RECENT PROGRESS ON THE KATO CONJECTURE

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Abstract: This is a survey paper on recent works [J], [JS2] and [KeS] in progress by Jannsen, Kerz and the author on the Kato conjecture on the cohomological Hasse principle. In [J] and [JS2], general approaches are proposed to solve the conjecture for schemes over a finite field assuming resolution of singularities. Based on the idea in [JS2], a new approach is proposed in [KeS] to solve the conjecture for schemes over a finite field or the ring of integers in a local field, restricted to the prime-to-characteristic part. A key ingredient in [KeS], which replaces resolution of singularities, is a recently announced result on refined alterations due to Gabber (see [Il2]). We will give an outline of the proof. As an application, it implies a finiteness result on higher Chow groups of arithmetic schemes using the Bloch-Kato conjecture whose proof has been announced by Rost and Voevodsky ([SJ] and [V2], see also [HW], [V3], [W1] and [W2]).

Contents

1. Statements of the Kato conjectures

We start with a review on the following fundamental fact in number theory. Let k be a global field, namely either a finite extension of $\mathbb{Q}$ or a function field in one variable over a finite field. For simplicity we assume that $\mathrm{ch}(k) > 0$ or k is totally imaginary. Let P be the set of all finite places of k, and denote by k_v the completion of k at $v \in P$. For a field L let $\mathrm{Br}(L)$ be its Brauer group, and identify the Galois cohomology group $H^1(L, \mathbb{Q}/\mathbb{Z})$ with the group of continuous characters on the absolute Galois group of L with values in $\mathbb{Q}/\mathbb{Z}$.

(1-1) For $v \in P$ there is a natural isomorphism

$$\mathrm{Br}(k_v) \xrightarrow[\alpha_v]{\simeq} H^1(F_v, \mathbb{Q}/\mathbb{Z}) \xrightarrow[\beta_v]{\simeq} \mathbb{Q}/\mathbb{Z},$$

where F_v be the residue field of v and α_v is the residue map and β_v is the evaluation of characters at the Frobenius element.

(1-2) There is an exact sequence

$$0 \to \mathrm{Br}(k) \to \bigoplus_{v \in P} H^1(F_v, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z} \to 0,$$

where the first map is the composite of the restriction maps and α_v and the second map is the sum of β_v. The injectivity of the first map is the so-called Hasse principle for central simples algebras of k, which is a celebrated theorem of Hasse-Brauer-Noether.

In [K] K. Kato proposed a fascinating framework of conjectures that generalizes the above facts to higher dimensional arithmetic schemes. In order to review these conjectures, we introduce some notations. Let L be a field with $p = \mathrm{ch}(L)$. Let n be an integer $n > 0$ and write $n = mp^r$ with $(p, m) = 1$. We define the following Galois cohomology groups:

$$H^i(L, \mathbb{Z}/n\mathbb{Z}(j)) = H^i(L, \mu_m^{\otimes j}) \oplus H^{i-j}(L, W_r\Omega^i_{L,log}) \tag{1.1}$$

where μ_m is the Galois module of n-th roots of unity and $W_r\Omega^i_{L,log}$ is the logarithmic part of the de Rham-Witt sheaf $W_r\Omega^i_L$ [Il], I 5.7. Note that there is a canonical identification $H^2(L, \mathbb{Z}/n\mathbb{Z}(1)) = Br(L)[n]$ where $[n]$ denotes the n-torsion part.

Now let X be a scheme of finite type over $\mathbb{F}_p$ or the integer ring of a number field or a (p-adic) local field. Kato introduced the following complex $KC_\bullet(X, \mathbb{Z}/n\mathbb{Z})$ which we call the Kato complex:

$$\cdots \to \bigoplus_{x \in X_{(a)}} H^{a+1}(x, \mathbb{Z}/n\mathbb{Z}(a)) \to \bigoplus_{x \in X_{(a-1)}} H^a(x, \mathbb{Z}/n\mathbb{Z}(a-1)) \to \cdots$$
$$\cdots \to \bigoplus_{x \in X_{(1)}} H^2(x, \mathbb{Z}/n\mathbb{Z}(1)) \to \bigoplus_{x \in X_{(0)}} H^1(x, \mathbb{Z}/n\mathbb{Z}) \tag{1.2}$$

where $X_{(a)} = \{x \in X \mid \dim \overline{\{x\}} = a\}$ and the term $\bigoplus_{x \in X_{(a)}}$ is put in degree a. We will also use the complexes:

$$KC_\bullet(X, \mathbb{Q}/\mathbb{Z}) = \varinjlim_n KC_\bullet(X, \mathbb{Z}/n\mathbb{Z}),$$
$$KC_\bullet(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = \varinjlim_n KC_\bullet(X, \mathbb{Z}/\ell^n\mathbb{Z}),$$

where ℓ is a prime. Their homology groups

$$KH_a(X, \Lambda) := H_a(KC_\bullet(X, \Lambda)) \quad (\Lambda = \mathbb{Z}/n\mathbb{Z},\ \mathbb{Q}/\mathbb{Z},\ \mathbb{Q}_\ell/\mathbb{Z}_\ell) \tag{1.3}$$

is called the Kato homology of X with coefficient Λ (It is indeed a homology theory in the sense of Definition 3.1 below).

Now let X be a projective smooth connected curve over a finite field $\mathbb{F}_q$ with the function field $k = \mathbb{F}_q(X)$, or $X = \mathrm{Spec}(\mathcal{O}_k)$ for the integer ring $\mathcal{O}_k$ of a number field or a local field. Then the Kato complex $KC_\bullet(X, \mathbb{Q}/\mathbb{Z})$ is identified with the following complex:

$$Br(k) \longrightarrow \bigoplus_{x \in X_{(0)}} H^1(x, \mathbb{Q}/\mathbb{Z})$$

Hence the above facts (1-1) and (1-2) are equivalent to the following:

$$KH_1(X, \mathbb{Q}/\mathbb{Z}) = 0 \quad \text{and} \quad KH_0(X, \mathbb{Q}/\mathbb{Z}) = \begin{cases} 0 & \text{if } k \text{ is local,} \\ \mathbb{Q}/\mathbb{Z} & \text{if } k \text{ is global.} \end{cases}$$

Kato [K] proposed the following vast generalizations of these facts.

Conjecture 1.1. *Let X be a connected proper smooth variety over a finite field $\mathbb{F}_q$. Then*

$$KH_a(X,\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} \begin{cases} 0 & \text{if } a \neq 0, \\ \mathbb{Z}/n\mathbb{Z} & \text{if } a = 0. \end{cases}$$

Conjecture 1.2. *Let X be a connected regular scheme proper and flat over* $\mathrm{Spec}(\mathcal{O}_k)$ *where $\mathcal{O}_k$ is the integer ring of a number field k. Assume*

(∗) *either n is odd or k is totally imaginary.*

Then

$$KH_a(X,\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} \begin{cases} 0 & \text{if } a \neq 0, \\ \mathbb{Z}/n\mathbb{Z} & \text{if } a = 0. \end{cases}$$

We note that the assumption (∗) may be removed by modifying $KH_a(X,\mathbb{Q}/\mathbb{Z})$ (see [JS1] Conjecture C on page 482).

Conjecture 1.3. *Let X be a regular scheme proper and flat over* $\mathrm{Spec}(\mathcal{O}_k)$ *where $\mathcal{O}_k$ is the integer ring of a local field k. Then*

$$KH_a(X,\mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{for } a \geq 0.$$

2. Known results and announcement of new results

As is already noticed, the Kato conjectures in case $\dim(X) = 1$ rephrase the classical fundamental facts on the Brauer group of a global field and a local field.

Kato [K] proved Conjectures 1.1, 1.2, and 1.3 in case $\dim(X) = 2$. He deduced it from higher class field theory for X proved in [KS2] and [Sa1]. For X of dimension 2, the vanishing of $KH_2(X,\mathbb{Z}/n\mathbb{Z})$ in Conjectures 1.1 had been earlier established in [CTSS] (prime-to-p-part), and completed by M. Gros [Gr] for the p-part.

We note that it is easy to show the isomorphism for $a = 0$ in the conjectures (see (3.4) in §3). So we are only concerned with the isomorphisms for $a > 0$. The first result after [K] is the following:

Theorem 2.1. *(Saito* [Sa2]*) Let X be a smooth projective 3-fold over a finite field F. Then $KH_3(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ for any prime $\ell \neq \mathrm{ch}(F)$.*

This result was immediately generalized to the following:

Theorem 2.2. *(Colliot-Thélène* [CT]*, Suwa* [Sw]*) Let X be a smooth projective variety over a finite field F. Then*

$$KH_a(X,\mathbb{Q}/\mathbb{Z}) = 0 \quad \text{for } 0 < a \leq 3$$

[CT] handled the prime-to-p part where $p = \mathrm{ch}(F)$, and Suwa in [Sw] later adapted the technique of [CT] to handle the p-part. A tool in [Sa2] is a class field theory of surfaces over local fields, while the technique in [CT] is global and different from that in [Sa2].

The arithmetic version of the above theorem was established in the following:

Theorem 2.3. *(Jannsen-Saito* [JS1]*) Let X be a regular projective flat scheme over $S =$* $\mathrm{Spec}(\mathcal{O}_k)$ *where k is a number field or a $\mathfrak{p}$-adic field. Assume that k is totally imaginary if k is a number field. Fix a prime p. Assume that for any closed point $v \in S$, the reduced part of $X_v = X \times_S v$ is a simple normal crossing divisor on X and that X_v is reduced if $v|p$. Then we have*

$$KH_a(X,\mathbb{Q}_p/\mathbb{Z}_p) = 0 \quad \text{for } 0 < a \leq 3$$

Recently general approaches to Conjecture 1.1 were proposed assuming resolution of singularities.

Theorem 2.4. *(Jannsen* [J]*, Jannsen-Saito* [JS2]*) Let X be a projective smooth variety of dimension d over a finite field F. Let $t \geq 1$ be an integer. Then we have*

$$KH_a(X, \mathbb{Q}/\mathbb{Z}) = 0 \quad \text{for } 0 < a \leq t$$

if either $t \leq 4$ or $(\mathbf{RS})_d$*, or* $(\mathbf{RES})_{t-2}$ *(see below) holds.*

$(\mathbf{RS})_d$: For any X integral and proper of dimension$\leq d$ over F, there exists a proper birational morphism $\pi : X' \to X$ such that X' is smooth over F. For any U smooth of dimension$\leq d$ over F, there is an open immersion $U \hookrightarrow X$ such that X is projective smooth over F with $X - U$, a simple normal crossing divisor on X.

$(\mathbf{RES})_t$: For any smooth projective variety X over F, any simple normal crossing divisor Y on X with $U = X - Y$, and any integral closed subscheme $W \subset X$ of dimension$\leq t$ such that $W \cap U$ is regular, there exists a projective smooth X' over F and a birational proper map $\pi : X' \to X$ such that $\pi^{-1}(U) \simeq U$, and that $Y' = X' - \pi^{-1}(U)$ is a simple normal crossing divisor on X', and that the proper transform of W in X' is regular and intersects transversally with Y'.

We note that a proof of $(\mathbf{RES})_2$ is given in [CJS] based on an idea of Hironaka, which enables us to obtain the unconditional vanishing of the Kato homology with $\mathbb{Q}/\mathbb{Z}$-coefficient in degree $a \leq 4$.

Finally the above approach has been improved to remove the assumptions $(\mathbf{RS})_d$ and $(\mathbf{RES})_t$ on resolution of singularities, at least if we are restricted to the prime-to-ch(F) part:

Theorem 2.5. *(Kerz-Saito* [KeS]*) Let X be a projective smooth variety over a finite field F. For a prime $\ell \neq \mathrm{ch}(F)$, we have $KH_a(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ for $a > 0$.*

A key to the proof is the following refinement of de Jong's alteration theorem due to Gabber (see [Il2]).

Theorem 2.6. *(Gabber) Let F be a perfect field and X be a variety over F. Let $W \subset X$ be a proper closed subscheme. Let ℓ be a prime different from* ch(F)*. Then there exists a projective morphism $\pi : X' \to X$ such that*

- *X' is smooth over F and the reduced part of $\pi^{-1}(W)$ is a simple normal crossing divisor on X.*
- *π is generically finite of degree prime to ℓ,*

The same technique proves the following arithmetic version as well:

Theorem 2.7. *(Kerz-Saito* [KeS]*) Let X be a regular projective flat scheme over a henselian discrete valuation ring with finite residue field F. Then, for a prime $\ell \neq \mathrm{ch}(F)$, we have $KH_a(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ for $a \geq 0$.*

Finally we remark that one can prove the above results with $\mathbb{Z}/\ell^n\mathbb{Z}$-coefficient instead of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-coefficient by using the Bloch-Kato conjecture: For a prime ℓ and a field L, we have the symbol map

$$h^t_{L,\ell} : K^M_t(L)/\ell \to H^t(L, \mathbb{Z}/\ell\mathbb{Z}(t))$$

where $K_t^M(L)$ denotes the Milnor K-group of L. It is conjectured that $h_{L,\ell}^t$ is surjective. The conjecture is called the Bloch-Kato conjecture in case $l \neq \mathrm{ch}(L)$. For a scheme X, we introduce the following condition:

$(\mathbf{BK})_{X,\ell}^t$: For any field L finitely generated over a residue field of X, $h_{L,\ell}^t$ is surjective.

The surjectivity of $h_{L,\ell}^t$ is known if $t = 1$ (the Kummer theory) or $t = 2$ (Merkurjev-Suslin [MS]) or $\ell = \mathrm{ch}(L)$ (Bloch-Gabber-Kato [BK]) or $\ell = 2$ (Voevodsky [V1]). It is conjectured (the Bloch-Kato conjecture) that $h_{L,\ell}^t$ is always an isomorphism for any field L. Recently a complete proof of the conjecture has been announced by Rost and Voevodsky ([SJ] and [V2], see also [HW], [V3], [W1] and [W2]).

Theorem 2.8. *(see [JS2] §5) Let X and ℓ be as in either of Theorem 2.5 or Theorem 2.7. Assume $(\mathbf{BK})_{X,\ell}^t$ holds. Then we have $KH_a(X, \mathbb{Z}/\ell^n\mathbb{Z}) = 0$ for $0 < a \leq t$.*

3. Outline of Proof of Theorem 2.5

In this section we give an outline of the proof of Theorem 2.5. We fix a finite field F with $p = \mathrm{ch}(F)$ and work in the category $\mathcal{C}$ of schemes separated of finite type over F. We first recall the following:

Definition 3.1. Let $\mathcal{C}_*$ be the category with the same objects as $\mathcal{C}$, but morphisms are just the proper maps in $\mathcal{C}$. Let Mod be the category of modules. A homology theory $H = \{H_a\}_{a\in\mathbb{Z}}$ on $\mathcal{C}$ is a sequence of covariant functors:

$$H_a(-) : \ \mathcal{C}_* \to Mod$$

satisfying the following conditions:

(i) For each open immersion $j : V \hookrightarrow X$ in $\mathcal{C}$, there is a map $j^* : H_a(X) \to H_a(V)$, associated to j in a functorial way.

(ii) If $i : Y \hookrightarrow X$ is a closed immersion in X, with open complement $j : V \hookrightarrow X$, there is a long exact sequence (called localization sequence)

$$\cdots \xrightarrow{\partial} H_a(Y) \xrightarrow{i_*} H_a(X) \xrightarrow{j^*} H_a(V) \xrightarrow{\partial} H_{a-1}(Y) \longrightarrow \cdots.$$

(The maps ∂ are called the connecting morphisms.) This sequence is functorial with respect to proper maps or open immersions, in an obvious way.

It is an easy exercise to check that the Kato homology (1.3)

$$KH(-, \Lambda) = \{KH_a(-, \Lambda)\}_{a\in\mathbb{Z}}$$

provides us with a homology theory on $\mathcal{C}$.

Given a homology theory H on $\mathcal{C}$, we have the spectral sequence of homological type associated to every $X \in Ob(\mathcal{C})$, called the niveau spectral sequence (cf. [BO]):

$$E_{a,b}^1(X) = \bigoplus_{x\in X_{(a)}} H_{a+b}(x) \ \Rightarrow \ H_{a+b}(X) \quad \text{with } H_a(x) = \varinjlim_{V\subseteq\overline{\{x\}}} H_a(V). \tag{3.1}$$

Here the limit is over all open non-empty subschemes $V \subseteq \overline{\{x\}}$. This spectral sequence is covariant with respect to proper morphisms in $\mathcal{C}$ and contravariant with respect to open immersions. The functoriality of the spectral sequence is a direct consequence of that of the homology theory H.

In what follows we assume $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n prime to p or $\Lambda = \mathbb{Q}_\ell/\mathbb{Z}_\ell$ with $\ell \neq p$.

Step 1: Kato homology and étale homology

We now consider the étale homology on $\mathcal{C}$ given by

$$H_a(X,\Lambda) := H^{-a}(X_{\text{ét}}, Rf^!\Lambda) \quad \text{for } f : X \to \mathrm{Spec}(F) \text{ in } \mathcal{C}.$$

Here $Rf^!$ is the right adjoint of $Rf_!$ defined in [SGA 4], XVIII, 3.1.4. This is a homology theory in the sense of Definition 3.1. For X smooth of pure dimension d over F, we have (cf. [BO] and [JS1], Th.2.14)

$$H_a^{\text{ét}}(X,\Lambda) = H_{\text{ét}}^{2d-a}(X,\Lambda(d))\,, \tag{3.2}$$

where, for an integer $r > 0$, $\Lambda(r)$ is the Tate twist by the étale sheaf of roots of unity.

We then look at the spectral sequence (3.1) arising from this homology theory. The first step of the proof is the following lemma:

Lemma 3.2. *For $X \in Ob(\mathcal{C})$, we have $E^1_{a,b}(X) = 0$ if $b < -1$ and there is a natural isomorphism of complexes*

$$KC_\bullet(X,\Lambda) \simeq E^1_{\bullet,-1}(X)$$

where the right hand side denotes the complex

$$\cdots \to E^1_{a,-1}(X) \xrightarrow{d^1} E^1_{a-1,-1}(X) \xrightarrow{d^1} \cdots \xrightarrow{d^1} E^1_{1,-1}(X) \xrightarrow{d^1} E^1_{0,-1}(X).$$

In particular we have a natural isomorphism

$$KH_a(X,\Lambda) \simeq E^2_{a,-1}(X).$$

The first assertion follows easily from the fact $\mathrm{cd}(F) = 1$ and the second from [JSS], Theorem 1.1.1.

By the above lemma we get the edge homomorphism

$$\epsilon^a_X \;:\; H_{a-1}(X,\Lambda) \to KH_a(X,\Lambda) = E^2_{a,-1}(X) \tag{3.3}$$

which is an isomorphism for $a = 0$ by the first assertion of the lemma. For X connected smooth projective of dimension d over F, it gives rise to canonical isomorphisms

$$KH_0(X,\Lambda) \simeq H_{-1}(X,\Lambda) \simeq H^{2d+1}(X_{\text{ét}},\Lambda(d)) \simeq H^1(F,\Lambda) \simeq \Lambda, \tag{3.4}$$

where the second isomorphism is due to (3.2) and the third is the trace map and the last is the natural isomorphism in (1-1).

Step 2: Log-pairs and Graphication

Let the assumption be as above. Let $\mathcal{S} \subset \mathcal{C}$ be the subcategory of irreducible smooth projective schemes over $\mathrm{Spec}(F)$.

Definition 3.3. A log-pair is a couple $\Phi = (X,Y)$ where $X \in Ob(\mathcal{S})$, and $Y = \emptyset$ or Y is a divisor with simple normal crossings on X. We call $U = X - Y$ the complement of Φ and denote sometimes $\Phi = (X,Y;U)$.

Let $\Phi = (X,Y)$ be a log-pair and let $Y_1,\dots,Y_N$ be the irreducible components of Y. For an integer $a \geq 1$ write

$$Y^{(a)} = \coprod_{1 \leq i_i < \cdots < i_a \leq N} Y_{i_1,\dots,i_a} \quad (Y_{i_1,\dots,i_a} = Y_{i_1} \cap \cdots \cap Y_{i_a}). \tag{3.5}$$

We also denote $Y^{(0)} = X$. For $1 \leq \nu \leq a$ the proper morphism

$$\delta_\nu : Y^{(a)} \to Y^{(a-1)} \tag{3.6}$$

is induced by the inclusions $Y_{i_1,\dots,i_a} \hookrightarrow Y_{i_1,\dots,\hat{i_\nu},\dots,i_a}$.

Definition 3.4. The graph complex of a log-pair $\Phi = (X, Y)$ is the complex:

$$G_\bullet(\Phi,\Lambda) \ : \ \Lambda^{\pi_0(Y^{(d)})} \xrightarrow{\partial} \Lambda^{\pi_0(Y^{(d-1)})} \xrightarrow{\partial} \cdots \to \Lambda^{\pi_0(Y^{(1)})} \xrightarrow{\partial} \Lambda^{\pi_0(X)}.$$

where $\pi_0(Y^{(a)})$ is the set of connected components of $Y^{(a)}$ and $\Lambda^{\pi_0(Y^{(a)})}$ is put in degree a. Here $\partial : \Lambda^{\pi_0(Y^{(a)})} \to \Lambda^{\pi_0(Y^{(a-1)})}$ is defined as

$$\partial = \sum_{\nu=1}^{a} (-1)^\nu \partial_\nu \ , \tag{3.7}$$

$$\partial_\nu : \Lambda^{\pi_0(Y^{(a)})} \to \Lambda^{\pi_0(Y^{(a-1)})} \ ; \ (a_i)_{i\in\pi_0(Y^{(a)})} \to \Big(\sum_{\delta_\nu(i)=j} a_i\Big)_{j\in\pi_0(Y^{(a-1)})}$$

where $\delta_\nu : \pi_0(Y^{(a)}) \to \pi_0(Y^{(a-1)})$ is induced by the map (3.6).

Now a key construction (see [JS2], (3.2)) is to define a natural map of complexes

$$\gamma_\Phi : KC_\bullet(U,\Lambda) \to G_\bullet(\Phi,\Lambda) \quad \text{for } \Phi = (X,Y;U)$$

which induces the natural homomorphism

$$\gamma_\Phi^a : KH_a(U,\Lambda) \to GH_a(\Phi,\Lambda) := H_a(G_\bullet(\Phi,\Lambda)). \tag{3.8}$$

which we call the graphication of the Kato homology. In order to control γ_Φ^a, we use

$$\gamma\epsilon_\Phi^a \ : \ H_{a-1}(U,\Lambda) \to GH_a(\Phi,\Lambda) \tag{3.9}$$

which is defined as the composite of γ_Φ^a with ϵ_U^a (cf. (3.3)). We note that the right hand side of (3.9) is nonzero only if $0 \le a \le d = \dim(X)$ while the left side could be nonzero for any a with $0 \le a \le 2d + 1$.

Definition 3.5. A log-pair $\Phi = (X, Y)$ is clean in degree q for a non-negative integer $q \le \dim(X)$ if $\gamma\epsilon_\Phi^a$ is injective for $a = q$ and surjective for $a = q + 1$.

Now the following theorem ([JS2], Lemmas 3.3 and 3.4) is crucial.

Theorem 3.6. *Take $\Lambda = \mathbb{Q}_\ell/\mathbb{Z}_\ell$. Let $\Phi = (X,Y;U)$ be an ample log-pair, which means by definition that one of the irreducible components of Y is an ample divisor on X. Then Φ is clean in degree q for all $q \le \dim(X)$.*

The proof of the above theorem hinges on the affine Lefschetz theorem and the Weil conjecture proved by Deligne [D].

The above theorem implies that for an ample log-pair Φ and for any integer a with $0 \le a \le \dim(X)$, γ_Φ^a is surjective, and an isomorphism if ϵ_U^a is surjective. This is already a big step in the proof of Theorem 2.5.

We will need a variant of the above construction.

Definition 3.7. The reduced graph complex of a log-pair $\Phi = (X, Y)$ is the complex:

$$G_\bullet(\hat\Phi,\Lambda) \ : \ \Lambda^{\pi_0(Y^{(d)})} \xrightarrow{\partial} \Lambda^{\pi_0(Y^{(d-1)})} \xrightarrow{\partial} \cdots \to \Lambda^{\pi_0(Y^{(1)})},$$

where $\Lambda^{\pi_0(Y^{(a)})}$ is put in degree a. We have evident maps of complexes

$$\Lambda^{\pi_0(X)}[0] \to G_\bullet(\Phi,\Lambda) \xrightarrow{\iota} G_\bullet(\hat\Phi,\Lambda)$$

which induce an isomorphism

$$\iota : GH_a(\Phi,\Lambda) \xrightarrow{\cong} GH_a(\hat\Phi,\Lambda) \quad \text{for } a \ne 1, \tag{3.10}$$

and an exact sequence

$$0 \to GH_1(\Phi,\Lambda)) \xrightarrow{\iota} GH_1(\hat{\Phi},\Lambda) \to \Lambda \to 0. \tag{3.11}$$

In the same way as above, one may also define the natural homomorphism ([KeS])

$$\gamma^a_{\hat{\Phi}} : KH_{a-1}(Y,\Lambda) \to GH_a(\hat{\Phi},\Lambda) \tag{3.12}$$

which fits into the commutative diagram

$$\begin{array}{ccc} KH_a(U,\Lambda) & \xrightarrow{\partial} & KH_{a-1}(Y,\Lambda) \\ \downarrow{\scriptstyle \gamma^a_{\Phi}} & & \downarrow{\scriptstyle \gamma^a_{\hat{\Phi}}} \\ GH_a(\Phi,\Lambda) & \xrightarrow{\iota} & GH_a(\hat{\Phi},\Lambda) \end{array}$$

where ∂ is the boundary map for the Kato homolgy.

Step 3: Pullback map for Kato homology

Another key ingredient to the proof of Theorem 2.5 is the construction of the pullback map for Kato homology, which is stated in the following form:

Lemma 3.8. *For any dominant morphism $f : X \to Y$ where $X,\ Y \in Ob(\mathcal{C})$ are integral smooth of the same dimension over F, we have the pullback maps for all q:*

$$f^* : H_q(Y,\Lambda) \to H_q(X,\Lambda), \quad f^* : KH_q(Y,\Lambda) \to KH_q(X,\Lambda)$$

which satisfy the following conditions:

- *For a dominant morphism $g : Y \to Z$ with Z integral smooth over F of the same dimension, we have $(g \cdot f)^* = f^* \cdot g^*$.*
- *The following diagram is commutative (cf. (3.3))*

$$\begin{array}{ccc} H_{q-1}(Y,\Lambda) & \xrightarrow{\epsilon^q_Y} & KH_q(Y,\Lambda) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ H_{q-1}(X,\Lambda) & \xrightarrow{\epsilon^q_X} & KH_q(X,\Lambda) \end{array}$$

- *If f is proper, the composite map*

$$KH_q(Y,\Lambda) \xrightarrow{f^*} KH_q(X,\Lambda) \xrightarrow{f_*} KH_q(Y,\Lambda)$$

  is the multiplication by the degree $[F(X) : F(Y)]$ of the extension of the function fields.

In [KeS] the above lemma is shown based on the intersection theory on cycle modules due to Rost [R] (the theory was originally developed over a field but it may be extended to a more general base).

Step 4: The condition $(\mathbf{LG})_q$

Let $q \geq 1$ be an integer. For a log-pair $\Phi = (X, Y; U)$ consider the condition:

$(\mathbf{LG})_q$: The composite map

$$\partial\epsilon^a_{\Phi} : H_{a-1}(U,\Lambda) \xrightarrow{\epsilon^a_U} KH_a(U,\Lambda) \xrightarrow{\partial} KH_{a-1}(Y,\Lambda)$$

is injective for $a = q$ and surjective for $a = q + 1$.

Lemma 3.9. *Let $q \geq 1$ be an integer. Let $\Phi = (X, Y; U)$ be a log-pair which satisfies the condition $(\mathbf{LG})_q$. Let $j^* : KH_q(X, \Lambda) \to KH_q(U, \Lambda)$ be the pullback via $j : U \hookrightarrow X$ and $\epsilon_U^q : H_{q-1}(U, \Lambda) \to KH_q(U, \Lambda)$ be as in (3.3). Then the map j^* is injective and* $\mathrm{Image}(j^*) \cap \mathrm{Image}(\epsilon_U^q) = 0$.

Proof First we claim that j^* is injective. Indeed we have the exact sequence

$$KH_{q+1}(U, \Lambda) \xrightarrow{\partial} KH_q(Y, \Lambda) \to KH_q(X, \Lambda) \xrightarrow{j^*} KH_q(U).$$

Since $\partial\epsilon_U^{q+1}$ is surjective by the assumption, ∂ is surjective and the claim follows. By the above claim it suffices to show $\mathrm{Image}(j^*) \cap \mathrm{Image}(\epsilon_U^q) = 0$. We have the exact sequence

$$KH_q(X, \Lambda) \xrightarrow{j^*} KH_q(U, \Lambda) \xrightarrow{\partial} KH_{q-1}(Y, \Lambda)$$

Let $\beta \in H_{q-1}(U, \Lambda)$ and assume $\alpha = \epsilon_U^q(\beta) \in KH_q(U, \Lambda)$ lies in $\mathrm{Image}(j^*)$. It implies $\partial(\alpha) = \partial\epsilon_U^q(\beta) = 0$. Since $\partial\epsilon_U^q$ is injective by the assumption, this implies $\beta = 0$ so that $\alpha = 0$. $\square$

For integers $q \geq 1$ and $d \geq 0$, consider the condition:

$\mathbf{KC}(q, d)$: $KH_a(X, \Lambda) = 0$ for all $X \in Ob(\mathcal{S})$ with $\dim(X) \leq d$ and $0 < a \leq q$.

Lemma 3.10. *Take $\Lambda = \mathbb{Q}_\ell/\mathbb{Z}_\ell$. Fix integers d and q with $d \geq q \geq 1$, and assume $\mathbf{KC}(q, d-1)$. For any log-pair $\Phi = (X, Y)$ with $d = \dim(X)$, there exists a log-pair $\Phi' = (X, Y')$ with $Y \subset Y'$ such that Φ' satisfies the condition $(\mathbf{LG})_q$.*

Proof It follows from Bertini's theorem ([AK] and [P]) that for any log-pair $\Phi = (X, Y)$ with $\dim(X) = d$, one can take $Z \subset X$, a smooth section of a sufficiently ample line bundle on X, such that $\Phi' = (X, Y \cup Z)$ is an ample log-pair so that it is clean in degree q for all $q \leq \dim(X)$ by Theorem 3.6. Hence it suffices to show that if $\Phi = (X, Y; U)$ is clean in degree q, then it satisfies the condition $(\mathbf{LG})_q$. We consider the commutative diagram

$$\begin{array}{ccccc} H_q(U,\Lambda) & \xrightarrow{\epsilon_U^{q+1}} & KH_{q+1}(U,\Lambda) & \xrightarrow{\partial} & KH_q(Y,\Lambda) \\ & & \downarrow{\scriptstyle\gamma_\Phi^{q+1}} & & \downarrow{\scriptstyle\gamma_{\hat\Phi}^{q+1}} \\ & & GH_{q+1}(\Phi,\Lambda) & \xrightarrow[\iota]{\simeq} & GH_{q+1}(\hat\Phi,\Lambda) \end{array}$$

where ι is an isomorphism by (3.10), and $\gamma\epsilon_\Phi^{q+1} = \gamma_\Phi^{q+1} \circ \epsilon_U^{q+1}$ is surjective by the assumption. Moreover, $\gamma_{\hat\Phi}^{q+1}$ is an isomorphism. Indeed we have the spectral sequence (Mayer-Vietoris for homology of closed coverings):

$$E^1_{s,t} = KH_t(Y^{(s)}, \Lambda) \Rightarrow KH_{s+t-1}(Y, \Lambda) \quad \text{(cf. (3.5))}$$

where we put $E^1_{s,t} = 0$ for $s \leq 0$, so that $\mathbf{KC}(q, d-1)$ implies $KH_t(Y^{(s)}, \Lambda) = 0$ for $0 < t \leq q$ and $s > 0$. The assertion follows easily from this and (3.4). Now a diagram chase shows that $\partial\epsilon_U^{q+1}$ is surjective.

Next we consider the commutative diagram

$$\begin{array}{ccccc} H_{q-1}(U,\Lambda) & \xrightarrow{\epsilon_U^{q}} & KH_{q}(U,\Lambda) & \xrightarrow{\partial} & KH_{q-1}(Y,\Lambda) \\ & & \downarrow{\scriptstyle\gamma_\Phi^{q}} & & \downarrow{\scriptstyle\gamma_{\hat\Phi}^{q-1}} \\ & & GH_{q}(\Phi,\Lambda) & \xrightarrow[\iota]{\hookrightarrow} & GH_{q}(\hat\Phi,\Lambda) \end{array}$$

where ι is injective by (3.10) and (3.11), and $\gamma\epsilon^q_\Phi = \gamma^q_\Phi \circ \epsilon^q_U$ is injective by the assumption. As before one can show by using $\mathbf{KC}(q, d-1)$ that $\gamma^{q-1}_{\hat{\Phi}}$ is an isomorphism. This shows $\partial\epsilon^q_U$ is injective and the proof is complete. □

Step 5: Enters Gabber's theorem to end the proof

We fix a prime $\ell \neq \mathrm{ch}(F)$ and take $\Lambda = \mathbb{Q}_\ell/\mathbb{Z}_\ell$. We finish the proof of Theorem 2.5 by the induction on $d = \dim(X) \geq 0$. The case $d = 0$ is trivial. Assume $d \geq 1$ and that $\mathbf{KC}(q, d-1)$ holds for $1 \leq q \leq d$. Let $X \in Ob(\mathcal{S})$ with $d = \dim(X)$. Let $\alpha \in KH_q(X, \Lambda)$. By recalling that

$$\epsilon^q_X : H_{q-1}(X, \Lambda) \to KH_q(X, \Lambda) = E^2_{q,-1}(X)$$

is an edge homomorphism and by looking at the differentials

$$d^r_{q,-1} : E^r_{q,-1}(X) \to E^r_{q-r,r-2}(X),$$

we conclude that there exists a closed subscheme $W \subset X$ such that $\dim(W) \leq q-2$, and that putting $U = X - W$, the pullback $\alpha_{|U} \in KH_q(U, \Lambda)$ of α via $U \to X$ lies in the image of ϵ^q_U, namely there exists $\beta \in H_{q-1}(U, \Lambda)$ such that $\alpha_{|U} = \epsilon^q_U(\beta)$. Take $\pi : X' \to X$ as in Theorem 2.6 and put $U' = \pi^{-1}(U)$ and $Y' = \pi^{-1}(W)_{red}$ which is a divisor with simple normal crossings on X'. By Lemma 3.10 there is a log-pair $\Phi = (X', Y''; V)$ with $Y' \subset Y''$ which satisfies the condition $(\mathbf{LG})_q$. Thanks to Step 3, we have the commutative diagram

$$\begin{array}{ccccc}
KH_q(X,\Lambda) & \xrightarrow{\pi^*} & KH_q(X',\Lambda) & & \\
\downarrow & & \downarrow & & \\
KH_q(U,\Lambda) & \xrightarrow{\pi^*} & KH_q(U',\Lambda) & \xrightarrow{j^*} & KH_q(V,\Lambda) \\
\uparrow{\scriptstyle \epsilon^q_U} & & \uparrow{\scriptstyle \epsilon^q_{U'}} & & \uparrow{\scriptstyle \epsilon^q_V} \\
H_{q-1}(U,\Lambda) & \xrightarrow{\pi^*} & H_{q-1}(U',\Lambda) & \xrightarrow{j^*} & H_{q-1}(V,\Lambda)
\end{array}$$

where $j : V \to U$ is the open immersion. Put $\alpha' = \pi^*(\alpha) \in KH_q(X', \Lambda)$ and $\beta' = \pi^*(\beta) \in H_{q-1}(U', \Lambda)$. Let $\alpha'_{|V} \in KH_q(V, \Lambda)$ (resp. $\beta'_{|V} \in H_{q-1}(V, \Lambda)$) be the pullback of α' (resp. β') via $V \hookrightarrow X'$ (resp. $V \hookrightarrow U'$). By the diagram we get $\alpha'_{|V} = \epsilon^q_V(\beta'_{|V}) \in KH_q(V, \Lambda)$. By Lemma 3.9 this implies $\alpha' = 0$. Since the composite

$$KH_q(X, \Lambda) \xrightarrow{\pi^*} KH_q(X', \Lambda) \xrightarrow{\pi_*} KH_q(X, \Lambda)$$

is the multiplication of the degree of π which is prime to ℓ, we get $\alpha = 0$. □

4. Applications

Let X be a smooth scheme over a field F and let

$$H^q_M(X, \mathbb{Z}(r)) = CH^r(X, 2r-q) = H_{2r-q}(z^r(X, \bullet)) \quad (q, r \geq 0)$$

be the motivic cohomology of X defined as Bloch's higher Chow group, where $z^r(X, \bullet)$ is Bloch's cycle complex [B1]. Recall that $\mathrm{CH}^r(X, 0)$ is the classical Chow group $\mathrm{CH}^r(X)$. A 'folklore conjecture', generalizing the analogous conjecture of Bass on K-groups, is that in case F is finite, $H^q_M(X, \mathbb{Z}(r))$ should be finitely generated. Except the case of $r = 1$ or $\dim(X) = 1$ (Quillen), the only known case is that of $H^{2d}_M(X, \mathbb{Z}(d)) = \mathrm{CH}^d(X) = \mathrm{CH}_0(X)$

where $d = \dim(X)$. It is a consequence of higher dimensional class field theory ([B2], [KS1] and [CTSS]).

One way to approach the problem is to look at an étale cycle map constructed by Geisser and Levine [GL] :

$$\rho^{r,q}_{X,\mathbb{Z}/n\mathbb{Z}} : \mathrm{CH}^r(X,q;\mathbb{Z}/n\mathbb{Z}) \to H^{2r-q}_{\text{ét}}(X,\mathbb{Z}/n\mathbb{Z}(r)), \tag{4.1}$$

Here

$$\mathrm{CH}^r(X,q;\mathbb{Z}/n\mathbb{Z}) = H_q(z^r(X,\bullet)\otimes^{\mathbb{L}}\mathbb{Z}/n\mathbb{Z}),$$

is the higher Chow group with finite coefficients which fits into an exact sequence

$$0 \to \mathrm{CH}^r(X,q)/n \to \mathrm{CH}^r(X,q;\mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^r(X,q-1)[n] \to 0,$$

and $\mathbb{Z}/n\mathbb{Z}(r)$ is the complex of étale sheaves on X:

$$\mathbb{Z}/n\mathbb{Z}(r) = \mu_m^{\otimes r} \oplus W_\nu\Omega^r_{X,log}[-r],$$

where $n = mp^r$ and $(p,m) = 1$ with $p = \mathrm{ch}(F)$ (cf. (1.1)). The same construction has been carried out for a regular scheme X of finite type over a Dedekind domain by Levine [L] (see also [Ge]), assuming that n is invertible on X.

We recall the following result due to Suslin-Voevodsky [SV] and Geisser-Levine [GL]:

Theorem 4.1. *Let the assumption be as above (the case over a Dedekind domain is included). Assume* $(\mathbf{BK})^t_{X,\ell}$ *for all* $t \geq 0$ *(cf. §1). Then* $\rho^{r,q}_{X,\mathbb{Z}/\ell^n\mathbb{Z}}$ *is an isomorphism for* $r \leq q$ *and injective for* $r = q+1$.

Now we turn our attention to $\rho^{r,q}_{X,\mathbb{Z}/n\mathbb{Z}}$ in case $r \geq d := \dim(X)$. We assume that X is a regular scheme over either a finite field F or a henselian discrete valuation ring with finite residue field F. In case $r > d$ it is easily shown (see [JS2], Lemma 6.2) that $\rho^{r,q}_{X,\mathbb{Z}/\ell^n\mathbb{Z}}$ is an isomorphism assuming $(\mathbf{BK})^{q+1}_{X,\ell}$. An interesting phenomenon emerges for $\rho^{r,q}_{X,\mathbb{Z}/\ell^n\mathbb{Z}}$ with $r = d$. $(\mathbf{BK})^{q+1}_{X,\ell}$ implies a long exact sequence: (see [JS2], Lemma 6.2)

$$KH_{q+2}(X,\mathbb{Z}/\ell^n\mathbb{Z}) \to \mathrm{CH}^d(X,q;\mathbb{Z}/\ell^n\mathbb{Z}) \xrightarrow{\rho^{d,q}_{X,\mathbb{Z}/\ell^n\mathbb{Z}}} H^{2d-q}_{\text{ét}}(X,\mathbb{Z}/\ell^n\mathbb{Z}(d))$$
$$\to KH_{q+1}(X,\mathbb{Z}/\ell^n\mathbb{Z}) \to \mathrm{CH}^d(X,q-1;\mathbb{Z}/\ell^n\mathbb{Z}) \xrightarrow{\rho^{d,q-1}_{X,\mathbb{Z}/\ell^n\mathbb{Z}}} \dots$$

Hence Theorem 2.8 implies the following:

Theorem 4.2. *(*[KeS]*) Let* X *be a regular projective scheme over either a finite field* F *or a henselian discrete valuation ring with finite residue field* F. *Let* $q \geq 0$ *be an integer and* $n > 0$ *be an integer prime to* $\mathrm{ch}(F)$ *and assume* $(\mathbf{BK})^{q+2}_{X,\ell}$ *for all prime* $\ell|n$. *Let* $d = \dim(X)$. *Then*

$$\rho^{d,q}_{X,\mathbb{Z}/n\mathbb{Z}} : \mathrm{CH}^d(X,q;\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H^{2d-q}_{\acute{e}t}(X,\mathbb{Z}/n\mathbb{Z}(d)).$$

In particular $\mathrm{CH}^d(X,q;\mathbb{Z}/n\mathbb{Z})$ *is finite.*

The above theorem implies the following affirmative result on the finiteness conjecture on motivic cohomology:

Corollary 4.3. *Let* X *be a quasi-projective scheme over either a finite field* F *or a henselian discrete valuation ring with finite residue field* F. *Let* $n > 0$ *be an integer prime to* $\mathrm{ch}(F)$ *and assume* $(\mathbf{BK})^t_{X,\ell}$ *for all primes* $\ell|n$ *and integers* $t \geq 0$. *Then* $\mathrm{CH}^r(X,q;\mathbb{Z}/n\mathbb{Z})$ *is finite for all* $r \geq \dim(X)$ *and* $q \geq 0$.

Proof For simplicity we only treat the case over a finite field F. We may assume $n = \ell^m$ for a prime $\ell \neq \mathrm{ch}(F)$. We proceed by the induction on $\dim(X)$. First we remark that the localization sequence for higher Chow groups implies that for a dense open subscheme $U \subset X$, the finiteness of $\mathrm{CH}^r(X, q; \mathbb{Z}/n\mathbb{Z})$ for all $r \geq \dim(X)$ and q is equivalent to that of $\mathrm{CH}^r(U, q; \mathbb{Z}/n\mathbb{Z})$. Thus it suffices to show the assertion for any smooth variety U over F. If U is an open subscheme of a smooth projective variety X over F, the assertion holds for X by Theorem 4.2 and hence for U by the above remark. In general Gabbers's theorem 2.6 implies that there exist an open subscheme V of a smooth projective variety X over F, an open subscheme W of U, and a finite étale morphism $\pi : V \to W$ of degree prime to ℓ. We know that the assertion holds for V so that it holds for W by a standard norm argument. This completes the proof by the above remark. □

Finally we note that the above corollary implies the following affirmative result on the Bass conjecture. Let $K'_i(X, \mathbb{Z}/n\mathbb{Z})$ be Quillen's higher K-groups with finite coefficients constructed from the category of coherent sheaves on X (which coincide with the algebraic K-groups with finite coefficients constructed from the category of vector bundles when X is regular).

Corollary 4.4. *Under the assumption of Corollary 4.3, $K'_i(X, \mathbb{Z}/n\mathbb{Z})$ is finite for $i \geq \dim(X) - 2$.*

Proof Theorem 4.1 implies that $\mathrm{CH}^r(X, q; \mathbb{Z}/n\mathbb{Z})$ is finite for $r \leq q + 1$. Hence the assertion follows from the Atiyah-Hirzebruch spectral sequence (see [L] for its construction in the most general case):

$$E_2^{p,q} = \mathrm{CH}^{-q/2}(X, -p-q; \mathbb{Z}/n\mathbb{Z}) \Rightarrow K'_{-p-q}(X, \mathbb{Z}/n\mathbb{Z})$$

(note $E_2^{p,q}$ may be nonzero only if $q \leq 0$ and $p + q \leq 0$). □

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