see Prop. 5.4 and Lem. 5.5).

The main theorem is:

Theorem (Thm. 4.2)**.** *Let p be the exponential characteristic of k, let n be an integer coprime to p, and let R be $\mathbb{Z}/n\mathbb{Z}$ or $\mathbb{Z}[1/p]$.*

1. *If k is algebraically closed, α^* is an isomorphism.*
2. *If k is finite and $C \in DM^{\mathrm{eff}}_{gm}(k,R)$, there is a canonical long exact sequence*

$$\cdots \to H_i^M(C,R) \xrightarrow{\alpha^*} H_i^{M,\acute{e}t}(C,R) \to H_{i+1}^W(C,\Lambda) \to H_{i-1}^M(C,R) \to \cdots$$

which is functorial in C where

$$\Lambda = \begin{cases} \mathbb{Q}/\mathbb{Z}[1/p] & \text{if } R=\mathbb{Z}[1/p], \\ \mathbb{Z}/n\mathbb{Z} & \text{if } R=\mathbb{Z}/n\mathbb{Z}. \end{cases}$$

In view of the remarks above the theorem, the first part of the theorem implies that if k is algebraically closed, then there are canonical isomorphisms for $X \in \mathcal{S}ch/k$:

$$H_i^S(X,\mathbb{Z}/n\mathbb{Z}) \cong H^i(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z})^*, \text{ and}$$
$$\mathrm{CH}_0(X,i;\mathbb{Z}/n\mathbb{Z}) \cong H_c^i(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z})^*.$$

One can identify these isomorphisms with those in [SV1] and [Su]. (This identification is essentially a consequence of the Suslin homology/cohomology pairing and the étale homology/cohomology pairing being compatible. See [Ke2, Prop. 15 and 17] for the details). Our proof of the above theorem is simpler than those in [SV1] and [Su], but depends on the Beilinson-Lichtenbaum conjecture due to Suslin-Voevodsky [SV3] and Geisser-Levine [GL, Thm.1.5] which relies on the Bloch-Kato conjecture proved by Rost-Voevodsky (see the proof of Lem. 5.6).

For a finite field k and $R = \Lambda = \mathbb{Z}/n\mathbb{Z}$ with n coprime to p, the map α^* gives rise to maps

$$H_i^S(X, \mathbb{Z}/n\mathbb{Z}) \to H^{i+1}(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})^*, \quad \mathrm{CH}_0(X, i; \mathbb{Z}/n\mathbb{Z}) \to H_c^{i+1}(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})^*,$$

and the second part of the above theorem relates the kernel and cokernel of these maps with weight homology and weight homology with compact support of X introduced in §3. In particular, if X is smooth over k, we get canonical isomorphisms

$$H_i^S(X, \mathbb{Z}/n\mathbb{Z}) \cong H^{i+1}(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})^* \quad \text{for } i \in \mathbb{Z}.$$

See Cor. 4.3 for the details and Rem. 4.4 for relation of these results with [Ge2].

In order to show the second part of the above theorem, we introduce new homology functors which seem to be of independent interest. The idea is to introduce an endofunctor $\Theta : DM^{\text{eff}}(k, R) \to DM^{\text{eff}}(k, R)$ equipped with a natural transformation $\Theta \to id$, which gives rise to distinguished triangles (Equation 6.2)

$$\Theta C \to C \to \alpha_*\alpha^*C \to \Theta C[1]$$

for $C \in DM^{\text{eff}}_{\text{ét}}(k, R)$, allowing one to define the "co-étale" part ΘC of a motive C. Then the second part of the above theorem is equivalent to the assertion that if k is finite, there are canonical isomorphisms for $C \in DM^{\text{eff}}_{gm}(k, R)$:

$$H_i^M(\Theta(C)), R) \xrightarrow{\cong} H^W_{i+1}(C, \Lambda) \quad \text{for } i \in \mathbb{Z}, \tag{1.6}$$

which are natural in C (see Theorem 6.3). To show this, we use $d_{\leq 0}DM^{\text{eff}}(k, R)$, the localizing subcategory of $DM^{\text{eff}}(k, R)$ generated by motives of smooth varieties of dimension zero (cf. [V, §3.4]). By [A-BV] there is a projection functor $L\pi_0 : DM^{\text{eff}}(k, R) \to d_{\leq 0}DM^{\text{eff}}(k, R)$, the "field of constants" part (cf. Equation (6.6)). Similarly as above, we get again distinguished triangles (Equation 6.8)

$$C_{red} \to C \to L\pi_0(C) \to C_{red}[1]$$

for $C \in DM^{\text{eff}}_{\text{ét}}(k, R)$, allowing one to define the "co-dimension-zero" or "reduced" part C_{red} of a motive C. Surprisingly, if k is a finite field, $H_i^M(\Theta(C_{red}), R) = 0$ for all $i \in \mathbb{Z}$, or equivalently the canonical morphism

$$\Theta(C) \to \Theta(L\pi_0(C))$$

induces isomorphisms $H_i^M(\Theta(C), R) \cong H_i^M(\Theta(L\pi_0(C)), R)$ for all $i \in \mathbb{Z}$ (Claim 6.5(2), Equation (6.9)). Heuristically, this can be interpreted as saying that over a finite field, all the difference between the motivic homology and the étale motivic homology is contained in the field of constants. Finally the desired isomorphisms (1.6) are obtained by showing canonical isomorphisms for $C \in DM^{\text{eff}}_{gm}(k, R)$:

$$H_i^M(\Theta(L\pi_0(C)), R) \xrightarrow{\cong} H^W_{i+1}(C, \Lambda) \quad \text{for } i \in \mathbb{Z}, \tag{1.7}$$

which are natural in C (see (6.10)).

Outline. In Section 2 we present our notation and conventions, and then the definition of our weight homology of motives functor (Def. 2.6). We also discuss the relationship between some "birational" and "weight" properties homological functors might satisfy in Lemma 2.7. Our weight homology functor satisfies all of them (Theorem 2.8).

In Section 3 we specialise to the case of weight homology of motives of schemes and motives of schemes with compact support. It is in this section that we show that our theory recovers the Gillet-Soulé weight homology over a perfect field (Thm. 3.1), and Geisser's Kato-Suslin homology over a finite field if the coefficient group is finite torsion prime to the characteristic of the base field (Thm. 3.6).

In Section 4 we present our main comparison theorem between motivic and étale motivic homology (Theorem 4.2) and give some consequences. The theorem is proved in the subsequent sections. Section 5 is the comparison in the case of motives of smooth schemes with compact support (Theorem 5.1).

In Section 6 we present the notions of "co-étale" and "reduced" motives and use them to prove the general case of Theorem 4.2 (see Theorem 6.3).

In Appendix A we identify that the étale realisation of motives of singular schemes (and motives with compact support of singular schemes) are what one expects them to be.

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Warning. We almost always work under the hypothesis that either the base field satisfies resolution of singularities, or alternatively, that the coefficients (A, Λ, R, $\mathbb{Z}/n\mathbb{Z}$, or Γ depending on the statement) are $\mathbb{Z}[\frac{1}{p}]$-linear. We will remind the reader from time to time.

2. Weight homology of motives

2.1. **Notation and conventions.** Through-out, k will be a perfect field of exponential characteristic p and n a positive integer prime to p. We will remind the reader occasionally. For a ring R, we write $DM^{\text{eff}}(k,R)$ for the tensor triangulated category of (unbounded) effective motives constructed by Voevodsky [V]. We will use $\mathcal{S}m/k$ (resp. $\mathcal{S}ch/k$) to denote the category of smooth (resp. separated finite type) k-schemes. If p is invertible in R, or if we assume strong resolution of singularities the category $DM^{\text{eff}}(k,R)$ is equipped with functors

$$M : \mathcal{S}ch/k \to DM^{\text{eff}}(k,R)\ ; \qquad X \mapsto M(X) \tag{2.1}$$

$$M^c : \mathcal{S}ch^{\text{prop}}/k \to DM^{\text{eff}}(k,R)\ ; \qquad X \mapsto M^c(X) \tag{2.2}$$

where $\mathcal{S}ch^{\text{prop}}/k$ is the subcategory of $\mathcal{S}ch/k$ with all objects but only proper morphisms [Ke1, Chap. 5] (or [V] for the version without conditions on the coefficients, but which assumes resolution of singularities).

We now focus on the subcategory $DM^{\text{eff}}_{gm}(k,R)$ of compact objects of $DM^{\text{eff}}(k,R)$. Let $\mathcal{S}mCor/k$ be Voevodsky's category of smooth correspondances and $K^b(\mathcal{S}mCor/k)$ be the homotopy category of bounded complexes in $\mathcal{S}mCor/k$. Then there is a natural functor (cf. [V])

$$\pi : K^b(\mathcal{S}mCor/k) \to DM^{\text{eff}}_{gm}(k,R) \tag{2.3}$$

such that $M(X) = \pi(X[0])$ for $X \in \mathcal{S}m$. We also consider the category of (covariant) effective Chow motives $\text{Chow}^{\text{eff}}(k,R)$. One description of $\text{Chow}^{\text{eff}}(k,R)$ is as the smallest idempotent complete full subcategory of $DM^{\text{eff}}(k,R)$ containing the objects $M(X)$ for all X smooth and projective ([Ke1, 5.5.11(4)] or [V, 4.2.6]). From this description there is a tautological inclusion

$$\iota : \text{Chow}^{\text{eff}}(k,R) \to DM^{\text{eff}}_{gm}(k,R). \tag{2.4}$$

2.2. **Weight structures and weight homology.** One of the major accomplishments of Bondarko's theory of *weight structures* is the following which extends Gillet-Soulé's weight complex of a variety to a weight complex for any motive (cf. [Bo2, Prop. 6.6.2]).

Theorem 2.1 (Bondarko [Bo2]). *Let k be a perfect field of exponential characteristic p. There exists a functor*

$$t : DM^{\text{eff}}_{gm}(k,\mathbb{Z}[\tfrac{1}{p}]) \to K^b(\mathit{Chow}^{\text{eff}}(k,\mathbb{Z}[\tfrac{1}{p}]))$$

which is triangulated and such that the composition

$$\mathit{Chow}^{\text{eff}}(k,\mathbb{Z}[\tfrac{1}{p}]) \xrightarrow{\iota} DM^{\text{eff}}_{gm}(k,\mathbb{Z}[\tfrac{1}{p}]) \xrightarrow{t} K^b(\mathit{Chow}^{\text{eff}}(k,\mathbb{Z}[\tfrac{1}{p}]))$$

is the canonical inclusion of $\mathit{Chow}^{\text{eff}}(k,\mathbb{Z}[\frac{1}{p}])$ into its bounded homotopy category as complexes concentrated in degree zero. The functor t is called the weight complex functor.

Remark 2.2. Bondarko shows that if $\mathbb{Q}$ is used instead of $\mathbb{Z}[\frac{1}{p}]$, the functor t is an equivalence if and only if the Beilinson-Parshin conjecture holds ([Bo, §8.3.2]).

Definition 2.3. Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ be an abelian category.

(1) A *homological functor* $\mathcal{H} : \mathcal{T} \to \mathcal{A}$ is an additive functor (preserving all small sums that exist in $\mathcal{T}$) such that for any distinguished triangle $T_1 \to T_2 \to T_3 \xrightarrow{+} T_1[1]$, the sequence

$$\cdots \to \mathcal{H}(T_1) \to \mathcal{H}(T_2) \to \mathcal{H}(T_3) \to \mathcal{H}(T_1[1]) \to \mathcal{H}(T_1[1]) \to \cdots$$

is exact.

(2) A homological functor $\mathcal{H} : DM^{\text{eff}}_{gm}(k,R) \to \mathcal{A}$ is $\mathcal{S}m\mathcal{P}roj$*-acyclic* if:

(SPA) for every smooth projective X we have $\mathcal{H}(M(X)[i]) = 0$ for $i \neq 0$.

Theorem 2.1 has the following consequence.

Theorem 2.4. *Let k be a perfect field of exponential characteristic p. Composition with the inclusion $\mathit{Chow}^{\text{eff}}(k,\mathbb{Z}[\frac{1}{p}]) \xrightarrow{\iota} DM^{\text{eff}}_{gm}(k,\mathbb{Z}[\frac{1}{p}])$ induces an equivalence of categories*

$$\iota_* : \left\{ \begin{array}{c} \mathcal{S}m\mathcal{P}roj\text{-acyclic} \\ \text{homological functors} \\ DM^{\text{eff}}_{gm}(k,\mathbb{Z}[\frac{1}{p}]) \to \mathcal{A} \end{array} \right\} \to \left\{ \begin{array}{c} \text{additive functors} \\ \mathit{Chow}^{\text{eff}}(k,\mathbb{Z}[\frac{1}{p}]) \to \mathcal{A} \end{array} \right\} \tag{2.5}$$

Furthermore, this equivalence induces canonical isomorphisms

$$\mathcal{H}(M(V_\bullet)[i]) \cong H^i\Big((\iota_*\mathcal{H})(V_\bullet)\Big) \tag{2.6}$$

for an object $\mathcal{H}$ of the left category, where $V_\bullet$ is a bounded complex of smooth projective varieties in $\mathcal{S}m\mathcal{C}or/k$, by $M(V_\bullet)$ we mean its image in $DM^{\text{eff}}_{gm}(k,\mathbb{Z}[\frac{1}{p}])$, and $(\iota_*\mathcal{H})(V_\bullet)$ is the complex in $\mathcal{A}$ obtained by applying $\iota_* H$ term-wise. These isomorphisms are natural in $\mathcal{H}$ and in $V_\bullet \in \mathit{Comp}^b(\mathcal{S}m\mathcal{P}roj\mathcal{C}or/k)$.

Proof. *ι_* is essentially surjective:* Every additive functor $\Gamma : \mathrm{Chow}^{\mathrm{eff}}(k, \mathbb{Z}[\frac{1}{p}]) \to \mathcal{A}$ has a canonical extension to a homological functor $K^b(\mathrm{Chow}^{\mathrm{eff}}(k, \mathbb{Z}[\frac{1}{p}])) \to \mathcal{A}$, which sends a complex $C^\bullet$ to $H^0\Gamma C^\bullet$, the zeroth cohomology of the complex of abelian groups obtained by applying Γ term-wise to $C^\bullet$. Let $\iota^*\Gamma = H^0\Gamma \circ t$ denote the composition of this extension with the weight complex functor. The association ι^* is clearly functorial in Γ. Furthermore, the functors $\iota^*\Gamma$ are $\mathcal{S}mProj$-acyclic since for every smooth projective X, the complex $tM(X)$ is concentrated in degree zero. So ι^* gives a right inverse to ι_* by Theorem 2.1, hence ι_* is essentially surjective.

The isomorphisms of Equation (2.6)*:* Let $\mathcal{S}mProjCor/k$ denote the full subcategory of $\mathcal{S}mCor/k$ whose objects are smooth projective varieties. Then we have a commutative square of functors

$$\begin{array}{ccc} \mathcal{S}mProjCor/k & \xrightarrow{\iota} & K^b(\mathcal{S}mProjCor/k) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \mathrm{Chow}^{\mathrm{eff}}(k, \mathbb{Z}[\frac{1}{p}]) & \xrightarrow[\iota]{} & DM^{\mathrm{eff}}_{gm}(k, \mathbb{Z}[\frac{1}{p}]) \end{array} \tag{2.7}$$

and the isomorphisms of Equation (2.6) are a consequence of Lemma 2.5 below applied to $\mathcal{B} = \mathcal{S}mProjCor/k$.

ι_* is full faithful: Finally, we notice that the two vertical functors π in the square of Equation (2.7) are full ([Ke1, 5.5.11(4)] or [V, 4.2.6]). Due to the lower row being idempotent completions, the functors π are not essentially surjective, but every object in their respective targets is a direct summand of an object in their images. So the association π_* sending an additive functor Γ on $\mathrm{Chow}^{\mathrm{eff}}(k, \mathbb{Z}[\frac{1}{p}])$ (resp. $DM^{\mathrm{eff}}_{gm}(k, \mathbb{Z}[\frac{1}{p}])$) to $\pi_*\Gamma = \Gamma \circ \pi$ is fully faithful. Consequently, the fully faithfullness of the ι_* of Theorem 2.4 follows from the fully faithfullness of the ι_* of Lemma 2.5 (with $\mathcal{A} = \mathcal{S}mProjCor/k$). □

Lemma 2.5. *Suppose that $\mathcal{B}$ is an additive category, and that $\mathcal{H} : K^b(\mathcal{B}) \to \mathcal{A}$ is a $\mathcal{B}$-acyclic homological functor, i.e., a homological functor satisfying the condition:*

($\mathcal{B}$A) $\mathcal{H}(B[i]) = 0$ *for any $B \in \mathcal{B}$ whenever $i \neq 0$.*

Let $\iota_*\mathcal{H}$ denote the induced additive functor $\mathcal{B} \to \mathcal{A}$. Then for every complex $C^\bullet$ in $K^b(\mathcal{B})$ there are canonical isomorphisms

$$\mathcal{H}(C^\bullet[i]) \cong H^i\Big((\iota_*\mathcal{H})(C^\bullet)\Big) \tag{2.8}$$

which are natural in $\mathcal{H}$ and $C^\bullet$, where and $(\iota_*\mathcal{H})(C^\bullet)$ is the complex of abelian groups obtained by applying ι_*H term-wise. Consequently, the assignment $\mathcal{H} \mapsto \iota_*\mathcal{H}$ is an equivalence from the category of $\mathcal{B}$-acyclic homological functors $K^b(\mathcal{B}) \to \mathcal{A}$ to the category of additive functors $\mathcal{B} \to \mathcal{A}$.

Proof. This proof is by induction on the length of the complex. For complexes concentrated in one degree the isomorphisms (2.8) are equalities. Suppose we have the isomorphisms (2.8) for the subcategory of complexes of length $< d$ and morphisms concentrated in a range of degrees $< d$, and let $C^\bullet$ be a complex concentrated in degrees $a, \dots, a+d$. There is a canonical distinguished triangle

$$\tau C^\bullet \to C^\bullet \to C^a[-a] \to \tau C^\bullet[1]$$

where $\tau C^\bullet$ is the same as $C^\bullet$ in degrees $a+1, a+2, \dots, a+d$, and zero elsewhere.

Now for $i \neq 0$ we have $\mathcal{H}(C^a[i]) = 0$ and so the isomorphisms (2.8) for $C^\bullet$ for $i \neq a, a+1$ come from those for $\tau C^\bullet$. For the case $i = a, a+1$, notice that since $C^\bullet$

(resp. $\tau C^\bullet$) is zero in degree $a-1$ (resp. a) we have

$$H^a\Big((\iota_*\mathcal{H})(C^\bullet)\Big) = ker\Big(\mathcal{H}(C^a) \to \mathcal{H}(C^{a+1})\Big) \text{ and}$$
$$H^{a+1}\Big((\iota_*\mathcal{H})(\tau C^\bullet)\Big) = ker\Big(\mathcal{H}(C^{a+1}) \to \mathcal{H}(C^{a+2})\Big).$$

With this in mind, the long exact sequence of the above distinguished triangle together with the isomorphisms (2.8) of C^a and $\tau C^\bullet$ then lead to the exact sequence

$$0 \to \mathcal{H}(C[a]) \to \mathcal{H}(C^a) \to ker\Big(\mathcal{H}(C^{a+1}) \to \mathcal{H}(C^{a+2})\Big) \to \mathcal{H}(C[a+1]) \to 0$$

from which we obtain the isomorphisms (2.8) for $C^\bullet$ and $i = a, a+1$. One can easily check the naturality in $\mathcal{H}$ and $C^\bullet$ of the isomorphisms constructed in this way. □

Definition 2.6. For an additive functor $\Gamma : \mathrm{Chow}^{\mathrm{eff}}(k, \mathbb{Z}[\frac{1}{p}]) \to \mathcal{A}$, let

$$H_0^W(-,\Gamma) : DM_{gm}^{\mathrm{eff}}(k, \mathbb{Z}[\tfrac{1}{p}]) \to \mathcal{A}$$

be the $\mathcal{S}mProj$-acyclic (Def. 2.3) homological functor associated to Γ by Theorem 2.4. We put

$$H_i^W(-,\Gamma) = H_0^W(-[i],\Gamma) \quad \text{for } i \in \mathbb{Z}.$$

For an abelian group A, we put

$$H_i^W(-,A) = H_i^W(-,\Gamma_A),$$

where $\Gamma_A : \mathrm{Chow}^{\mathrm{eff}}(k, \mathbb{Z}[\frac{1}{p}]) \to \mathcal{A}B$ is the additive functor such that $\Gamma_A(X) = A^{c(X)}$ for X smooth projective, where $c(X)$ is the set of connected components of X.

2.3. **Additional properties of homological functors.** We will observe a wide class of homological functors on $DM_{gm}^{\mathrm{eff}}(k,R)$ satisfies stronger conditions than $\mathcal{S}mProj$-acyclicity (Def. 2.3) (see Theorem 2.8 below):

Lemma 2.7. *Suppose either resolution of singularities holds, or R is a $\mathbb{Z}[\frac{1}{p}]$-algebra. Let $\mathcal{H} : DM_{gm}^{\mathrm{eff}}(k,R) \to \mathcal{A}$ be a homological functor. The following three properties are equivalent.*[3]

- (PI) *The canonical morphism $\mathcal{H}(M(\mathbb{P}^1_X)[i]) \to \mathcal{H}(M(X)[i])$ is an isomorphism for all smooth projective X, and all $i \in \mathbb{Z}$.*
- (TL) *We have $\mathcal{H}(M(1)[i]) = 0$ for all $M \in DM_{gm}^{\mathrm{eff}}(k,R)$ and $i \in \mathbb{Z}$.*
- (BI) *for any dense open immersion $U \to X$ of smooth schemes the induced morphism $\mathcal{H}(M(U)[i]) \to \mathcal{H}(M(X)[i])$ is an isomorphism for all $i \in \mathbb{Z}$.*

Moreover, a $\mathcal{S}mProj$-acyclic homological functor $\mathcal{H}$ satisfies the above equivalent properties if and only if it is $\mathcal{S}m$-acyclic, *i.e., if and only if:*

- ($\mathcal{S}m$A) *For every smooth (not necessarily projective) X we have $\mathcal{H}(M(X)[i]) = 0$ for $i \neq 0$.*

Proof. (PI) ⇒ (TL). Since motives of smooth projective varieties generate $DM_{gm}^{\mathrm{eff}}(k,R)$ as a triangulated category it follows that $\mathcal{H}(M(\mathbb{P}^1) \otimes M') \to \mathcal{H}(M')$ is an isomorphism for all objects $M' \in DM_{gm}^{\mathrm{eff}}(k,R)$. Now the isomorphism $M(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$ implies the claim.

(TL) ⇒ (BI). This follows from the generalised Gysin distinguished triangle (for $U \to X$ a dense open immersion of smooth schemes of dimension d with closed complement Z)

$$M(U) \to M(X) \to M^c(Z)^*(d)[2d] \to M(U)[1]$$

[3](PI) = $\mathbb{P}^1$-Invariance, (TL) = Tate Local, (BI) = Birational Invariance.

([Ke1, Prop 5.5.5, Thm 5.5.14(3)], cf. [V, Prop. 4.1.5, Thm 4.3.7(3)]).

(BI) $\Rightarrow$ (PI). This follows from a very neat geometric argument of Colliot-Thélène. See [KaSu, Appendix A].

Now suppose that $\mathcal{H}$ is $\mathcal{S}mProj$-acyclic.

(BI) $\Rightarrow$ ($\mathcal{S}m$A). If resolution of singularities holds, this is a direct consequence of the existence of smooth compactifications. Otherwise: By Nagata's compactification theorem and Gabber's alterations theorem [Il] for every smooth scheme U and every $l \neq p$ there is a roof of morphisms $U \leftarrow U' \to X'$ where X' is smooth and projective, $U' \to X'$ is a dense open immersion, and $U' \to U$ is finite flat surjective of degree prime to l. Since $\mathcal{H}$ is assumed to be $\mathcal{S}mProj$-acyclic, the property (BI) implies that ($\mathcal{S}m$A) is satisfied for the scheme U'. But $M(-)$ is functorial in $\mathcal{S}mCor/k$ so by the trace formula, $M(f)M({}^tf) = \deg f \cdot \mathrm{id}_{M(U)}$, for the finite flat morphism f, the motive $M(U)$ is a direct summand of $M(U')$, at least $\mathbb{Z}_{(l)}$-linearly. But we were free to choose $l \neq p$, and so $M(U)$ is a direct summand of $M(U')$ as long as we use $\mathbb{Z}[\frac{1}{p}]$-linear coefficients (cf. [Ke1, Appendix A2]). Hence, Property (TL) for U' implies Property (TL) for U.

($\mathcal{S}m$-acyclic) $\Rightarrow$ (TL). Let $X \in \mathcal{S}m/k$. Since $M(\mathbb{P}^1_X) \cong M(X) \oplus M(X)(1)[2]$, the property ($\mathcal{S}m$A) implies that $\mathcal{H}(M(X)(1)[i]) = 0$ unless $i = 2$. But the isomorphism $M(\mathbb{A}^1_X - X) \cong M(X) \oplus M(X)(1)[1]$ with ($\mathcal{S}m$A) then implies that $\mathcal{H}(M(X)(1)[i]) = 0$ unless $i = 1$. That is, $\mathcal{H}(M(X)(1)[i]) = 0$ for all i. Since objects of the for $M(X)$ for $X \in \mathcal{S}m/k$ generate the triangulated category $DM^{\mathrm{eff}}_{gm}(k, R)$ it follows that $\mathcal{H}(M(1)[i]) = 0$ for all $M \in DM^{\mathrm{eff}}_{gm}(k, R)$ and $i \in \mathbb{Z}$. □

Theorem 2.8. *Suppose either resolution of singularities holds, or A is a $\mathbb{Z}[\frac{1}{p}]$-module. The homological functor $H^W_0(-, A)$ from Definition 2.6 satisfies all properties of Lemma 2.7. That is, $H^W_0(-, A)$ is $\mathcal{S}m$-acyclic.*

Proof. Obviously $\Gamma_A(X) = \Gamma_A(\mathbb{P}^1_X)$ for X smooth projective so that $H^W_0(-, A)$ satisfies (PI). It also satisfies (N) by definition. Hence Theorem 2.8 follows from Lemma 2.7. □

3. Weight homology of schemes

As always, let k be a perfect field of exponential characteristic p. In this section we fix an additive functor $\Gamma : \mathrm{Chow}^{\mathrm{eff}}(k, \mathbb{Z}[\frac{1}{p}]) \to \mathcal{A}$.

3.1. **Weight homology with compact support of schemes.** For $X \in \mathcal{S}ch/k$ and $i \in \mathbb{Z}$, we put

$$H^W_{i,c}(X, \Gamma) = H^W_0(M^c(X)[i], \Gamma). \tag{3.1}$$

Let $\mathcal{S}ch^{\mathrm{prop}}/k$ be the category of the same objects as $\mathcal{S}ch/k$ but only proper morphisms. By [V, §2.2] (if resolution of singularities holds) or [Ke1], this gives rise to covariant functors

$$H^W_{i,c}(-, \Gamma) \ : \ \mathcal{S}ch^{\mathrm{prop}} \to \mathcal{A} \quad (i \in \mathbb{Z})$$

satisfying the following properties:

(**Wc**1) (Nilpotent invariance) For $X \in \mathcal{S}ch/k$ with its reduced structure $X_{\mathrm{red}} \hookrightarrow X$, the map $H^W_{i,c}(X_{\mathrm{red}}, \Gamma) \to H^W_{i,c}(X, \Gamma)$ is an isomorphism for any $i \in \mathbb{Z}$.

(**Wc**2) (Localization axiom) For $X \in \mathcal{S}ch/k$ and a closed subscheme $Z \subset X$, one has a functorial long exact sequence

$$\cdots \to H^W_{i,c}(Z, \Gamma) \to H^W_{i,c}(X, \Gamma) \to H^W_{i,c}(X - Z, \Gamma) \to H^W_{i-1,c}(Z, \Gamma) \cdots$$

(**Wc**3) (Pull-back by quasi-finite flat morphisms) For a quasi-finite flat morphism $f : Y \to X$ in $\mathcal{S}ch/k$, one has a canonical map

$$f^* : H^W_{i,c}(X, \Gamma) \to H^W_{i,c}(Y, \Gamma)$$

compatible with functoriality for proper morphisms in $\mathcal{S}ch/k$ and with localization sequences in (**Wc**2) in an obvious sense.

(**Wc**4) (Pull-back by morphisms between smooth schemes) This is a consequence of duality. For a morphism $f : Y \to X$ in $\mathcal{S}m/k$ such that $\dim(Y) = \dim(X)$, one has a canonical map

$$f^* : H^W_{i,c}(X, \Gamma) \to H^W_{i,c}(Y, \Gamma)$$

compatible with functoriality for proper morphisms in $\mathcal{S}ch/k$ and with localization sequences in (**Wc**2) in an obvious sense.

(**Wc**5) For $X \in \mathcal{S}ch/k$ projective smooth, we have

$$H^W_{i,c}(X, \Gamma) = \begin{cases} \Gamma(X) & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

(**Wc**6) If the groups $\Gamma(X)$ are finitely generated for all X smooth and projective (e.g., $\Gamma = \Gamma_{\mathbb{Z}/n}$) then the groups $H^W_{i,c}(X, \Gamma)$ are finitely generated for all $X \in \mathcal{S}ch/k$.

Recall the definition of the Gillet-Soulé weight homology: For $X \in Sch/k$, let $X \to \overline{X}$ be an open immersion into a projective variety and Z the closed complement. Take $\overline{\mathcal{X}}_\bullet \to \overline{X}$ a proper cdh hypercover (or proper $l$dh-hypercover [Il], [Ke1]) such that each $\overline{\mathcal{X}}_i$ is smooth over k and let $\mathcal{Z}_\bullet \to Z$ be a refinement of the induced hypercover of Z such that each $\mathcal{Z}_i$ is smooth. We then obtain a complex concentrated in positive homological degrees in the additive envelope of the category of smooth projective varieties equipped with an augmentation

$$\mathcal{V}_\bullet = Cone(\mathcal{Z}_\bullet \to \overline{\mathcal{X}}_\bullet) \to X. \tag{3.2}$$

By definition, the Gillet-Soulé weight complex $W(X)$ is the image of $\mathcal{V}_\bullet$ in the homotopy category of complexes of Chow motives $K^b(\mathrm{Chow}^{\mathrm{eff}}(k, \mathbb{Z}[\frac{1}{p}]))$, and for an additive functor $\Gamma : \mathrm{Chow}^{\mathrm{eff}}(k, \mathbb{Z}[\frac{1}{p}]) \to \mathcal{A}B$ one defines

$$H^{GS}_i(X, \Gamma) = H_i(\Gamma(\mathcal{V}_\bullet)). \tag{3.3}$$

Theorem 3.1. *There are isomorphisms*

$$H^{GS}_i(X, \Gamma) \cong H^W_{i,c}(X, \Gamma) \tag{3.4}$$

natural in $X \in \mathcal{S}ch/k$.

Remark 3.2. A remarkable point is that for $H^{GS}_i(-, \Gamma)$, the properties (**Wc**3) and (**Wc**4) are not obvious from its construction.

Proof. The isomorphisms (3.4) come from Equation (2.6) and the morphism $M(\mathcal{V}_\bullet) \to M^c(X)$ in $DM^{\mathrm{eff}}_{gm}(k, \mathbb{Z}[\frac{1}{p}])$ induced by Equation (3.2) which is an isomorphism by [Ke1, Thm 5.3.1] (cf. [V, Thm 4.1.2]). □

Example 3.3. Let $E \in \mathcal{S}ch/k$ be projective which is a simple normal crossing divisor on a smooth $P \in \mathcal{S}ch/k$. Let $E_1, \dots, E_N$ be the irreducible components of E and put for an integer $a \geq 0$

$$E^{[a]} = \coprod_{1 \leq i_0 < \cdots < i_a \leq N} E_{i_0,\dots,i_a} \quad (E_{i_0,\dots,i_a} = E_{i_0} \cap \cdots \cap E_{i_a}).$$

For $\Gamma : \mathrm{Chow}^{\mathrm{eff}}(k,R) \to \mathcal{A}$ as above, consider a homological complex in $\mathcal{A}$

$$\Gamma(E) : \cdots \to \Gamma(E^{[a]}) \xrightarrow{\partial_a} \Gamma(E^{[a-1]}) \xrightarrow{\partial_{a-1}} \cdots \xrightarrow{\partial_1} \Gamma(E^{[0]}),$$

where $\Gamma(E^{[a]})$ is in degree a and ∂_a is the alternating sum of the maps induced by inclusions $E_{i_0,\dots,i_a} \hookrightarrow E_{i_0,\dots,\widehat{i_\nu},\dots a}$ for $\nu = 0,\dots,a$. Then, using (**Wc**2) and (**Wc**5), one can show isomorphisms

$$H^W_{a,c}(E,\Gamma) \cong H_a(\Gamma(E)) \quad \text{for } a \in \mathbb{Z}.$$

3.2. **Weight homology of schemes.** For $X \in \mathcal{S}ch/k$ and $i \in \mathbb{Z}$, we put (cf. Definition 2.6)

$$H^W_i(X,\Gamma) = H^W_i(M(X),\Gamma). \tag{3.5}$$

By [V, §2.2] (if resolution of singularities holds) or [Ke1] this gives rise to covariant functors

$$H^W_i(-,\Gamma) \;:\; \mathcal{S}ch/k \to \mathcal{A} \quad (i \in \mathbb{Z})$$

satisfying the following properties:

(**W**0) (Nilpotent invariance) For $X \in \mathcal{S}ch/k$ with its reduced structure $X_{\mathrm{red}} \hookrightarrow X$, the map $H^W_i(X_{\mathrm{red}},\Gamma) \to H^W_i(X,\Gamma)$ is an isomorphism for any $i \in \mathbb{Z}$.

(**W**1) (Homotopy invariance) For $X \in \mathcal{S}ch/k$, the map $H^W_i(X\times\mathbb{A}^1,\Gamma) \to H^W_i(X,\Gamma)$ induced by the projection is an isomorphism for any $i \in \mathbb{Z}$.

(**W**2) (Mayer-Vietoris axiom) For $X \in \mathcal{S}ch/k$ and an open covering $X = U \cup V$, one has a functorial long exact sequence

$$\cdots \to H^W_i(U\cap V,\Gamma) \to H^W_i(U,\Gamma)\oplus H^W_i(V,\Gamma) \to H^W_i(X,\Gamma) \to H^W_{i-1}(U\cap V,\Gamma) \to \cdots$$

(**W**3) (Abstract blow-up) Consider a cartesian diagram in $\mathcal{S}ch/k$:

$$\begin{array}{ccc} p^{-1}(Z) & \longrightarrow & X_Z \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

where $p_Z : X_Z \to Z$ is proper and $Z \to X$ is a closed immersion such that $p^{-1}(X-Z) \to X - Z$ is an isomorphism. Then one has a functorial long exact sequence

$$\cdots \to H^W_i(p_Z^{-1}(Z),\Gamma) \to H^W_i(X_Z,\Gamma)\oplus H^W_i(Z,\Gamma) \to H^W_i(X,\Gamma) \to H^W_{i-1}(p_Z^{-1}(Z),\Gamma) \to \cdots$$

(**W**4) For $X \in \mathcal{S}m/k$ smooth projective, we have

$$H^W_i(X,\Gamma) = \begin{cases} \Gamma(X) & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

By Theorem 2.8, if $\Gamma = \Gamma_A$ for an abelian group A, we have the stronger property:

(**W**5) For $X \in \mathcal{S}m/k$ smooth, we have

$$H^W_i(X,A) = \begin{cases} A^{c(X)} & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

where $c(X)$ is the set of connected components of X.

(**W**6) For a proper $X \in \mathcal{S}ch/k$, we have $H^W_{i,c}(X,\Gamma) = H^W_i(X,\Gamma)$.

(**W**7) If the groups $\Gamma(X)$ are finitely generated for all X smooth (e.g., $\Gamma = \Gamma_{\mathbb{Z}/n}$) then the groups $H^W_i(X,\Gamma)$ are finitely generated for all $X \in \mathcal{S}ch/k$ if

In the rest of this section, we assume that k is a finite field with $p = \mathrm{ch}(k)$. Recall ([Ge2, §8]) that since k is finite, one can define Geisser's Kato-Suslin homology $H_i^{KS}(X, A)$ of $X \in \mathcal{S}ch/k$ with coefficients in an abelian group A as the ith homology of the complex which we will denote by

$$R\Gamma^{KS}(X, A) = \mathrm{Tot}\left(C_*^X(k) \otimes A \to C_*^X(\overline{k}) \otimes A \overset{\mathrm{id}-\varphi}{\to} C_*^X(\overline{k}) \otimes A \right) \tag{3.6}$$

where $\overline{k}$ is the algebraic closure of the finite field k, the Frobenius of k is denoted by φ, and the presheaf of complexes $C_*^X(-)$ is the Suslin complex $\underline{C}_*(-)$ [SV1, §7] of the presheaf $c_{\mathrm{equi}}(X/k)$ [SV2]. Consider the property (W5) for Geisser's theory:

$(\mathcal{S}m\mathrm{A})_{KS}$ For $X \in \mathcal{S}m/k$ smooth, we have

$$H_i^{KS}(X, A) = \begin{cases} A^{c(X)} & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

where $c(X)$ is the set of connected components of X.

Proposition 3.4 ([Ge2])**.**

(1) *If $A = \mathbb{Z}/n$ with n coprime to p, then $(\mathcal{S}mA)_{KS}$ is satisfied.*
(2) *If A is $\mathbb{Z}[1/p]$-module, then $(\mathcal{S}mA)_{KS}$ holds if the following condition holds:*
  (P_0) *Rational Suslin homology is $\mathcal{S}m$-acyclic. That is, for all X smooth over a finite field, the rational Suslin homology groups $H_i^S(X, \mathbb{Q})$ are zero for $i \neq 0$.*

Remark 3.5. (P_0) is equivalent to the (part) of Parshin's conjecture (see [Ge3]): For X smooth and proper over k, $\mathrm{CH}_0(X, i)$ is torsion for $i \neq 0$.

Proof. Prop. 3.4(2) is Geisser's [Ge2, Thm. 8.2]) (Indeed he shows also the converse implication for $A = \mathbb{Q}$). We give an alternative proof of Prop. 3.4. Notice that $R\Gamma^{KS}(X, A)$ is by definition the cone of the morphism

$$C_*^X(k) \otimes A \to \mathrm{Tot}\left(C_*^X(\overline{k}) \otimes A \overset{\mathrm{id}-\varphi}{\to} C_*^X(\overline{k}) \otimes A \right).$$

The ith homology of the source is by definition $H_i^S(X, A)$. The ith homology of the target is what Geisser defines to be the *arithmetic homology* and is denoted by $H_{i+1}^{\mathrm{ar}}(X, A)$ ([Ge2, §7.1]). So now $(\mathcal{S}m\mathrm{A})_{KS}$ is equivalent to asking that the canonical morphisms $H_i^S(X, A) \to H_{i+1}^{\mathrm{ar}}(X, A)$ be isomorphisms for $i \neq 0$. By definition we have an exact sequence

$$0 \to H_{i+1}^S(\overline{X}, A)_G \to H_{i+1}^{\mathrm{ar}}(X, A) \to H_i^S(\overline{X}, A)^G \to 0, \tag{3.7}$$

where $\overline{X} = X \otimes_k \overline{k}$ with an algebraic closure $\overline{k}$ of k. By (1.4) and Th. 4.2 together with (**W**5), we have isomorphisms for X smooth,

$$\begin{aligned} H_i^S(X, A) &\cong H_i^M(M(X), A) \cong H_i^{M,\text{ét}}(M(X), A), \\ H_i^S(\overline{X}, A) &\cong H_i^M(M(\overline{X}), A) \cong H_i^{M,\text{ét}}(M(\overline{X}), A). \end{aligned} \tag{3.8}$$

Letting G_k be the absolute Galois group of k, we have a spectral sequence

$$E_{i,j}^2 = H^{-i}(G_k, H_j^{M,\text{ét}}(M(\overline{X}), A)) \Rightarrow H_{i+j}^{M,\text{ét}}(M(X), R), \tag{3.9}$$

Assuming either that A is torsion or that (P_0) holds, $H_j^{M,\text{ét}}(M(\overline{X}), A)$ are torsion for $j \neq 0$ and hence (3.9) gives rise to

$$0 \to H_{i+1}^{M,\text{ét}}(M(\overline{X}), A)_G \to H_i^M(M(X), A) \to H_i^{M,\text{ét}}(M(\overline{X}), A)^G \to 0.$$

In view of (3.8), this provides an exact sequence

$$0 \to H_{i+1}^S(\overline{X}, A)_G \to H_i^S(X, A) \to H_i^S(\overline{X}, A)^G \to 0.$$

Combined with (3.7), this proves the desired isomorphism $H_i^S(X,A) \cong H_{i+1}^{\mathrm{ar}}(X,A)$. □

Theorem 3.6. *Assume k is finite and either A is a torsion prime to p or A is a $\mathbb{Z}[\frac{1}{p}]$-module and (P_0) holds. Then there are isomorphisms natural in $X \in \mathcal{S}ch/k$:*

$$H_i^{KS}(X,A) \cong H_i^W(X,A), \tag{3.10}$$

where $H_i^W(X,A)$ is the weight homology from §3.2.

Proof. Since the presheaves $c_{\mathrm{equi}}(X/k)$ are functorial in $X \in \mathcal{S}mCor/k$, the functor $R\Gamma^{KS}(-,A)$ is as well. In fact, using the total complex functor Tot, the functor $R\Gamma^{KS}(-,A)$ can be extended to a functor on the homotopy category $K^b(\mathcal{S}mCor/k)$ of bounded complexes in $\mathcal{S}mCor/k$, inducing a triangulated functor $K^b(\mathcal{S}mCor/k) \to K^b(\mathcal{A}B)$. For every $X \in \mathcal{S}m/k$ and open subsets $U, V \subseteq X$, the functor $R\Gamma^{KS}(-,A)$ sends the complexes $(\mathbb{A}^1_X \to X)$ and $(U \cap V \to U \oplus V \to U \cup V)$ to acyclic complexes since this is true for $C_*^{(-)}(k) \otimes A$ and $C_*^{(-)}(\bar{k}) \otimes A$. Hence $R\Gamma^{KS}(-,A)$ factors via a triangulated functor (cf. (2.3))

$$R\Gamma^{KS}(-,A) : DM_{gm}^{\mathrm{eff}}(k,R) \to K^b(\mathcal{A}B).$$

By the usual cdh/ldh descent arguments one can show a canonical isomorphism

$$H_i^{KS}(X,A) \cong H^{-i}(R\Gamma^{KS}(M(X),A)) \quad \text{for } X \in \mathcal{S}ch/k.$$

By Prop. 3.4 the homological functor $R\Gamma^{KS}(-,A)$ is $\mathcal{S}mProj$-acyclic. Hence it now follows from the equivalence of Theorem 2.5 that $H^{-i}R\Gamma^{KS}(-,A)$ is canonically isomorphic to the homological functor $H_i^W(-,A)$ on $DM_{gm}^{\mathrm{eff}}(k,R)$. □

4. Motivic homology and étale motivic homology

We continue with k a perfect field of exponential characteristic p. We let $DM_{\text{ét}}^{\mathrm{eff}}(k,R)$ denote the étale version of $DM^{\mathrm{eff}}(k,R)$ (i.e., constructed with étale sheaves with transfers instead of Nisnevich ones) and

$$\alpha^* : DM^{\mathrm{eff}}(k,R) \to DM_{\text{ét}}^{\mathrm{eff}}(k,R) \tag{4.1}$$

the canonical functor induced by étale sheafification. The functor α^* admits a right adjoint which is denoted by α_*.

Lemma 4.1. *The right adjoint α_* to α^* (Equation (4.1)) is fully faithful.*

Proof. It suffices to show that the counit map $\alpha^*\alpha_* C \to C$ is an isomorphism for $C \in DM_{\text{ét}}^{\mathrm{eff}}(k,R)$. We may assume $C = F[0]$ for a single étale sheaf F. Then the assertion is true for tautological reasons: Indeed, $H^0(\alpha_* F)$ is just F where we only remember that it is a Nisnevich sheaf so that $\alpha^* H^0(\alpha_* F) \to F$ is an isomorphism. On the other hand, for $q > 0$, the étale sheaf associated to $H^q(\alpha_* F)$ is 0, i.e. $\alpha^* H^q(\alpha_* F) = 0$. This completes the proof of the lemma. □

The motivic homology and étale motivic homology with R-coefficients of an object C in $DM^{\mathrm{eff}}(k,R)$ are defined as

$$H_i^M(C,R) = \hom_{DM^{\mathrm{eff}}(k,R)}(R[i],C), \quad \text{and} \tag{4.2}$$

$$H_i^{M,\text{ét}}(C,R) = \hom_{DM^{\mathrm{eff}}(k,R)}(R[i],\alpha_*\alpha^* C) \tag{4.3}$$

where $R[i] = M(k)[i]$ is a shift of the unit for the tensor structures.

The unit $\mathrm{id} \to \alpha_*\alpha^*$ of the adjunction (α^*, α_*) induces a natural map

$$\alpha^* : H_i^M(C,R) \to H_i^{M,\text{ét}}(C,R)$$

which is functorial in $C \in DM^{\text{eff}}(k,R)$.

For $C \in DM^{\text{eff}}_{gm}(k,R)$ we put

$$H_i^W(C,\Lambda) = H_0^W(C[i],\Lambda),$$

where $H_0^W(-,\Lambda) : DM^{\text{eff}}_{gm}(k,R) \to \mathcal{AB}$ is the weight homology functor from Definition 2.6 for $A = \Lambda$.

Theorem 4.2. *Let k be a field of exponential characteristic p and let n be an integer prime to p. Suppose either $R = \Lambda = \mathbb{Z}/n$, or $R = \mathbb{Z}[1/p]$ and $\Lambda = \mathbb{Q}/\mathbb{Z}[1/p]$.*

(1) *If k is algebraically closed, $\alpha^* : H_i^M(C,R) \to H_i^{M,\acute{e}t}(C,R)$ is an isomorphism.*
(2) *If k is finite and $C \in DM^{\text{eff}}_{gm}(k,R)$, there is a canonical long exact sequence*

$$\cdots \to H_i^M(C,R) \xrightarrow{\alpha^*} H_i^{M,\acute{e}t}(C,R) \to H_{i+1}^W(C,\Lambda) \to H_{i-1}^M(C,R) \to \cdots$$

which is functorial in C.

The proof of the theorem will be given in the following sections.

Take $R = \Lambda = \mathbb{Z}/n\mathbb{Z}$ with $(n,p) = 1$. Then for $C = M(X)$ (resp. $M^c(X)$), the groups $H_i^{M,\text{ét}}(C,\mathbb{Z}/n\mathbb{Z})$ are identified with the dual of étale cohomology (resp. étale cohomology with compact support) by Proposition 5.4 and Lemma 5.5 below. Thus (1.4) and (1.5) and Theorem 4.2(2) imply the following:

Corollary 4.3. *Assume k is finite. For an integer n prime to p, there are long exact sequences functorial in $X \in \mathcal{S}ch/k$:*

$$(4.4)\quad \cdots \to H_i^S(X,\mathbb{Z}/n\mathbb{Z}) \to H^{i+1}(X_{\acute{e}t},\mathbb{Z}/n\mathbb{Z})^* \to H_{i+1}^W(X,\mathbb{Z}/n\mathbb{Z}) \to H_{i-1}^S(X,\mathbb{Z}/n\mathbb{Z}) \to \cdots ,$$

$$(4.5)\quad \cdots \to \mathrm{CH}_0(X,i;\mathbb{Z}/n\mathbb{Z}) \to H_c^{i+1}(X_{\acute{e}t},\mathbb{Z}/n\mathbb{Z})^* \to H_{i+1,c}^W(X,\mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}_0(X,i-1;\mathbb{Z}/n\mathbb{Z}) \to \cdots ,$$

where $H_i^W(X,\mathbb{Z}/n\mathbb{Z})$ and $H_{i,c}^W(X,\mathbb{Z}/n\mathbb{Z})$ are the weight homology and the weight homology with compact support (cf. (3.5) *and* (3.1)*). In particular, if X is smooth,* (**W**5) *implies canonical isomorphisms*

$$(4.6)\quad H_i^S(X,\mathbb{Z}/n\mathbb{Z}) \cong H^{i+1}(X_{\acute{e}t},\mathbb{Z}/n\mathbb{Z})^* \quad \textit{for } i \in \mathbb{Z}.$$

Remark 4.4. Under the assumption of Cor. 4.3, Geisser [Ge2] shows an isomorphism

$$(4.7)\quad H_i^{\text{ar}}(X,\mathbb{Z}/n\mathbb{Z}) \cong H^i(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z})^*,$$

where $H_i^{\text{ar}}(X,\mathbb{Z}/n\mathbb{Z})$ is the arithmetic homology (see the proof of Prop. 3.4). Under (4.7) and the identification of Theorem 3.6, the long exact sequence (4.4) is identified with the long exact sequence from [Ge2, §8]:

$$(4.8)\quad \cdots \to H_i^S(X,\mathbb{Z}/n\mathbb{Z}) \to H_{i+1}^{\text{ar}}(X,\mathbb{Z}/n\mathbb{Z}) \to H_{i+1}^{KS}(X,\mathbb{Z}/n\mathbb{Z}) \to H_{i-1}^S(X,\mathbb{Z}/n\mathbb{Z}) \to \cdots$$

As a consequence of the long exact sequence (4.4) and property (W7) we obtain an alternative proof of the prime-to-p part of [Ge2, Thm. 6.1]:

Theorem 4.5 (cf. [Ge2, Thm. 6.1])**.** *For k a finite field and n prime to the characteristic, the groups $H_i^S(X, \mathbb{Z}/n\mathbb{Z})$ are finitely generated for all i and all $X \in \mathcal{S}ch/k$.*

Remark 4.6. We have a natural isomorphism

$$H^1(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})^* \cong \pi_1^{\text{ab}}(X)/n,$$

where $\pi_1^{\text{ab}}(X)$ is the abelian fundamental group of X. Hence (4.4) gives an exact sequence

$$(4.9) \qquad H_2^W(X, \mathbb{Z}/n\mathbb{Z}) \to H_0^S(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\rho_X} \pi_1^{\text{ab}}(X)/n \to H_1^W(X, \mathbb{Z}/n\mathbb{Z}),$$

Functoriality implies that for a closed point $x \in X$, the composite of ρ_X with

$$\mathbb{Z} = H_0^S(X, \mathbb{Z}) \to H_0^S(X, \mathbb{Z}/n\mathbb{Z})$$

sends $1 \in \mathbb{Z}$ to the Frobenius element at x in $\pi_1^{\text{ab}}(X)/n$. Thus (4.9) and (4.6) provide a generalisation of tame class field theory for smooth schemes over a finite field studied in [ScSp]. A generalisation of [ScSp] in a different direction is studied in [GeSc].

Example 4.7. Here is an example of computation of $H_i^W(X, \Lambda)$ for $X \in \mathcal{S}ch/k$. Assume that there is a cartesian diagram:

$$\begin{array}{ccc} p^{-1}(Z) & \longrightarrow & X_Z \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

where $p_Z : X_Z \to Z$ is proper and $Z \to X$ is a closed immersion such that X_Z is smooth, $p^{-1}(X - Z) \to X - Z$ is an isomorphism and the reduced part E of $p^{-1}(Z)$ is a simple normal crossing divisor on X_Z. Assume additionally that Z_{red} is proper smooth (e.g. $\dim(Z) = 0$). Then, using (**W**0), (**W**3), (**W**4), (**Wc**1) and Example 3.3, we get an isomorphism:

$$H_{a+1}^W(X, \Lambda) \cong H_a(\Gamma_E(\Lambda)) \quad \text{for } a \geq 1.$$

$$H_1^W(X, \Lambda) \cong \operatorname{Ker}\big(\Lambda^{c(E)} \to \Lambda^{c(Z)} \oplus \Lambda^{c(X_Z)}\big),$$

where under the same notation as Example 3.3, $\Gamma_E(\Lambda)$ is the complex

$$\cdots \to \Lambda^{c(E^{[a]})} \xrightarrow{\partial_a} \Lambda^{c(E^{[a-1]})} \xrightarrow{\partial_{a-1}} \cdots \xrightarrow{\partial_1} \Lambda^{c(E^{[0]})}.$$

5. Étale cycle map for motivic homology

In this section we prove the following theorem.

Theorem 5.1. *Let X be smooth over k and n a positive integer invertible in k. For an integer $r \geq 0$, the map*

$$\alpha^* : H_i^M(M^c(X)(r), \mathbb{Z}/n\mathbb{Z}) \to H_i^{M,ét}(M^c(X)(r), \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism under one of the following conditions:

(*i*) *k is algebraically closed.*
(*ii*) *k is finite and $r > 0$.*
(*iii*) *k is finite, X is proper over k, and $r = 0$ and $i \neq -1$.*

We need the following result:

Proposition 5.2. *If k has finite cohomological dimension then:*

(1) *If R is a $\mathbb{Q}$-algebra, α^* in Equation* (4.1) *is an equivalence of $\otimes$-triangulated categories.*
(2) *If $p = \text{ch}(k) > 0$ and R is annhilated by a power of p, then $DM_{ét}^{\text{eff}}(k, R) = 0$.*

(3) *For n prime to p, let $D_{ét}(k,\mathbb{Z}/n\mathbb{Z})$ be the derived category of sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules over the small étale site $Et(k)$ of* $\mathrm{Spec}(k)$*. Then the functor*

$$\Phi^{ét} : DM^{\mathrm{eff}}_{ét}(k,\mathbb{Z}/n\mathbb{Z}) \to D_{ét}(k,\mathbb{Z}/n\mathbb{Z})$$

induced by the association $\mathcal{F} \mapsto \mathcal{F}(\bar{k})$ for an étale sheaf $\mathcal{F}$ on $\mathcal{S}m/k$, is an equivalence of $\otimes$-triangulated categories.

Proof. In the bounded above case, this is due to Voevodsky ([V, Prop. 3.3.2 and 3.3.3]). The general case is shown by the same argument. □

In the appendix we show that the functors $\Phi^{ét}, \alpha^*, M$, and M^c are all compatible in the way one would expect:

Proposition 5.3. *Suppose that k is algebraically closed. For $X \in \mathcal{S}ch/k$ with structural morphism $\pi : X \to \mathrm{Spec}(k)$, there are canonical functorial isomorphisms*

$$\mathrm{Rhom}(\Phi^{ét}\alpha^* M(X), \mathbb{Z}/n\mathbb{Z}) = R\pi_*(\mathbb{Z}/n\mathbb{Z})_X, \quad \textit{and}$$
$$\mathrm{Rhom}(\Phi^{ét}\alpha^* M^c(X)), \mathbb{Z}/n\mathbb{Z}) = R\pi_!(\mathbb{Z}/n\mathbb{Z})_X.$$

By Propositions 5.2 and A.2, we get the following.

Proposition 5.4. *For $X \in \mathcal{S}ch/k$ and $i, r \in \mathbb{Z}$, there are natural isomorphisms*

$$H_i^{M,ét}(M(X)(r), \mathbb{Z}/n\mathbb{Z}) \cong H_i^{ét}(X, \mathbb{Z}/n\mathbb{Z}(r)),$$
$$H_i^{M,ét}(M^c(X)(r), \mathbb{Z}/n\mathbb{Z}) \cong H_{i,c}^{ét}(X, \mathbb{Z}/n\mathbb{Z}(r)).$$

where

$$H_i^{ét}(X, \mathbb{Z}/n\mathbb{Z}(r)) = \mathrm{Hom}_{D_{ét}(k,\mathbb{Z}/n\mathbb{Z})}(R\pi_*(\mathbb{Z}/n\mathbb{Z})_X, \mu_n^{\otimes r}[-i]),$$
$$H_{i,c}^{ét}(X, \mathbb{Z}/n\mathbb{Z}(r)) = \mathrm{Hom}_{D_{ét}(k,\mathbb{Z}/n\mathbb{Z})}(R\pi_!(\mathbb{Z}/n\mathbb{Z})_X, \mu_n^{\otimes r}[-i]). \tag{5.1}$$

called the étale homology *and* étale homology with compact support *of X with $\mathbb{Z}/n\mathbb{Z}$-coefficients respectively.*

We now “convert” étale homology into cohomology:

Lemma 5.5. *Let $X \in \mathcal{S}ch/k$ and let n be a positive integer invertible in k.*

(1) *If X is smooth over k of pure dimension d, there are canonical isomorphisms*

$$H_i^{ét}(X, \mathbb{Z}/n\mathbb{Z}(r)) \cong H_c^{2d-i}(X_{ét}, \mu_n^{\otimes d+r}),$$
$$H_{i,c}^{ét}(X, \mathbb{Z}/n\mathbb{Z}(r)) \cong H^{2d-i}(X_{ét}, \mu_n^{\otimes d+r}).$$

(2) *If k is algebraically closed there are canonical isomorphisms*

$$H_i^{ét}(X, \mathbb{Z}/n\mathbb{Z}(r)) \cong H^i(X_{ét}, \mu_n^{\otimes -r})^*,$$
$$H_{i,c}^{ét}(X, \mathbb{Z}/n\mathbb{Z}(r)) \cong H_c^i(X_{ét}, \mu_n^{\otimes -r})^*.$$

(3) *If k is finite there are canonical isomorphisms*

$$H_i^{ét}(X, \mathbb{Z}/n\mathbb{Z}(r)) \cong H^{i+1}(X_{ét}, \mu_n^{\otimes -r})^*,$$
$$H_{i,c}^{ét}(X, \mathbb{Z}/n\mathbb{Z}(r)) \cong H_c^{i+1}(X_{ét}, \mu_n^{\otimes -r})^*.$$

Proof. By duality, we have natural isomorphisms

$$\underline{\mathrm{hom}}_{D_{\mathrm{ét}}(k,\mathbb{Z}/n\mathbb{Z})}(R\pi_*(\mathbb{Z}/n\mathbb{Z})_X, \mathbb{Z}/n\mathbb{Z}) \cong R\pi_! R\pi^! \mathbb{Z}/n\mathbb{Z},$$
$$\underline{\mathrm{hom}}_{D_{\mathrm{ét}}(k,\mathbb{Z}/n\mathbb{Z})}(R\pi_!(\mathbb{Z}/n\mathbb{Z})_X, \mathbb{Z}/n\mathbb{Z}) \cong R\pi_* R\pi^! \mathbb{Z}/n\mathbb{Z}.$$

where $\underline{\mathrm{hom}}_{D_{\mathrm{ét}}(k,\mathbb{Z}/n\mathbb{Z})}$ denotes the inner Hom in $D_{\mathrm{ét}}(k,\mathbb{Z}/n\mathbb{Z})$. For $X \in \mathcal{S}m/k$ of pure dimension d, we have $R\pi^!\mathbb{Z}/n\mathbb{Z} = \mu_n^{\otimes d}[2d]$ and (1) and (2) follow from this. If k is finite, (3) follows from the Tate duality for Galois cohomology of finite fields. □

5.1. **Proof of Theorem 5.1.** Let X be smooth of pure dimension d over a perfect field k. It is shown in [Iv2] that the following diagram is commutative:

$$\begin{array}{ccc}
H_i^M(M^c(X)(r), \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\alpha^*} & H_i^{M,\text{ét}}(M^c(X)(r), \mathbb{Z}/n\mathbb{Z}) \\
\simeq \downarrow (*1) & & \simeq \downarrow (*2) \\
\mathrm{CH}_{-r}(X, i+2r; \mathbb{Z}/n\mathbb{Z}) & & H^{2d-i}(X_{\text{ét}}, \mu_n^{\otimes d+r})^* \\
\downarrow = & \nearrow_{cl_X} & \\
\mathrm{CH}^{d+r}(X, i+2r; \mathbb{Z}/n\mathbb{Z}), & &
\end{array}$$

where cl_X is the étale cycle map defined by Geisser-Levine [GL], and $(*1)$ comes from Equation (1.5) and $(*2)$ from Equation (5.4) and Lemma 5.5(1).

Similarly, when k is algebraically closed, in [Ke2] it is shown that (for $r \geq 0$) this same diagram is commutative but where now cl_X is Suslin's isomorphism from [Su] (cf. [Ke1, Thm. 5.6.1]).

Hence Theorem 5.1 follows from the Theorem 5.6 below.

Theorem 5.6. *Under the assumption of Theorem 5.1, the Geisser-Levine cycle map*

$$cl_X^{d+r,q} : \mathrm{CH}^{d+r}(X, q; \mathbb{Z}/n\mathbb{Z}) \to H^{2(d+r)-q}(X_{\acute{e}t}, \mathbb{Z}/n\mathbb{Z}(d+r))^*$$

are isomorphisms for all $q \in \mathbb{Z}$.

Proof. The case (ii) (resp. (iii)) is shown in [JS, Lem.6.2] (resp. [KeS1, Thm. (9.3]). The case (i) is a consequence of [Su] and [Ke1, Theo. 5.6.1]. Here we give an alternative proof of (i) following the proof of [JS, Lem.6.2]. It is simpler but depends on the Beilinson-Lichtenbaum conjecture due to Suslin-Voevodsky [SV3] and Geisser-Levine [GL, Thm.1.5] which relies on the Bloch-Kato conjecture proved by Rost-Voevodsky.

By the localization theorem for higher Chow groups ([B] and [L]), we have the niveau spectral sequence

$${}^{CH}E^1_{a,b} = \bigoplus_{x \in X_a} \mathrm{CH}^{a+r}(x, a+b; \mathbb{Z}/n\mathbb{Z}) \Rightarrow \mathrm{CH}^{d+r}(X, a+b; \mathbb{Z}/n\mathbb{Z}).$$

By the purity for étale cohomology, we have the niveau spectral sequence

$${}^{\text{ét}}E^1_{a,b} = \bigoplus_{x \in X_a} H^{a-b+2r}_{\text{ét}}(x, \mathbb{Z}/n\mathbb{Z}(a+r)) \Rightarrow H^{2(d+r)-a-b}_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(d+r)).$$

The cyle map $cl_X^{d+r,a+b}$ preserves the induced filtrations and induces maps on $E^\infty_{a,b}$ compatible with the map on E^1-terms induced by the cycle maps for $x \in X_a$:

$$cl_x^{a+r,a+b} \ : \ \mathrm{CH}^{a+r}(x, a+b; \mathbb{Z}/n\mathbb{Z}) \to H^{a-b+2r}_{\text{ét}}(x, \mathbb{Z}/n\mathbb{Z}(a+r)).$$

By the Beilinson-Lichtenbaum conjecture, $cl_x^{a+r,a+b}$ is an isomorphism if $b \geq r$. On the other hand, we have ${}^{CH}E^1_{a,b} = 0$ for $b < r$ since $\mathrm{CH}^i(L, j; \mathbb{Z}/n\mathbb{Z}) = 0$ for a field L and integers $i > j$, and we have ${}^{\text{ét}}E^1_{a,b} = 0$ for $b < 2r$ since $cd(\kappa(x)) = a$ for $x \in X_a$ by the assumption that k is algebraically closed. Noting $r \geq 0$, these imply that the cycle maps induce an isomorphism of the spectral sequences and hence the desired isomorphism of Theorem 5.6. □

6. Reduced motives and co-étale motives

In case k is finite, the weight homology functor in Definition 2.6 is compared with a different homology functor which is defined as the homology of the "co-étale" part of motives (see Theorem 6.3(2)). To introduce this, we need some preliminaries. In this section we still work over a perfect field k.

6.1. **A triangulated lemma.** We will apply the following lemma twice; in Equation (6.1) and Equation (6.7).

Lemma 6.1. *Suppose that $B : \mathcal{T} \to \mathcal{S}$ is a fully faithful exact functor of triangulated categories which admits a left adjoint $A : \mathcal{S} \to \mathcal{T}$. Then there is a triangulated endofunctor*

$$C : \mathcal{S} \to \mathcal{S}$$

with natural transformations $c : C \to id_{\mathcal{S}}$ and $d : BA \to C[1]$ such that for any object $X \in \mathcal{S}$,

$$C(X) \xrightarrow{c_X} X \xrightarrow{\eta_X} BA(X) \xrightarrow{d_X} C(X)[1]$$

is a distinguished triangle of $\mathcal{S}$, where η_X is the unit map of the adjunction. If small direct sums are representable in $\mathcal{S}, \mathcal{T}$ and A, B commute with them, so does C.

Proof. Take $X \in \mathcal{S}$ and choose an object $C \in \mathcal{S}$ fitting in a distinguished triangle

$$C \to X \xrightarrow{\eta_X} BA(X) \xrightarrow{+1}$$

We want to show that C is determined by X uniquely up to unique isomorphism. For this it suffices to see that, for any $Y \in \mathcal{S}$, one has $\mathrm{Hom}_{\mathcal{S}}(C, BA(Y)) = 0$. By adjunction, we are reduced to showing that $A(C) = 0$, equivalently the map $A(\eta_X) : A(X) \to ABA(X)$ is an isomorphism. This holds since $A(\eta_X)$ is split by the map $\varepsilon_{A(X)} : ABA(X) \to A(X)$, where ε is the counit of the adjunction, and the latter is an isomorphism by the full faithfulness of B. The last statement is also obvious. □

The first application of Lemma 6.1 is to the adjunction

$$(\alpha^*, \alpha_*) : DM^{\mathrm{eff}}(k, R) \rightleftarrows DM^{\mathrm{eff}}_{\text{ét}}(k, R),$$

we get an endomorphism

$$\Theta : DM^{\mathrm{eff}}(k, R) \to DM^{\mathrm{eff}}(k, R) \; ; \qquad C \to \Theta(C) \tag{6.1}$$

equipped with natural transformations $\alpha_*\alpha^*[-1] \to \Theta \to \mathrm{id}$ such that

$$\Theta(C) \to C \to \alpha_*\alpha^* C \xrightarrow{+1} \tag{6.2}$$

is a distinguished triangle for every $C \in DM^{\mathrm{eff}}(k, R)$. One could call $\Theta(C)$ the *co-étale part* of C since $\hom_{DM^{\mathrm{eff}}(k,R)}(\Theta(C), C') = 0$ for all C' in the image of $\alpha_* : DM^{\mathrm{eff}}_{\text{ét}}(k, R) \to DM^{\mathrm{eff}}(k, R)$. Notice that a consequence of Proposition 5.2(1) and (2) is that $\Theta(C)$ is necessarily torsion prime to $\mathrm{ch}(k)$:

Lemma 6.2. *For every $C \in DM^{\mathrm{eff}}(k, R)$ the canonical morphism*

$$\operatorname*{hocolim}_n \underline{\hom}(\mathbb{Z}/n, \Theta(C)) \xrightarrow{\sim} \Theta(C), \tag{6.3}$$

is an isomorphism where $\underline{\hom}$ is the inner Hom *in $DM^{\mathrm{eff}}(k, R)$ and n ranges over all positive integers invertible in k.*

Note also that

$$\Theta(\underline{\hom}(\mathbb{Z}/n, C)) \simeq \underline{\hom}(\mathbb{Z}/n, \Theta(C)). \tag{6.4}$$

In view of (6.2) Theorem 4.2 is a consequence of the following.

Theorem 6.3. *Let $C \in DM^{\mathrm{eff}}(k, R)$. Suppose either $R = \Lambda = \mathbb{Z}/n$, or $R = \mathbb{Z}[1/p]$ and $\Lambda = \mathbb{Q}/\mathbb{Z}[1/p]$.*

1. *If k is algebraically closed, $H^M_i(\Theta(C)), R) = 0$ for $i \in \mathbb{Z}$.*
2. *If k is finite, there are canonical isomorphisms for $C \in DM^{\mathrm{eff}}_{gm}(k, R)$:*

$$H^M_i(\Theta(C)), R) \xrightarrow{\cong} H^W_{i+1}(C, \Lambda) \quad \textit{for } i \in \mathbb{Z},$$

which are natural in C.

6.2. **Reduced motives.** For $r \geq 0$, let $d_{\leq r}DM^{\text{eff}}(k,R)$ be the localizing subcategory of $DM^{\text{eff}}(k,R)$ generated by motives of smooth varieties of dimension $\leq r$ (cf. [V, §3.4]). We have a natural equivalence of categories

$$d_{\leq 0}DM^{\text{eff}}(k,R) \cong D(\mathcal{A}_R),$$

where $\mathcal{A}_R$ is the category of additive contravariant functors on the category of permutation $R[G_k]$-modules [V, Prop. 3.4.1].

In [A-BV], Ayoub and Barbieri-Viale construct the left adjoint

$$L\pi_0 : DM^{\text{eff}}(k,R) \to d_{\leq 0}DM^{\text{eff}}(k,R) \tag{6.5}$$

to the inclusion functor $d_{\leq 0}DM^{\text{eff}}(k,R) \to DM^{\text{eff}}(k,R)$. It satisfies

$$L\pi_0(M(X)) = M(\pi_0(X)) \quad \text{for } X \in \mathcal{S}m, \tag{6.6}$$

where $\pi_0(X)$ is the "scheme of constants" of X, i.e.

$$\pi_0(X) = \coprod_{i \in I} \text{Spec}(L_i),$$

where $\{X_i\}_{i\in I}$ are the connected components of X and L_i is the algebraic closure of k in the function field of X_i. The functor $L\pi_0$ commutes with small direct sums. By Lemma 6.1 there is a triangulated endofunctor

$$(-)_{\text{red}} : DM^{\text{eff}}(k,R) \to DM^{\text{eff}}(k,R) \ ; \qquad C \to C_{\text{red}} \tag{6.7}$$

with natural transformations $L\pi_0[-1] \to (-)_{\text{red}} \xrightarrow{\iota} \text{id}$ such that

$$C_{\text{red}} \xrightarrow{\iota} C \to L\pi_0(C) \xrightarrow{+1} \tag{6.8}$$

is a distinguished triangle for every $C \in DM^{\text{eff}}(k,R)$.

Definition 6.4. We call C_{red} the *reduced part* of C. We may say that C is *reduced* if the canonical morphism $C_{\text{red}} \to C$ is an isomorphism, or equivalently if $L\pi_0 C = 0$.

6.3. **Proof of Theorem 6.3.** We start with the following:

Claim 6.5. *Let $C \in DM^{\text{eff}}(k,R)$.*

1. *If k is algebraically closed, we have $H_i^M(\Theta(C),R) = 0$ for $i \in \mathbb{Z}$.*
2. *If k is finite, we have $H_i^M(\Theta(C_{\text{red}}),R) = 0$ for $i \in \mathbb{Z}$.*

Proof. Reduction to $R = \mathbb{Z}/n\mathbb{Z}$ and $C = M(X)$ with X smooth and proper over k: Since $M(k)$ is a compact object of $DM^{\text{eff}}(k,R)$, the functor

$$H_i^M(-,R) : DM^{\text{eff}}(k,R) \to \mathcal{AB}$$

commutes with infinite direct sums, hence with hocolim_n. Therefore, in light of the isomorphisms

$$\Theta(C) \xrightarrow{\sim} \underset{n}{\text{hocolim}}\, \underline{\text{hom}}(\mathbb{Z}/n, \Theta(C)) \xrightarrow{\sim} \underset{n}{\text{hocolim}}\ \Theta(\underline{\text{hom}}(\mathbb{Z}/n, C))$$

of Lemma 6.3 and Equation 6.4 we may assume C is killed by some n and $R = \mathbb{Z}/n\mathbb{Z}$.

Reduction to $C = M(X)$: Since α^*, α_* and $L\pi_0$ commute with infinite direct sums, the statement of the claim is true for a direct sum $\bigoplus_{\alpha\in A} C_\alpha$ if it is true for all C_α. Moreover, if $C' \to C \to C'' \xrightarrow{+1}$ is an exact triangle in $DM^{\text{eff}}(k,\mathbb{Z}/n)$ and the statement of the claim is true for C' and C'', then it is true for C. By Gabber's refinement of de Jong's theorem [Il], motives of smooth projective varieties are dense in $DM^{\text{eff}}(k,\mathbb{Z}/n\mathbb{Z})$ in the sense that the smallest triangulated subcategory containing their shifts and stable under infinite direct sums is equal to $DM^{\text{eff}}(k,\mathbb{Z}/n\mathbb{Z})$ (see [Ke1, Prop.5.5.3]). Thus it suffices to show Claim 6.5 for $R = \mathbb{Z}/n\mathbb{Z}$ and $C = M(X)$, where X is a smooth and proper over k.

Proof of Claim 6.5(1): Since we have reduced to the case $C = M(X)$ with X smooth proper over k and $R = \mathbb{Z}/n\mathbb{Z}$, Claim 6.5(1) follows from Theorem 5.1.

Proof of Claim 6.5(2): Now we prove the second assertion of the claim, where k is finite and $C = M(X)$ with X smooth proper over k. We have a commutative diagram

$$\begin{array}{ccccccc} \cdots \to & H_i^M(M(X)_{\mathrm{red}}, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & H_i^M(M(X), \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & H_i^M(L\pi_0 M(X), \mathbb{Z}/n\mathbb{Z}) & \to \cdots \\ & \downarrow \alpha^* & & \downarrow \alpha^* & & \downarrow \alpha^* & \\ \cdots \to & H_i^{M,\text{ét}}(M(X)_{\mathrm{red}}, \mathbb{Z}/n\mathbb{Z}) & \to & H_i^{M,\text{ét}}(M(X), \mathbb{Z}/n\mathbb{Z}) & \to & H_i^{M,\text{ét}}(L\pi_0 M(X), \mathbb{Z}/n\mathbb{Z}) & \to \cdots \end{array}$$

We want to show that α^* on the left hand side is an isomorphism. By Theorem 5.1 and Equation (6.6), α^* in the middle and on the right hand side are isomorphisms for $i \neq -1$. By Equation (6.6) and Equation (1.5) we have

$$H_{-1}^M(M(X), \mathbb{Z}/n\mathbb{Z}) = H_{-1}^M(L\pi_0 M(X), \mathbb{Z}/n\mathbb{Z}) = 0.$$

Hence we are reduced to showing that the map

$$H_{-1}^{M,\text{ét}}(M(X), \mathbb{Z}/n\mathbb{Z}) \to H_{-1}^{M,\text{ét}}(L\pi_0 M(X), \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism. By (5.4) and Lemma 5.5, this follows from the fact that the trace map

$$H^{2d+1}(X_{\text{ét}}, \mu_n^{\otimes d}) \to H^1(\pi_0(X)_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism. This completes the proof of Claim 6.5. □

It remains to show Theorem 6.3(2). Assume k is finite. By definition we have a distinguished triangle

$$\Theta(C_{\mathrm{red}}) \to \Theta(C) \to \Theta(L\pi_0(C)) \xrightarrow{+1}.$$

Hence Claim 6.5(2) implies

$$H_i^M(\Theta(C), R) \cong H_i^M(\Theta(L\pi_0(C))), R) \tag{6.9}$$

hence we are reduced to showing

$$H_i^M(\Theta(L\pi_0(C))), R) \cong H_{i+1}^W(C, \Lambda). \tag{6.10}$$

By Theorem 2.4 this follows from the following.

Claim 6.6. *Suppose k is finite. For $X \in \mathcal{S}m$,*

$$H_i^M(\Theta(L\pi_0(M(X))), R) = \begin{cases} \Lambda^{c(X)} & \text{for } i = -1 \\ 0 & \text{for } i \neq -1 \end{cases}$$

where $c(X)$ is the set of connected components of X.

Proof. Writing

$$\widetilde{H}_i^M(C, R) = H_i^M(\Theta(L\pi_0(C))), R) \quad \text{for } C \in DM_{gm}^{\mathrm{eff}}(k, R), \tag{6.11}$$

this gives a homological functor on $DM_{gm}^{\mathrm{eff}}(k, R)$. Note that by Equations (6.3) and (6.4), we have for $C \in DM^{\mathrm{eff}}(k, R)$,

$$\Theta(C) = \Theta(C_{\mathrm{tor}}) \quad \text{with } C_{\mathrm{tor}} = \operatorname*{hocolim}_n \underline{\mathrm{hom}}(\mathbb{Z}/n, C). \tag{6.12}$$

We may assume X is geometrically connected over a field L finite over k. By Equation (6.6), we have $L\pi_0(M(X)) = M(\mathrm{Spec}(L))$. We easily see $M(\mathrm{Spec}(L))_{\mathrm{tor}} = \Lambda_L[-1]$, where

$$\Lambda_L = \begin{cases} M(\mathrm{Spec}(L)) \otimes \mathbb{Q}/\mathbb{Z}[1/p] & \text{if } R = \mathbb{Z}[1/p], \\ M(\mathrm{Spec}(L)) & \text{if } R = \mathbb{Z}/n\mathbb{Z}. \end{cases}$$

Hence we get a distinguished triangle

$$\Theta(L\pi_0(M(X))) \to \Lambda_L[-1] \to \alpha_*\alpha^*\Lambda_L[-1] \xrightarrow{+1}$$

which induces a long exact sequence

$$\cdots \to \widetilde{H}^M_{i+1}(M(X), R) \to H^M_i(\mathrm{Spec}(L), R) \xrightarrow{\alpha^*} H^{M,\text{ét}}_i(\mathrm{Spec}(L), R) \\ \to \widetilde{H}^M_i(M(X), R) \to \cdots .$$

By Theorem 5.1, α^* is an isomorphism for $i \neq -1$. For $i = -1$, we have $H^M_{-1}(\mathrm{Spec}(L), R) = 0$ while by Lemma 5.5 we have

$$H^{M,\text{ét}}_{-1}(\mathrm{Spec}(L), R) \cong H^1(\mathrm{Spec}(k)_{\text{ét}}, \Lambda) \cong \Lambda,$$

where the second isomorphism is the trace map in the Tate duality for Galois cohomology of finite fields. This completes the proof of Claim 6.6. □

Appendix A. Étale realisations of non-smooth motives

We now show the compatibility of the functors

$$\alpha^* : DM^{\text{eff}}(k, R) \to DM^{\text{eff}}_{\text{ét}}(k, R)$$
$$\Phi^{\text{ét}} : DM^{\text{eff}}_{\text{ét}}(k, \mathbb{Z}/n\mathbb{Z}) \to D_{\text{ét}}(k, \mathbb{Z}/n\mathbb{Z})$$
$$M : \mathcal{S}ch/k \to DM^{\text{eff}}(k, R)\ ; \qquad X \mapsto M(X)$$
$$M^c : \mathcal{S}ch^{\text{prop}}/k \to DM^{\text{eff}}(k, R)\ ; \qquad X \mapsto M^c(X)$$

from Equation (4.1), Proposition 5.2, and Equations (2.1) and (2.2) respectively.

We begin by recalling results of Suslin-Voevodsky that we will use. Let $\mathbb{Z}/n\mathbb{Z} \to \mathcal{I}^*$ be an injective resolution of the constant qfh-sheaf associated to $\mathbb{Z}/n\mathbb{Z}$ in $\mathcal{S}hv_{\text{qfh}}(\mathcal{S}ch/k)$. Recall that every qfh-sheaf has a canonical structure of transfers (Yoneda's Lemma together with [SV2, Thm. 4.2.12(1)]). So the two categories $\mathcal{S}hv_{\text{qfh}}(\mathcal{C}or/k)$ and $\mathcal{S}hv_{\text{qfh}}(\mathcal{S}ch/k)$ are canonically equivalent.

For a scheme $X \in \mathcal{S}ch/k$ let $[X]$ denote its corresponding object in $\mathcal{C}or/k$ as well as the presheaf with transfers on $\mathcal{C}or/k$ that it represents. The two natural transformations

$$\eta : \mathrm{id} \to \underline{C}_*(-), \qquad \varepsilon : \mathbb{Z}/n\mathbb{Z} \to \mathcal{I}^*$$

in $\mathcal{C}omp(\mathcal{S}hv_{\text{qfh}}(\mathcal{C}or/k))$ together with the global sections functor

$$\Gamma(k, -) : \mathcal{S}hv_{\text{qfh}}(\mathcal{C}or/k) \to \mathcal{AB}$$

induce canonical natural transformations

$$\mathrm{hom}(C_*([X]), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\varepsilon} \mathrm{hom}(C_*([X]), \mathcal{I}^*(k)) \tag{A.1}$$

$$\mathrm{hom}(\underline{C}_*([X]), \mathcal{I}^*) \xrightarrow{\Gamma(k,-)} \mathrm{hom}(C_*([X]), \mathcal{I}^*(k)) \tag{A.2}$$

$$\mathrm{hom}(\underline{C}_*([X]), \mathcal{I}^*) \xrightarrow{\eta} \mathrm{hom}([X], \mathcal{I}^*) \cong \mathcal{I}^*(X) \tag{A.3}$$

of complexes of abelian groups.

Theorem A.1 (Suslin-Voevodsky). *Suppose that the field k is algebraically closed, n is invertible in k, and $X \in \mathcal{S}ch/k$. The complex $\mathcal{I}^*$ and the sheaves $\mathcal{I}^i$ are all étale acyclic. That is, one has*

$$a_{ét}(\underline{H}^i(\mathcal{I}^*)(-)) = 0 \text{ and } H^i_{ét}(-, \mathcal{I}^j) = 0 \qquad \text{for all } i \neq 0, j \geq 0.$$

Moreover, the two morphisms (A.1), (A.2), *and* (A.3) *of complexes of abelian groups are all quasi-isomorphisms.*

In fact, if $[X]$ is replaced by a bounded complex in $\mathcal{C}omp^b(\mathcal{C}or/k)$ via the obvious use of the total complex functor, they remain quasi-isomorphisms.

Proof. The assertion that $\mathcal{I}^*$ is acyclic is [SV1, Thm. 10.2]. The assertion that the $\mathcal{I}^i$ are acyclic can be proven using verbatim the techniques of [V, Prop. 3.1.7] with "Nisnevich" replaced by "étale".

The claim that $\Gamma(k,-)$ and η induce a quasi-isomorphisms is [SV1, Thm. 7.6] (see [Ge1] for a discussion on why the characteristic zero hypothesis in Suslin-Voevodsky's statement is not necessary). The morphism induced by ε is a quasi-isomorphism because the complex $C_*([X])$ is a complex of free abelian groups, i.e, a complex of projective objects.

The usual spectral sequence arguments allow us to pass from a single $[X]$ to a bounded complex. □

Proposition A.2. *Suppose that the field k is algebraically closed and n is invertible in k. For $X \in \mathcal{S}ch/k$ with the natural morphism $\pi : X \to \operatorname{Spec}(k)$, there are canonical functorial isomorphisms*

$$\mathrm{Rhom}(\Phi^{ét}\alpha^* M(X), \mathbb{Z}/n\mathbb{Z}) = R\pi_*(\mathbb{Z}/n\mathbb{Z})_X, \quad \textit{and} \tag{A.4}$$

$$\mathrm{Rhom}(\Phi^{ét}\alpha^* M^c(X)), \mathbb{Z}/n\mathbb{Z}) = R\pi_!(\mathbb{Z}/n\mathbb{Z})_X. \tag{A.5}$$

Proof. We begin with the non-compact support case. Recall that by definition, the motive of a scheme $X \in \mathcal{S}ch/k$ is the image of the complex of presheaves with transfers $\underline{C}_*([X])$ in $DM^{\mathrm{eff}}(k)$. The image of this complex in $DM^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(k)$ is just the étale sheafification ([V2, Cor. 5.29]) and since k is separably closed, the image of this in $D_{\mathrm{\acute{e}t}}(k, \mathbb{Z}/n\mathbb{Z})$ is just its evaluation at k. Since the terms of this complex are free abelian groups, they are projective objects of $\mathcal{S}hv_{\mathrm{\acute{e}t}}(Et/k) \cong \mathcal{AB}$ and so $\mathrm{Rhom}(\Phi^{\mathrm{\acute{e}t}}\alpha^* M(X), \mathbb{Z}/n\mathbb{Z})$ is represented by the complex $\hom(C_*([X]), \mathbb{Z}/n\mathbb{Z})$.

On the other hand, the complex $R\pi_*(\mathbb{Z}/n\mathbb{Z})_X$ is represented by $\mathcal{I}^*(X)$ where $\mathcal{I}^*$ is any complex of sheaves on $(\mathcal{S}ch/k)_{\mathrm{\acute{e}t}}$ for which each $\mathcal{I}^i$ is an acyclic étale sheaf and which is quasi-isomorphic to $\mathbb{Z}/n\mathbb{Z}$ as a complex of étale sheaves. For example, the $\mathcal{I}^*$ which we chose above serves this purpose by Theorem A.1.

So the zig-zag of quasi-isomorphisms

$$\hom(C_*([X]), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\varepsilon} \xleftarrow{\Gamma(k,-)} \xrightarrow{\eta} \mathcal{I}^*(X)$$

of Theorem A.1 provides the isomorphism in $D_{\mathrm{\acute{e}t}}(k, \mathbb{Z}/n\mathbb{Z})$ we are looking for.

Now let $\mathcal{S}ch^{comp}(k)$ be the category whose objects are closed immersions $Z \to \overline{X}$ between proper schemes and morphisms are cartesian squares. There is a canonical functor towards $\mathcal{S}ch^{prop}(k)$ (the category with the same objects as $\mathcal{S}ch(k)$ but only proper morphisms) which sends a closed immersion $Z \to \overline{X}$ to its open complement $\overline{X} - Z$. The functor $\mathcal{S}ch^{comp}(k) \to \mathcal{S}ch^{prop}(k)$ is full and essentially surjective by Nagata's compactification theorem. Furthermore, by the localisation distinguished triangle ([Ke1, Prop. 5.5.5] or [V, Prop. 4.1.5]) the functor $M : \mathcal{S}ch^{comp}(k) \to DM^{\mathrm{eff}}(k)$ (which sends a closed immersion $Z \to X$ to the image of the complex $\mathrm{Tot}(\underline{C}_*([Z]) \to \underline{C}_*([\overline{X}])))$ factors through $M^c : \mathcal{S}ch^{prop}/k \to DM^{\mathrm{eff}}(k)$ (via the obvious canonical natural transformation induced by the short exact sequence $0 \to [Z] \to [\overline{X}] \to z_{equi}(X/k, 0)$). That is, $M^c(X)$ is represented by the complex $\underline{C}_*([X^c])$ where we define $[X^c]$ to be the two term complex $(\dots \to 0 \to [Z] \to [\overline{X}] \to 0 \to \dots)$.

On the étale side, the situation is the same. The object $R\pi_!(\mathbb{Z}/n\mathbb{Z})_X$ is (by definition) represented by the complex $\mathrm{Tot}(\mathcal{I}^*(\overline{X}) \to \mathcal{I}^*(Z))$ where $\mathcal{I}^*$ is as above.

So the isomorphism of Equation (A.5) can be obtained from the zig-zag of quasi-isomorphisms

$$\hom(C_*([X^c]), \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\varepsilon} \xleftarrow{\Gamma(k,-)} \xrightarrow{\eta} \mathcal{I}^*(X^c).$$

□

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