

K-THEORY OF NON-ARCHIMEDEAN RINGS I

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ABSTRACT. We introduce a variant of homotopy K -theory for Tate rings, which we call *analytic K -theory*. It is homotopy invariant with respect to the analytic affine line viewed as an ind-object of closed disks of increasing radii. Under a certain regularity assumption we prove an analytic analog of the Bass fundamental theorem and we compare analytic K -theory with continuous K -theory, which is defined in terms models. Along the way we also prove some results about the algebraic K -theory of Tate rings.

1. INTRODUCTION

In this note we introduce and study a new version of K -theory for non-archimedean rings, which takes the topology into account. As a motivation, we first recall some facts about classical topological K -theory before we describe our construction in more detail in Subsection 1.2.

1.1. Topological K -theory. Let X be a compact Hausdorff space. The (complex) topological K -theory of X is usually defined as the mapping spectrum

$$K^{\text{top}}(X) = \text{map}(\Sigma^\infty X_+, KU),$$

where KU is the spectrum representing complex K -theory. The connective covering spectrum of $K^{\text{top}}(X)$ is denoted by $k^{\text{top}}(X)$. Note that one can recover $K^{\text{top}}(X)$ as

$$(1.1) \quad K^{\text{top}}(X) \cong \text{colim}(k^{\text{top}}(X) \rightarrow \Omega k^{\text{top}}(\Sigma_+ X) \rightarrow \Omega^2 k^{\text{top}}(\Sigma_+^2 X) \rightarrow \dots)$$

Here for a topological space Y we write $\Sigma_+ Y = Y_+ \wedge S^1$ (as unpointed spaces).

For a ring A let $K(A)$ be the non-connective algebraic K -theory spectrum of A and $k(A)$ be its connective cover. For a simplicial ring A let $k(A)$ be the degreewise connective algebraic K -theory spectrum of A , i.e. the realization of the simplicial spectrum $[m] \mapsto k(A_m)$. Let $C(X)$ be the ring of complex continuous functions on the compact Hausdorff space X and let $j \mapsto \Delta_{\text{top}}^j$ be the cosimplicial space of standard simplices.

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An important observation is that topological K -theory can be obtained from algebraic K -theory by making it homotopy invariant. Recall that there is a canonical morphism of spectra $k(C(X)) \rightarrow k^{\text{top}}(X)$. As k^{top} is homotopy invariant, it is natural to study the induced morphism

$$\psi: k(C(X \times \Delta_{\text{top}})) \rightarrow k^{\text{top}}(X),$$

where the domain is the realization of the simplicial spectrum given by $[m] \mapsto k(C(X \times \Delta_{\text{top}}^m))$. Our definition of non-archimedean analytic K -theory below is motivated by the fact that ψ is an equivalence of spectra.

1.2. Analytic K -theory and Karoubi–Villamayor K -theory. There exist several articles generalizing aspects of either classical homotopy theory of the real interval $[0, 1] \subset \mathbb{R}$ or of \mathbb{A}^1 -homotopy theory from algebraic geometry to the non-archimedean setting, for example [KV71], [Kar71], [Cal85], [Ayo15]. In all those works the approach has been to use the rigid analytic closed unit ball \mathbb{B}_κ over the non-archimedean complete valued field κ (or some variant of it in the work of Karoubi and Villamayor) as a building block for homotopy theory.

The novel aspect we suggest in this note is to use the affinoid exhaustion of the whole analytic line

$$(1.2) \quad (\mathbb{A}^1)_\kappa^{\text{an}} = \text{colim}_\rho \mathbb{B}_\kappa(\rho)$$

instead of the rigid analytic closed unit ball $\mathbb{B}_\kappa = \mathbb{B}_\kappa(1)$ to construct a homotopy theory, or more precisely a non-archimedean K -theory. Here $\mathbb{B}_\kappa(\rho)$ is the rigid analytic closed ball of radius ρ centered at the origin. So in some sense this new homotopy theory keeps more information than the previous approaches mentioned above. Our approach is motivated by the observation that the Picard group of a smooth affinoid algebra is homotopy invariant in our sense, while it is not invariant under \mathbb{B}_κ -homotopy in general, see [KST16].

More concretely, we mimic the description of topological K -theory explained in Subsection 1.1 in order to construct a new analytic K -theory of affinoid spaces. Instead of actually performing the limit in (1.2) we work with pro-systems of algebraic K -groups.

Let κ be a complete discretely valued field and let X be an affinoid space over κ . For $1 \leq \rho \in |\kappa^\times|$ let Δ_ρ^m be the standard rigid m -simplex of radius ρ , i.e.

$$\Delta_\rho^m = \text{Sp } \kappa\langle X_0, \dots, X_m \rangle_\rho / (X_0 + \dots + X_m - 1)$$

where $\kappa\langle X_0, \dots, X_m \rangle_\rho$ is the Tate algebra of formal power series which converge on a closed polydisc of radius ρ .

By varying $\rho \geq 1$ we obtain an ind-object $\rho \mapsto \Delta_\rho$ in the category of cosimplicial affinoid spaces over κ . We suggest to define connective analytic K -theory of the affinoid space X as the pro-spectrum

$$k^{\text{an}}(X) = \text{“lim”}_\rho k(\mathcal{O}(X \times \Delta_\rho)).$$

Here $\mathcal{O}(Y)$ denotes the ring of rigid analytic functions of an affinoid space Y . There exists also a Karoubi–Villamayor analog of $k^{\text{an}}(X)$ defined as the pro-simplicial set

$$KV^{\text{an}}(X) = \text{“}\lim_{\rho}\text{” BGL}(\mathcal{O}(X \times \Delta_{\rho})).$$

For $X = \text{Sp } A$ we also write $k^{\text{an}}(A)$ for $k^{\text{an}}(X)$ and similarly $KV^{\text{an}}(A)$ for $KV^{\text{an}}(X)$. We write $k_i^{\text{an}}(A)$ resp. $KV_i^{\text{an}}(A)$ for the homotopy pro-group π_i of $k^{\text{an}}(A)$ resp. of $KV^{\text{an}}(A)$.

This definition of non-archimedean analytic K -theory has very pleasant properties if A is regular. For technical reasons we also have to assume a certain weak form of resolution of singularities for a ring of definition of A , denoted $(\dagger)_A$, see Subsection 3.2 for details. Condition $(\dagger)_A$ is satisfied for any regular affinoid algebra A if κ has residue characteristic zero by a variant of Hironaka’s resolution of singularities.

Theorem 1.1. *For a regular affinoid algebra A which satisfies condition $(\dagger)_A$ we obtain:*

- (i) $k_0^{\text{an}}(A)$ is isomorphic to the constant pro-group $K_0(A)$.
- (ii) There are isomorphisms of pro-groups

$$k_1^{\text{an}}(A) \cong \text{“}\lim_{0 < r < 1}\text{” } K_1(A)/(1 + A(r)) = \text{“}\lim_{\rho > 1}\text{” } \text{GL}(A)/\text{GL}(A)_{\rho},$$

where $\text{GL}(A)_{\rho}$ is the normal subgroup generated by matrices g satisfying the unipotence condition $\lim_n \|(g - 1)^n\| \rho^n = 0$ and where $A(r) = \{a \in A \mid \|a\| < r\}$ for $r > 0$.

- (iii) For $i > 0$ there is an isomorphism $KV_i^{\text{an}}(A) \xrightarrow{\cong} k_i^{\text{an}}(A)$ of pro-groups.
- (iv) k^{an} satisfies the analytic analog of the Bass fundamental theorem of algebraic K -theory, i.e. for $i > 0$ there is a canonical exact sequence

$$0 \rightarrow k_i^{\text{an}}(A) \rightarrow k_i^{\text{an}}(A\langle t \rangle) \times k_i^{\text{an}}(A\langle t^{-1} \rangle) \rightarrow k_i^{\text{an}}(A\langle t, t^{-1} \rangle) \rightarrow k_{i-1}^{\text{an}}(A) \rightarrow 0$$

together with a canonical splitting of the surjection on the right.

For an affinoid κ -algebra A the choice of a surjection $\kappa\langle t_1, \dots, t_n \rangle_1 \twoheadrightarrow A$ gives rise to a residue norm $\|\cdot\|$ on A . Up to equivalence, it does not depend on the chosen surjection. This is the norm occurring in (ii). The results of Theorem 1.1 are obtained at different places throughout the paper. We collect the references in Section 1.4 below.

Remark 1.2. Note that (i) and (ii) of Theorem 1.1 are analogous to classical properties of topological K -theory, see [Ros05, Sec. 3.2]. We will see in a sequel to this note that (iv) can be interpreted as the Mayer–Vietoris sequence of k^{an} corresponding to the standard admissible covering of the rigid space $(\mathbb{P}_A^1)^{\text{an}}$.

Theorem 1.1(iv) implies that we can use a similar delooping process as in (1.1) to define a non-connective analytic K -theory spectrum $K^{\text{an}}(A)$ such

that $k^{\text{an}}(A)$ is the connective covering of $K^{\text{an}}(A)$ and such that the analytic Bass fundamental theorem holds in general, i.e. the analog of the exact sequence in Theorem 1.1(iv) for K_i^{an} exists for all $i \in \mathbb{Z}$.

1.3. Continuous K -theory. Let A be an affinoid algebra over the complete discretely valued field κ . Let $\kappa_0 \subset \kappa$ be the ring of integers and $\pi \in \kappa_0$ be a prime element. Choose a κ_0 -algebra A_0 flat and topologically of finite type over κ_0 with $A_0[1/\pi] \cong A$, see [Bos14]. Such an A_0 is also called a ring of definition of A . Morrow [Mor16] defines continuous K -theory of A as

$$K^{\text{cont}}(A) = \text{cofib}(K(A_0 \text{ on } (\pi)) \rightarrow \text{“lim”}_n K(A_0/(\pi^n))),$$

where K stands for the non-connective algebraic K -theory spectrum and where the cofibre is taken in the ∞ -category of pro-spectra, recalled in Section 2. Using pro-excision one shows that the pro-group $K_i^{\text{cont}}(A)$ defined as $\pi_i(K^{\text{cont}}(A))$ for $i \in \mathbb{Z}$ does not depend on the choice of the ring of definition A_0 up to canonical isomorphism.

In fact, variants of the pro-group $K_i^{\text{cont}}(A)$ have been studied since the introduction of algebraic K -theory, for example we have:

- (i) The canonical map $K_0(A) \xrightarrow{\cong} K_0^{\text{cont}}(A)$ is an isomorphism of pro-groups, see Proposition 5.10. For $i \leq 0$ the pro-groups $K_i^{\text{cont}}(A)$ are constant and agree with the negative “topological” K -groups studied in [KV71, Sec. 7] and in [Cal85].
- (ii) The canonical map $\text{“lim”}_n K_1(A)/(1 + \pi^n A_0) \xrightarrow{\cong} K_1^{\text{cont}}(A)$ is an isomorphism of pro-groups, see Lemma 5.5. In particular, for a complete discretely valued field κ with ring of integers κ_0 we get $K_1^{\text{cont}}(\kappa) \cong \text{“lim”}_n \kappa^\times / (1 + \pi^n \kappa_0)$.
- (iii) For a complete discretely valued field κ with ring of integers κ_0 the canonical map $\text{“lim”}_n K_2(\kappa)/\{1 + \pi^n \kappa_0, \kappa_0^\times\} \xrightarrow{\cong} K_2^{\text{cont}}(\kappa)$ is an isomorphism of pro-groups, see Lemma 5.6. In particular, if the residue field of κ_0 is finite, the norm-residue symbol induces an isomorphism $K_2^{\text{cont}}(\kappa) \cong \mu \subset \kappa^\times$, where μ are the roots of unity (Moore’s Theorem), see [Mil71, App.].

The key observation of our note is that for regular affinoid algebras continuous K -theory agrees with analytic K -theory. This unifies the two existing approaches to the K -theory of non-archimedean rings which originate from [KV71].

Theorem 1.3. *For a regular affinoid algebra A which satisfies condition $(\dagger)_A$ the pro-spectra $K^{\text{an}}(A)$ and $K^{\text{cont}}(A)$ are weakly equivalent.*

For the definition of weakly equivalent pro-spectra see Subsection 2.2. In particular, this means that the pro-groups $K_i^{\text{an}}(A)$ and $K_i^{\text{cont}}(A)$ are naturally isomorphic. This theorem is proven as Theorem 6.18 below.

1.4. Overview of sections. In Section 2 we collect various facts about pro-objects in ∞ -categories, and in particular about pro-spectra and pro-spaces. Most of our results are formulated for adic rings or Tate rings — we use affinoid algebras in the introduction only for simplicity of presentation. Basic properties of these non-archimedean rings are discussed in Section 3. There we also introduce normed rings which are used in the study of the Karoubi-Villamayor version of our theory in Section 7. In Section 4 we discuss some general preliminaries about algebraic K -theory and we study the low degree algebraic K -groups of Tate rings. Continuous K -theory is introduced in Section 5. Beside the basic properties mentioned in 1.3 above, we also prove an analytic version of the Bass fundamental theorem and study homotopy invariance of continuous K -theory. In Section 6 we introduce the analytic K -theory discussed in 1.2. The main result there is the comparison with continuous K -theory for a regular affinoid algebra A satisfying $(\dagger)_A$.

We end this overview with the *proof of Theorem 1.1*. Assertion (i) is Lemma 6.13(ii). By Theorem 1.3 we can identify analytic K -theory with continuous K -theory. Using this identification, Lemma 5.5 gives the first isomorphism of part (ii) and Proposition 5.10(i) gives (iv). Assertion (iii) is precisely Lemma 7.5. Now Lemma 7.3 gives the second isomorphism in (ii). \square

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2. TOPOLOGICAL PRELIMINARIES

In the study of K -theory of non-archimedean rings, pro-spectra play an important role. The theory of ∞ -categories provides a well developed and flexible framework to deal with homotopy theoretic questions about pro-spectra. In the following, we therefore collect some facts about pro-objects in ∞ -categories in general and about pro-spectra. General references for pro-objects are [AM69] or [Isa02]. The ∞ -categorical version is discussed for example in [Lur17b, §A.8.1]. We write \mathbf{Spc} for the ∞ -category of spaces.

2.1. Pro-objects in ∞ -categories. Let \mathbf{C} be an ∞ -category. For technical reasons we assume that \mathbf{C} is accessible and admits finite limits. A pro-object in \mathbf{C} is represented by a diagram $X: I \rightarrow \mathbf{C}$ where I is a small cofiltered ∞ -category. We also use the notation $X = \text{“}\lim\limits_J X_i\text{”}$ and call this a level-representation of X . The pro-objects are the objects of an ∞ -category $\text{Pro}(\mathbf{C})$. If $Y = \text{“}\lim\limits_J Y_j\text{”}$ is a second pro-object, the mapping space is given by the classical formula

$$(2.1) \quad \text{Map}\left(\text{“}\lim\limits_I X_i\text{”}, \text{“}\lim\limits_J Y_j\text{”}\right) \simeq \lim\limits_J \text{colim}\limits_I \text{Map}(X_i, Y_j),$$

where the limit and colimit are computed in the ∞ -category of spaces. A precise definition is

$$(2.2) \quad \mathbf{Pro}(\mathbf{C}) = \mathbf{Fun}^{\text{lex,acc}}(\mathbf{C}, \mathbf{Spc})^{\text{op}}$$

where $\mathbf{Fun}^{\text{lex,acc}}(\mathbf{C}, \mathbf{Spc})$ is the full subcategory of $\mathbf{Fun}(\mathbf{C}, \mathbf{Spc})$ spanned by those functors which are accessible (i.e. preserve small κ -filtered colimits for some regular cardinal κ) and left exact (i.e. commute with finite limits). Here the pro-object X from above corresponds to the functor $\text{colim}_I \text{Map}(X_i, -): \mathbf{C} \rightarrow \mathbf{Spc}$, see [Lur17b, Def. A.8.1.1, Rem. A.8.1.5].

Any functor $f: \mathbf{C} \rightarrow \mathbf{D}$ induces a functor $\mathbf{Pro}(f): \mathbf{Pro}(\mathbf{C}) \rightarrow \mathbf{Pro}(\mathbf{D})$ given by “ \lim_I ” $X_i \mapsto$ “ \lim_I ” $f(X_i)$, see [Lur17b, Ex. A.8.1.8].

Finite limits and colimits in $\mathbf{Pro}(\mathbf{C})$ can be computed level-wise: Let K be a finite simplicial set. The functor $\lim_K: \mathbf{Fun}(K, \mathbf{C}) \rightarrow \mathbf{C}$ induces a functor $\mathbf{Pro}(\mathbf{Fun}(K, \mathbf{C})) \rightarrow \mathbf{Pro}(\mathbf{C})$, the ‘level-wise limit’. If \mathbf{C} admits finite colimits, we similarly have a level-wise colimit. Any pro-object of K -diagrams induces a K -diagram of pro-objects via the natural functor $\mathbf{Pro}(\mathbf{Fun}(K, \mathbf{C})) \rightarrow \mathbf{Fun}(K, \mathbf{Pro}(\mathbf{C}))$.

Lemma 2.1. *Let $F \in \mathbf{Pro}(\mathbf{Fun}(K, \mathbf{C}))$. Then the level-wise limit of F is a limit of the induced diagram $K \rightarrow \mathbf{Pro}(\mathbf{C})$. If \mathbf{C} admits finite colimits, the same is true for the colimit.*

Proof. This can be checked as in the 1-categorical case (see [AM69, App., Prop. 4.1]) using formula (2.1) and the fact that in \mathbf{Spc} finite limits commute with filtered colimits [Lur09, Prop. 5.3.3.3]. \square

We also have level-representations for certain diagrams. In the following Lemma $\mathbf{N}(A)$ denotes the nerve of a partially ordered set A viewed as category.

Lemma 2.2. *Let A be a finite partially ordered set. The natural functor*

$$\mathbf{Pro}(\mathbf{Fun}(\mathbf{N}(A), \mathbf{C})) \rightarrow \mathbf{Fun}(\mathbf{N}(A), \mathbf{Pro}(\mathbf{C}))$$

is essentially surjective.

Proof. If \mathbf{C} is small, this follows directly from [Lur09, Prop. 5.3.5.15]. The general case can be reduced to this as in the proof of [Lur17b, Prop. A.8.1.6]. \square

Lemma 2.3. *The ∞ -category $\mathbf{Pro}(\mathbf{C})$ is complete. If \mathbf{C} admits finite resp. all small colimits, then so does $\mathbf{Pro}(\mathbf{C})$.*

Proof. Lemma 2.1 implies in particular that $\mathbf{C} \rightarrow \mathbf{Pro}(\mathbf{C})$ commutes with finite limits and colimits. In particular, $\mathbf{Pro}(\mathbf{C})$ has a final object. Combining Lemmas 2.2 and 2.1 we see that $\mathbf{Pro}(\mathbf{C})$ admits pullbacks. Now [Lur09, Cor. 4.4.2.4] implies that $\mathbf{Pro}(\mathbf{C})$ has all finite limits. If \mathbf{C} admits finite colimits, then by the same argument $\mathbf{Pro}(\mathbf{C})$ has all finite colimits. Since $\mathbf{Pro}(\mathbf{C})$ also has all small cofiltered limits, it follows that $\mathbf{Pro}(\mathbf{C})$ is complete by [Lur09, Prop. 4.4.2.6]. In order to show that $\mathbf{Pro}(\mathbf{C})$ is cocomplete

if \mathbf{C} is, it now suffices to show that $\text{Pro}(\mathbf{C})$ has small coproducts [Lur09, Prop. 4.4.2.6]. This is shown in Lemma 2.4 below. \square

Recall that an accessible ∞ -category [Lur09, Def. 5.4.2.1] which is cocomplete is called presentable [Lur09, Def. 5.5.0.1]. A presentable ∞ -category is automatically complete [Lur09, Cor. 5.5.2.4].

Lemma 2.4. *Assume that \mathbf{C} is presentable. Let A be a set, and let $(X^\alpha)_{\alpha \in A}$ be a family of objects in $\text{Pro}(\mathbf{C})$. Choose level representations*

$$X^\alpha = \text{“lim”}_{I_\alpha} X_i^\alpha.$$

Define the cofiltered ∞ -category $K = \prod_{\alpha \in A} I_\alpha$. Then

$$X = \text{“lim”}_{k \in K} \prod_{\alpha \in A} X_{k_\alpha}^\alpha$$

is a coproduct of $(X^\alpha)_{\alpha \in A}$.

Proof. (See [Isa02, §8] for the classical situation.) Let $Y = \text{“lim”}_J Y_j$ be any pro-object. Then

$$\begin{aligned} \text{Map}(X, Y) &\simeq \lim_J \text{colim}_K \prod_A \text{Map}(X_{k_\alpha}^\alpha, Y_j) \\ &\simeq \lim_J \prod_{\alpha \in A} \text{colim}_{I_\alpha} \text{Map}(X_{k_\alpha}^\alpha, Y_j) && \text{(see below)} \\ &\simeq \prod_{\alpha \in A} \lim_J \text{colim}_{I_\alpha} \text{Map}(X_{k_\alpha}^\alpha, Y_j) \\ &\simeq \prod_{\alpha \in A} \text{Map}(X^\alpha, Y). \end{aligned}$$

The second equivalence holds since $\text{colim}_K \prod_A \simeq \prod_{\alpha \in A} \text{colim}_{I_\alpha}$ in the ∞ -category of spaces. Indeed, since products and filtered colimits in spaces commute with homotopy groups, this follows from the corresponding fact in the category of sets, where it can be checked directly. \square

Lemma 2.5. *If \mathbf{C} is stable, then so is $\text{Pro}(\mathbf{C})$.*

For the notion of a stable ∞ -category see [Lur17a, Def. 1.1.1.9].

Proof. By Lemma 2.1 $\text{Pro}(\mathbf{C})$ is pointed. By Lemma 2.3 it has all finite limits and colimits. Since by Lemma 2.1 again the suspension functor in $\text{Pro}(\mathbf{C})$ can be computed level-wise, it is an equivalence. We conclude using [Lur17a, Cor. 1.4.2.27]. \square

2.2. Weak equivalences of pro-spectra. We let \mathbf{Sp} be the presentable stable ∞ -category of spectra [Lur17a, §1.4.3]. It carries a t-structure whose heart is equivalent to the category \mathbf{Ab} of abelian groups. In particular, an abelian group A gives a spectrum $H(A)$ with $\pi_0(H(A)) \cong A$, $\pi_i(H(A)) = 0$ for $i \neq 0$.

We denote by $\mathbf{Sp}^+ \subseteq \mathbf{Sp}$ the full stable subcategory of spectra which are bounded above, and denote the inclusion by ι . By [Lur17b, Ex. A.8.1.8] we get an adjunction

$$\iota^* : \mathrm{Pro}(\mathbf{Sp}) \rightleftarrows \mathrm{Pro}(\mathbf{Sp}^+) : \mathrm{Pro}(\iota)$$

where ι^* is induced using (2.2) by composition with ι , and $\mathrm{Pro}(\iota)$ is fully faithful. In other words, $\mathrm{Pro}(\mathbf{Sp}^+)$ is a localization of $\mathrm{Pro}(\mathbf{Sp})$ in the sense of [Lur09, Def. 5.2.7.2]. In particular, $\mathrm{Pro}(\mathbf{Sp}^+)$ also admits all small colimits.

In terms of level representations ι^* has the following description: Denote by $\tau_{\leq n} : \mathbf{Sp} \rightarrow \mathbf{Sp}_{\leq n}$ the truncation functor. It is left adjoint to the inclusion.

Lemma 2.6. *Let “ \lim_j ” $X_i \in \mathrm{Pro}(\mathbf{Sp})$. Then we have a natural equivalence*

$$\iota^* (“\lim_j” X_i) \simeq “\lim_{\mathbb{N} \times I}” \tau_{\leq n} X_i$$

in $\mathrm{Pro}(\mathbf{Sp}^+)$.

Proof. In terms of left exact accessible functors $\iota^* (“\lim_j” X_i)$ is given by $\mathrm{colim}_I \mathrm{Map}(X_i, -) \in \mathrm{Fun}^{\mathrm{lex}, \mathrm{acc}}(\mathbf{Sp}^+, \mathbf{Spc})$ and this colimit can be computed object-wise (see [Lur09, Cor. 5.1.2.3]). Consider $Y \in \mathbf{Sp}^+$. Note that if $Y \in \mathbf{Sp}_{\leq m}$ then $\mathrm{Map}(\tau_{\leq n} X_i, Y) \simeq \mathrm{Map}(X_i, Y)$ for all $n \geq m$. Hence

$$\mathrm{colim}_I \mathrm{Map}(X_i, Y) \simeq \mathrm{colim}_{\mathbb{N} \times I} \mathrm{Map}(\tau_{\leq n} X_i, Y),$$

which implies the claim. \square

Definition 2.7. We call a map of pro-spectra $X \rightarrow Y$ a *weak equivalence* if $\iota^* X \rightarrow \iota^* Y$ is an equivalence in $\mathrm{Pro}(\mathbf{Sp}^+)$. We say that the pro-spectra X, Y are *weakly equivalent* if there is an equivalence of $\iota^* X$ and $\iota^* Y$ in $\mathrm{Pro}(\mathbf{Sp}^+)$.

We say that X is *weakly contractible* if it is weakly equivalent to a point. Finally, we call $X \rightarrow Y \rightarrow Z$ a *weak fibre sequence* if $\iota^* X \rightarrow \iota^* Y \rightarrow \iota^* Z$ is a fibre sequence in $\mathrm{Pro}(\mathbf{Sp}^+)$.

The functor $\pi_i : \mathbf{Sp} \rightarrow \mathbf{Ab}$ induces functors

$$\pi_i : \mathrm{Pro}(\mathbf{Sp}) \rightarrow \mathrm{Pro}(\mathbf{Ab}) \quad \text{and} \quad \pi_i : \mathrm{Pro}(\mathbf{Sp}^+) \rightarrow \mathrm{Pro}(\mathbf{Ab}),$$

given by $X = “\lim_j” X_j \mapsto \pi_i(X) = “\lim_j” \pi_i(X_j)$, called the *i -th homotopy pro-group*. Abusing notation we write $\tau_{\leq n} : \mathrm{Pro}(\mathbf{Sp}) \rightarrow \mathrm{Pro}(\mathbf{Sp})$ for the functor $\mathrm{Pro}(\tau_{\leq n})$.

Lemma 2.8. *For a map $f : X \rightarrow Y$ in $\mathrm{Pro}(\mathbf{Sp})$ the following are equivalent:*

- (1) *f is a weak equivalence.*
- (2) *$\tau_{\leq n} f$ is an equivalence for every $n \in \mathbb{Z}$.*
- (3) *There exists an $m \in \mathbb{Z}$ such that $\tau_{\leq m} f$ is an equivalence and f induces isomorphisms $\pi_n(X) \xrightarrow{\sim} \pi_n(Y)$ in $\mathrm{Pro}(\mathbf{Ab})$ for all $n \in \mathbb{Z}$.*

Proof. Since the functor $\tau_{\leq n}$ factors through $\iota^* : \mathrm{Pro}(\mathbf{Sp}) \rightarrow \mathrm{Pro}(\mathbf{Sp}^+)$, (1) implies (2). Clearly, (2) implies (3). It remains to show that (3) implies

(1). Choose a level representation “ $\lim_I X_i \rightarrow \lim_I Y_i$ ”. By assumption “ $\lim_I \tau_{\leq m} X_i \rightarrow \lim_I \tau_{\leq m} Y_i$ ” is an equivalence. Using the fibre sequence

$$\Sigma^n H(\pi_n(X_i)) \rightarrow \tau_{\leq n} X_i \rightarrow \tau_{\leq n-1} X_i$$

(similarly for Y_i) and that “ $\lim_I \pi_n(X_i) \rightarrow \lim_I \pi_n(Y_i)$ ” is an isomorphism, we inductively show that we have equivalences “ $\lim_I \tau_{\leq n} X_i \xrightarrow{\cong} \lim_I \tau_{\leq n} Y_i$ ” for all $n \in \mathbb{Z}$. Lemma 2.6 now implies that f is a weak equivalence. \square

2.3. Pro-spaces. Let \mathbf{Spc}_*^0 denote the ∞ -category of pointed connected spaces.

Definition 2.9. A morphism $X \rightarrow Y$ in $\text{Pro}(\mathbf{Spc}_*^0)$ is called a *weak equivalence* if it induces an isomorphism between homotopy pro-groups $\pi_i(X) \rightarrow \pi_i(Y)$ for every $i > 0$.

Isaksen [Isa01] has a general notion of weak equivalences between pro-simplicial sets. There is a canonical functor from the ∞ -category associated to the category of pro-objects in pointed connected simplicial sets to $\text{Pro}(\mathbf{Spc}_*^0)$ and this functor detects weak equivalences by [Isa01, Cor. 7.5].

Let $f: X \rightarrow Y$ be a morphism between pro-objects of connected spectra and $\Omega^\infty f: \Omega^\infty X \rightarrow \Omega^\infty Y$ the induced map on pro-objects of infinite loop spaces. Then by Lemma 2.8 f is a weak equivalence in the sense of Definition 2.7 if and only if $\Omega^\infty f$ is a weak equivalence in $\text{Pro}(\mathbf{Spc}_*^0)$.

Lemma 2.10. *If $X \rightarrow Y$ is a weak equivalence in $\text{Pro}(\mathbf{Spc}_*^0)$ then the induced map on integral homology pro-groups $H_i(X) \rightarrow H_i(Y)$ is an isomorphism in $\text{Pro}(\mathbf{Ab})$.*

This can be deduced from [Isa01, Thm. 7.3].

The following version of Whitehead’s theorem is only stated for pro-systems indexed by a countable category. This suffices for our purposes.

Proposition 2.11. *Let $f: X \rightarrow Y$ be a morphism of pro-systems of pointed connected nilpotent spaces which can be indexed by a countable category. If the induced map $f_*: H_i(X) \rightarrow H_i(Y)$ is an isomorphism in $\text{Pro}(\mathbf{Ab})$ for every $i > 0$, then f is a weak equivalence in $\text{Pro}(\mathbf{Spc}_*^0)$.*

This can be deduced from [BK72, Prop. III.6.6, Cor. III.5.4(ii)].

3. NON-ARCHIMEDEAN RINGS

3.1. Adic rings. In this note we call a complete topological ring A an *adic ring* if a system of neighborhoods of zero is given by the ideals $\{(\pi^n) \mid n > 0\}$ for some element $\pi \in A$. For an element $\pi \in A$ with this property we also say that A is π -adic.

For such an adic ring A and element $\pi \in A$ we denote by

$$A\langle t_1, \dots, t_m \rangle = \lim_n A[t_1, \dots, t_m]/(\pi^n)$$

the completion of the polynomial ring. Fix $a_0 \in A$. By

$$(3.1) \quad A\langle \Delta_{a_0}^m \rangle = A\langle t_0, \dots, t_m \rangle / (t_0 + \dots + t_m - a_0)$$

we denote the standard m -simplex of ‘diameter a_0 ’. So $A\langle \Delta_{a_0}^m \rangle$ is the π -adic completion of

$$(3.2) \quad A[\Delta_{a_0}^m] = A[t_0, \dots, t_m] / (t_0 + \dots + t_m - a_0).$$

Note that there exists an A -algebra isomorphism

$$A\langle x_1, \dots, x_m \rangle \xrightarrow{\cong} A\langle \Delta_{a_0}^m \rangle \quad x_i \mapsto t_i.$$

For varying m and fixed a_0 the $A\langle \Delta_{a_0}^m \rangle$ form in the usual way a simplicial ring which we denote by $A\langle \Delta_{a_0} \rangle$. If we choose another element $a_1 \in A$ we get a morphism of simplicial A -algebras

$$\Psi_{a_1}^{a_0}: A\langle \Delta_{a_1 a_0} \rangle \rightarrow A\langle \Delta_{a_0} \rangle$$

by mapping each variable t_i to $a_1 t_i$. The same applies to the polynomial rings $A[\Delta_{a_0}^m]$.

In the following proposition we summarize some facts about excellent rings. Let κ be a complete discretely valued field with ring of integers κ_0 . We abbreviate $A\langle t_1, \dots, t_m \rangle$ by $A\langle \underline{t} \rangle$.

Proposition 3.1. *For a noetherian complete adic ring A we have:*

- (i) *For an element $\pi \in A$ as above the ring A is quasi-excellent if and only if $A/(\pi)$ is quasi-excellent*
- (ii) *If A is quasi-excellent, then $A\langle \underline{t} \rangle$ is quasi-excellent, and the ring homomorphism $A \rightarrow A\langle \underline{t} \rangle$ is regular.*
- (iii) *If A is a κ_0 -algebra of topologically finite type, it is excellent.*

Proof. Part (i) is due to Gabber, see [ILO14, Thm. I.9.2]. Part (ii) follows immediately from part (i) and [Sta17, 0AH2]. For part (iii) note that a κ_0 -algebra A which is topologically of finite type is a quotient of a ring $\kappa_0\langle \underline{t} \rangle$, which is quasi-excellent (by (ii)) and which is universally catenary as it is regular (again by (ii)). \square

3.2. Tate rings. For the convenience of the reader we recall some facts about Tate rings in this section, see [Hub93, Sec. 1]. Tate rings are generalizations of affinoid algebras and we are going to state most of our results about K -theory in the language of Tate rings.

Definition 3.2. A topological ring A is called a *Tate ring* if there exists an open subring $A_0 \subset A$ which is a complete adic ring in the sense of Subsection 3.1, i.e. it has the π -adic topology for some $\pi \in A_0$ and $A = A_0[1/\pi]$. We call such a subring A_0 a *ring of definition* of A .

An element π as in Definition 3.2 is clearly invertible and topologically nilpotent in A .

Remark 3.3. The following observations about a Tate ring A are easy to prove.

- (i) For an invertible and topologically nilpotent $\pi \in A$ and a ring of definition $A_0 \subset A$ there exists $n > 0$ such that $\pi^n \in A_0$ and A_0 has the π^n -adic topology.
- (ii) If A_0 and A'_0 are both rings of definition of the Tate ring A then $A_0 \cap A'_0$ and $A_0 \cdot A'_0$ are also rings of definition of A .
- (iii) For a power bounded element $a_0 \in A$ there exists a ring of definition $A_0 \subset A$ with $a_0 \in A_0$.

Quite often in this note we need to assume the existence of a ‘nice’ ring of definition of a Tate ring A . We usually consider the following condition on A :

- (\dagger) $_A$ There exists a ring of definition $A_0 \subset A$ which is noetherian and there exists a proper morphism $p: X \rightarrow \text{Spec}(A_0)$ such that X is regular and such that p is an isomorphism over $\text{Spec}(A)$.

There is also the following weaker version of (\dagger) $_A$:

- (\ddagger) $_A$ There exists a ring of definition $A_0 \subset A$ which is noetherian and quasi-excellent.

Remark 3.4. By [Gro67, Sec. IV.7.9] condition (\dagger) $_A$ implies condition (\ddagger) $_A$. If A is regular and if one has resolution of singularities available, for example if $\mathbb{Q} \subset A_0$ [Tem08], then condition (\ddagger) $_A$ conversely implies condition (\dagger) $_A$.

We define the Tate algebra over the Tate ring A by

$$A\langle t_1, \dots, t_m \rangle = (A_0\langle t_1, \dots, t_m \rangle) \otimes_{A_0} A.$$

It is easy to see that the Tate algebra does not depend on the choice of the ring of definition A_0 . For a power bounded element $a_0 \in A$ we define the standard m -simplex with ‘diameter a_0 ’ by the same formula as in (3.1), as well as its polynomial version (3.2). Given another power bounded $a_1 \in A$ there exists a well-defined map

$$\Psi_{a_1}^{a_0}: A\langle \Delta_{a_1 a_0} \rangle \rightarrow A\langle \Delta_{a_0} \rangle$$

mapping each variable t_i to $a_1 t_i$. In these constructions one uses Remark 3.3(iii) to reduce to the analogous constructions for adic rings.

Lemma 3.5. *Consider a Tate ring A which satisfies condition (\dagger) $_A$ (resp. (\ddagger) $_A$). Then $A\langle \underline{t} \rangle$ satisfies condition (\dagger) $_{A\langle \underline{t} \rangle}$ (resp. (\ddagger) $_{A\langle \underline{t} \rangle}$). Moreover the ring homomorphism $A \rightarrow A\langle \underline{t} \rangle$ is regular.*

Here we abbreviate $A\langle t_1, \dots, t_m \rangle$ by $A\langle \underline{t} \rangle$ again.

Proof. This is immediately clear from Proposition 3.1. Note that in the situation of (\dagger) $_A$ the scheme $X \otimes_{A_0} A_0\langle \underline{t} \rangle$ is regular as the ring homomorphism $A_0 \rightarrow A_0\langle \underline{t} \rangle$ is regular by Proposition 3.1(ii) and X is regular. \square

Lemma 3.6. *If A is an affinoid algebra over a complete discretely valued field κ , then condition (\ddagger) $_A$ holds. If moreover A is regular and the residue field of κ has characteristic zero, then condition (\dagger) $_A$ holds.*

Proof. For an affinoid algebra A over κ choose a ring A_0 which is flat and topologically of finite type over κ_0 . Then A_0 is a ring of definition for A and it is excellent by Proposition 3.1(iii). For $\mathbb{Q} \subset A_0$ we know that the scheme $\text{Spec}(A_0)$ has a resolution of singularities [Tem08]. \square

3.3. Normed rings. When we study the Karoubi–Villamayor analog of our analytic K -theory in Section 7 it is convenient to do it for a more general class of non-archimedean rings than Tate rings, namely normed rings. This material is more or less well-known, but for the reader’s convenience we recall some details in this subsection.

Definition 3.7. Let A be an associative (not necessarily commutative or unital) ring. A *non-archimedean norm* on A is a map $\|\cdot\|: A \rightarrow \mathbb{R}_{\geq 0}$ satisfying

- (1) $\|x\| = 0 \Leftrightarrow x = 0$,
- (2) $\|-x\| = \|x\|$,
- (3) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$,
- (4) $\|xy\| \leq \|x\|\|y\|$,

for all $x, y \in A$.

In the following, all norms will be non-archimedean, and for simplicity we will usually drop the adjective ‘non-archimedean’. A ring equipped with a norm is called a *normed ring*. The normed ring $(A, \|\cdot\|)$ is called *complete* if A is complete for the metric induced by the norm, and it is called *unital* if A is unital and $\|1\| = 1$.

Example 3.8. (i) Any associative ring A can be equipped with the trivial norm $\|\cdot\|$ given by $\|a\| = 1$ if $a \neq 0$, $\|0\| = 0$.

(ii) If $(A, \|\cdot\|)$ is a normed ring, one can equip its unitalization $\tilde{A} = A \rtimes \mathbb{Z}$ with the norm

$$\|(a, n)\| = \max\{\|a\|, \|n\|_{\mathbb{Z}}\}$$

where $\|\cdot\|_{\mathbb{Z}}$ is the trivial norm on \mathbb{Z} . Then $(\tilde{A}, \|\cdot\|)$ is a unital normed ring and the inclusion $A \hookrightarrow \tilde{A}$ is isometric. It is complete if A was.

(iii) Let A be a Tate ring with a ring of definition $A_0 \subset A$ and an element $\pi \in A_0$ which is a topologically nilpotent unit in A . Fix $0 < \epsilon < 1$. We define the gauge norm associated to A_0, π , and ϵ by

$$\|a\|_{A_0, \pi, \epsilon} = \epsilon^{\max\{i \in \mathbb{Z} \mid a \in \pi^i A_0\}}.$$

A different choice of A_0, π , and ϵ gives an equivalent norm in the sense defined below.

Definition 3.9. A homomorphism between normed rings $\phi: (A, \|\cdot\|_A) \rightarrow (B, \|\cdot\|_B)$ is called *s-bounded* for some real number $s > 0$ if there is a $C > 0$ such that $\|\phi(x)\|_B \leq C\|x\|_A^s$ for all $x \in A$. It is called *bounded* if it is *s-bounded* for some s .

Two norms $\|\cdot\|, \|\cdot\|'$ on a ring A are called *equivalent* if the identities $(A, \|\cdot\|) \rightarrow (A, \|\cdot\|')$ and $(A, \|\cdot\|') \rightarrow (A, \|\cdot\|)$ are bounded.

Let $(A, \|\cdot\|)$ be a complete normed ring, and let $\rho > 0$ be a real number. We define the ring of ρ -convergent power series with variables t_1, \dots, t_m and with coefficients in A

$$A\langle t_1, \dots, t_m \rangle_\rho = \left\{ \sum_I a_I t^I \mid \|a_I\| \rho^{|I|} \xrightarrow{|I| \rightarrow \infty} 0 \right\},$$

where we use standard multi-index notation. The variables commute with every element of the ring. We set

$$\left\| \sum_I a_I t^I \right\|_\rho = \max\{\|a_I\| \rho^{|I|}\}.$$

This defines a norm $\|\cdot\|_\rho$ on $A\langle t \rangle_\rho$ and, equipped with this norm, $A\langle t \rangle_\rho$ becomes again a complete normed ring. (See [BGR84, Ch. 1, 1.4.1] for the case of a commutative unital complete normed ring and radius of convergence $\rho = 1$. Everything carries over to our situation *mutatis mutandis*.)

Let $(A, \|\cdot\|)$ be a complete normed ring and let $\rho \geq 1$. We define

$$A\langle \Delta_\rho^m \rangle = A\langle t_0, \dots, t_m \rangle_\rho / I_{n,\rho}$$

where $I_{n,\rho}$ is the twosided ideal

$$I_{n,\rho} = \{t_0 f + \dots + t_m f - f \mid f \in A\langle t_0, \dots, t_n \rangle_\rho\}.$$

It is then clear that $A\langle \Delta_\rho \rangle$ with the usual face and degeneracy maps is a simplicial ring. For radii $\rho' \geq \rho \geq 1$ there is a natural restriction map $A\langle \Delta_{\rho'} \rangle \rightarrow A\langle \Delta_\rho \rangle$ of simplicial A -algebras.

Remark 3.10. For a Tate ring A as in Example 3.8(iii) and for $\rho = \epsilon^{-j}$ with $j \geq 0$ there exists an isomorphism of simplicial A -algebras

$$\Theta: A\langle \Delta_{\pi^j} \rangle \xrightarrow{\simeq} A\langle \Delta_\rho \rangle, \quad t_i \mapsto \pi^j t_i.$$

For $\rho' = \epsilon^{-j-1}$ we obtain a commutative square of simplicial rings

$$\begin{array}{ccc} A\langle \Delta_{\pi^{j+1}} \rangle & \xrightarrow{\Psi_{\pi^j}} & A\langle \Delta_{\pi^j} \rangle \\ \Theta \Big\downarrow \simeq & & \simeq \Big\downarrow \Theta \\ A\langle \Delta_{\rho'} \rangle & \longrightarrow & A\langle \Delta_\rho \rangle. \end{array}$$

4. ALGEBRAIC K -THEORY OF NON-ARCHIMEDEAN RINGS

In this section we summarize some background material on algebraic K -theory and we study algebraic K -theory of non-archimedean rings.

4.1. K -theory spectra and the plus-construction. Let R be a unital ring. We denote by

$$k(R) \in \mathbf{Sp}_{\geq 0}$$

the connective algebraic K -theory spectrum associated to the exact category of finitely generated projective (right) R -modules. Later we will also use

the non-connective algebraic K -theory spectrum $K(R)$. Its connected cover $\tau_{\geq 1}k(R)$ is denoted by $K^{\text{GL}}(R)$. We have a natural equivalence

$$\text{BGL}(R)^+ \simeq \Omega^\infty K^{\text{GL}}(R)$$

where the left-hand side is Quillen's plus construction.

Now let R be a simplicial ring.

Definition 4.1. We write $k(R)$ for the *degree-wise K -theory* of R , i.e. the geometric realization of the simplicial spectrum $[p] \mapsto k(R_p)$. Similarly, we write $K^{\text{GL}}(R)$ for the geometric realization of the simplicial spectrum $[p] \mapsto K^{\text{GL}}(R_p)$.

Note that in general $K^{\text{GL}}(R)$ is not the connected cover of $k(R)$. The infinite loop space of $K^{\text{GL}}(R)$ can again be described as a plus construction. We write $\text{BGL}(R)$ for the diagonal of the bisimplicial set $[p] \mapsto \text{BGL}(R_p)$. Note that in general the diagonal is weakly equivalent to the geometric realization of a bisimplicial set. Let $(-)^+ : \mathbf{sSet} \rightarrow \mathbf{sSet}$ be the functorial plus construction as defined in [DGM13, Def. 3.1.1.5].

Lemma 4.2. *There are natural weak equivalences*

$$\text{BGL}(R)^+ \simeq \text{diag}([p] \mapsto \text{BGL}(R_p)^+)$$

and

$$\text{BGL}(R)^+ \simeq \Omega^\infty K^{\text{GL}}(R).$$

Proof. The first equivalence follows from [DGM13, Lemma 3.1.3.1]. It implies the second one, since geometric realization commutes with Ω^∞ for connective spectra [Lur17a, Prop. 1.4.3.9]. \square

This can be simplified even more for connected simplicial rings:

Lemma 4.3. *Assume that the simplicial ring R is connected. Then there is a natural weak equivalence*

$$\text{BGL}(R) \simeq \Omega^\infty K^{\text{GL}}(R).$$

Proof. Since R is connected, every elementary matrix in $\text{GL}(R)$ can be connected with the identity matrix by a 1-simplex in $\text{GL}(R)$. This implies that $\pi_1 \text{BGL}(R) \cong \pi_0 \text{GL}(R)$ is abelian, and hence $\text{BGL}(R) \rightarrow \text{BGL}(R)^+$ is a weak equivalence. Now the claim follows from Lemma 4.2. \square

4.2. Pro- \mathbb{A}^1 -homotopy invariance for algebraic K -theory. The observations of this subsection resulted from a discussion with M. Morrow. Let R be a commutative noetherian ring and let $\pi \in R$. Let \mathbf{Sch}_R be the category of schemes of finite type over R . In this subsection we study the K -theoretic properties of the pro-system of R -algebras “ $\lim_{n \in \mathbb{N}} S_n$ with S_n the polynomial ring $R[t] = R[t_1, \dots, t_m]$ for all $n \in \mathbb{N}$ and with transition maps $S_{n+1} \rightarrow S_n$ characterized by $t_i \mapsto \pi t_i$ ($1 \leq i \leq m$). We shall abbreviate this pro-system also as “ $\lim_{t \rightarrow \pi t} R[t]$ ”.

Let F be a contravariant functor from \mathbf{Sch}_R to $\text{Pro}(\mathbf{Sp})$. We define a new functor $N^\pi F$ from \mathbf{Sch}_R to $\text{Pro}(\mathbf{Sp})$ by

$$(4.1) \quad N^\pi F(X) = \text{cofib}(F(X) \xrightarrow{p^*} \text{“lim”}_{t \rightarrow \pi t} F(X \otimes_R R[t])),$$

where $p : X \otimes_R R[t] \rightarrow X$ is the canonical map. Note that this p^* has a left inverse given by the zero section.

We are interested in criteria, which guarantee that $N^\pi F(X)$ is contractible resp. weakly contractible, see Definition 2.7. For X in \mathbf{Sch}_R we write X_n for the scheme $X \otimes_R R/(\pi^n)$.

Lemma 4.4. *Let X be in \mathbf{Sch}_R such that there exists $n > 0$ with $X = X_n$. Then $N^\pi F(X)$ is contractible.*

Proof. This is clear as the map $X \otimes_R R[t]$ given by $t \mapsto \pi^n t$ factors through X . \square

A morphism $f : \tilde{X} \rightarrow X$ in \mathbf{Sch}_R is called *admissible* if f is proper and an isomorphism over $X \otimes_R R[1/\pi]$. Consider the following conditions on the functor F :

(P1) $_F$ For any admissible morphism $f : \tilde{X} \rightarrow X$ the functor F satisfies pro-descent, i.e. we have a weak equivalence

$$\text{“lim”}_n F(X, X_n) \rightarrow \text{“lim”}_n F(\tilde{X}, \tilde{X}_n).$$

(P2) $_F$ For any regular scheme X in \mathbf{Sch}_R the pro-spectrum $N^\pi F(X)$ is weakly contractible.

Here for a morphism of schemes $Y' \rightarrow Y$ in \mathbf{Sch}_R we use the notation

$$F(Y, Y') = \text{fib}(F(Y) \rightarrow F(Y')).$$

Lemma 4.5. *Let F be a functor satisfying condition (P1) $_F$ and let $p : \tilde{X} \rightarrow X$ be an admissible morphism in \mathbf{Sch}_R . Then p induces a weak equivalence*

$$N^\pi F(X) \xrightarrow{\simeq} N^\pi F(\tilde{X}).$$

Proof. Consider the commutative diagram of fibre sequences of pro-spectra

$$\begin{array}{ccccc} \text{“lim”}_n N^\pi F(X, X_n) & \xrightarrow{\simeq} & N^\pi F(X) & \longrightarrow & \text{“lim”}_n N^\pi F(X_n) \\ \simeq \downarrow & & \downarrow & & \downarrow \\ \text{“lim”}_n N^\pi F(\tilde{X}, \tilde{X}_n) & \xrightarrow{\simeq} & N^\pi F(\tilde{X}) & \longrightarrow & \text{“lim”}_n N^\pi F(\tilde{X}_n) \end{array}$$

The pro-spectra on the right are both contractible by Lemma 4.4, so the left horizontal maps are equivalences. The left vertical map is a weak equivalence by condition (P1) $_F$. \square

Proposition 4.6. *Let F be a functor satisfying the conditions (P1) $_F$ and (P2) $_F$. Then for any X in \mathbf{Sch}_R such that there exists an admissible morphism $\tilde{X} \rightarrow X$ with \tilde{X} regular the pro-spectrum $N^\pi F(X)$ is weakly contractible.*

Proof. By Lemma 4.5 the map $N^\pi F(X) \rightarrow N^\pi F(\tilde{X})$ is a weak equivalence of pro-spectra. As $N^\pi F(\tilde{X})$ is weakly contractible by $(P2)_F$ the same is true for $N^\pi F(X)$. \square

Using that non-connective algebraic K -theory spectrum K satisfies the conditions $(P1)_F$ resp. $(P2)_F$ by [KST18, Thm. A] resp. [Wei13, Thm. V.6.3] we get:

Corollary 4.7. *For X as in Proposition 4.6 we get a weak equivalence of pro-spectra*

$$K(X) \xrightarrow{\simeq} \text{“}\lim_{t \rightarrow \pi t}\text{” } K(X \otimes_R R[t]).$$

We finish this subsection with a variant of the above observations. Let now R be a commutative not necessarily noetherian ring and $\pi \in R$. For a covariant functor F from R -algebras to $\text{Pro}(\mathbf{Sp})$ we define by the same method as in (4.1) a new functor $N^\pi F$ from R -algebras to $\text{Pro}(\mathbf{Sp})$.

We consider the following excision property for F :

$(E)_F$ For any extension of π -torsion free R -algebras $S_1 \subset S_2$ such that there exists $m > 0$ with $\pi^m S_2 \subset S_1$ we have a weak equivalence

$$\text{“}\lim_n\text{” } F(S_1, S_1/(\pi^n)) \xrightarrow{\simeq} \text{“}\lim_n\text{” } F(S_2, S_2/(\pi^n)).$$

The same argument as in the proof of Lemma 4.5, replacing condition $(P1)_F$ by condition $(E)_F$ gives:

Lemma 4.8. *Let F satisfy condition $(E)_F$ and let $S_1 \subset S_2$ be an extension as in condition $(E)_F$. Then the map of pro-spectra*

$$N^\pi F(S_1) \xrightarrow{\simeq} N^\pi F(S_2)$$

is a weak equivalence.

Remark 4.9. (i) Algebraic K -theory satisfies the condition $(E)_K$ by work of Suslin, Geisser–Hesselholt and Morrow, see [Mor14].

(ii) Combining (i) and Lemma 4.8 we get a weak equivalence of pro-spectra

$$N^\pi K(S_1) \xrightarrow{\simeq} N^\pi K(S_2).$$

4.3. K_0 and K_1 of non-archimedean rings. In this subsection we discuss some results about K_0 and K_1 of non-archimedean rings.

We define a continuous version of K_1 of a Tate ring A as the pro-group

$$K_1^{\text{cont}}(A) = \text{“}\lim_n\text{” } K_1(A)/(1 + \pi^n A_0).$$

Here A_0 is a ring of definition and $\pi \in A_0$ is a topologically nilpotent unit in A . Following Morrow [Mor16] we shall in Section 5 give a definition of the continuous K -theory pro-group $K_i^{\text{cont}}(A)$ for any i . Note that the sequence

$$0 \rightarrow 1 + \pi^n A_0 \rightarrow K_1(A_0) \rightarrow K_1(A_0/(\pi^n)) \rightarrow 0$$

is exact [Wei13, Lem. III.2.4].

Let F be a covariant functor from the category of Tate rings to an abelian category. Using a Bass type construction, compare [Wei13, Sec. III.4], we define a new functor ΣF on the same categories by

$$\Sigma F(A) = \text{coker}(F(A\langle t \rangle) \times F(A\langle t^{-1} \rangle) \rightarrow F(A\langle t, t^{-1} \rangle)),$$

Lemma 4.10. *For any Tate ring A the natural map $\Sigma K_1(A) \rightarrow \Sigma K_1^{\text{cont}}(A)$ is an isomorphism of pro-groups.*

Proof. We are going to show that the product map

$$(\check{C}_n) \quad (1 + \pi^n A_0\langle t \rangle) \times (1 + \pi^n A_0\langle t^{-1} \rangle) \rightarrow (1 + \pi^n A_0\langle t, t^{-1} \rangle)$$

is surjective for all $n > 0$. The cokernel of \check{C}_n is the Čech cohomology group $H^1(\mathcal{U}, 1 + \pi^n \mathcal{O})$, where \mathcal{U} is the standard covering of the formal π -adic scheme $(\mathbb{P}_{A_0}^1)^\wedge$ and \mathcal{O} denotes its structure sheaf. Together with the vanishing of $H^1(\mathcal{U}, \mathcal{O}/\pi\mathcal{O}) \cong H^1(\mathbb{P}_{A_0/(\pi)}^1, \mathcal{O}_{\mathbb{P}_{A_0/(\pi)}^1})$, see [Gro67, Prop. III.2.1.12], the long exact sequence of Čech cohomology corresponding to the exact sequence

$$0 \rightarrow 1 + \pi^{n+1}\mathcal{O} \rightarrow 1 + \pi^n\mathcal{O} \rightarrow \mathcal{O}/\pi\mathcal{O} \rightarrow 0$$

implies that the map

$$\text{coker}(\check{C}_{n+1}) \rightarrow \text{coker}(\check{C}_n)$$

is surjective. A simple convergence argument, as in the final part of the proof of [KST16, Prop. 18], shows however that the $\lim_n \text{coker}(\check{C}_n)$ vanishes. \square

In analogy with [Gru68, p. 77] we define a canonical homomorphism

$$s : K_0(A) \rightarrow K_1(C), \quad [P] \mapsto [P \otimes_A C \xrightarrow{t} P \otimes_A C].$$

Here and in the following we let $C = A\langle t, t^{-1} \rangle$.

Proposition 4.11. *For any Tate ring A the map s induces an isomorphism $\bar{s} : K_0(A) \xrightarrow{\sim} \Sigma K_1(A)$.*

Proof. Gruson constructs a left inverse r to \bar{s} , see [Gru68, Sec. IV.4]. This map r has the following geometric description: The adic space $(\mathbb{P}_A^1)^{\text{an}}$ has the open covering $\mathcal{U} = \{\text{Sp } A\langle t \rangle, \text{Sp } A\langle t^{-1} \rangle\}$. The set of analytic vector bundles up to isomorphism $\text{Vec}((\mathbb{P}_A^1)^{\text{an}})$ is in bijection with the set of algebraic vector bundles up to isomorphism $\text{Vec}(\mathbb{P}_A^1)$ by GAGA [Hub96].

Consider an element $\xi \in \Sigma K_1(A)$ induced by a matrix $M \in \text{GL}_n(C)$. One defines $r(\xi)$ as the composition of the maps

$$\begin{aligned} \text{GL}_n(C) &\rightarrow H^1(\mathcal{U}, \text{GL}_n) \rightarrow \text{Vec}((\mathbb{P}_A^1)^{\text{an}}) \cong \text{Vec}(\mathbb{P}_A^1) \rightarrow \\ &K_0(\mathbb{P}_A^1) \xleftarrow[\simeq]{(p^*, ([\mathcal{O}(1)] - [\mathcal{O}(0)]) \cdot p^*)} K_0(A) \times K_0(A) \xrightarrow{\text{pr}_2} K_0(A), \end{aligned}$$

where pr_2 is the projection to the second component.

In order to finish the proof of Proposition 4.11 we thus have to show that \bar{s} is surjective. The argument is parallel to the proof of [Gru68, Prop. IV.6]. We start with an element $\xi \in \Sigma K_1(A)$ represented by a matrix $M \in \text{GL}_n(C)$. Using the density of Laurent polynomials in C and Lemma 4.10, we can

assume without loss of generality that the entries of M lie in $A[t, t^{-1}]$. For any integer $N > 0$ the matrix $t^N M$ represents the element $\xi + \bar{s}(A^{nN}) \in \Sigma K_1(A)$. So after making this replacement, we can assume without loss of generality that the entries of M lie in $A[t]$.

As explained in parts c. and d. of the proof of [Gru68, Prop. IV.6] we can furthermore assume without loss of generality that $M = b_0 + b_1 t$ with $b_0, b_1 \in M(n \times n; A)$ and $b_0 + b_1 = 1$.

By [Gru68, Lem. IV.6] there exists a decomposition of A -modules $A^n = P_0 \oplus Q_0$, which is stable under the action of b_0 and b_1 and such that b_1 is topologically nilpotent on P_0 and b_0 is topologically nilpotent on Q_0 . Let $P = P_0 \otimes_A C$, $Q = Q_0 \otimes_A C$ and let M' be the matrix which acts by M on P and by $t^{-1}M$ on Q . Then the element induced by $M'|_P$ in $K_1(C)$ comes from $K_1(A\langle t \rangle)$ and the element induced by $M'|_Q$ in $K_1(C)$ comes from $K_1(A\langle t^{-1} \rangle)$, so M' represents zero in $\Sigma K_1(A)$. On the other hand M' represents the element $\xi - \bar{s}([Q_0]) \in \Sigma K_1(A)$. \square

Remark 4.12. The non-archimedean negative K -groups $\Sigma^i K_0(A)$ are studied by Calvo [Cal85], she denotes them $K_{-i}^{\text{top}}(A)$. Karoubi established a result analogous to Proposition 4.11 using a different convergence condition in [Kar71].

4.4. Analytic Bass delooping. In this subsection we study an abstract spectral version of the analytic Bass construction which we used in Proposition 4.11. It is characterized by a universal property (see Lemma 4.15).

Consider a functor E from adic rings or from Tate rings to $\text{Pro}(\mathbf{Sp})$. We define

$$\Gamma E(A) = E(A\langle t \rangle) \sqcup_{E(A)} E(A\langle t^{-1} \rangle)$$

and

$$\Lambda E(A) := \text{fib}(\Gamma E(A) \rightarrow E(A\langle t, t^{-1} \rangle)).$$

Similarly, for a functor F from adic rings or from Tate rings to an abelian category we write

$$\Gamma F(A) := \text{coker}(F(A) \rightarrow F(A\langle t \rangle) \oplus F(A\langle t^{-1} \rangle)).$$

For E as above we then have

$$\pi_i(\Gamma E(A)) \cong \Gamma(\pi_i E)(A)$$

and a long exact sequence

$$(4.2) \quad \cdots \rightarrow \Gamma(\pi_{i+1} E)(A) \rightarrow \pi_{i+1}(E(A\langle t, t^{-1} \rangle)) \rightarrow \pi_i(\Lambda E(A)) \rightarrow \Gamma(\pi_i E)(A) \rightarrow \cdots$$

Connective algebraic K -theory refines to a functor from commutative rings to E_∞ -ring spectra, and we can consider functors E as above which carry a K -module structure. Fix a map from the sphere spectrum to $\Omega K(\mathbb{Z}[t, t^{-1}])$ representing the class of t in $K_1(\mathbb{Z}[t, t^{-1}])$. If E carries a K -module structure,

we use that map to construct a map $\lambda: E \rightarrow \Lambda E$ given on an adic or Tate ring A by the composition

$$\lambda: E(A) \xrightarrow{\sim t} \Omega E(A\langle t, t^{-1} \rangle) \xrightarrow{\partial} \Lambda E(A),$$

where the second map comes from the definition of $\Lambda E(A)$ as a fibre.

Definition 4.13. We write \mathbf{M} for the ∞ -category of functors from adic or Tate rings to $\text{Pro}(\mathbf{Sp})$ equipped with a K -module structure and $\mathbf{M}^B \subseteq \mathbf{M}$ for the full subcategory spanned by those functors E for which $\lambda: E \rightarrow \Lambda E$ is an equivalence.

Note that the ∞ -category \mathbf{M}^B does not depend on the particular map representing t , since a different choice yields a homotopic λ . The functors in \mathbf{M}^B satisfy an analytic version of the Bass fundamental theorem:

Proposition 4.14 (Bass fundamental theorem). *For $E \in \mathbf{M}^B$ we have exact sequences*

$$0 \longrightarrow E_i(A) \longrightarrow E_i(A\langle t \rangle) \oplus E_i(A\langle t^{-1} \rangle) \longrightarrow E_i(A\langle t, t^{-1} \rangle) \xrightarrow{\sim t} E_{i-1}(A) \longrightarrow 0$$

where the split is induced by cup-product with t and, as usual, $E_i(A) = \pi_i(E(A))$, etc. In particular, there is an isomorphism $\lambda: E_i(A) \rightarrow \Sigma E_{i+1}(A)$ of pro-groups.

Here Σ is as defined in Subsection 4.3.

Proof. Since the map $A \rightarrow A\langle t \rangle$ is split, we get short exact sequences

$$(4.3) \quad 0 \rightarrow E_i(A) \rightarrow E_i(A\langle t \rangle) \oplus E_i(A\langle t^{-1} \rangle) \rightarrow \Gamma E_i(A) \rightarrow 0.$$

On the other hand, we have a long exact sequence

$$\cdots \rightarrow \Lambda E_i(A) \rightarrow \Gamma E_i(A) \rightarrow E_i(A\langle t, t^{-1} \rangle) \xrightarrow{\partial} \Lambda E_{i-1}(A) \rightarrow \cdots$$

Since $E \in \mathbf{M}^B$, the composition $E_{i-1}(A) \xrightarrow{\sim t} E_i(A\langle t, t^{-1} \rangle) \xrightarrow{\partial} \Lambda E_{i-1}(A)$ is an isomorphism. Hence the long exact sequence yields short exact sequences

$$(4.4) \quad 0 \rightarrow \Gamma E_i(A) \rightarrow E_i(A\langle t, t^{-1} \rangle) \xrightarrow{\partial} E_{i-1}(A) \rightarrow 0$$

split by $\sim t$. Splicing the exact sequences (4.3) and (4.4) together gives the result. \square

Lemma 4.15. *The inclusion $\mathbf{M}^B \subseteq \mathbf{M}$ admits a left adjoint which we denote by $E \mapsto E^B$. Explicitly, it can be described by*

$$(4.5) \quad E^B(A) \simeq \text{colim}(E(A) \xrightarrow{\lambda} \Lambda E(A) \xrightarrow{\Lambda(\lambda)} \Lambda^2 E(A) \rightarrow \cdots).$$

Proof. Define a functor $(-)^B: \mathbf{M} \rightarrow \mathbf{M}^B$ by (4.5). It comes with an obvious natural transformation $\alpha: \text{id} \rightarrow (-)^B$. By construction, the map $\lambda: E^B \rightarrow \Lambda E^B$, which is induced by the system of maps $\lambda: \Lambda^k E \rightarrow \Lambda^{k+1} E$, is an equivalence. Hence $E^B \in \mathbf{M}^B$. Similarly, both maps $\alpha_{E^B}, (\alpha_E)^B: E^B \rightarrow (E^B)^B$ are equivalences. The recognition principle [Lur09, Prop. 5.2.7.4] implies that $(-)^B$ is indeed a left adjoint of the inclusion $\mathbf{M}^B \subseteq \mathbf{M}$. \square

In Section 6 we will use the following simple observation.

Lemma 4.16. *Assume that there is an integer N such that $E_i(A) = 0$ for every $i > N$ and every adic resp. Tate ring A and that for any A the pro-spectrum $E(A)$ is constant, i.e. is given by an ordinary spectrum. Then $E^B(A) \simeq 0$ for every such A .*

Proof. From the long exact sequence (4.2) we deduce that also $\Lambda E_i(A) = 0$ for $i > N$ and every A . Since E has values in spectra, we deduce from (4.5) that

$$E_i^B(A) \simeq \operatorname{colim}_k \Lambda^k E_i(A) = 0$$

for $i > N$ and every A . But by Proposition 4.14 the spectrum $E_i^B(A)$ is a direct summand of $E_{i+1}^B(A\langle t, t^{-1} \rangle)$. By a downwards induction we conclude that $E_i^B(A) = 0$ for all $i \in \mathbb{Z}$. \square

5. CONTINUOUS K -THEORY

5.1. Construction. Let A_0 be a complete π -adic ring for some $\pi \in A_0$.

Definition 5.1. Continuous K -theory of A_0 is defined as the pro-spectrum

$$K^{\operatorname{cont}}(A_0) = \text{“}\lim_n\text{” } K(A_0/(\pi^n)).$$

Clearly, $K^{\operatorname{cont}}(A_0)$ is independent of the choice of π up to canonical equivalence of pro-spectra.

Lemma 5.2. *We have*

$$K_i^{\operatorname{cont}}(A_0) = \begin{cases} \text{“}\lim_n\text{” } K_1(A_0)/(1 + \pi^n A_0), & \text{for } i = 1, \\ K_0(A_0) = K_0(A_0/(\pi)), & \text{for } i = 0, \\ K_i(A_0/(\pi)), & \text{for } i < 0. \end{cases}$$

Proof. The first line is [Wei13, Lem. III.2.4], the second and third lines are a consequence of [Wei13, Lemma II.2.2]. \square

For a topological ring R whose topology is generated by the powers of some ideal $I \subset R$ we write

$$(5.1) \quad \bar{K}(R) = K(R \text{ on } I)$$

for the non-connective K -theory spectrum with support on I , see [TT90, Def. 6.4]. By $\bar{k}(R) = \tau_{\geq 0} \bar{K}(R)$ we denote the connective cover of $\bar{K}(R)$.

Let now A be a Tate ring with a ring of definition $A_0 \subset A$ and let $\pi \in A_0$ be a topologically nilpotent unit in A .

Definition 5.3. Continuous K -theory of A (with respect to the ring of definition A_0) is defined as the pro-spectrum

$$K^{\operatorname{cont}}(A; A_0) = \operatorname{cofib}(\bar{K}(A_0) \rightarrow K^{\operatorname{cont}}(A_0)).$$

We also set

$$K^{\operatorname{cont}}(A) = \operatorname{colim}_{A_0} K^{\operatorname{cont}}(A; A_0),$$

where the colimit is over the partially ordered set of rings of definition $A_0 \subset A$.

Note that for a fixed Tate ring A the partially ordered set of rings of definition $A_0 \subset A$ is directed, see Remark 3.3(ii).

We are going to show in Proposition 5.4, following [Mor16], that

$$K^{\text{cont}}(A; A_0) \simeq K^{\text{cont}}(A)$$

depends only on A up to weak equivalence. For this we need some preparations. By the localization theorem [TT90, Thm. 7.4] we have a fibre sequence of spectra

$$(5.2) \quad \bar{K}(A_0) \rightarrow K(A_0) \rightarrow K(A),$$

so there exists a canonical morphism $K(A) \rightarrow K^{\text{cont}}(A)$. Moreover, the pro-spectrum $K^{\text{cont}}(A; A_0)$ is part of a cartesian square

$$(5.3) \quad \begin{array}{ccc} K(A_0) & \longrightarrow & K(A) \\ \downarrow & & \downarrow \\ K^{\text{cont}}(A_0) & \longrightarrow & K^{\text{cont}}(A; A_0). \end{array}$$

Proposition 5.4. *For a Tate ring A with ring of definition A_0 the map $K^{\text{cont}}(A; A_0) \xrightarrow{\sim} K^{\text{cont}}(A)$ is a weak equivalence.*

Proof. By Remark 3.3 it is sufficient to show that for $A_0 \subset A'_0$ two rings of definition of A the map $K^{\text{cont}}(A; A_0) \rightarrow K^{\text{cont}}(A; A'_0)$ is a weak equivalence of pro-spectra. For this consider the commutative cube of pro-spectra

$$(5.4) \quad \begin{array}{ccccc} K(A_0) & \longrightarrow & K(A) & & \\ \downarrow & \searrow & \downarrow & \swarrow & \\ & & K(A'_0) & \longrightarrow & K(A) \\ & & \downarrow & & \downarrow \\ K^{\text{cont}}(A_0) & \longrightarrow & K^{\text{cont}}(A; A_0) & \xrightarrow{\alpha} & K^{\text{cont}}(A; A'_0) \\ & \searrow & \downarrow & & \downarrow \\ & & K^{\text{cont}}(A'_0) & \longrightarrow & K^{\text{cont}}(A; A'_0). \end{array}$$

The back square is the same as the cartesian square (5.3) and the front square is the corresponding cartesian square with A_0 replaced by A'_0 . The left square is weakly cartesian by Remark 4.9(i). So the right square is weakly cartesian as well or in other words α is a weak equivalence of pro-spectra. \square

In Subsection 4.3 we gave another definition of the pro-group $K_1^{\text{cont}}(A)$ for a Tate ring A . In fact both definitions are compatible.

Lemma 5.5. *The natural map of pro-groups*

$$\text{“}\lim_n\text{” } K_1(A)/(1 + \pi^n A_0) \xrightarrow{\sim} K_1^{\text{cont}}(A)$$

is an isomorphism.

Proof. This follows from the five lemma and the commutative diagram with exact columns

$$\begin{array}{ccc}
\bar{K}_1(A_0) & \xlongequal{\quad} & \bar{K}_1(A_0) \\
\downarrow & & \downarrow \\
\text{“}\varinjlim_n\text{” } K_1(A_0)/(1 + \pi^n A_0) & \xrightarrow{\cong} & K_1^{\text{cont}}(A_0) \\
\downarrow & & \downarrow \\
\text{“}\varinjlim_n\text{” } K_1(A)/(1 + \pi^n A_0) & \longrightarrow & K_1^{\text{cont}}(A) \\
\downarrow & & \downarrow \\
\bar{K}_0(A_0) & \xlongequal{\quad} & \bar{K}_0(A_0) \\
\downarrow & & \downarrow \\
K_0(A_0) & \xrightarrow{\cong} & K_0(A_0/(\pi))
\end{array}$$

Here the second and the fifth horizontal arrows are isomorphisms by Lemma 5.2. \square

An argument parallel to the proof of Lemma 5.5, using Theorems 2.5 and 2.13 from [Ste73], shows:

Lemma 5.6. *For a Tate ring which has a local ring of definition $A_0 \subset A$ the map*

$$\text{“}\varinjlim_n\text{” } K_2(A)/\{1 + \pi^n A_0, A_0^\times\} \xrightarrow{\cong} K_2^{\text{cont}}(A)$$

is an isomorphism of pro-groups.

Example 5.7. Assume that the Tate ring A admits a ring of definition A_0 which is noetherian and regular. Then $\bar{K}_i(A_0) = 0$ for $i < 0$. By definition of $K^{\text{cont}}(A)$ and Lemma 5.2 we thus get isomorphisms

$$K_i(A_0/(\pi)) \xrightarrow{\cong} K_i^{\text{cont}}(A) \quad \text{for } i < 0.$$

In particular, the negative continuous K -groups need not vanish even if A admits a regular ring of definition.

A concrete example is the following. Let κ be a discretely valued field with ring of integers κ_0 and uniformizer π . Set $A_0 = \kappa_0\langle x, y \rangle / (y^2 - x^3 + x^2 - \pi)$ and $A = A_0[1/\pi]$, a thickened version of a nodal curve. Then A_0 is regular, but $K_{-1}^{\text{cont}}(A) \cong K_{-1}(A_0/(\pi)) \cong \mathbb{Z}$ (see [Wei13, Exc. III.4.12]).

Continuous K -theory of a Tate ring A can also be calculated in terms of a Raynaud type geometric model of A . By this we mean the following: Assume that there exists a noetherian ring of definition $A_0 \subset A$ and let $\pi \in A_0$ be a topologically nilpotent unit in A . Assume given an “admissible blow-up” or more precisely a proper morphism $p: X \rightarrow \text{Spec}(A_0)$ which is an

isomorphism over $\text{Spec}(A)$. For $n > 0$ set $X_n = X \otimes_{A_0} A_0/(\pi^n)$ and define continuous K -theory of X as the pro-spectrum

$$K^{\text{cont}}(X) = \text{“}\lim_n\text{” } K(X_n).$$

Proposition 5.8. *There is a weak fibre sequence of pro-spectra.*

$$\bar{K}(X) \rightarrow K^{\text{cont}}(X) \rightarrow K^{\text{cont}}(A).$$

Here $\bar{K}(X)$ stands for the algebraic K -theory spectrum of X with support in X_1 .

Proof. The proof is very similar to the proof of Proposition 5.4. Let for the moment W be the cofibre of $\bar{K}(X) \rightarrow K^{\text{cont}}(X)$. Then we get a commutative cube of pro-spectra

$$(5.5) \quad \begin{array}{ccccc} K(A_0) & \longrightarrow & K(A) & & \\ & \searrow & \downarrow & \cong & \\ & & K(X) & \longrightarrow & K(A) \\ & & \downarrow & & \downarrow \\ K^{\text{cont}}(A_0) & \longrightarrow & K^{\text{cont}}(A; A_0) & \xrightarrow{\alpha} & W \\ & \searrow & \downarrow & & \\ & & K^{\text{cont}}(X) & \longrightarrow & W \end{array}$$

The back square is the same as the cartesian square (5.3). The front square is cartesian by the same reasoning. The left square is weakly cartesian by [KST18, Thm. A]. So the right square is weakly cartesian as well or in other words α is a weak equivalence of pro-spectra. \square

Remark 5.9. Proposition 5.8 can be used to show that continuous K -theory satisfies Mayer–Vietories for finite affinoid coverings of affinoid spaces as indicated in [Mor16].

5.2. Bass fundamental theorem. Let A be a Tate ring with a ring of definition A_0 and let $\pi \in A_0$ be a topologically nilpotent unit in A . As both $\bar{K}(A_0)$ and $K^{\text{cont}}(A_0)$ carry a module structure over $K(A_0)$ and $K^{\text{cont}}(A)$ carries a compatible module structure over $K(A)$ the techniques of Subsection 4.4 apply. In particular we obtain a commutative diagram of fibre sequences

$$\begin{array}{ccccc} \bar{K}(A_0) & \longrightarrow & K^{\text{cont}}(A_0) & \longrightarrow & K^{\text{cont}}(A; A_0) \\ \lambda_1 \downarrow & & \lambda_2 \downarrow & & \lambda_3 \downarrow \\ \Lambda \bar{K}(A_0) & \longrightarrow & \Lambda K^{\text{cont}}(A_0) & \longrightarrow & \Lambda K^{\text{cont}}(A; A_0) \end{array}$$

in which the vertical maps are equivalences of pro-spectra. Indeed, in order to show that λ_1 and λ_2 are equivalences one can replace in the definition of Λ the rings $A_0\langle t^\pm \rangle$ and $A_0\langle t, t^{-1} \rangle$ by their dense subrings $A_0[t^\pm]$ and $A_0[t, t^{-1}]$ in view of [TT90, Prop. 3.19], so these equivalences λ_1 and λ_2

are a consequence of [TT90, Thm. 6.6]. This means the canonical map $K^{\text{cont}}(A) \rightarrow (K^{\text{cont}})^B(A)$ is an equivalence, so we deduce that the Bass fundamental theorem holds for K^{cont} , see Proposition 4.14.

Proposition 5.10. *Continuous K -theory of a Tate ring A satisfies:*

- (i) $\lambda: K_i^{\text{cont}}(A) \rightarrow \Sigma K_{i+1}^{\text{cont}}(A)$ is an isomorphism of pro-groups.
- (ii) The canonical map $K_0(A) \rightarrow K_0^{\text{cont}}(A)$ is an isomorphism of pro-groups.
- (iii) The pro-groups $K_i^{\text{cont}}(A)$ are constant for $i \leq 0$ and agree with the groups $K_i^{\text{top}}(A)$ defined and studied in [Cal85].

Proof. Part (i) is clear from what is explained above. Part (ii) follows from part (i), Lemma 4.10 and Proposition 4.11. Part (iii) is clear in view of part (ii) and the Bass fundamental theorem for K^{cont} . \square

Remark 5.11. Combining Proposition 5.10(ii) and Lemma 5.5 with the exact sequence in **Ab**

$$0 \rightarrow \lim^1 K_1^{\text{cont}}(A) \rightarrow \pi_0 \lim K^{\text{cont}}(A) \rightarrow \lim^0 K_0^{\text{cont}}(A) \rightarrow 0,$$

we deduce that $K_0(A) \xrightarrow{\cong} \pi_0 \lim K^{\text{cont}}(A)$ is an isomorphism. Here \lim^0 and \lim^1 are the usual limit functors from the category $\text{Pro}(\mathbf{Ab})$ to **Ab** and $\lim: \text{Pro}(\mathbf{Sp}) \rightarrow \mathbf{Sp}$ is the right adjoint to the inclusion.

Using Gabber's rigidity theorem we can compute continuous K -theory of Tate rings with finite coefficients. Let ℓ be a positive integer. If F is a functor from adic rings or from Tate rings to spectra or pro-spectra, we write $F(A; \mathbb{Z}/\ell)$ for the (level-wise) smash product of $F(A)$ with the mod- ℓ Moore spectrum \mathbb{S}/ℓ .

Let A be a Tate ring with a ring of definition A_0 , and let ℓ be a positive integer which is invertible in A_0 .

Lemma 5.12. *The canonical map $K(A) \rightarrow K^{\text{cont}}(A)$ induces isomorphisms $K_i(A; \mathbb{Z}/\ell) \xrightarrow{\cong} K_i^{\text{cont}}(A; \mathbb{Z}/\ell)$ and $K_0(A)/\ell \xrightarrow{\cong} K_0^{\text{cont}}(A)/\ell$ for $i \geq 1$.*

Proof. The statement about K_0 follows directly from Proposition 5.10(ii). Since smashing with \mathbb{S}/ℓ is an exact functor, the cartesian square (5.3) and Proposition 5.4 yield the weakly cartesian square

$$\begin{array}{ccc} K(A_0; \mathbb{Z}/\ell) & \longrightarrow & K(A; \mathbb{Z}/\ell) \\ \downarrow & & \downarrow \\ K^{\text{cont}}(A_0; \mathbb{Z}/\ell) & \longrightarrow & K^{\text{cont}}(A; \mathbb{Z}/\ell) \end{array}$$

Since A_0 is π -adically complete (where π generates the topology on A_0), Gabber's rigidity theorem [Gab92, Thm. 1] implies that the left vertical map induces an isomorphism on π_i for every $i > 0$. This implies that $K_i(A; \mathbb{Z}/\ell) \rightarrow K_i^{\text{cont}}(A; \mathbb{Z}/\ell)$ is an isomorphism for $i > 1$ and injective for $i = 1$. In order to show that it is also surjective for $i = 1$, we consider the

following diagram in which the exact rows come from the Bockstein sequence and $[\ell]$ denotes the ℓ -torsion subgroup:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1(A)/\ell & \longrightarrow & K_1(A; \mathbb{Z}/\ell) & \longrightarrow & K_0(A)[\ell] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_1^{\text{cont}}(A)/\ell & \longrightarrow & K_1^{\text{cont}}(A; \mathbb{Z}/\ell) & \longrightarrow & K_0^{\text{cont}}(A)[\ell] & \longrightarrow & 0 \end{array}$$

The right vertical map is an isomorphism by Proposition 5.10(ii). The left vertical map is surjective by Lemma 5.5. Hence the middle vertical map is surjective, too. \square

5.3. Analytic pro-homotopy invariance. Let A be an adic ring resp. a Tate ring and let $\pi \in A$ be an element generating the topology on A resp. a topologically nilpotent unit. Let \mathbf{C}_A be the comma category of adic rings resp. Tate rings over A . In the following we fix an integer $m > 0$ and we write $A\langle \underline{t} \rangle = A\langle t_1, \dots, t_m \rangle$. For a functor F from \mathbf{C}_A to $\text{Pro}(\mathbf{Sp})$ we define the new functor $\hat{N}^\pi F = \hat{N}_m^\pi F$ on the same categories by

$$\hat{N}^\pi F(A) = \text{cofib}(F(A) \rightarrow \text{“lim”}_{\underline{t} \rightarrow \pi \underline{t}} F(A\langle \underline{t} \rangle)).$$

Recall that the polynomial version N^π of \hat{N}^π has been studied in Subsection 4.2.

Let A be a Tate ring, $A_0 \subset A$ be a ring of definition and $\pi \in A_0$ an element which is a topologically nilpotent unit in A .

Lemma 5.13. *We have:*

- (i) $\hat{N}^\pi K^{\text{cont}}(A_0)$ is a contractible pro-spectrum.

If moreover A is regular and satisfies condition $(\dagger)_A$ we have:

- (ii) $\hat{N}^\pi \bar{K}(A_0)$ is a weakly contractible pro-spectrum.
 (iii) There is a natural weak equivalence

$$\hat{N}^\pi K(A) \simeq \text{cofib}(K(A_0) \rightarrow \text{“lim”}_{\underline{t} \rightarrow \pi \underline{t}} K(A_0[[\underline{t}]])).$$

- (iv) In the case of one variable, i.e. $m = 1$, there is a natural isomorphism of pro-abelian groups

$$\pi_1(\hat{N}^\pi K(A)) \cong \text{“lim”}_{[\pi]} W(A_0),$$

where $W(A_0) = (1 + tA_0[[t]])^\times$ is the ring of big Witt vectors and the right hand side denotes the pro-system

$$W(A_0) \xleftarrow{[\pi]} W(A_0) \xleftarrow{[\pi]} W(A_0) \xleftarrow{[\pi]} \dots$$

with $[\pi] = (1 - \pi t)^{-1} \in W(A_0)$.

Proof. Part (i) is clear as for fixed $j > 0$ the map

$$A_0/(\pi^j) \rightarrow \varinjlim_{\underline{t} \rightarrow \pi \underline{t}} A\langle \underline{t} \rangle / (\pi^j)$$

is an isomorphism of pro-rings. For (ii) we use that the canonical map $N^\pi \bar{K}(A_0) \xrightarrow{\cong} \hat{N}^\pi \bar{K}(A_0)$ is an isomorphism by excision [TT90, Thm. 7.1], where the left side is as defined in Subsection 4.2. From localization theory [TT90, Thm. 7.4] we get a fibre sequence

$$(5.6) \quad N^\pi \bar{K}(A_0) \rightarrow N^\pi K(A_0) \rightarrow N^\pi K(A).$$

The right pro-spectrum in (5.6) is contractible as A is regular and K -theory of regular rings is \mathbb{A}^1 -homotopy invariant [Wei13, Thm. V.6.3]. Up to weak equivalence the pro-spectrum in the middle of (5.6) is independent of the choice of the ring of definition A_0 by Remark 4.9. So we can assume by condition $(\dagger)_A$ that A_0 is noetherian and that there exists a proper morphism $\tilde{X} \rightarrow \text{Spec}(A_0)$ which is an isomorphism over $\text{Spec}(A)$ and such that \tilde{X} is regular. Corollary 4.7 finally implies that $N^\pi K(A_0)$ is weakly contractible. Together this implies that also the pro-spectrum on the left of (5.6) is weakly contractible proving (ii). For (iii), note that the natural map

$$\hat{N}^\pi \bar{K}(A_0) \rightarrow \hat{N}^\pi K(A)$$

is a weak equivalence by (ii). From the commutative diagram

$$\begin{array}{ccc} A_0\langle \underline{t} \rangle & \longrightarrow & A_0[[\underline{t}]] \\ \underline{t} \rightarrow \pi \underline{t} \downarrow & \swarrow & \downarrow \underline{t} \rightarrow \pi \underline{t} \\ & A_0\langle \underline{t} \rangle & \longrightarrow & A_0[[\underline{t}]] \end{array}$$

we obtain an isomorphism of pro-simplicial rings

$$\varinjlim_{\underline{t} \rightarrow \pi \underline{t}} A_0\langle \underline{t} \rangle \simeq \varinjlim_{\underline{t} \rightarrow \pi \underline{t}} A_0[[\underline{t}]].$$

This proves (iii). Part (iv) follows from part (iii) and the isomorphism

$$K_1(A_0[[t]])/K_1(A_0) \simeq (1 + tA_0[[t]])^\times,$$

see [Wei13, Lem. III.2.4]. □

From the fibre sequence

$$\hat{N}^\pi \bar{K}(A_0) \rightarrow \hat{N}^\pi K^{\text{cont}}(A_0) \rightarrow \hat{N}^\pi K^{\text{cont}}(A)$$

and Lemma 5.13 we immediately deduce:

Proposition 5.14. *For a regular Tate ring A which satisfies condition $(\dagger)_A$ the canonical map*

$$K^{\text{cont}}(A) \xrightarrow{\cong} \varinjlim_{\underline{t} \rightarrow \pi \underline{t}} K^{\text{cont}}(A\langle \underline{t} \rangle)$$

is a weak equivalence of pro-spectra.

As a corollary to Proposition 5.14 and Proposition 5.10(ii) we recover the following pro-homotopy invariance result for K_0 , which has already been shown in [KST16] using a more direct approach.

Corollary 5.15. *For a regular Tate ring A which satisfies condition $(\dagger)_A$ the canonical map*

$$K_0(A) \xrightarrow{\cong} \varinjlim_{t \rightarrow \pi t} K_0(A\langle t \rangle)$$

is an isomorphism of pro-groups.

For K_1 we at least have the following weak pro-homotopy invariance result:

Proposition 5.16. *If A is regular and satisfies condition $(\dagger)_A$, then*

$$R^i \lim_{t \rightarrow \pi t} \pi_1 \hat{N}^\pi K(A) = 0 \quad \text{for } i \geq 0.$$

Proof. We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & W(A_0) & \xrightarrow{[\pi]^{n+1}} & W(A_0) & \longrightarrow & W(A_0)/[\pi]^{n+1} \longrightarrow 0 \\ & & \downarrow [\pi] & & \downarrow = & & \downarrow \\ 0 & \longrightarrow & W(A_0) & \xrightarrow{[\pi]^n} & W(A_0) & \longrightarrow & W(A_0)/[\pi]^n \longrightarrow 0 . \end{array}$$

From this we get

$$R^i \lim_{t \rightarrow \pi t} W(A_0) \cong \begin{cases} \ker(W(A_0) \rightarrow \widehat{W(A_0)}_{[\pi]}), & \text{for } i = 0, \\ \text{coker}(W(A_0) \rightarrow \widehat{W(A_0)}_{[\pi]}), & \text{for } i = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\widehat{W(A_0)}_{[\pi]}$ is the $[\pi]$ -adic completion of $W(A_0)$. Hence the assertion follows from Lemma 5.13 and the following lemma. \square

Lemma 5.17. *The ring $W(A_0)$ is $[\pi]$ -adically complete.*

Proof. We have

$$f \in [\pi]^n \cdot W(A_0) \quad \text{for } f = 1 + \sum_{i \geq 1} a_i t^i \in W(A_0)$$

if and only if there are $c_i \in A_0$, $i \geq 1$, such that

$$a_i = \pi^{ni} c_i \quad \text{for } i \geq 1.$$

The lemma follows from this by direct computations. \square

6. ANALYTIC K -THEORY

6.1. Basic constructions. Let A be a π -adic ring for some $\pi \in A$. In order to define analytic K -theory we consider the system of simplicial rings

$$(6.1) \quad \cdots \longrightarrow A\langle \Delta_{\pi^3} \rangle \xrightarrow{\Psi_\pi^{\pi^2}} A\langle \Delta_{\pi^2} \rangle \xrightarrow{\Psi_\pi^{\pi^1}} A\langle \Delta_{\pi^1} \rangle \xrightarrow{\Psi_\pi^{\pi^0}} A\langle \Delta_1 \rangle,$$

see Subsection 3.1. We would like to see that the pro-system of simplicial rings (6.1) does not depend on the choice of π up to canonical isomorphism. Indeed, let $\varpi \in A$ be another element generating the topology of A and consider the analog of the pro-system (6.1) with π replaced by ϖ . We have $\varpi^n = a\pi$ and that $\pi^n = b\varpi$ for some positive integer n and $a, b \in A$. In view of the commutative square

$$\begin{array}{ccc} A\langle\Delta_{\varpi^{n(j+1)}}\rangle & \xrightarrow{\Psi_{\varpi^n}^{\varpi^{nj}}} & A\langle\Delta_{\varpi^{nj}}\rangle \\ \Psi_{a^{j+1}}^{\pi^{j+1}} \downarrow & & \downarrow \Psi_{a^j}^{\pi^j} \\ A\langle\Delta_{\pi^{j+1}}\rangle & \xrightarrow{\Psi_{\pi}^{\pi^j}} & A\langle\Delta_{\pi^j}\rangle \end{array}$$

and the analogous square with π and ϖ interchanged and a replaced by b we get a canonical isomorphism of pro-simplicial rings Φ_{ϖ}^{π} such that the diagram

$$\begin{array}{ccc} \text{“}\lim_j\text{” } A\langle\Delta_{\pi^j}\rangle & \xrightarrow[\simeq]{\Phi_{\varpi}^{\pi}} & \text{“}\lim_j\text{” } A\langle\Delta_{\varpi^j}\rangle \\ & \searrow & \swarrow \\ & A\langle\Delta_1\rangle & \end{array}$$

commutes.

If A is a Tate ring and $\pi \in A$ is a topologically nilpotent unit, one similarly constructs a pro-simplicial ring $\text{“}\lim_j\text{” } A\langle\Delta_{\pi^j}\rangle$ and one shows that it does not depend on the choice of π up to canonical isomorphism.

In the remainder of this subsection we assume that A is either an adic ring or a Tate ring. Let F be a functor from the category of adic rings or from the category of Tate rings to connective spectra. We set

$$F^{(j)}(A) = F(A\langle\Delta_{\pi^j}\rangle)$$

and define a new functor F^{an} to the ∞ -category of pro-spectra by

$$F^{\text{an}}(A) = \text{“}\lim_j\text{” } F^{(j)}(A).$$

Here $F(A\langle\Delta_{\pi^j}\rangle)$ denotes the geometric realization of the simplicial spectrum $[p] \mapsto F(A\langle\Delta_{\pi^j}^p\rangle)$.

Lemma 6.1. *For any A and any functor F as above, there is a convergent spectral sequence of pro-abelian groups*

$$E_{pq}^1 = \text{“}\lim_j\text{” } \pi_q F(A\langle\Delta_{\pi^j}^p\rangle) \implies \pi_{p+q} F^{\text{an}}(A)$$

which yields an isomorphism of pro-groups

$$\pi_0 F^{\text{an}}(A) = \text{“}\lim_j\text{” } \text{coker}(\pi_0 F(A\langle\Delta_{\pi^j}^1\rangle) \xrightarrow{\partial_0 - \partial_1} \pi_0 F(A)).$$

Proof. For fixed j we have the classical convergent E^1 -spectral sequence of a simplicial spectrum (see e.g. [Lur17a, Prop. 1.2.4.5])

$$\pi_q F(A\langle\Delta_{\pi^j}^p\rangle) \implies \pi_{p+q} F^{(j)}(A).$$

Since F has values in connective spectra, these are bounded uniformly in j . They hence yield the desired convergent spectral sequence in the abelian category $\text{Pro}(\mathbf{Ab})$. The asserted isomorphism is an immediate consequence. \square

Lemma 6.2. *For fixed j , the maps of simplicial rings $A\langle t \rangle \langle \Delta_{\pi^j} \rangle \rightarrow A\langle t \rangle \langle \Delta_{\pi^j} \rangle$ induced by $t \mapsto \pi^j t$ and $t \mapsto 0$, respectively, are simplicially homotopic.*

Proof. For fixed n and $0 \leq \ell \leq n$ we have ring homomorphisms

$$h_\ell: A\langle t \rangle \langle \Delta_{\pi^j}^n \rangle \rightarrow A\langle t \rangle \langle \Delta_{\pi^j}^{n+1} \rangle$$

induced by $h_\ell(t) = t(t_0 + \cdots + t_\ell)$ and $h_\ell(t_i) = s_\ell(t_i)$, where s_ℓ is the ℓ -th degeneracy map and the t_i are the coordinates on $\Delta_{\pi^j}^n$. These give the desired homotopy. \square

Given variables $\underline{t} = (t_1, \dots, t_m)$, we let

$$\hat{N}_m^\pi F(A) = \text{cofib}(F(A) \rightarrow \text{“lim”}_{\underline{t} \mapsto \pi \underline{t}} F(A\langle \underline{t} \rangle))$$

as in Subsection 5.3.

Proposition 6.3. *For each j , the natural map*

$$F^{(j)}(A) \rightarrow \text{“lim”}_{\underline{t} \mapsto \pi \underline{t}} F^{(j)}(A\langle \underline{t} \rangle)$$

is an equivalence of pro-spectra. In particular, the pro-spectrum $\hat{N}_m^\pi F^{\text{an}}(A)$ is contractible for any $m \geq 0$.

Proof. The map $A\langle t \rangle \rightarrow A$, $t \mapsto 0$, induces a left inverse. As any functor preserves simplicial homotopies, Lemma 6.2 implies that each j -fold composition of transition maps $F^{(j)}(A\langle t \rangle) \rightarrow F^{(j)}(A\langle t \rangle)$ factors through $F^{(j)}(A)$ up to homotopy. This implies the first claim. Taking “lim” over j implies that

$$F^{\text{an}}(A) \xrightarrow{\simeq} \text{“lim”}_{\underline{t} \mapsto \pi \underline{t}} F^{\text{an}}(A\langle \underline{t} \rangle)$$

is also an equivalence. By induction on m this gives the second assertion. \square

If $G \rightarrow F$ is the connected covering of F , we consider the fibre sequence

$$G \rightarrow F \rightarrow F_0$$

applied to the pro-simplicial ring “lim” $_j A\langle \Delta_{\pi^j} \rangle$, where $F_0 = H(\pi_0 F)$ denotes the Eilenberg–MacLane spectrum living in homotopy degree 0.

Lemma 6.4. *For an adic ring resp. Tate ring A there is a fibre sequence of pro-spectra*

$$G^{\text{an}}(A) \rightarrow F^{\text{an}}(A) \rightarrow \text{“lim”}_j F_0(A\langle \Delta_{\pi^j} \rangle).$$

If $\hat{N}_m^\pi F_0(A)$ is contractible for any number of variables $m \geq 0$, then there is a weak fibre sequence of pro-spectra

$$G^{\text{an}} \rightarrow F^{\text{an}} \rightarrow F_0(A).$$

Note that $\hat{N}_m^\pi F_0(A)$ is contractible if and only if the natural map $F_0(A) \rightarrow \text{“lim”}_j F_0(A\langle\Delta_{\pi^j}\rangle)$ is an equivalence.

Proof. For every j and m there is a fibre sequence of spectra

$$G(A\langle\Delta_{\pi^j}^m\rangle) \rightarrow F(A\langle\Delta_{\pi^j}^m\rangle) \rightarrow F_0(A\langle\Delta_{\pi^j}^m\rangle).$$

Since fibre sequences of spectra are preserved by geometric realization, this induces a fibre sequence

$$G^{(j)}(A) \rightarrow F^{(j)}(A) \rightarrow F_0(A\langle\Delta_{\pi^j}\rangle)$$

for every j . The first claim follows by taking “lim” over j .

For the second claim we use that the homotopy pro-groups of the pro-spectrum “lim” $_j F_0(A\langle\Delta_{\pi^j}\rangle)$ are computed as the homotopy pro-groups of the pro-simplicial abelian group “lim” $_j (\pi_0 F)(A\langle\Delta_{\pi^j}\rangle)$. \square

6.2. Analytic K -theory of adic rings. We apply the construction of the previous subsection to the connective K -theory of adic rings. Let A be an adic ring, and let $\pi \in A$ generate the topology of A .

Definition 6.5. The *analytic K -theory of A* is defined as the pro-spectrum

$$k^{\text{an}}(A) = \text{“lim”}_j k(A\langle\Delta_{\pi^j}\rangle).$$

By the discussion in Subsection 6.1 this is independent of the choice of π up to canonical equivalence. In the following, we also use the connective covering

$$k^{\text{cont}}(A) = \text{“lim”}_n k(A/(\pi^n))$$

of the continuous K -theory $K^{\text{cont}}(A)$ of A . As a bridge between continuous and analytic K -theory we further define the ‘mixed’ version

$$k^{\text{cont,an}} = \text{“lim”}_{j,n} k(A\langle\Delta_{\pi^j}\rangle/(\pi^n)).$$

Lemma 6.6. *The natural map*

$$k^{\text{cont}}(A) \xrightarrow{\cong} k^{\text{cont,an}}(A)$$

is an equivalence of pro-spectra.

Proof. For fixed n the map $A/(\pi^n) \rightarrow \text{“lim”}_j A\langle\Delta_{\pi^j}\rangle/(\pi^n)$ is an isomorphism of pro-simplicial rings. Hence it induces an equivalence of pro-spectra

$$k(A/(\pi^n)) \xrightarrow{\cong} \text{“lim”}_j k(A\langle\Delta_{\pi^j}\rangle/(\pi^n))$$

The claim follows by taking the limit over n . \square

Proposition 6.7. *The map*

$$k^{\text{an}}(A) \rightarrow k^{\text{cont,an}}(A)$$

is a weak equivalence of pro-spectra.

Proof. We first reduce to connected K -theory. For fixed j , n , and m we have, as in the discussion preceding Lemma 6.4, a map of fibre sequences of spectra

$$\begin{array}{ccccc} K^{\mathrm{GL}}(A\langle\Delta_{\pi^j}^m\rangle) & \longrightarrow & k(A\langle\Delta_{\pi^j}^m\rangle) & \longrightarrow & K_0(A\langle\Delta_{\pi^j}^m\rangle) \\ \downarrow & & \downarrow & & \downarrow \cong \\ K^{\mathrm{GL}}(A\langle\Delta_{\pi^j}^m\rangle/(\pi^n)) & \longrightarrow & k(A\langle\Delta_{\pi^j}^m\rangle/(\pi^n)) & \longrightarrow & K_0(A\langle\Delta_{\pi^j}^m\rangle/(\pi^n)). \end{array}$$

The right vertical map is an isomorphism because $A\langle\Delta_{\pi^j}^m\rangle$ is π -adically complete, see [Wei13, Lemma II.2.2].

It is now enough to show that for each j the map

$$K^{\mathrm{GL}}(A\langle\Delta_{\pi^j}\rangle) \rightarrow \text{“}\lim_n\text{” } K^{\mathrm{GL}}(A\langle\Delta_{\pi^j}\rangle/(\pi^n))$$

is a weak equivalence of pro-spectra. We check this on the associated infinite loop spaces. By Lemma 4.2 this means we have to show that the bottom horizontal map in the commutative diagram

$$\begin{array}{ccc} \mathrm{BGL}(A\langle\Delta_{\pi^j}\rangle) & \xrightarrow{\phi} & \text{“}\lim_n\text{” } \mathrm{BGL}(A\langle\Delta_{\pi^j}\rangle/(\pi^n)) \\ \downarrow & & \downarrow \\ \mathrm{BGL}(A\langle\Delta_{\pi^j}\rangle)^+ & \xrightarrow{\phi^+} & \text{“}\lim_n\text{” } \mathrm{BGL}(A\langle\Delta_{\pi^j}\rangle/(\pi^n))^+ \end{array}$$

is a weak equivalence in $\mathrm{Pro}(\mathbf{Spc}_*^0)$ (see Subsection 2.3). Since the map to the plus-construction is acyclic, the vertical maps induce isomorphisms on integral homology.

Claim 6.8. The map ϕ is a weak equivalence in $\mathrm{Pro}(\mathbf{Spc}_*^0)$.

By Lemma 2.10 this claim implies that ϕ^+ induces an isomorphism on integral homology. Since ϕ^+ is a morphism between countable pro-systems of connected H-spaces, Proposition 2.11 implies that ϕ^+ is a weak equivalence. \square

Proof of the Claim. We have to show that ϕ induces an isomorphism on homotopy pro-groups. For this we use that

$$\pi_i(\mathrm{BGL}(A\langle\Delta_{\pi^j}\rangle)) \cong \pi_{i-1}(\mathrm{GL}(A\langle\Delta_{\pi^j}\rangle)) \quad \text{for } i \geq 1.$$

For each n , the natural map

$$\mathrm{GL}(A\langle\Delta_{\pi^j}\rangle) \rightarrow \mathrm{GL}(A\langle\Delta_{\pi^j}\rangle/(\pi^n))$$

is a surjective homomorphism of simplicial groups and hence a Kan fibration. Its fibre is given by the simplicial group of finite matrices of arbitrary size

$$1 + \pi^n \mathrm{M}(A\langle\Delta_{\pi^j}\rangle)$$

and it suffices to show that the inclusion

$$1 + \pi^{n+j} \mathrm{M}(A\langle\Delta_{\pi^j}\rangle) \hookrightarrow 1 + \pi^n \mathrm{M}(A\langle\Delta_{\pi^j}\rangle)$$

is null-homotopic. We let $\Delta[1]$ be the simplicial 1-simplex, and we consider $t_1 \in A\langle\Delta_{\pi^j}^1\rangle$ as a map of simplicial sets

$$t_1: \Delta[1] \rightarrow A\langle\Delta_{\pi^j}\rangle$$

satisfying $t_1(0) = 0$, $t_1(1) = \pi^j$. A contracting homotopy is then given by

$$\begin{aligned} (1 + \pi^{n+j} M(A\langle\Delta_{\pi^j}\rangle)) \times \Delta[1] &\rightarrow 1 + \pi^n M(A\langle\Delta_{\pi^j}\rangle), \\ (1 + a, s) &\mapsto 1 + t(s) \cdot (\pi^{-j}a). \end{aligned} \quad \square$$

Since by Lemma 6.6 the map of pro-spectra $k^{\text{cont}}(A) \xrightarrow{\cong} k^{\text{cont,an}}(A)$ is an equivalence, it has an essentially unique inverse. Composing this with the weak equivalence $k^{\text{an}}(A) \rightarrow k^{\text{cont,an}}(A)$ of Proposition 6.7 gives an essentially unique weak equivalence of pro-spectra $k^{\text{an}}(A) \rightarrow k^{\text{cont}}(A)$ which makes the diagram

$$\begin{array}{ccc} k^{\text{an}}(A) & \xrightarrow{\cong} & k^{\text{cont}}(A) \\ & \searrow \cong & \swarrow \cong \\ & k^{\text{cont,an}}(A) & \end{array}$$

commutative. Summarizing we have:

Theorem 6.9. *For any π -adic ring A , there is a natural weak equivalence of pro-spectra*

$$k^{\text{an}}(A) \xrightarrow{\cong} k^{\text{cont}}(A).$$

Remark 6.10. In fact, this result holds more generally. We needed the completeness of a π -adic ring only in the proof of Proposition 6.7. More precisely, we used that for a π -adically complete ring A the map $K_0(A) \rightarrow K_0(A/(\pi))$ is an isomorphism and $\text{GL}(A) \rightarrow \text{GL}(A/(\pi))$ is surjective with kernel $1 + \pi M(A)$. But these two facts also hold if A is only assumed to be henselian with respect to (π) . Hence we get the same result if we work with henselian rings and replace the simplicial ring $A\langle\Delta_{\pi^j}\rangle$ in the construction of analytic K -theory with the henselisation of $A[\Delta_{\pi^j}]$ along the ideal (π) .

6.3. Analytic K -theory of Tate rings. Let now A be a Tate ring and $\pi \in A$ a topologically nilpotent unit.

Definition 6.11. The *connective analytic K -theory of A* is defined as the pro-spectrum

$$k^{\text{an}}(A) = \text{“}\varinjlim\text{” } k(A\langle\Delta_{\pi^j}\rangle).$$

Again, this is well-defined up to canonical equivalence. As in the previous subsection we will compare analytic K -theory and continuous K -theory. For this we need some auxiliary constructions. If $A_0 \subseteq A$ is a ring of definition

with $\pi \in A_0$, we set

$$\begin{aligned}\bar{k}^{\text{an}}(A_0) &= \text{“lim”}_j \bar{k}(A_0 \langle \Delta_{\pi^j} \rangle), \\ \tilde{k}(A; A_0) &= \text{cofib}(\bar{k}(A_0) \rightarrow k(A_0)), \\ \tilde{k}^{\text{cont}}(A; A_0) &= \text{cofib}(\bar{k}(A_0) \rightarrow k^{\text{cont}}(A_0)), \\ \tilde{k}^{\text{an}}(A; A_0) &= \text{cofib}(\bar{k}^{\text{an}}(A_0) \rightarrow k^{\text{an}}(A_0)), \\ \tilde{k}^{\text{cont,an}}(A; A_0) &= \text{cofib}(\bar{k}^{\text{an}}(A_0) \rightarrow k^{\text{cont,an}}(A_0))\end{aligned}$$

(see (5.1) for the definition of \bar{k}). Note that the above spectra are all connective and that $\tau_{\geq 1} \tilde{k}(A; A_0) \simeq \tau_{\geq 1} k(A) \simeq K^{\text{GL}}(A)$. There is a natural map $\tilde{k}^{\text{cont}}(A; A_0) \rightarrow K^{\text{cont}}(A; A_0)$. It is immediate from the constructions that this map induces an equivalence on connected covers $\tau_{\geq 1} \tilde{k}^{\text{cont}}(A; A_0) \rightarrow \tau_{\geq 1} K^{\text{cont}}(A; A_0)$. In view of Proposition 5.4 we have a natural weak equivalence

$$\tau_{\geq 1} \tilde{k}^{\text{cont}}(A; A_0) \xrightarrow{\simeq} \tau_{\geq 1} K^{\text{cont}}(A).$$

Recall condition $(\dagger)_A$ from subsection 3.2 (page 11).

Lemma 6.12. *Assume that A is regular and satisfies condition $(\dagger)_A$. Then there is a natural zig-zag of weak equivalences of pro-spectra*

$$\tilde{k}^{\text{cont}}(A; A_0) \xrightarrow{\simeq} \tilde{k}^{\text{cont,an}}(A; A_0) \xleftarrow{\simeq} \tilde{k}^{\text{an}}(A; A_0).$$

Proof. Lemma 5.13(ii) and the convergent spectral sequence from Lemma 6.1

$$E_{pq}^1 = \text{“lim”}_j \bar{k}_q(A_0 \langle \Delta_{\pi^j}^p \rangle) \implies \bar{k}_{p+q}^{\text{an}}(A_0)$$

imply that the natural map $\bar{k}(A_0) \rightarrow \bar{k}^{\text{an}}(A_0)$ is a weak equivalence. Combining this with Lemma 6.6 and the definitions of the tilde versions yields the left weak equivalence of the assertion. The right-hand weak equivalence follows from Proposition 6.7. \square

Lemma 6.13. *Let A, A_0 be as above.*

- (i) *There is a natural isomorphism $\tilde{k}_0(A; A_0) \cong \tilde{k}_0^{\text{an}}(A; A_0)$.*
- (ii) *If A is regular and satisfies condition $(\dagger)_A$, then we have a natural isomorphism*

$$K_0(A) \cong k_0^{\text{an}}(A)$$

and a natural weak equivalence

$$\tau_{\geq 1} \tilde{k}^{\text{an}}(A; A_0) \simeq \tau_{\geq 1} k^{\text{an}}(A).$$

Proof. Recall that k and \tilde{k} have common connected cover K^{GL} . For fixed m we have a commutative diagram of pro-groups

$$\begin{array}{ccc} K_0(A_0) & \xrightarrow{\cong} & \text{“lim”}_j K_0(A_0 \langle \Delta_{\pi^j}^m \rangle) \\ \downarrow & & \downarrow \\ \tilde{k}_0(A; A_0) & \longrightarrow & \text{“lim”}_j \tilde{k}_0(A \langle \Delta_{\pi^j}^m \rangle, A_0 \langle \Delta_{\pi^j}^m \rangle). \end{array}$$

The top horizontal map is an isomorphism by Lemmas 5.2 and 5.13(i). The vertical maps are epimorphisms since they are level-wise surjective by definition of \tilde{k} . Hence the bottom horizontal map is an epimorphism, too. Since it has a left inverse, it is an isomorphism. Now Lemma 6.4 implies that we have a weak fibre sequence

$$(6.2) \quad K^{\mathrm{GL},\mathrm{an}}(A) \rightarrow \tilde{k}^{\mathrm{an}}(A; A_0) \rightarrow \tilde{k}_0(A; A_0).$$

Since $K^{\mathrm{GL},\mathrm{an}}(A)$ is connected, this gives (i).

Now assume that A satisfies $(\dagger)_A$. By Corollary 5.15 the map $K_0(A) \rightarrow \varinjlim_j K_0(A\langle\Delta_{\pi^j}^m\rangle)$ is an isomorphism for each m . By Lemma 6.4 again, we have a weak fibre sequence

$$(6.3) \quad K^{\mathrm{GL},\mathrm{an}}(A) \rightarrow k^{\mathrm{an}}(A) \rightarrow K_0(A).$$

This gives the first statement of (ii).

There is a natural map from (6.2) to (6.3). By what we have shown so far, it induces a weak equivalence $\tau_{\geq 1}\tilde{k}^{\mathrm{an}}(A; A_0) \xrightarrow{\cong} \tau_{\geq 1}k^{\mathrm{an}}(A)$. \square

The continuous K -theory $K^{\mathrm{cont}}(A)$ of A can be recovered from $\tilde{k}^{\mathrm{cont}}(A; A_0)$ via the Bass construction:

Lemma 6.14. *There is a natural equivalence of pro-spectra*

$$(\tilde{k}^{\mathrm{cont}})^B(A; A_0) \xrightarrow{\cong} K^{\mathrm{cont}}(A; A_0).$$

Proof. Since $K^{\mathrm{cont}}(A; A_0)$ (viewed as functor in A_0) satisfies the Bass fundamental theorem (see first paragraph of Subsection 5.2), the natural map $K^{\mathrm{cont}}(A; A_0) \rightarrow (K^{\mathrm{cont}})^B(A; A_0)$ is an equivalence and we have an induced map $(\tilde{k}^{\mathrm{cont}})^B(A; A_0) \rightarrow K^{\mathrm{cont}}(A; A_0)$. The cofibre $\mathrm{cofib}(\tilde{k}^{\mathrm{cont}}(A; A_0) \rightarrow K^{\mathrm{cont}}(A; A_0))$ is equivalent to $\mathrm{cofib}(\tau_{\leq -1}\tilde{k}^B(A_0) \rightarrow \tau_{\leq -1}K^{\mathrm{cont}}(A_0))$. By nil-invariance of negative algebraic K -theory of rings, the latter is a constant pro-spectrum, which is moreover 0-truncated. Now Lemma 4.16 implies that the Bass construction applied to the cofibre (again viewed as functor in A_0) vanishes, whence $(\tilde{k}^{\mathrm{cont}})^B(A; A_0) \rightarrow K^{\mathrm{cont}}(A; A_0)$ is an equivalence. \square

Definition 6.15. The *non-connective analytic K -theory* of the Tate ring A is defined by the analytic Bass construction applied to $K^{\mathrm{GL},\mathrm{an}}$:

$$K^{\mathrm{an}}(A) = (K^{\mathrm{GL},\mathrm{an}})^B(A).$$

Remark 6.16. Another candidate for the non-connective analytic K -theory would be the pro-spectrum $(k^{\mathrm{an}})^B(A)$. If A is regular and satisfies condition $(\dagger)_A$, both constructions yield weakly equivalent pro-spectra by the lemma below. In general, it seems however that $K^{\mathrm{an}}(A)$ has better formal properties. For example, one can prove that it has descent for admissible coverings, as we will explain in a sequel to this note.

Choose a ring of definition $A_0 \subset A$.

Lemma 6.17.(i) *The natural map*

$$K^{\text{an}}(A) = (K^{\text{GL,an}})^B(A) \rightarrow (\tilde{k}^{\text{an}})^B(A; A_0)$$

is a weak equivalence.(ii) *If A is regular and satisfies $(\dagger)_A$, then the natural map*

$$(\tilde{k}^{\text{an}})^B(A; A_0) \rightarrow (k^{\text{an}})^B(A)$$

is an equivalence.

Proof. Apply the analytic Bass construction to the weak fibre sequence (6.2). Then Lemma 4.16 implies (i). In the situation of (ii), Lemma 6.13(ii) implies that the cofibre of $\tilde{k}^{\text{an}}(A; A_0) \rightarrow k^{\text{an}}(A)$ is a constant pro-spectrum concentrated in homotopy degree 0. Lemma 4.16 again implies the assertion. \square

Theorem 6.18. *If the Tate ring A is regular and satisfies $(\dagger)_A$, there is a natural zig-zag of weak equivalences*

$$K^{\text{cont}}(A) \xrightarrow{\simeq} K^{\text{cont,an}}(A) \xleftarrow{\simeq} K^{\text{an}}(A).$$

Here $K^{\text{cont,an}}(A)$ is defined by first applying the Bass construction (4.5) to $\tilde{k}^{\text{cont,an}}(A; A_0)$ and then taking the colimit over the poset of rings of definition $A_0 \subset A$ as in Definition 5.3.

Proof. Fix a ring of definition $A_0 \subset A$. We apply the Bass construction (4.5) to the zig-zag of Lemma 6.12. By the previous Lemma $K^{\text{an}}(A) \rightarrow (\tilde{k}^{\text{an}})^B(A; A_0)$ is a weak equivalence of pro-spectra. Combining this with Lemma 6.14 we get the natural zig-zag of weak equivalences

$$K^{\text{cont}}(A; A_0) \xrightarrow{\simeq} (\tilde{k}^{\text{cont,an}})^B(A; A_0) \xleftarrow{\simeq} K^{\text{an}}(A).$$

Taking the colimit over the rings of definition $A_0 \subset A$ gives the result. \square

Corollary 6.19. *Assume that A is regular and satisfies $(\dagger)_A$. Then the pro-spectra $k^{\text{an}}(A)$ and $\tau_{\geq 0}K^{\text{an}}(A)$ are weakly equivalent. In particular, we have an isomorphism of pro-groups*

$$K_0(A) \cong K_0^{\text{an}}(A).$$

Proof. By Lemma 6.17 the canonical map $K^{\text{an}}(A) \rightarrow (k^{\text{an}})^B(A)$ is a weak equivalence and we claim that $k^{\text{an}}(A) \rightarrow (k^{\text{an}})^B(A)$ induces a weak equivalence on connective covers. We first consider connected covers. For this we choose a ring of definition $A_0 \subset A$. By Theorem 6.18 the pro-spectra $\tau_{\geq 1}K^{\text{an}}(A)$ and $\tau_{\geq 1}K^{\text{cont}}(A)$ are weakly equivalent. The latter is clearly equivalent to $\tau_{\geq 1}\tilde{k}^{\text{cont}}(A; A_0)$, which is weakly equivalent to $\tau_{\geq 1}\tilde{k}^{\text{an}}(A; A_0)$ by Lemma 6.12. Lemma 6.13 finally gives the desired weak equivalence with $\tau_{\geq 1}k^{\text{an}}(A)$. One checks that all the weak equivalences involved are compatible with the canonical maps $K^{\text{an}}(A) \rightarrow (k^{\text{an}})^B(A) \leftarrow k^{\text{an}}(A)$. This gives the claim on connected covers. For the π_0 statement we look at the composition $K_0(A) \rightarrow k_0^{\text{an}}(A) \rightarrow (k^{\text{an}})_0^B(A) \cong K_0^{\text{cont}}(A)$. The first map is an

isomorphism by Lemma 6.13, and the composition is an isomorphism by Proposition 5.10. Hence the second map is an isomorphism, as desired. \square

Remark 6.20. Let A be a Tate ring with ring of definition A_0 and ℓ a positive integer which is invertible in A_0 . A theorem of Weibel [TT90, Thm. 9.5] says that K -theory of $\mathbb{Z}[1/\ell]$ -algebras with \mathbb{Z}/ℓ -coefficients is homotopy invariant. Using this one can show that the analogs of Lemmas 6.12, 6.13, and 6.17 with \mathbb{Z}/ℓ -coefficients are true without assuming that A is regular. Hence the pro-spectra $K^{\text{cont}}(A; \mathbb{Z}/\ell)$ and $K^{\text{an}}(A; \mathbb{Z}/\ell)$ are weakly equivalent. Combining this with Lemma 5.12, we also get isomorphisms $K_i(A; \mathbb{Z}/\ell) \cong K_i^{\text{an}}(A; \mathbb{Z}/\ell)$ for $i > 0$.

7. ANALYTIC KV -THEORY

In this section we briefly explain the main points of the KV -analog of our analytic K -theory. Here $A = (A, \|\cdot\|)$ is a complete normed ring as in Subsection 3.3. In particular, A need not be commutative nor unital. If A is not unital, we set $\text{GL}(A) := \ker(\text{GL}(\tilde{A}) \rightarrow \text{GL}(\mathbb{Z}))$ using the unitalization \tilde{A} . As in Subsection 6.1 we get a pro-simplicial ring

$$\text{“lim”}_\rho A\langle\Delta_\rho\rangle.$$

Definition 7.1. The *analytic KV -theory* of A is defined as the pro-simplicial set

$$KV^{\text{an}}(A) = \text{“lim”}_\rho \text{BGL}(A\langle\Delta_\rho\rangle).$$

Its i -th homotopy pro-group is denoted by $KV_i^{\text{an}}(A)$.

We may also view $KV^{\text{an}}(A)$ as a pro-space, i.e. as an object of the ∞ -category $\text{Pro}(\mathbf{Spc}_*^0)$. Directly from the definition we get:

Lemma 7.2. *There are natural isomorphisms*

$$KV_i^{\text{an}}(A) \cong \text{“lim”}_\rho \pi_{i-1} \text{GL}(A\langle\Delta_\rho\rangle).$$

If $a = (a_{ij})$ is a matrix with coefficients in A , we set $\|a\| = \max\{\|a_{ij}\|\}$. This defines a norm on the ring of matrices $M(A)$.

Lemma 7.3. *There is a natural isomorphism*

$$KV_1^{\text{an}}(A) \cong \text{“lim”}_\rho \text{GL}(A) / \text{GL}(A)_\rho,$$

where $\text{GL}(A)_\rho$ is the subgroup of $\text{GL}(A)$ generated by matrices g such that $\lim_{n \rightarrow \infty} \|(g-1)^n\| \rho^n = 0$.

Proof. By the previous Lemma 7.2, it suffices to show that $\pi_0 \text{GL}(A\langle\Delta_\rho\rangle) \cong \text{GL}(A) / \text{GL}(A)_\rho$. We can identify $\pi_0 \text{GL}(A\langle\Delta_\rho\rangle) = \text{GL}(A) / G$ where G is the subgroup consisting of matrices g such that there exists a matrix $g(t) \in \text{GL}(A\langle t \rangle_\rho)$ with $g(0) = 1$ and $g(1) = g$. The same proof as in [KV71, App. 3] shows that $G = \text{GL}(A)_\rho$. \square

Remark 7.4. An s -bounded homomorphism between complete normed rings $\phi: A \rightarrow B$ (see Definition 3.9) induces a map of simplicial rings $A\langle\Delta_{\rho^{1/s}}\rangle \rightarrow B\langle\Delta_{\rho}\rangle$ and hence a map of pro-spaces $KV^{\text{an}}(A) \rightarrow KV^{\text{an}}(B)$. This implies that, up to natural equivalence, $KV^{\text{an}}(A)$ depends only on the equivalence class of the norm. Thus by Example 3.8(iii), the analytic KV -theory of a Tate ring is well defined.

Lemma 7.5. *If A is a regular Tate ring satisfying $(\dagger)_A$, then we have natural weak equivalences*

$$KV^{\text{an}}(A) \simeq \Omega^{\infty} \tau_{\geq 1} k^{\text{an}}(A) \simeq \Omega^{\infty} \tau_{\geq 1} K^{\text{an}}(A).$$

Proof. Since for each $\rho \geq 1$ the simplicial ring $A\langle\Delta_{\rho}\rangle$ is connected, Lemma 4.3 implies that $KV^{\text{an}}(A)$ is naturally equivalent to “ \lim_{ρ} ” $\Omega^{\infty} K^{\text{GL}}(A\langle\Delta_{\rho}\rangle)$. By Remark 3.10 the latter is equivalent to $\Omega^{\infty} K^{\text{GL,an}}(A)$. Hence the claim follows from (6.3) and Corollary 6.19. \square

The same proof as that of Proposition 6.3 shows:

Proposition 7.6. *The inclusions $A \subset A\langle t \rangle_{\sigma}$ induce an equivalence of pro-spaces*

$$KV^{\text{an}}(A) \xrightarrow{\sim} \text{“}\lim_{\sigma}\text{” } KV^{\text{an}}(A\langle t \rangle_{\sigma}).$$

We write $\text{GL}(A\langle t_1, \dots, t_n \rangle_{\rho})_0$ for the subgroup consisting of matrices f such that $f(0, \dots, 0) = 1$.

Definition 7.7. A bounded homomorphism $\phi: A \rightarrow B$ of complete normed rings is called a *pro-GL-fibration* if for every n the induced map

$$\text{“}\lim_{\rho}\text{” } \text{GL}(A\langle t_1, \dots, t_n \rangle_{\rho})_0 \rightarrow \text{“}\lim_{\rho}\text{” } \text{GL}(B\langle t_1, \dots, t_n \rangle_{\rho})_0$$

is an epimorphism of pro-sets.

Remarks 7.8. (i) A pro-GL-fibration is surjective. (For $b \in B$ consider $\begin{pmatrix} 1 & t_1 b \\ 0 & 1 \end{pmatrix}$.)

(ii) If $I \subset A$ is a closed ideal, we equip A/I with the residue norm. If I is moreover nilpotent, then $A \rightarrow A/I$ is a pro-GL-fibration. (Each $I\langle\Delta_{\rho}^n\rangle$ is nilpotent, hence $\text{GL}(A\langle\Delta_{\rho}^n\rangle) \rightarrow \text{GL}(A/I\langle\Delta_{\rho}^n\rangle)$ is surjective.)

Let $X = \text{“}\lim_j\text{” } X^{(j)}$ be a pro-simplicial set. Write $X_n = \text{“}\lim_j\text{” } X_n^{(j)}$ for the pro-set of its n -simplices. We call X *weakly discrete* if, for every n , the unique degeneracy map $s := s_0^n: X_0 \rightarrow X_n$ is an isomorphism of pro-sets.

Lemma 7.9. *If $X \rightarrow Y$ is a map of pro-simplicial sets such that each $X_n \rightarrow Y_n$ is an epimorphism of pro-sets, and if X is weakly discrete, then so is Y .*

Proof. Consider the commutative diagram of pro-sets

$$\begin{array}{ccc} X_0 & \xrightarrow{s} & X_n \\ \downarrow & \cong & \downarrow \\ Y_0 & \xrightarrow{s} & Y_n. \end{array}$$

Since $X_0 \rightarrow X_n$ is an isomorphism and $X_n \rightarrow Y_n$ is an epimorphism, $s: Y_0 \rightarrow Y_n$ is also an epimorphism. Since any composition of face maps $Y_n \rightarrow Y_0$ is a left-inverse, s is indeed an isomorphism. \square

Lemma 7.10. *If X is a pointed weakly discrete pro-simplicial set which is level-wise Kan, then $\pi_0(X) \cong X_0$ and the higher homotopy pro-groups vanish.*

Proof. The analog of Lemma 7.10 for truncated pro-simplicial sets is clear as they are weakly discrete if and only if they are equivalent to pro-systems of discrete simplicial sets. As the map $X \rightarrow X^\natural$ to the Postnikov tower, see [AM69, Ch. 4], is a weak equivalence under our assumption on X , one immediately reduces the lemma to the truncated case. \square

Analytic KV -theory satisfies excision for pro-GL-fibrations:

Proposition 7.11. *If $A \rightarrow B$ is a pro-GL-fibration of complete normed rings with kernel I , there is a long exact sequence of pro-abelian groups*

$$\cdots \rightarrow KV_i^{\text{an}}(I) \rightarrow KV_i^{\text{an}}(A) \rightarrow KV_i^{\text{an}}(B) \rightarrow KV_{i-1}^{\text{an}}(I) \rightarrow \cdots$$

ending in $KV_1^{\text{an}}(B) \rightarrow K_0(I) \rightarrow K_0(A) \rightarrow K_0(B)$.

Proof. Let G_ρ denote the image of the simplicial group $\text{GL}(A\langle\Delta_\rho\rangle)$ in $\text{GL}(B\langle\Delta_\rho\rangle)$. For each ρ we have a short exact sequence of simplicial groups

$$(7.1) \quad 1 \rightarrow \text{GL}(I\langle\Delta_\rho\rangle) \rightarrow \text{GL}(A\langle\Delta_\rho\rangle) \rightarrow G_\rho \rightarrow 1.$$

Since $A \rightarrow B$ is a pro-GL-fibration,

$$\text{“lim”}_\rho \text{GL}(A\langle\Delta_\rho^n\rangle) \times \text{GL}(B) \rightarrow \text{“lim”}_\rho \text{GL}(B\langle\Delta_\rho^n\rangle)$$

is an epimorphism of pro-sets. This implies that

$$\text{GL}(B) \rightarrow \text{“lim”}_\rho \text{GL}(B\langle\Delta_\rho^n\rangle)/G_{\rho,n}$$

is an epimorphism, too. Hence the pro-simplicial set $\text{“lim”}_\rho \text{GL}(B\langle\Delta_\rho\rangle)/G_\rho$ is weakly discrete by Lemma 7.9. For each ρ , the projection

$$\text{GL}(B\langle\Delta_\rho\rangle) \rightarrow \text{GL}(B\langle\Delta_\rho\rangle)/G_\rho$$

is a Kan fibration between Kan complexes with fibre G_ρ . Together with Lemma 7.10 this implies that

$$\text{“lim”}_\rho \pi_n(G_\rho) \cong \text{“lim”}_\rho \pi_n(\text{GL}(B\langle\Delta_\rho\rangle)) = KV_{n+1}^{\text{an}}(B)$$

for $n > 0$ and that we have an exact sequence

$$(7.2) \quad 0 \rightarrow \text{“lim”}_\rho \pi_0(G_\rho) \rightarrow \text{“lim”}_\rho \pi_0(\text{GL}(B\langle\Delta_\rho\rangle)) \rightarrow \text{“lim”}_\rho \text{GL}(B\langle\Delta_\rho^0\rangle)/G_{\rho,0} \rightarrow 0.$$

Note that $G_{\rho,0} = \text{im}(\text{GL}(A) \rightarrow \text{GL}(B))$ is a normal subgroup of $\text{GL}(B\langle\Delta_\rho^0\rangle) = \text{GL}(B)$ as $A \rightarrow B$ is surjective by Remark 7.8 and hence every elementary

matrix in $\mathrm{GL}(B)$ can be lifted to $\mathrm{GL}(A)$. By [Wei13, Exc. II.2.3] we have an exact sequence

$$(7.3) \quad 0 \rightarrow \mathrm{GL}(B)/\mathrm{im}(\mathrm{GL}(A) \rightarrow \mathrm{GL}(B)) \rightarrow K_0(I) \rightarrow K_0(A) \rightarrow K_0(B).$$

Splicing together the long exact sequence of homotopy groups associated to (7.1), the sequence (7.2), and (7.3), we get the desired result. \square

Corollary 7.12. *If $I \subset A$ is a closed, nilpotent ideal, then $KV_i^{\mathrm{an}}(A) \cong KV_i^{\mathrm{an}}(A/I)$ for all $i \geq 1$.*

Proof. Since I is nilpotent, $K_0(I) = 0$. By Remark 7.8(iii) and Proposition 7.11, it suffices to prove that $\mathrm{GL}(I\langle\Delta_\rho\rangle)$ is contractible for every ρ . Since $I\langle\Delta_\rho^n\rangle$ is nilpotent for each n and ρ , $\mathrm{GL}(I\langle\Delta_\rho\rangle) \cong 1 + \mathrm{M}(I\langle\Delta_\rho\rangle)$ and the contractibility is shown as in the proof of Claim 6.8. \square

Since the maps $A\langle s \rangle_\rho \xrightarrow{s \rightarrow 0} A$ have a section, they are pro-GL-fibrations. By Propositions 7.11 and 7.6 it follows that

$$(7.4) \quad \text{“lim”}_\rho KV_i^{\mathrm{an}}(sA\langle s \rangle_\rho) = 0$$

for every $i \geq 1$.

Lemma 7.13. *For $\rho > 1$, the map $sA\langle s \rangle_\rho \xrightarrow{s \rightarrow 1} A$ is a pro-GL-fibration.*

Proof. Write $t = (t_1, \dots, t_n)$. Fix $\rho' \geq 1$. There is a unique continuous homomorphism $A\langle t \rangle_{\rho\rho'} \rightarrow A\langle s \rangle_\rho\langle t \rangle_{\rho'}$ sending t_i to st_i . Given a matrix $g = g(t) \in \mathrm{GL}(A\langle t \rangle_{\rho\rho'})$ with $g(0) = 1$, we get a matrix $g(st) \in \mathrm{GL}(sA\langle s \rangle_\rho\langle t \rangle_{\rho'})$ whose image in $\mathrm{GL}(A\langle t \rangle_{\rho'})$ under the map $s \mapsto 1$ coincides with that of g , concluding the proof. \square

For $\rho > 1$ we set

$$\Omega_\rho A := \ker(sA\langle s \rangle_\rho \xrightarrow{s \rightarrow 1} A) = s(s-1)A\langle s \rangle_\rho.$$

Corollary 7.14. *We have natural isomorphisms*

$$KV_i^{\mathrm{an}}(A) \cong \text{“lim”}_\rho KV_{i-1}^{\mathrm{an}}(\Omega_\rho A)$$

for $i \geq 2$, and

$$KV_1^{\mathrm{an}}(A) \cong \text{“lim”}_\rho \ker(K_0(\Omega_\rho A) \rightarrow K_0(sA\langle s \rangle_\rho)).$$

Proof. For every $\rho > 1$ we have a long exact excision sequence

$$\begin{aligned} \cdots \rightarrow KV_i^{\mathrm{an}}(sA\langle s \rangle_\rho) \rightarrow KV_i^{\mathrm{an}}(A) \rightarrow KV_{i-1}^{\mathrm{an}}(\Omega_\rho A) \rightarrow KV_{i-1}^{\mathrm{an}}(sA\langle s \rangle_\rho) \rightarrow \\ \cdots \rightarrow KV_1^{\mathrm{an}}(A) \rightarrow K_0(\Omega_\rho A) \rightarrow K_0(sA\langle s \rangle_\rho) \rightarrow K_0(A). \end{aligned}$$

Now apply (7.4). \square

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