RAMIFICATION THEORY OF RECIPROCITY SHEAVES, I ZARISKI-NAGATA PURITY

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Abstract. We prove a Zariski-Nagata purity theorem for the motivic ramification filtration of a reciprocity sheaf. An important tool in the proof is a generalization of the Kato-Saito reciprocity map from geometric global class field theory to all reciprocity sheaves. As a corollary we obtain cut-by-curves and cut-by-surfaces criteria for various ramification filtrations. In some cases this reproves known theorems, in some cases we obtain new results.

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1. Introduction

In this paper we prove a Zariski-Nagata purity theorem for the motivic ramification filtration of a reciprocity sheaf, using a generalization of the Kato-Saito reciprocity map [KS86] to all reciprocity sheaves. As a corollary we obtain cut-by-curves and cut-by-surfaces criteria for various ramification filtrations.

1.1. Let X be a smooth scheme over a perfect field k and let D be an effective Cartier divisor on X such that its support has simple normal crossings and denote by $U = X \setminus D$ its complement. Denote by

$$H^1(U) := H^1(U_{\text{ét}}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cts}}(\pi_1^{\mathrm{ab}}(U), \mathbb{Q}/\mathbb{Z}), \tag{1.1.1}$$

the group of torsion characters of the abelianized fundamental group $\pi_1^{\mathrm{ab}}(U)$ of U. There are three different ways to express that a character $\chi \in H^1(U)$ has ramification bounded by D:

(i) For any generic point $\eta \in D$, we have $\mathrm{Art}_{K_\eta}(\chi_{|K_\eta}) \le v_\eta(D)$, where $K_\eta = \mathrm{Frac}(\mathcal{O}^h_{X,\eta})$ is the quotient field of the henselization of X at η, $v_\eta(D)$ is the multiplicity of η in D, and Art_{K_η} is the Kato-Matsuda Artin conductor for characters of the Galois group of K_η. (In the notation of [Mat97, Definition 3.2.5], $\mathrm{Art}_{K_\eta} = 1 + \mathrm{Sw}'_{K_\eta}$).

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(ii) For any k-morphism $h : C \to X$ with C a smooth curve, such that $h^{-1}(U)$ is dense in C, and any closed point $x \in h^{-1}(D)$, we have $\mathrm{Art}_{L_x}(\chi_{|L_x}) \leq v_x(h^*D)$, where $L_x = \mathrm{Frac}(\mathcal{O}^h_{C,x})$ and $v_x(h^*D)$ is the multiplicity of x in the pullback h^*D. [1]

(iii) Let $P^{(D)}_X$ be the blow-up of $X \times X$ in the center $\Delta_X \cap (D \times D)$ with the strict transforms of $X \times D$ and $D \times X$ removed, where Δ_X denotes the diagonal. Note that we have a natural open immersion $U \times U \hookrightarrow P^{(D)}_X$. Then,

$$p_1^*(\chi) - p_2^*(\chi) \in \mathrm{Im}\left(H^1(P^{(D)}_X) \hookrightarrow H^1(U \times U)\right),$$

where $p_i : U \times U \to U$ denotes the projection to the ith factor.

It is known that the above conditions are equivalent: the equivalence of (i) and (ii) follows from the ramification theory developed in [Kat89] and [Mat97] (see [KS16, Corollary 2.8]). The condition (iii) is introduced in Abbes-Saito's non-logarithmic ramification theory ([TSai17, Section 2, Subsection 3.1]) and the equivalence of (i) and (iii) follows from [AS11, Proposition 8.8], [TSai17, Proposition 2.27] and [Yat17, Theorem 0.1].

One aim of this paper (and its sequel [RSb]) is to generalize the statements above to all reciprocity sheaves and prove that they are still equivalent, see Theorem 1.4 below.

1.2. We recall the notion of reciprocity sheaves introduced by Kahn, Saito and Yamazaki, which is closely related to the theory of modulus sheaves with transfers developed by the same authors in joint work with Miyazaki, see [KSY16], [KSY22], [KMSY21a], and [KMSY21b]. Fix a perfect field k. Let **Cor** be the category whose objects are the smooth separated k-schemes and finite correspondences as morphisms. Let **PST** be the category of presheaves with transfers, i.e., the presheaves of abelian groups on **Cor**. A reciprocity presheaf F is such an object of **PST** that any section $a \in F(U)$ has *a modulus*: this condition is a generalization of Voevodsky's $\mathbf{A}^1$-invariance and is better understood using the theory of modulus presheaves with transfers, which we now recall. A *modulus pair* (X, D) consists of a separated scheme X of finite type over k and a (possibly empty) effective Cartier divisor D on X such that the complement $U = X \setminus D$ is smooth. The category of modulus pairs has as morphisms the finite correspondences between the smooth complements satisfying a certain admissibility condition with respect to the divisors. A presheaf on the category of modulus pairs is called a *modulus presheaf with transfers*, we denote by $\underline{\mathbf{M}}\mathbf{PST}$ the corresponding category. If the modulus pair (X, D) is proper, i.e., X is proper over k, then we can associate to it a particular presheaf with transfers, which is denoted by $h_0(X, D)$. It satisfies the following two properties, where we write $Z_K = Z \otimes_k K$ for a k-scheme Z and a function field K over k and where we extend a presheaf with transfers to essentially smooth k-schemes by taking colimits, see (2.2.3):

(a) $h_0(X, D)$ is a quotient of the representable presheaf with transfers $\mathbb{Z}_{\mathrm{tr}}(U) = \mathbf{Cor}(-, U)$, where $U = X \setminus D$, and

(b) for any function field K over k, the map $\mathbb{Z}_{\mathrm{tr}}(U)(K) \to h_0(X, D)(K)$ from (a) induces an isomorphism

$$\mathrm{CH}_0(X_K, D_K) \xrightarrow{\simeq} h_0(X, D)(K),$$

[1] Note that Art_{L_x} is the classical Artin conductor since the residue field of L_x is perfect.

via the identification of $\mathbb{Z}_{\mathrm{tr}}(U)(K)$ with $Z_0(U_K)$, the group of zero-cycles on U_K. Here $\mathrm{CH}_0(X_K|D_K)$ is the Chow group of zero-cycles with modulus introduced in [KS16], which is the quotient of $Z_0(U_K)$ by the subgroup generated by $\mathrm{div}_C(f)$, where C is an integral curve in U_K and f is a rational function on the normalization $\tilde{C}$ of the closure of C in X_K which satisfies $f \equiv 1 \bmod D_{\widetilde{C}}$, i.e., at the points of $D_{\widetilde{C}}$ the function $f-1$ is regular and contained in the ideal sheaf of $D_{\widetilde{C}}$, where $D_{\widetilde{C}} = D_K \times_{X_K} \widetilde{C}$.

In fact the Zariski sheafification of $h_0(X,D)$ is uniquely determined by (a) and (b).[2] A key role is played by the functor

$$\underline{\omega}^{\mathbf{CI}} : \mathbf{PST} \longrightarrow \underline{\mathbf{M}}\mathbf{PST},$$

given by

$$\underline{\omega}^{\mathbf{CI}} F(X,D) := \varinjlim_{(Y,E)} \mathrm{Hom}_{\mathbf{PST}}(h_0(Y,E),F), \quad \text{for } F \in \mathbf{PST},$$

where (X,D) is a modulus pair and the colimit is over the cofiltered ordered set of compactifications (Y,E) of (X,D), see [KMSY21a, 1.8]. Note that by (a) we have $\underline{\omega}^{\mathbf{CI}} F(X,D) \subset F(U)$. A modulus for $a \in F(U)$ is a proper modulus pair (X,D) as above such that a is contained in $\underline{\omega}^{\mathbf{CI}} F(X,D)$. We arrive at the following definition.

Definition 1.3. Let $F \in \mathbf{PST}$ and write $\widetilde{F} := \underline{\omega}^{\mathbf{CI}} F \in \underline{\mathbf{M}}\mathbf{PST}$. Following [KSY22], we say that F is a *reciprocity sheaf* if for any smooth k-scheme U the restriction F_U is a sheaf on the small Nisnevich site of U and any section $a \in F(U)$ has a modulus, i.e.,

$$F(U) = \bigcup_{(X,D)} \widetilde{F}(X,D), \tag{1.3.1}$$

where the union is over all proper modulus pairs (X,D) with $U = X \setminus D$. We denote by $\mathbf{RSC}_{\mathrm{Nis}} \subset \mathbf{PST}$ the full subcategory of reciprocity sheaves.

We say that a reciprocity sheaf F has *level* $n \geq 0$, if for any smooth k-scheme X and any $a \in F(\mathbf{A}^1 \times X)$ the following implication holds:

$$a_{\mathbf{A}^1_z} \in F(z) \subset F(\mathbf{A}^1_z), \quad \text{for all } z \in X_{(\leq n-1)} \Longrightarrow a \in F(X) \subset F(\mathbf{A}^1 \times X),$$

where $a_{\mathbf{A}^1_z}$ denotes the restriction of a to $\mathbf{A}^1_z = \mathbf{A}^1 \times z \subset \mathbf{A}^1 \times X$, $X_{(\leq n-1)}$ denotes the set of points in X whose closure has dimension $\leq n-1$, and for a smooth scheme S we identify $F(S)$ with its image in $F(\mathbf{A}^1 \times S)$ via pullback along the projection map. [3]

Any $\mathbf{A}^1$-invariant Nisnevich sheaf with transfers is a reciprocity sheaf of level 0. A relevant reciprocity sheaf to the introduction is $F = H^1$ from 1.1 and it has level 1. See [RS21, Part 2], [BRS22, §11.1], and 6.11 for other examples.

Heuristically, a presheaf with transfers F satisfies (1.3.1) if any section $a \in F(U)$ has "bounded ramification along boundaries of compactifications of U" and moreover F is $\mathbf{A}^1$-invariant if any $a \in F(U)$ has "tame ramification at boundaries". A manifestation of this viewpoint has been given in [RS21], where *the motivic conductor* associated to a reciprocity sheaf F is introduced. It is a collection of maps

$$c^F = \{c^F_L : F(L) \to \mathbb{N}\}_L$$

[2] This follows from the results in [KSY22] together with [BS19, Theorem 3.3].

[3] This is equivalent to the motivic conductor of F having level n in the language of [RS21].

where L runs through all henselian discrete valuation fields of geometric type over k, defined for $a \in F(L)$ by

$$c_L^F(a) = \min\{n \in \mathbb{N} \mid a \in \widetilde{F}(\operatorname{Spec} \mathcal{O}_L, n \cdot s_L)\},$$

where $s_L \in \operatorname{Spec} \mathcal{O}_L$ is the closed point. It is shown in [RS21] that c^F recovers classically known conductors: for $F = H^1$ in 1.1, $c_L^F = \mathrm{Art}_L$ from (i); for $F = \mathrm{Conn}^1$, the group of rank 1 connections, c_L^F is the irregularity (up to a shift); if F is represented by a commutative algebraic group and $\mathrm{trdeg}(L/k) = 1$, c_L^F is Rosenlicht-Serre's conductor. This is why for $F \in \mathbf{RSC}_{\mathrm{Nis}}$ we call

$$\widetilde{F}(X, D) \subset F(U), \quad \text{with } (X, D) \text{ proper modulus pairs with } X \setminus D = U, \tag{1.3.2}$$

the *motivic ramification filtration on* $F(U)$.

We say that *resolutions of singularities hold over* k *in dimension* $\leq n$ if for any integral projective k-scheme Z of dimension $\leq n$ and any effective Cartier divisor E on Z, there exists a proper birational morphism $h : Z' \to Z$ such that Z' is regular and $|h^{-1}(E)|$ has simple normal crossings. This is known to hold if $\mathrm{char}(k) = 0$ by Hironaka or if $n \leq 3$ by [CP09]. The following is a consequence of the main result of this paper, see Corollary 6.10.

Theorem 1.4. *Let $\overline{X}$ be a smooth projective k-scheme and $\overline{D}$ be an effective Cartier divisor on $\overline{X}$ whose support has simple normal crossings. Let $X \subset \overline{X}$ be a non-empty open subscheme, $D = \overline{D} \cap X$ and $U = X \setminus |D|$. Let F be a reciprocity sheaf of level $n \geq 0$. Assume resolutions of singularities hold over k in dimension $\leq n$. For $\chi \in F(U)$, the following statements are equivalent:* [4]

- *(i) For any generic point $\eta \in D$, we have $c_{K_\eta}^F(\chi_{|K_\eta}) \leq v_\eta(D)$ (cf. 1.1(i)).*
- *(ii) For any k-morphism $h : Z \to X$ with Z smooth quasi-projective of dimension $\leq n$ such that the support of $D_Z = h^*D$ has simple normal crossings and for any generic point $x \in D_Z$, we have $c_{L_x}^F(\chi_{|L_x}) \leq v_x(D_Z)$, where $L_x = \mathrm{Frac}(\mathcal{O}_{Z,x}^h)$ (cf. 1.1(ii)).*
- *(iv) $\chi \in \widetilde{F}(X, D)$.*

Building on the present article and [RSa], we will show in [RSb] the equivalence of (iv) and the following statement:

- (iii) Let $P_X^{(D)}$ and $p_1, p_2 : U \times U \rightrightarrows U$ be as in 1.1(iii). Then,

$$p_1^*(\chi) - p_2^*(\chi) \in \mathrm{Im}\left(F(P_X^{(D)}) \xrightarrow{j^*} F(U \times U)\right).$$

Note that the equivalence of (i) and (iv) proven in the present paper reads as the equality

$$\widetilde{F}(X, D) = F_{\mathrm{gen}}(X, D). \tag{1.4.1}$$

Here by definition

$$F_{\mathrm{gen}}(X, D) := \mathrm{Ker}\left(F(U) \to \bigoplus_\eta \frac{F(\operatorname{Spec} \mathcal{O}_{X,\eta}^h \setminus \eta)}{\widetilde{F}(\operatorname{Spec} \mathcal{O}_{X,\eta}^h, D_\eta^h)}\right),$$

where the direct sum is over the generic points η of D and $D_\eta^h = D \times_X \operatorname{Spec} \mathcal{O}_{X,\eta}^h$. This can be viewed as a Zariski-Nagata purity for the motivic ramification filtration.

[4] In case $X = \overline{X}$, the implications $(i) \Leftrightarrow (iii) \Leftrightarrow (iv) \Rightarrow (ii)$ hold without assuming resolution of singularities in dimension $\leq n$.

If the support of D is smooth, this also follows from [Sai20, Corollary 8.6(2)] and if D is reduced, it follows from [Sai, Theorem 2.3]. The general case treated here is new. On the other hand, the equivalence of (iv) and (ii) in the case $n = 1$ is viewed as a cut-by-curves criterion for the motivic ramification filtration. See 6.11 for examples where we obtain unconditional results, in particular the cut-by-surfaces criterion for the ramification of fppf-torsors under infinitesimal finite unipotent group schemes over k is new.

1.5. It is usually very difficult to compute Chow groups with modulus used in the definition of the motivic ramification filtration (see the description of $h_0(X, D)$ especially 1.2(b) above) so that it seems hopeless to compute the filtration plainly from its definition. On the other hand, condition (iii) from above, which is considered in [RSb], is easy to check and it provides an effective method to compute the filtration. In the proof of Theorem 1.4, we use another effective method for the computation, called *reciprocity pairing*, see (1.5.3) and Theorem 1.6 below.

Let K be a function field over k and let X be a reduced and projective K-scheme of pure dimension d. We denote by $K^M_{r,X}$ $(r \geq 0)$ the Nisnevich sheafification of the improved Milnor K-theory from [Ker10]. For a nowhere dense closed subscheme $D \subset X$, we consider the following Nisnevich sheaves for $r \geq 1$[5]

$$V_{r,X|D} := \mathrm{Im}(\mathcal{O}^\times_{X|D} \otimes_{\mathbb{Z}} K^M_{r-1,X} \to K^M_{r,X}) \quad \text{with } \mathcal{O}^\times_{X|D} := \mathrm{Ker}(\mathcal{O}^\times_X \to i_*\mathcal{O}^\times_D).$$

Let $U \subset X$ be a regular dense open subscheme such that $U \cap D = \emptyset$. For a closed point $x \in U$, the Gersten resolution ([Ker10, Proposition 10(8)]) yields an isomorphism

$$\theta_x : \mathbb{Z} \xrightarrow{\simeq} H^d_x(U_{\mathrm{Nis}}, K^M_{d,U}) \cong H^d_x(X_{\mathrm{Nis}}, V_{d,X|D}),$$

which induces a surjective map (cf. [KS86, Theorem 2.5]):

$$\theta_U = \sum_x \theta_x : Z_0(U) = \bigoplus_{x \in U} \mathbb{Z} \twoheadrightarrow H^d(X_{\mathrm{Nis}}, V_{d,X|D}), \tag{1.5.1}$$

where x runs through all closed points of U. Consider the pairing

$$(-,-)_{U \subset X/K} : F(U) \otimes Z_0(U) \to F(K), \quad a \otimes [x] \mapsto (g_x)_* i^*_x(a), \tag{1.5.2}$$

where $i_x : x \to U$ is the closed immersion and $(g_x)_* : F(x) \to F(K)$ is the transfer map for the finite map $g_x : x \to \operatorname{Spec} K$ induced by $X \to \operatorname{Spec} K$. A key result used in the proof of Theorem 1.4 is Proposition 6.7, stating that (1.5.2) induces a pairing

$$(-,-)_{(X,D)/K} : F_{\mathrm{gen}}(X, D) \otimes H^d(X_{\mathrm{Nis}}, V_{d,X|D}) \to F(K), \tag{1.5.3}$$

where $F_{\mathrm{gen}}(X, D)$ is defined similarly as in (1.4.1), in particular it depends only on the points of codimension at most 1 in the normalization of X.

In Examples and Remarks 6.9, it is explained that the pairing (1.5.2) recovers some classically known pairings: in case k is finite and $F(X) = H^1(X)$ from 1.1, it recovers the reciprocity map constructed in [KS86, (3.7)]. In case $\mathrm{char}(k) = 0$ and $F(X)$ is the group of isomorphism classes of absolute rank one connections on X relative to k, this pairing is a higher-dimensional version of the pairing constructed in [BE01, 4.] for X a curve. See also 6.9(3) for a pairing involving the non-$\mathbf{A}^1$-invariant part of étale motivic cohomology with mod-p^n coefficients induced by the pairing (1.5.3), which is reminiscent of the pairings constructed in [JSZ18] and [GK], though in different cohomological degrees.

[5] See 6.3 for a comparison of $V_{r,X|D}$ with the sheaf $K^M_r(\mathcal{O}_X, I_D) = \mathrm{Ker}(K^M_{r,X} \to K^M_{r,D})$ used in [KS86, (1.3)], where $I_D \subset \mathcal{O}_X$ is the ideal sheaf for D.

Theorem 1.6 (Theorem 6.8)**.** *Let F be a reciprocity sheaf. Let X be a smooth projective k-scheme of pure dimension d and D an effective Cartier divisor on X whose support has simple normal crossings. Denote by $j : U = X \setminus D \hookrightarrow X$ the open immersion. For $a \in F(U)$, the following conditions are equivalent:*

(i) $a \in \widetilde{F}(X, D)$;

(ii) $a \in F_{\mathrm{gen}}(X, D)$;

(iii) for any function field K over k, the map

$$(a_K, -)_{U_K \subset X_K/K} : Z_0(U_K) \to F(K)$$

induced by the pairing (1.5.2) factors through $H^d(X_{K,\mathrm{Nis}}, V_{d,X_K|D_K})$, where $a_K \in F(U_K)$ with $U_K = U \otimes_k K$ is the pullback of a.

Note that the assumption on the smoothness of X in the above theorem cannot be relaxed to just requiring the complement $X \setminus D$ to be smooth. Indeed, in Theorem 7.1 we show as an application of the above that a normal Cohen-Macaulay scheme Y which is of finite type over a field of characteristic zero has rational singularities if and only if there exists locally on Y an effective Cartier divisor E whose support contains the singular locus of Y, such that the natural map $\widetilde{\Omega^d}(Y, E) \to \Omega^d_{\mathrm{gen}}(Y, E)$ is bijective.

We sketch the strategy of the proof of Theorem 1.6, which implies Theorem 1.4. The implication *(i)* $\Rightarrow$ *(ii)* is direct from the definition. The implication of *(iii)* $\Rightarrow$ *(i)* is a formal consequence of 1.2(b) above, [Sai20, Theorem 3.1] and a commutative diagram

$$\begin{tikzcd} & Z_0(U_K) \arrow[dl, "\pi"'] \arrow[dr, "\theta_{U_K}"] & \\ \mathrm{CH}_0(X_K|D_K) \arrow[rr, "cyc_{X_K|D_K}"] & & H^d(X_{K,\mathrm{Nis}}, V_{d,X_K|D_K}) \end{tikzcd}$$

where π is the quotient map, θ_{U_K} is the map (1.5.1) and $cyc_{X_K|D_K}$ is a cycle class map constructed in [RS23], cf. Proposition 6.4.

Thus the most essential point is the implication *(ii)* $\Rightarrow$ *(iii)* in Theorem 1.6. This requires to construct the pairing (1.5.3). To this end we extend the formalism of pushforward for cohomology of reciprocity sheaves from [BRS22] to that for some ad hoc version of compactly supported cohomology, which is done in section 4. Using this pushforward, we construct an approximation of the desired pairing, see (6.5.3) and Lemma 6.6. To show that this pairing induces (1.5.3), we use a description of the cohomology group $H^d(X_{\mathrm{Nis}}, V_{d,X|D})$ in terms of the sections of $V_{d,X|D}$ over henselizations of X along Paršin chains from [KS86], which is recalled in section 5. This reduces the problem to a purely local situation in which case it follows from Theorem 3.7, which relies on a weak version of the Brylinski-Kato formula for reciprocity sheaves, see Corollary 3.3 and see Remark 3.4 for an explanation why this is called (weak) Brylinski-Kato formula[6].

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Notation 1.7. For a noetherian ring R, $\mathrm{Frac}(R)$ denotes its total ring of fractions.

[6]The full Brylinski-Kato formula for reciprocity sheaves will be proven in [RSb]

In the following k denotes a field and $\mathbf{Sm}$ the category of separated schemes which are smooth and of finite type over k. For k-schemes X and Y we write $X \times Y := X \times_k Y$. For $n \geq 0$ we write $\mathbf{P}^n := \mathbf{P}^n_k$, $\mathbf{A}^n := \mathbf{A}^n_k$. We say a field L is a *henselian dvr of geometric type over* k, if there exists $U \in \mathbf{Sm}$ and $x \in U$ a 1-codimensional point, such that $L = \mathrm{Frac}(\mathcal{O}^h_{U,x})$; we denote the ring of integers of L by $\mathcal{O}_L$.

For a scheme X we denote by $X_{(i)}$ (resp. $X^{(i)}$) the set of i-dimensional (resp. i-codimensional) points of X. If $(R, \mathfrak{m})$ is a local ring, then we denote by

$$R\{x_1, \ldots, x_n\}$$

the henselization of the polynomial ring $R[x_1, \ldots, x_n]$ at the ideal $\mathfrak{m}R[x_1, \ldots, x_n] + (x_1, \ldots, x_n)$.

Let F be a Nisnevich sheaf on a scheme X and $x \in X$ a point. Then we denote by F_x its Zariski stalk and by $F^h_x = \varinjlim_{x \in U/X} F(U)$ the Nisnevich stalk, where the colimit is over all Nisnevich neighborhoods $U \to X$ of x.

2. Preliminaries

In this section k is a perfect field.

2.1. We recall some basic notions from [KMSY21a] and [KMSY21b]. A *modulus pair* is a pair $\mathcal{X} = (X, D)$, where X is a separated scheme of finite type over k, D is an effective Cartier divisor on X ($D = \emptyset$ is allowed) and $X \setminus |D|$ is smooth; $\mathcal{X}$ is said to be proper if X is. For modulus pairs $\mathcal{X} = (X, D)$ and $\mathcal{Y} = (Y, E)$ we denote by $\underline{\mathbf{M}}\mathbf{Cor}(\mathcal{X}, \mathcal{Y})$ the free abelian group generated by integral closed subschemes $V \subset X \setminus |D| \times Y \setminus |E|$ which are finite and surjective over a connected component of $X \setminus |D|$ and such that $\overline{V}^N \to X \times Y$ the normalization of the closure of V is proper over X and satisfies

$$D_{|\overline{V}^N} \geq E_{|\overline{V}^N}. \tag{2.1.1}$$

The category of modulus pairs with morphisms $\underline{\mathbf{M}}\mathbf{Cor}(\mathcal{X}, \mathcal{Y})$ and composition induced by the composition of finite correspondences is denoted by $\underline{\mathbf{M}}\mathbf{Cor}$; its full subcategory of proper modulus pairs by $\mathbf{MCor}$. We have a fully faithful functor $\mathbf{Cor} \to \underline{\mathbf{M}}\mathbf{Cor}$, $X \mapsto (X, \emptyset)$, and we will abbreviate notation and write

$$(X, \emptyset) =: X \quad \text{in } \underline{\mathbf{M}}\mathbf{Cor}\,. \tag{2.1.2}$$

Furthermore $\underline{\mathbf{M}}\mathbf{PST}$ (resp. $\underline{\mathbf{M}}\mathbf{NST}$) denotes the category of presheaves (resp. Nisnevich sheaves) on $\underline{\mathbf{M}}\mathbf{Cor}$, similarly with $\mathbf{MPST}$ and $\mathbf{MNST}$, for details see *loc. cit.*

There is a symmetric monoidal structure on $\underline{\mathbf{M}}\mathbf{Cor}$ defined by

$$(X, D) \otimes (Y, E) = (X \times Y, p_X^* D + p_Y^* D), \tag{2.1.3}$$

where p_X and p_Y define the projections to X and Y, respectively. It extends to a symmetric monoidal structure $\otimes_{\underline{\mathbf{M}}\mathbf{PST}}$ on $\underline{\mathbf{M}}\mathbf{PST}$.

2.2. We denote by $\mathbf{PST}$ (resp. $\mathbf{NST}$) the category of presheaves (resp. Nisnevich sheaves) with transfers on $\mathbf{Sm}$. Let $F \in \mathbf{PST}$. Following [KSY22, Definition 2.2.4] we say $a \in F(U)$ has *modulus* $\mathcal{X} = (X, D)$ (called SC-modulus in *loc. cit.*), if $\mathcal{X}$ is a proper modulus pair with $U = X \setminus |D|$ and the Yoneda map $a : \mathbb{Z}_{\mathrm{tr}}(U) :=$

$\mathbf{Cor}(-,U) \to F$ factors via the quotient $q : \mathbb{Z}_{\rm tr}(U) \twoheadrightarrow h_0(\mathcal{X})$, where $h_0(\mathcal{X})$ is the presheaf with transfers given on $S \in \mathbf{Sm}$ by

$$(2.2.1)\quad h_0(\mathcal{X})(S) := \mathrm{Coker}\left(\underline{\mathbf{M}}\mathbf{Cor}(\overline{\square} \otimes S, \mathcal{X}) \xrightarrow{i_0^* - i_1^*} \mathbb{Z}_{\rm tr}(U)(S)\right), \quad \overline{\square} = (\mathbf{P}^1, \infty),$$

with $i_e : \{0\} \to \mathbf{P}^1$ the inclusion of the e-section, $e \in \{0,1\}$.

We say F is a reciprocity presheaf if any $a \in F(U)$ has a modulus, for any $U \in \mathbf{Sm}$. The category of reciprocity presheaves is denoted by $\mathbf{RSC}$ and $\mathbf{RSC}_{\rm Nis} = \mathbf{RSC} \cap \mathbf{NST}$ denotes the category of reciprocity sheaves; both are abelian, for the latter see [Sai20, Theorem 0.1].

For $F \in \mathbf{RSC}_{\rm Nis}$ and $\mathcal{X} = (X, D)$ we denote by $\widetilde{F}(X, D)$ those sections $a \in F(X \setminus |D|)$ having a modulus of the form $(\overline{X}, \overline{D} + B)$, where $\overline{D}$ and B are effective Cartier divisors such that $X = \overline{X} \setminus |B|$ and $\overline{D}_{|X} = D$. Then $\mathcal{X} \mapsto \widetilde{F}(\mathcal{X})$ defines a Nisnevich sheaf on $\underline{\mathbf{M}}\mathbf{Cor}$ and with the notation from [KMSY21a, 2.4] and [KSY22, Proposition 2.3.7] we have

$$(2.2.2)\qquad \widetilde{F} = \tau_! \omega^{\mathbf{CI}} F,$$

which is also equal to $\underline{\omega}^{\mathbf{CI}} F$ from the introduction. Moreover $\widetilde{F}$ is $\overline{\square}$-invariant, semi-pure, and has M-reciprocity, i.e.,

$$\widetilde{F} \in \mathbf{CI}^{\tau, sp}_{\rm Nis} \subset \underline{\mathbf{M}}\mathbf{NST},$$

see e.g. [MS, 1.] for notation and the references there. We also recall that for any $G \in \mathbf{CI}^{\tau, sp}_{\rm Nis}$, we have $\underline{\omega}_! G \in \mathbf{RSC}_{\rm Nis}$ and $\underline{\omega}_! \widetilde{F} = F$, where

$$\underline{\omega}_! G(X) = G(X, \emptyset).$$

Finally if $(X, D) = \varprojlim_i (X_i, D_i)$ is a pro-modulus pair in the sense of [RS21, 3.7] we define

$$(2.2.3)\qquad G(X, D) = \varinjlim_i G(X_i, D_i).$$

If R is a regular noetherian k-algebra and $I \subset R$ is an ideal which is invertible defining a Cartier Divisor D_I on $\mathrm{Spec}\, R$, then $(\mathrm{Spec}\, R, D_I)$ is a pro-modulus pair (by a result of Popescu [Pop86]) and we set

$$(2.2.4)\qquad G(R, I^{-1}) := G(\mathrm{Spec}\, R, D_I).$$

Usually we will refer to a pro-modulus pair simply as modulus pair.

2.3. We recall some twists introduced in [RSY21, 5e] and [MS, 2.]. By [RSY21, Corollary 4.18] there is a lax monoidal structure on $\mathbf{RSC}_{\rm Nis}$ which in particular assigns to $F_1, F_2 \in \mathbf{RSC}_{\rm Nis}$ a new reciprocity sheaf denoted by

$$(F_1, F_2)_{\mathbf{RSC}_{\rm Nis}}$$

(denoted by $h_{0,\rm Nis}(\widetilde{F}_1 \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \widetilde{F}_2)$ in *loc. cit.*). Let $F \in \mathbf{RSC}_{\rm Nis}$. For $n \geq 0$ we define

$$(2.3.1)\qquad F\langle 0\rangle := F, \quad F\langle n+1\rangle := (F\langle n\rangle, \mathbf{G}_m)_{\mathbf{RSC}_{\rm Nis}},$$

and

$$(2.3.2)\qquad \gamma^0 F := F, \quad \gamma^1 F := \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, F), \quad \gamma^{n+1} F := \gamma^1(\gamma^n F).$$

For $F \in \mathbf{RSC}_{\rm Nis}$ and $n, m \geq 0$ we have

(1) $\gamma^n F$, $F\langle n\rangle \in \mathbf{RSC}_{\rm Nis}$ and $F\langle n+m\rangle = F\langle m\rangle\langle n\rangle$, $\gamma^{m+n} F = \gamma^n(\gamma^m F)$;

(2) $F \in \mathbf{HI}_{\mathrm{Nis}}$ (=$\mathbf{A}^1$-invariant sheaves in $\mathbf{NST}$) $\Rightarrow F\langle n\rangle = F \otimes_{\mathbf{HI}} \mathbf{G}_m^{\otimes_{\mathbf{HI}} n}$ (see [RSY21, Theorem 5.3]);
(3) there is a natural surjection $F \otimes_{\mathbf{NST}} K_n^M \twoheadrightarrow F\langle n\rangle$ in $\mathbf{NST}$ and an isomorphism $\gamma^n F = \underline{\mathrm{Hom}}_{\mathbf{PST}}(K_n^M, F)$, see [BRS22, Proposition 9.3], where K_n^M denotes the improved n-th Milnor K-sheaf from [Ker10].
(4) Furthermore we will use the following shorthand in the rest of the text

$$\gamma^n F\langle m\rangle := \gamma^n(F\langle m\rangle) = \underline{\mathrm{Hom}}_{\mathbf{PST}}(K_n^M, F\langle m\rangle).$$

The weak cancellation theorem in the form [MS, Theorem 5.2] yields

$$\gamma^n F\langle m\rangle \cong \begin{cases} F\langle m-n\rangle, & m \geq n, \\ \gamma^{m-n} F, & n \geq m. \end{cases}$$

3. A Brylinski-Kato formula and continuity of a pairing

In this section k is a perfect field. We fix $G \in \mathbf{CI}_{\mathrm{Nis}}^{\tau,sp}$ and we will use the shorthand notation

$$G(X) := \underline{\omega}_! G(X) := G(X, \emptyset). \tag{3.0.1}$$

Proposition 3.1. *Let $(R, \mathfrak{m})$ be a regular henselian local k-algebra of geometric type and set $\mathcal{X} = \operatorname{Spec} R\{t\}$ (see Notation 1.7), $\mathcal{Z} = V(t) \subset \mathcal{X}$. For $a \in R\{t\}$ and $n \geq 2$ denote by $f_a : \mathcal{X} \to \mathcal{X}$ the morphism induced by the unique local R-algebra morphism $R\{t\} \to R\{t\}$ with $t \mapsto t + at^n$. Then f_a defines a morphism of modulus pairs $f_a : (\mathcal{X}, n\mathcal{Z}) \to (\mathcal{X}, n\mathcal{Z})$ and the map induced by pullback along f_a*

$$f_a^* : \frac{G(\mathcal{X}, n\mathcal{Z})}{G(\mathcal{X})} \longrightarrow \frac{G(\mathcal{X}, n\mathcal{Z})}{G(\mathcal{X})}$$

is equal to the identity.

We first need the following lemma which is a version of [Sai20, Lemma 6.7].

Lemma 3.2. *Let R be an excellent normal henselian local ring with infinite residue field, set $S = \operatorname{Spec} R$, and denote by $s \in S$ its closed point. Let $\pi : X \to S$ be a smooth morphism of relative dimension one and let $Z \subset X$ be a closed irreducible subscheme which is finite over S and such that we have an isomorphism of residue fields $\kappa(s) \xrightarrow{\simeq} \kappa(x)$, where $x \in Z$ denotes the closed point. Then*

$$\mathcal{O}_{X,x}^h = \varinjlim_{(U, x_U) \to (X,x)} \mathcal{O}(U),$$

where the colimit is over all Nisnevich neighborhoods $u : (U, x_U) \to (X, x)$ satisfying the following properties

(1) U is affine;
(2) u induces an isomorphism $Z_U := u^{-1}(Z) \xrightarrow{\simeq} Z$;
(3) $(\pi \circ u : U \to S, Z_U)$ has a good compactification in the sense of [Sai20, Definition 2.1]. *(Recall that this means that $\pi \circ u$ factors as $U \xrightarrow{j} \overline{U} \xrightarrow{\overline{\pi}_U} S$ with j an open immersion, $\overline{\pi}_U$ proper, $\overline{U}$ is normal, $\overline{U} \setminus U$ is the support of an effective Cartier divisor U_∞, and $Z_U \sqcup U_\infty \subset$ affine open of $\overline{U}$.)*

Proof. First we show that (X, x) admits one Nisnevich neighborhood with the stated property. Note that by assumption $Z = \operatorname{Spec} R'$, with R' henselian local. By [Lev06, Theorem 10.0.1] (see also the explanations in [Lev06, 5.1]) we find a Nisnevich neighborhood $v : (V, x_V) \to (X, x)$ which admits a closed immersion $i : V \hookrightarrow \mathbf{A}_S^n$, such that its closure $\overline{V} \subset \mathbf{P}^n_S$ admits a finite and surjective map to $\mathbf{P}^1_S$ and such that $\overline{V} \setminus V$ is finite over S. In particular, $\overline{V} \setminus V = (\mathbf{P}^n_S \setminus \mathbf{A}^n_S) \cap \overline{V}$ is a Cartier divisor in $\overline{V}$ and $\overline{V}$ is equidimensional of relative dimension one over S. Denote by $Y \to \overline{V}$ the normalization; it is an isomorphism over V, moreover it is a finite morphism, since S is excellent. Since Z is henselian local, we can write $v^{-1}(Z) = Z_V \sqcup C$ with $Z_V \cong Z$ and the components of C are either finite over Z or don't meet the special fiber. Since $\overline{V} \setminus V$ is finite over S, the intersection of the closure $\overline{C} \subset Y$ with the special fiber Y_s is finite; hence by [Sai20, Lemma 12.1] $\overline{C}$ is finite over S. Since Z_V is finite over S, we have $\overline{C} \cap Z_V = \emptyset$. By [GLL15, Theorem 5.1] we find an effective ample Cartier divisor $H \subset Y$ such that $\overline{C} \subset H$, $H \cap (Z_V \cap Y_s) = \emptyset$, and H does not contain Y_s. By [Sai20, Lemma 12.1], we have $H \cap Z_V = \emptyset$ and H is finite over S. Similarly, we find an effective ample Cartier divisor $A \subset Y$ such that $H \cup (Y \setminus V) \cup Z_V \subset Y \setminus A$; note that $Y \setminus A$ is affine. We set $U := V \setminus (V \cap H)$ and $x_U := x_V$. Then $(U, x_U) \to (X, x)$ is a Nisnevich neighborhood which satisfies (1) - (3), where we can take $U \hookrightarrow Y$ as a good compactification of $(U \to S, Z_U)$.

If $\nu : (X', x') \to (X, x)$ is any Nisnevich neighborhood, we find a $Z' \subset \nu^{-1}(Z)$ mapping isomorphically to Z. Now replacing (X, Z, x) in the above discussion by (X', Z', x') shows that any Nisnevich neighborhood of (X, x) has a refinement satisfying (1) - (3). □

Recall from [Sai20, Definition 5.1] that a pair $(X \to S, Z)$ as in Lemma 3.2 is by definition a *nice V-pair* if it admits a good compactification, Z is reduced and étale over S, and nZ diagonally embedded into $nZ \times_S X$ is a principal divisor, for all $n \geq 1$.

Proof of Proposition 3.1. By the standard trace argument we may assume $R/\mathfrak{m}$ is infinite. Set $S = \operatorname{Spec} R$, $X = \operatorname{Spec} R[t]$, and $Z = V(t) \subset X$. The pair $(X \to S, Z)$ is a nice V-pair by [Sai20, Lemma 5.2]. Let $x \in X$ be the point corresponding to $\mathfrak{m} + (t)$. Let $A = R\{t\}$ and take $a \in A$. By Lemma 3.2 we find an $R[t]$-algebra $A_a \subset A$ satisfying the following conditions:

(i) $a \in A_a$ and $1 + at^{n-1} \in A_a^\times$;
(ii) $X_a = \operatorname{Spec}(A_a)$ is a Nisnevich neighborhood of (X, x);
(iii) the natural map $j_a : X_a \to X$ induces an isomorphism $Z_a := j_a^{-1}(Z) \cong Z$;
(iv) $(X_a \to S, Z_a)$ admits a good compactification.

By [Sai20, Lemma 4.2, Lemma 4.3], (X_a, Z_a) is a nice V-pair over S. Consider the S-morphism

$$\phi_a : X_a \to X \quad \text{induced by } R[t] \to A_a, \quad t \mapsto t + at^n = t(1 + at^{n-1}).$$

By (i) and (iii), $Z_a \cong \phi_a^{-1}(Z)$. Let

$$\lambda : \mathcal{X} \to X \quad \text{and} \quad \lambda_a : \mathcal{X} \to X_a$$

be the maps induced by the natural inclusions

$$R[t] \to A \quad \text{and} \quad A_a \to A,$$

respectively. Note that $\lambda = j_a \circ \lambda_a$. We obtain the following commutative diagrams

$$\begin{tikzcd} \frac{G(X,nZ)}{G(X)} \arrow[r, "\lambda^*"] \arrow[d, "j_a^*"] & \frac{G(\mathcal{X},n\mathcal{Z})}{G(\mathcal{X})} \arrow[d, "id"] \\ \frac{G(X_a,nZ_a)}{G(X_a)} \arrow[r, "\lambda_a^*"] & \frac{G(\mathcal{X},n\mathcal{Z})}{G(\mathcal{X})} \end{tikzcd} \qquad \begin{tikzcd} \frac{G(X,nZ)}{G(X)} \arrow[r, "\lambda^*"] \arrow[d, "\phi_a^*"] & \frac{G(\mathcal{X},n\mathcal{Z})}{G(\mathcal{X})} \arrow[d, "f_a^*"] \\ \frac{G(X_a,nZ_a)}{G(X_a)} \arrow[r, "\lambda_a^*"] & \frac{G(\mathcal{X},n\mathcal{Z})}{G(\mathcal{X})} \end{tikzcd}.$$

The horizontal arrows are isomorphisms by [Sai20, Corollary 2.21] and Lemma 3.2. Thus it suffices to show

$$\phi_a^* = j_a^* : \frac{G(X, nZ)}{G(X)} \to \frac{G(X_a, nZ_a)}{G(X_a)}. \tag{3.2.1}$$

Set

$$h_a = s - (t + at^n) \in \Gamma(X_a \times_S X, \mathcal{O}) = A_a[s],$$

where $X = \operatorname{Spec} R[s]$. Then $\operatorname{Div}(h_a)$ and $\operatorname{Div}(s-t) \subset X_a \times_S X$ are the graphs of ϕ_a and j_a, respectively. Since furthermore $\operatorname{Div}(s-1)$ is the graph of the composition of the projection $X_a \to S$ with the 1-section $S \hookrightarrow \mathbf{A}^1_S = X$, we see that $\operatorname{Div}(s-1)^*$ induces the zero map between the quotients in (3.2.1). Thus the equality in (3.2.1) follows by [Sai20, Theorem 2.10(2)] from the following claim.

Claim 3.2.1. $\frac{h_a}{s-1}$ and $\frac{s-t}{s-1}$ are admissible for $\big((X, nZ), (X_a, nZ_a)\big)$ (cf. [Sai20, Definition 2.3]), i.e., with $\theta_h = \operatorname{Div}(h)$, for $h \in \{\frac{h_a}{s-1}, \frac{s-t}{s-1}\}$, we have

(i) h is regular in a neighborhood of $X_a \times_S Z$;
(ii) $\theta_h \times_X nZ$ is the image of the diagonal map $nZ \to X_a \times_S X$ coming from the isomorphism $nZ_a \simeq nZ$ induced by j_a;
(iii) h extends to an invertible function on a neighborhood of $X_a \times_S \infty$ in $X_a \times_S \mathbf{P}^1_S$.

(i) is immediate in both cases. Note that $s-1$ is a unit on $X_a \times_S nZ$. Hence (ii) follows from the equality of ideals in $A_a[s, \frac{1}{s-1}]$:

$$(h, s^n) = (s - t, t^n, s^n).$$

This is immediate for $h = s - t/s - 1$; for $h = h_a/s - 1$ we set $u := 1 + at^{n-1} \in A_a^\times$ and then the above equality follows from

$$t^n = (t - \tfrac{s}{u})(t^{n-1} + t^{n-2}\tfrac{s}{u} + \ldots + (\tfrac{s}{u})^{n-1}) + (\tfrac{s}{u})^n \in (h_a, s^n).$$

Letting $\sigma = 1/s$, (iii) follows from

$$\tfrac{h_a}{s-1} = \tfrac{1-\sigma(t+at^n)}{1-\sigma} \quad \text{and} \quad \tfrac{s-t}{s-1} = \tfrac{1-\sigma t}{1-\sigma} \quad \text{in } A_a[\sigma].$$

This proves Claim 3.2.1 and completes the proof of Proposition 3.1. □

Corollary 3.3. *Let K be a function field over k and denote by (see Notation 1.7)*

$$\pi : \mathcal{X} := \operatorname{Spec} K\{x, t\} \to T := \operatorname{Spec} K\{t\}$$

the natural map induced by the inclusion $K\{t\} \hookrightarrow K\{x, t\}$, and set $\mathcal{Z} = V(t) \subset \mathcal{X}$. Let $a \in K\{t\}$ and $n \geq 2$. Consider the closed immersions

$$i : C := V(x - t) \hookrightarrow \mathcal{X} \quad \text{and} \quad i_a : C_a := V(x - t - at^n) \hookrightarrow \mathcal{X}.$$

Then the compositions

$$\pi_C = \pi \circ i : C \to T \quad \text{and} \quad \pi_{C_a} = \pi \circ i_a : C_a \to T$$

are isomorphisms and for any $g \in G(\mathcal{X}, n\mathcal{Z})$ we have

$$\pi_{C*} i^* g - \pi_{C_a *} i_a^* g \in G(T),$$

where π_{C*} is induced by the action of the transpose of the graph of π_C, which is equal to $(\pi_C^*)^{-1}$, similarly with π_{C_a*} .

Proof. It is direct to check that π_{C_a} and π_C are isomorphisms. Let $f_a : (\mathcal{X}, n\mathcal{Z}) \to (\mathcal{X}, n\mathcal{Z})$ be the $K\{x\}$-morphism induced by $t \mapsto t + at^n$. Set $V(x-t, t^n) =: n\mathcal{Z}_C \subset \mathcal{X}$. Then $C \cap n\mathcal{Z} = n\mathcal{Z}_C = C_a \cap n\mathcal{Z} = f_a^{-1}(C \cap n\mathcal{Z})$. Thus f_a restricts to a morphism $f_{C,a} : (C_a, n\mathcal{Z}_C) \to (C, n\mathcal{Z}_C)$. Denote by 0 the closed point of T and define the K-morphism

$$f_{T,a} : (T, n\cdot 0) \to (T, n\cdot 0) \quad \text{induced by} \quad K\{t\} \to K\{t\}, \quad t \mapsto t + at^n.$$

We have

$$f_{T,a} \circ \pi_{C_a} = \pi_C \circ f_{C,a} : (C_a, n\mathcal{Z}_C) \to (T, n\cdot 0)$$

and obtain the commutative diagram

$$\begin{tikzcd}
\frac{G(\mathcal{X},n\mathcal{Z})}{G(\mathcal{X})} \arrow[r, "i^*"] \arrow[d, "f_a^*"] & \frac{G(C,n\mathcal{Z}_C)}{G(C)} \arrow[r, "\pi_{C*}"] \arrow[d, "f_{C,a}^*"] & \frac{G(T,n\cdot 0)}{G(T)} \arrow[d, "f_{T,a}^*"] \\
\frac{G(\mathcal{X},n\mathcal{Z})}{G(\mathcal{X})} \arrow[r, "i_a^*"] & \frac{G(C_a,n\mathcal{Z}_C)}{G(C_a)} \arrow[r, "\pi_{C_a*}"] & \frac{G(T,n\cdot 0)}{G(T)}.
\end{tikzcd}$$

By Proposition 3.1 we find that f_a^* (resp. $f_{T,a}^*$) induces the identity on the left hand side (resp. on the right hand side) in the above diagram. This yields the statement. □

Remark 3.4. Let L be a henselian dvf of geometric type over k with $\mathcal{O}_L$ the ring of integers and $\mathfrak{m}$ the maximal ideal. Recall from [RS21, Def 4.14] that we can associate a conductor to G which at L is the function $c_L^G : G(L) \to \mathbb{N}_0$ given by (see (2.2.4) for notation)

$$c_L^G(g) = \min\{n \geq 0 \mid g \in G(\mathcal{O}_L, \mathfrak{m}^{-n})\}.$$

In the situation of Corollary 3.3, let $g \in G(\mathcal{X}, n\mathcal{Z})$. The intersections $C \cap \mathcal{Z} = C_a \cap \mathcal{Z}$ are equal to the closed point of $\mathcal{X}$ and the intersection multiplicity of C and C_a at this closed point is $\geq n$ and the corollary implies that $c_L^G(g_{|C}) = c_{L_a}^G(g_{|C_a})$, where L (resp. L_a) is the function field of C (resp. C_a).

In case $G = \widetilde{H^1}$ (see (2.2.2) for notation), where H^1 is the reciprocity sheaf $X \mapsto \mathrm{Hom}_{\mathrm{cont}}(\pi_1(X)^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}))$ (see [RS21, 8.1]), Corollary 3.3 gives back a special case of an Artin-version of the Brylinski-Kato formula in [Kat89, Theoremm 0.2] (see also [Bry83, Proposition 4]). It is a special case since in [Kat89] the curves are only assumed to intersect the divisor $\mathcal{Z}$ transversally and have intersection multiplicity $\geq n$, but do not need the specific form as in the corollary above; it is the Artin version, since in [Kat89] the statement is for the Swan conductor whereas c^{H^1} is equal to the Artin conductor, by [RS21, Theorem 8.8]. Note that by [RS21] we can apply Corollary 3.3 also for the (non-log) irregularity of rank one connections (in char 0), the Artin conductor of lisse $\overline{\mathbb{Q}}_\ell$-sheaves, the conductor of fppf-torsors under finite k-group schemes etc. We will prove the Brylinsky-Kato formula in full generality in [RSb] (without the restriction on the special choice of C and C_a).

3.5. By [BRS22, 4.3] we have a functor

$$h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}} : \underline{\mathbf{M}}\mathbf{PST}^\tau := \tau_!(\mathbf{MPST}) \to \mathbf{CI}_{\mathrm{Nis}}^{\tau,sp}, \tag{3.5.1}$$

which is the sheafified and semi-purified maximal cube invariant quotient and is left-adjoint to the inclusion functor $\mathbf{CI}_{\mathrm{Nis}}^{\tau,sp} \to \underline{\mathbf{M}}\mathbf{PST}^\tau$. We set

$$\gamma^1 G := \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\underline{\omega}^*\mathbf{G}_m, G), \quad G(1) := h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(G \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \underline{\omega}^*\mathbf{G}_m), \tag{3.5.2}$$

where $(\underline{\omega}^*\mathbf{G}_m)(X,D) = \mathbf{G}_m(X\setminus D)$. By the above and [BRS22, Lemma 1.9(1)] we have $\gamma^1 G$, $G(1)\in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. [7] Consider the natural pairing on $\underline{\mathbf{M}}\mathbf{Cor}$

$$\underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\underline{\omega}^*\mathbf{G}_m, G)\times \underline{\omega}^*\mathbf{G}_m \to \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\underline{\omega}^*\mathbf{G}_m, G)\otimes_{\underline{\mathbf{M}}\mathbf{PST}}\underline{\omega}^*\mathbf{G}_m \to G, \tag{3.5.3}$$

where the first map is the natural one and the second map is the counit of adjunction. Since $G\in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, this composition induces a pairing

$$\gamma^1 G\times \underline{\omega}^*\mathbf{G}_m \to (\gamma^1 G)(1)\to G. \tag{3.5.4}$$

By [MS, (2.2) and Corollary 2.2] we have for $\mathcal{X}=(X,D)\in \underline{\mathbf{M}}\mathbf{Cor}$

$$\gamma^1 G(\mathcal{X}) = \frac{G(\mathcal{X}\otimes(\mathbf{P}^1, 0+\infty))}{p_2^* G(\mathcal{X})}, \tag{3.5.5}$$

where $p_2 : \mathcal{X}\otimes(\mathbf{P}^1,0+\infty)\to\mathcal{X}$ is induced by the projection $X\times\mathbf{P}^1\to X$ (Note that $\gamma^1 G(\mathcal{X})$ is a direct summand of $G(\mathcal{X}\otimes(\mathbf{P}^1,0+\infty))$). For a henselian regular local k-algebra R of geometric type, (3.5.4) induces the bilinear pairing (cf. (3.0.1))

$$\gamma^1 G(R)\times R^\times \to (\gamma^1 G)(1)(R)\to G(R). \tag{3.5.6}$$

By [BRS22, Lemma 5.6] the precomposition of (3.5.6) with the natural map $G(\mathbf{P}^1_R, 0_R+\infty_R)\times R^\times\to \gamma^1 G(R)\times R^\times$ is given by

$$G(\mathbf{P}^1_R, 0_R+\infty_R)\times R^\times \to G(R),\quad (a,f)\mapsto \delta_f^* a - \delta_1^* a, \tag{3.5.7}$$

where $\delta_f : \operatorname{Spec} R\to \operatorname{Spec} R\times\mathbf{G}_m$ denotes the graph of the k-morphism $f\in \operatorname{Hom}(\operatorname{Spec} R, \mathbf{G}_m)$.

Lemma 3.6. *Let $(\mathcal{O}_L,\mathfrak{m})$ be a henselian dvr of geometric type over k. Then the pairing*

$$(\gamma^1 G)(L)\times L^\times \xrightarrow{(3.5.6)} G(L)$$

restricts to a pairing

$$(\gamma^1 G)(\mathcal{O}_L,\mathfrak{m}^{-n})\times U_L^{(n)}\to G(\mathcal{O}_L),\quad n\geq 1,$$

where $U_L^{(n)} = 1+\mathfrak{m}^n$.

Proof. The choice of a coefficient field $K\hookrightarrow\mathcal{O}_L$ and a local parameter $t\in\mathfrak{m}$ yields an isomorphism $\mathcal{O}_L\cong K\{t\}$. Set $T=\operatorname{Spec} K\{t\}$ and denote by 0 the closed point. Since any element in $U_L^{(n)}$ can be written in the form $(1+t+at^n)/(1+t)$, with $a\in K\{t\}$, it suffices to show

$$\delta^*_{1+t+at^n} = \delta^*_{1+t} : G((T,n\cdot 0)\otimes(\mathbf{P}^1,0+\infty))\to \frac{G(T\setminus 0)}{G(T)}. \tag{3.6.1}$$

Write $\mathbf{P}^1\setminus\{\infty\} = \operatorname{Spec} k[y]$ and set $x=y-1$. Then the henselization of the local ring of $\mathbf{P}^1_T$ at (t,x) is identified with $K\{x,t\}$. Let $\mathcal{X}=\operatorname{Spec} K\{x,t\}$ and $\mathcal{Z}=\{t=0\}\subset\mathcal{X}$. Then δ_{1+t+at^n} is the composition

$$T\xrightarrow{\simeq,\ \pi_{C_a}^{-1}} C_a \xrightarrow{i} \mathcal{X}\to\mathbf{P}^1_T,$$

where we use the notation from Corollary 3.3 and where the last map is the natural one; for δ_{1+t} the same holds with C_a replaced by C and i_a replaced by i. Hence (3.6.1) follows from this corollary. □

[7] The notation $\gamma^1 G$ is compatible with (2.3.2), in the sense that $\underline{\omega}_!\gamma^1 G = \gamma^1\underline{\omega}_! G$, by [MS, Proposition 2.10].

The following is the main result of this section, which will play a key role in the proof of the Zariski-Nagata purity Theorem 6.8.

Theorem 3.7. *Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$ and let $(\mathcal{O}_L, \mathfrak{m})$ be a henselian dvr of geometric type over k. Then (with the notation from 2.2, 2.3) the pairing*

$$F(L) \times L^{\times} \to (F \otimes_{\mathbf{NST}} \mathbf{G}_m)(L) \to F\langle 1\rangle(L)$$

induces a pairing

$$\widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-n}) \times U_L^{(n)} \to F\langle 1\rangle(\mathcal{O}_L).$$

Proof. Consider the diagram

$$\begin{tikzcd}
\widetilde{F} \times \underline{\omega}^*\mathbf{G}_m \arrow[r] \arrow[d, "\overline{\mathrm{unit}}\otimes\mathrm{id}_{\mathbf{G}_m}"'] & \widetilde{F}(1) \arrow[d, "\overline{\mathrm{unit}}(1)"'] \arrow[dr, equal] & \\
\gamma^1(\widetilde{F}(1)) \times \underline{\omega}^*\mathbf{G}_m \arrow[r] & (\gamma^1(\widetilde{F}(1)))(1) \arrow[r, "\overline{\mathrm{counit}}_{\widetilde{F}(1)}"'] & \widetilde{F}(1).
\end{tikzcd}$$

where we use the notation from (3.5.2), the horizontal pairings are induced by the natural map for $G \in \underline{\mathbf{M}}\mathbf{PST}$:

$$G \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \underline{\omega}^*\mathbf{G}_m \to h_{0,\mathrm{Nis}}^{\overline{\square},\mathrm{sp}}(G \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \underline{\omega}^*\mathbf{G}_m) = G(1), \tag{3.7.1}$$

the map $\overline{\mathrm{counit}}_{\widetilde{F}(1)}$ is induced by the counit of the adjunction

$$(-) \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \underline{\omega}^*\mathbf{G}_m \dashv \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\underline{\omega}^*\mathbf{G}_m, -)$$

(see (3.5.4)), and the vertical maps are induced by

$$G \xrightarrow{\mathrm{unit}} \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\underline{\omega}^*\mathbf{G}_m, G \otimes_{\underline{\mathbf{M}}\mathbf{PST}} \underline{\omega}^*\mathbf{G}_m) \xrightarrow{(3.7.1)} \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\underline{\omega}^*\mathbf{G}_m, G(1)),$$

where "unit" denotes the unit of the above adjunction. We claim the diagram commutes. Indeed for the square this holds by functoriality and for the triangle, this follows from the fact that the composition

$$\widetilde{F} \otimes \underline{\omega}^*\mathbf{G}_m \xrightarrow{\mathrm{unit}\otimes\mathrm{id}} \underline{\mathrm{Hom}}(\underline{\omega}^*\mathbf{G}_m, \widetilde{F} \otimes \underline{\omega}^*\mathbf{G}_m) \otimes \underline{\omega}^*\mathbf{G}_m \xrightarrow{\mathrm{counit}_{\widetilde{F}\otimes\underline{\omega}^*\mathbf{G}_m}} \widetilde{F} \otimes \underline{\omega}^*\mathbf{G}_m,$$

where all $\otimes$ and $\underline{\mathrm{Hom}}$ are in $\underline{\mathbf{M}}\mathbf{PST}$, is the identity for general reasons, e.g. [Mac71, IV, Theorem 1]. We have:

(i) $F\langle 1\rangle = \underline{\omega}_!(F(1))$, by definition [RSY21, 5.21];
(ii) $\underline{\omega}_!\gamma^1 G = \underline{\omega}_! \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\underline{\omega}^*\mathbf{G}_m, G) = \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, \underline{\omega}_! G) = \gamma^1\underline{\omega}_! G$ for $G \in \underline{\mathbf{M}}\mathbf{PST}$, where the middle equality holds by [MS, Proposition 2.10].

Thus, applying $\underline{\omega}_!$ to the diagram above and evaluating on L gives the diagram

$$\begin{tikzcd}
F(L) \times L^{\times} \arrow[r] \arrow[d] & F\langle 1\rangle(L) \arrow[d] \arrow[dr, equal] & \\
\gamma^1(\widetilde{F}(1))(L) \times L^{\times} \arrow[r] & \left(\gamma^1(\widetilde{F}(1)))(1)\right)(L) \arrow[r] & F\langle 1\rangle(L),
\end{tikzcd}$$

where we use the notation (3.0.1) in the bottom line. Hence the statement follows from Lemma 3.6 with $G = \widetilde{F}(1) \in \mathbf{CI}_{\mathrm{Nis}}^{\tau,sp}$. $\square$

Remark 3.8. We remark that the pairing (3.5.6) can be trivial. Indeed, e.g., by the projective bundle formula [BRS22, Theorem 6.3] we have $\gamma^1\tilde{\mathbf{G}}_a(R) = H^1(\mathbf{P}^1_R, \mathcal{O}_{\mathbf{P}^1_R}) = 0$, with $\mathbf{G}_a$ the additive group. However, the pairing (3.5.6) for $G = \tilde{\mathbf{G}}_a(1)$ is non-trivial. Indeed, assume $\mathrm{char}(k) = 0$, then $\tilde{\mathbf{G}}_a(1) = \widetilde{\Omega^1_{-/\mathbb{Z}}}$ by [BRS22, Theorem 11.1]. And using [RS21, Theorem 6.4] and (3.5.5) we find (for $n \geq 1$)

$$\gamma^1(\tilde{\mathbf{G}}_a(1))(\mathcal{O}_L, \mathfrak{m}^{-n}) = \frac{1}{t^{n-1}} \cdot \mathcal{O}_L \cdot \mathrm{dlog}(y),$$

where t is the local parameter of $\mathcal{O}_L$ and y is the standard coordinate of $\mathbf{P}^1$, and the pairing (3.5.7) is is given by

$$(\tfrac{1}{t^{n-1}} a \cdot \mathrm{dlog}(y), f) \mapsto \tfrac{1}{t^{n-1}} a\, \mathrm{dlog}(f),$$

which is clearly in $\Omega^1_{\mathcal{O}_L/\mathbb{Z}}$, if $f \in U_L^{(n)}$.

4. Pushforwards with compact supports

In this section k is a perfect field. We fix $F \in \mathbf{RSC}_{\mathrm{Nis}}$. For $X \in \mathbf{Sm}$ we denote by F_X the restriction of F to X_{Nis}.

4.1. Let $f : X \to Y$ be a projective morphism in $\mathbf{Sm}$ of relative dimension $r \in \mathbb{Z}$. In [BRS22, 9.5] there is constructed a pushforward

$$f_* : Rf_*(\gamma^b F\langle a + r\rangle_X)[r] \to \gamma^b F\langle a\rangle_Y, \tag{4.1.1}$$

for all $a, b \geq 0$ with $a + r \geq 0$, in $D(Y_{\mathrm{Nis}})$ the derived category of abelian Nisnevich sheaves on Y. Recall that the pushforward is essentially given by a Gysin map for a closed immersion of X into a projective bundle over Y followed by a projective trace map stemming from the projective bundle formula. By [BRS22, Thm 9.7] this pushforward has all the expected properties.

4.2. Let $\iota : Z \hookrightarrow X$ be a locally closed immersion between separated schemes. For G a sheaf of abelian groups on Z_{Nis}, we define $\iota_! G$ as the subsheaf of $\iota_* G$ on X_{Nis} given on $V \to X$ étale by

$$\iota_! G(V) = \{s \in G(Z \times_X V) \mid \mathrm{supp}(s) \text{ is proper over } V\}.$$

It follows from [SGA73, Exp. XVII, Prop 6.1.1] that $\iota_! G$ is a Nisnevich sheaf on X. If we choose a factorization

$$\iota : Z \xrightarrow{i} U \xrightarrow{j} X, \tag{4.2.1}$$

with i a closed immersion and j an open embedding, it follows directly from the definition, that we have

$$\iota_! G = j_! i_* G.$$

Hence, for $V \to X$ étale and $v \in V$ a point

$$\iota_! G(\mathcal{O}^h_{V,v}) = \begin{cases} G(\mathcal{O}^h_{V_Z,v}), & \text{if } v \in V_Z = V \times_X Z, \\ 0, & \text{else.} \end{cases}$$

From this formula we see that $\iota_!$ is exact and furthermore, if $\kappa : W \hookrightarrow Z$ is another locally closed immersion with W separated and H is a sheaf on W_{Nis}, then there is a canonical identification

$$(\iota\kappa)_! H = \iota_! \kappa_! H \quad \text{inside } (\iota\kappa)_* H = \iota_* \kappa_* H.$$

Since $\iota_!$ is an exact functor on the category of abelian Nisnevich sheaves, it extends to an exact triangulated functor between the derived categories

$$\iota_! : D(Z_{\mathrm{Nis}}) \to D(X_{\mathrm{Nis}}).$$

If $j : U \hookrightarrow Z$ is open, then $j_!$ is equal to the extension-by-zero functor from [AGV72, Exp. IV, Prop 11.3.1] which is left adjoint to j^{-1}. The adjunction extends to an adjunction $j_! : D(U_{\mathrm{Nis}}) \leftrightarrows D(X_{\mathrm{Nis}}) : j^{-1}$.

Lemma 4.3. *For a commutative diagram*

$$\begin{tikzcd} Z' \arrow[r, "\kappa"] \arrow[d, "g"] & X' \arrow[d, "f"] \\ Z \arrow[r, "\iota"] & X \end{tikzcd} \tag{4.3.1}$$

with f and g proper and ι, κ locally closed immersions, we have a natural equivalence

$$\iota_! g_* \xrightarrow{\sim} f_*\kappa_! : \mathrm{Sh}(Z'_{\mathrm{Nis}}) \to \mathrm{Sh}(X_{\mathrm{Nis}}). \tag{4.3.2}$$

Proof. Let $V \to X$ be étale and $s \in \iota_* g_* G(V) = G(V \times_X Z') = G(V' \times_{X'} Z') = f_*\kappa_* G(V)$ a section, where $V' = f^{-1}(V) = V \times_X X'$. Assume $s \in \iota_! g_* G(V)$, i.e., viewing s as a section in $g_*G(V \times_X Z)$, its support $S \subset V \times_X Z$ is proper over V. Viewing s as a section in $G(V \times_X Z')$, its support is equal to $S' = S \times_Z Z'$. As g is proper, the composition $S' \subset V \times_X Z' \to V \times_X Z \to V$ is also proper. Since this last map factors as $V \times_X Z' = V' \times_{X'} Z' \to V' \xrightarrow{f_{|V'}} V$, we conclude that S' is proper over V'. This defines the map (4.3.2). On the other hand, if $s \in f_*\kappa_! G(V)$, i.e., S' is proper over V', then S' is also proper over V, as f is proper, whence S is proper over V as well, i.e., $s \in \iota_! g_* G(V)$. This shows that (4.3.2) is an isomorphism. □

In (4.3.1), if C is a complex of Nisnevich sheaves on Z'_{Nis}, choose a K-injective resolution $C \to I$ and a K-injective resolution $\kappa_! I \to J$. We obtain a natural morphism $\iota_! g_* I \to f_*\kappa_! I \to f_* J$ which yields a natural transformation

$$\alpha_\sigma : \iota_! Rg_* \to Rf_*\kappa_! : D(Z'_{\mathrm{Nis}}) \to D(X_{\mathrm{Nis}}), \tag{4.3.3}$$

where σ denotes the square (4.3.1). It follows directly from the construction that this transformation is functorial in the following sense: assume given another square σ' as (4.3.1):

$$\begin{tikzcd} T' \arrow[r, "\mu"] \arrow[d, "h"] & Z' \arrow[d, "g"] \\ T \arrow[r, "\lambda"] & Z, \end{tikzcd}$$

then we have the equality

$$\alpha_{\sigma\circ\sigma'} = \alpha_\sigma \circ \alpha_{\sigma'} : (\iota\lambda)_! Rh_* \to \iota_! Rg_*\mu_! \to Rf_*(\kappa\mu)_!, \tag{4.3.4}$$

where $\sigma \circ \sigma'$ denotes the square obtained by concatenating σ and σ' in the obvious way. Furthermore, if ι and κ are closed immersions, (4.3.3) is equal to the isomorphism $\iota_* Rg_* \cong Rf_*\kappa_*$.

Definition 4.4. Let $\iota : Z \hookrightarrow X$ be a locally closed immersion in **Sm** of pure codimension c. For $a \geq c$ and $b \geq 0$ we define the *Gysin map* for $Z \hookrightarrow X$ by

$$g_{Z/X} : \iota_! \gamma^b F\langle a-c\rangle_Z[-c] \to \gamma^b F\langle a\rangle_X \quad \text{in } D(X_{\mathrm{Nis}})$$

by choosing a factorization (4.2.1) and defining $g_{Z/X}$ as the composition

$$j_!(i_*\gamma^b F\langle a-c\rangle_Z[-c]) \xrightarrow{j_!(i_*)} j_!(\gamma^b F\langle a\rangle_U) \xrightarrow{\text{nat}} \gamma^b F\langle a\rangle_X,$$

where i_* denotes the pushforward from (4.1.1) for $f = i$ and the map "nat" denotes the counit of the adjunction $j_! \dashv j^{-1}$. This definition is independent of the factorization (4.2.1) by Lemma 4.5(1) below.

Lemma 4.5. *Let $\iota : Z \hookrightarrow X$ be as in Definition 4.4 above.*

(1) $g_{Z/X}$ is independent of the choice of the factorization (4.2.1).

(2) Let $\lambda : W \to Z$ be a closed immersion in **Sm** *of pure codimension d and assume $a \geq c+d$, $b \geq 0$. Then the following diagram commutes:*

$$\begin{tikzcd} \iota_!\lambda_*\gamma^b F\langle a-d-c\rangle_W[-d-c] \arrow[r, "\iota_! g_{W/Z}"] \arrow[d, equal] & \iota_!\gamma^b F\langle a-c\rangle_Z[-d] \arrow[d, "g_{Z/X}"] \\ (\iota\lambda)_!\gamma^b F\langle a-d-c\rangle_W[-d-c] \arrow[r, "g_{W/X}"] & \gamma^b F\langle a\rangle_X \end{tikzcd}.$$

(3) Let $\tau : Z' \to Z$ be an open immersion and $a \geq c$, $b \geq 0$. Then the following diagram commutes:

$$\begin{tikzcd} \iota_!\tau_!\gamma^b F\langle a-c\rangle_{Z'}[-c] \arrow[r, "g_{Z'/X}"] \arrow[d] & \gamma^b F\langle a\rangle_X \\ \iota_!\gamma^b F\langle a-c\rangle_Z[-c] \arrow[ur, "g_{Z/X}"'] & \end{tikzcd},$$

where the vertical map is induced by the counit $\tau_!\tau^{-1} \to \mathrm{id}$.

(4) Assume we have a commutative triangle

$$\begin{tikzcd} & P \arrow[d, "\pi"] \\ Z \arrow[ur, "\kappa"] \arrow[r, "\iota"] & X \end{tikzcd}$$

with π smooth projective of relative dimension n and ι and κ locally closed immersions of pure codimension c and $n+c$, respectively. Then the following diagram commutes for $a \geq c$, $b \geq 0$

$$\begin{tikzcd} \iota_!\gamma^b F\langle a-c\rangle_Z[-c] \arrow[r, "g_{Z/X}"] \arrow[d, "(4.3.3)"'] & \gamma^b F\langle a\rangle_X \\ R\pi_*\kappa_!\gamma^b F\langle a-c\rangle_Z[-c] \arrow[r, "g_{Z/P}"] & R\pi_*\gamma^b F\langle a+n\rangle_P[n] \arrow[u, "\pi_*"'] \end{tikzcd}$$

Proof. In the following we write for short $G(n)_P := \gamma^b F\langle a+n\rangle_P[n]$.

(1). Let $Z \xrightarrow{i'} U' \xrightarrow{j'} X$ be another factorization. If j factors through an open immersion $\tau : U \to U'$, the assertion is reduced to the commutativity of

$$\begin{tikzcd} \tau_!\tau^{-1}i'_*G(-c)_Z \arrow[r, "\tau_!(i_*)"] \arrow[d] & \tau_!\tau^{-1}G(0)_{U'} \arrow[d] \\ i'_*G(-c)_Z \arrow[r, "i'_*"] & G(0)_{U'} \end{tikzcd},$$

where the vertical maps are induced by $\tau_!\tau^{-1} \to \mathrm{id}$ and we use $\tau^{-1}i'_* = i_*$ by [BRS22, Thm 9.7(3)]. Since the top arrow is equal to $\tau_!\tau^{-1}(i'_*)$, the diagram clearly commutes. The general case is reduced to the above case by considering the factorization $Z \to U_1 \cap U_2 \to X$. To show (2) we can choose the factorization $W \xrightarrow{i\lambda} U \xrightarrow{j} X$ to compute $g_{W/X}$. Hence the commutativity of the diagram follows directly from the equality of pushforwards $(i\lambda)_* = i_* \circ \lambda_*$, which holds by [BRS22, Thm 9.7(2)]. To show (3) consider the closed immersion $i' : Z' \to U' = U \setminus (Z \setminus Z')$ and the open immersion $\tau_U : U' \to U$. We can thus compute $g_{Z'/X}$ using the factorization $Z' \xrightarrow{i'} U' \xrightarrow{j\tau_U} X$. The following diagram is clearly commutative

$$\begin{tikzcd}
j_!\tau_{U!}\tau_U^{-1}i_*G(-c)_Z \arrow[r, "i_*"] \arrow[d] & j_!\tau_{U!}\tau_U^{-1}G(0)_U \arrow[r] \arrow[d] & G(0)_X \\
j_!i_*G(-c)_Z[-c] \arrow[r, "i_*"] & j_!G(0)_U, \arrow[ur] &
\end{tikzcd}$$

where the vertical maps are induced by the counit $\tau_!\tau^{-1} \to \mathrm{id}$. Since the pushforward is compatible with restriction to opens (see [BRS22, Thm 9.7(3)]), the top row in the diagram is equal to the composition

$$(j\tau_U i')_!G(-c)_{Z'} \xrightarrow{i'_*} (j\tau_U)_!G(0)_{U'} \to G(0)_X.$$

This proves (3). Finally we prove (4). A factorization (4.2.1) induces a diagram with cartesian square

$$\begin{tikzcd}
 & P_U \arrow[r, "j'"] \arrow[d, "\pi_U"] & P \arrow[d, "\pi"] \\
Z \arrow[ur, "i'"] \arrow[r, "i"] & U \arrow[r, "j"] & X,
\end{tikzcd}$$

with i, i' closed - and j, j' open immersions. By (4.3.4) the map (4.3.3) for the diagram

$$\begin{tikzcd}
Z \arrow[r, "\kappa"] \arrow[d, "="] & D \arrow[d, "\pi"] \\
Z \arrow[r, "\iota"] & X
\end{tikzcd}$$

is given by

$$\iota_! = j_!i_* = j_!R\pi_{U*}i'_* \to R\pi_*j'_!i'_* = R\pi_*\kappa_!, \tag{4.5.1}$$

where the middle map is by adjunction induced from the natural isomorphism $R\pi_{U*} \xrightarrow{\simeq} j^{-1}R\pi_*j'_!$. Consider the following diagram

$$\begin{tikzcd}
j_!i_*G(-c)_Z \arrow[r, "i_*"] \arrow[d, equal] & j_!G(0)_U \arrow[r] & G(0)_X \\
j_!R\pi_{U*}i'_*G(-c)_Z \arrow[r, "i'_*"] \arrow[d] & j_!R\pi_{U*}G(n)_{P_U} \arrow[r] \arrow[u, "\pi_{U*}"] \arrow[d] & R\pi_*G(n)_P \arrow[u, "\pi_*"'] \arrow[d, equal] \\
R\pi_*j'_!i'_*G(-c)_Z \arrow[r, "i'_*"] & R\pi_*j'_!G(n)_{P_U} \arrow[r] & R\pi_*G(n)_P.
\end{tikzcd}$$

The square in the top left corner commutes by functoriality of the pushforward (see [BRS22, Thm 9.7(2)]); the square on the top right corner commutes since

$\pi_{U*} = j^{-1}(\pi_*)$ (see [BRS22, Thm 9.7(3)]); the other squares clearly commute. Thus the big outer square commutes which yields the statement. $\square$

Definition 4.6. Let $f : X \to Y$ be a quasi-projective morphism in $\mathbf{Sm}$ of pure relative dimension $r \in \mathbb{Z}$. We fix a compactification

$$f : X \xrightarrow{j} \overline{X} \xrightarrow{\overline{f}} Y, \tag{4.6.1}$$

where j is a dense open immersion and $\overline{f}$ is projective. For $a \geq 0$ with $a + r \geq 0$ and $b \geq 0$ we define a pushforward

$$(\overline{f}, j)_* : R\overline{f}_* j_! \gamma^b F\langle a+r\rangle_X[r] \to \gamma^b F\langle a\rangle_Y$$

as follows: choose a factorization

$$\overline{f} : \overline{X} \xrightarrow{i} P \xrightarrow{p} Y, \tag{4.6.2}$$

with i a closed immersion and p projective smooth of relative dimension n. Then $(\overline{f}, j)_*$ is defined as the composition

$$\begin{aligned}(\overline{f}, j)_* : R\overline{f}_* j_! \gamma^b F\langle a+r\rangle_X[r] &= Rp_* \iota_! \gamma^b F\langle a+r\rangle_X[r] \\ &\xrightarrow{g_{X/P}} Rp_* \gamma^b F\langle a+n\rangle_P[n] \xrightarrow{p_*} \gamma^b F\langle a\rangle_Y,\end{aligned} \tag{4.6.3}$$

where $\iota = i \circ j : X \hookrightarrow P$ denotes the immersion, $g_{X/P}$ denotes the associated Gysin map from 4.4, and the map p_* is the pushforward recalled in 4.1. By Lemma 4.7 below $(\overline{f}, j)_*$ does not depend on the choice (4.6.2), however it might depend on the choice (4.6.1).

Lemma 4.7. *Assumptions as in Definition 4.6.*

(1) The map $(\overline{f}, j)_*$ is independent of the choice of the factorization (4.6.2).

(2) Assume j factors as a composition of open immersions

$$X \xrightarrow{\tau} X' \xrightarrow{j'} \overline{X}.$$

Then the following diagram commutes:

$$\begin{tikzcd} R\overline{f}_* j_! \gamma^b F\langle a+r\rangle_X[r] \arrow[r, "(\overline{f},j)_*"] \arrow[d] & \gamma^b F\langle a\rangle_Y \\ R\overline{f}_* j'_! \gamma^b F\langle a+r\rangle_{X'}[r], \arrow[ur, "(\overline{f},j')_*"'] & \end{tikzcd}$$

where the vertical map is induced by the natural transformation

$$j_!\tau^{-1} = j'_!\tau_!\tau^{-1} \to j'_!.$$

(3) Assume given a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "j'"] \arrow[d, "g"] & \overline{X'} \arrow[r, "\overline{f'}"] \arrow[d, "\overline{g}"] & Y \\ X \arrow[r, "j"] & \overline{X} \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

with $X, X', Y \in \mathbf{Sm}$, j, j' dense open immersions and g, $\overline{g}$, $\overline{f}$, $\overline{f'}$ projective. Set $s = \dim X' - \dim X$. Assume $a \geq \max\{0, -r, -r-s\}$. Then the following *diagram commutes*

$$\begin{tikzcd}
R\overline{f}_*j_!Rg_*\gamma^bF\langle a+r+s\rangle_{X'}[r+s] \arrow[r, "g_*"] \arrow[d, "(4.3.3)"] & R\overline{f}_*j_!\gamma^bF\langle a+r\rangle_X[r] \arrow[d, "(\overline{f},j)_*"] \\
R\overline{f'}_*j'_!\gamma^bF\langle a+r+s\rangle_{X'}[r+s] \arrow[r, "(\overline{f'},j')_*"] & \gamma^bF\langle a\rangle_Y,
\end{tikzcd}$$

where g_* is the pushforward from (4.1.1).

Proof. In the proof we set

$$G(m)_P := \gamma^bF\langle a+m\rangle_P[m].$$

(1). Let $\overline{X} \xrightarrow{i_P} P \xrightarrow{p} Y$, and $\overline{X} \xrightarrow{i_Q} Q \xrightarrow{q} Y$, be two factorizations of $\overline{f}$ with i_P and i_Q closed immersions and p and q smooth projective morphisms of relative dimension m and n, respectively. We obtain the following commutative diagram

$$\begin{tikzcd}
 & & P \arrow[dr, "p"] & \\
X \arrow[r, "j"] & \overline{X} \arrow[ur, "i_P"] \arrow[r] \arrow[dr, "i_Q"'] & PQ \arrow[u, "\pi_P"'] \arrow[d, "\pi_Q"'] \arrow[r, "\varrho"] & Y, \\
 & & Q \arrow[ur, "q"'] &
\end{tikzcd}$$

with $PQ = P \times_Y Q$ and π_P, π_Q the projections. This yields a diagram

$$\begin{tikzcd}
 & Rp_*G(m)_P \arrow[dr, "p_*"] & \\
R\overline{f}_*j_!G(r)_X \arrow[ur, "g_{X/P}"] \arrow[r, "g_{X/PQ}"] \arrow[dr, "g_{X/Q}"'] & R\varrho_*G(m+n)_{PQ} \arrow[u, "\pi_{P*}"'] \arrow[d, "\pi_{Q*}"'] & G(0)_Y. \\
 & Rq_*G(n)_Q \arrow[ur, "q_*"'] &
\end{tikzcd}$$

The big triangle on the right hand side commutes by the functoriality of the pushforward, see [BRS22, Thm 9.7(2)] and the two small triangles on the left hand side commute by Lemma 4.5(4). This implies the statement.

(2) follows from (1) and Lemma 4.5(3). Finally (3). We choose a factorization $\overline{X'} \to P' \to Y$ of $\overline{f'}$ as in (4.6.2). After replacing P' by $P \times_Y P'$ we obtain a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, hook, "i'"] \arrow[d, "g"'] \arrow[dr, "h"] & U' \arrow[r, hook, "k'"] \arrow[d, "\pi_U"] & P' \arrow[r, "p'"] \arrow[d, "\pi"'] & Y \\
X \arrow[r, hook, "i"] & U \arrow[r, hook, "k"] & P \arrow[ur, "p"'] &
\end{tikzcd},$$

where p, p', π, π_U are smooth projective, k, k' are open immersions, i, i' are closed immersions, the square in the middle is cartesian, $h = \pi_U i' = ig$, and $\overline{f'}j' = p'k'i'$, $\overline{f}j = pki$. Let r, s, n, and n' be the relative dimensions of f, g, p, and p', respectively, and set $t := r + s$. Consider the following diagram

$$\begin{tikzcd}
Rp_*k_!i_*G(r)_X \arrow[r, "i_*"] & Rp_*k_!G(n)_U \arrow[r, "\mathrm{nat}"] & Rp_*G(n)_P \arrow[r, "p_*"] & G(0)_Y \\
Rp_*k_!Rh_*G(t)_{X'} \arrow[u, "g_*"] \arrow[d, "(4.3.3)"'] \arrow[r, "i'_*"] & Rp_*k_!R\pi_{U*}G(n')_{U'} \arrow[u, "\pi_{U*}"'] \arrow[d, "(4.3.3)"'] \arrow[r, "\mathrm{nat}"] & Rp_*R\pi_*G(n')_{P'} \arrow[u, "\pi_*"] \arrow[ur, "p'_*"'] & \\
Rp_*R\pi_*k'_!i'_*G(t)_{X'} \arrow[r, "i'_*"] & Rp_*R\pi_*k'_!G(n')_{U'}. \arrow[ur, "\mathrm{nat}"'] & &
\end{tikzcd}$$

The top left square and the top right triangle commute by functoriality of the pushforward (see [BRS22, Thm 9.7(2)]); the square in the middle of the top row commutes since $\pi_{U*} = k^{-1}(\pi_*)$ (compatibility of the pushforward with restriction along smooth maps, see [BRS22, Thm 9.7(3)]); the bottom row of the diagram clearly commutes. Thus the whole diagram commutes. Noting $Rp_*k_!Rh_* = R\overline{f}_*j_!Rg_*$, this yields the statement in view of the definition of $(\overline{f}, j)_*$ and $(\overline{f}', j')_*$ and (4.3.4). $\square$

5. Recollections on Paršin chains, higher local rings, and cohomology

In this section we recall some definitions and results from [KS86, (1.6)]. Let X be a reduced noetherian separated scheme of dimension $d < \infty$, such that $X^{(d)} = X_{(0)}$. For $U \to X$ étale and a subscheme $T \subset X$, we write $U_T = U \times_X T$.

5.1. For $x, y \in X$ we write

$$y < x :\Longleftrightarrow \overline{\{y\}} \subsetneq \overline{\{x\}}, \text{ i.e., } y \in \overline{\{x\}} \text{ and } y \neq x.$$

A *chain* on X is a sequence

$$\underline{x} = (x_0, \ldots, x_n) \quad \text{with } x_0 < x_1 < \ldots < x_n. \tag{5.1.1}$$

The chain $\underline{x}$ is a *maximal Paršin chain* (of just a *maximal chain*) if $n = d$ and $x_i \in X_{(i)}$. Note that the assumptions on X imply $x_i \in \overline{\{x_{i+1}\}}^{(1)}$. We denote

$$\mathrm{c}(X) = \{\text{chains on } X\} \quad \text{and} \quad \mathrm{mc}(X) = \{\text{maximal chains on } X\}.$$

Let $0 \le r \le d$. A *maximal chain with break at* r is a chain (5.1.1) with $n = d - 1$ and $x_i \in X_{(i)}$, for $i < r$, and $x_i \in X_{(i+1)}$, for $i \ge r$. We denote

$$\mathrm{mc}_r(X) = \{\text{maximal chain with break at } r \text{ on } X\}.$$

For $\underline{x} = (x_0, \ldots, x_{d-1}) \in \mathrm{mc}_r(X)$, we denote by $\mathrm{b}(\underline{x})$ the set of $y \in X_{(r)}$ such that

$$\underline{x}(y) := (x_0, \ldots, x_{r-1}, y, x_r, \ldots, x_{d-1}) \in \mathrm{mc}(X). \tag{5.1.2}$$

Note that $\mathrm{b}(\underline{x})$ depends only on x_r and x_{r-1}.

5.2. Let $S \subset X$ be a finite subset contained in an affine open neighborhood of X. A *strict Nisnevich neighborhood of* S is an étale map $u : U \to X$ such that U is affine, the base change $u^{-1}(S) \to S$ of u is an isomorphism, and every connected component of U intersects $u^{-1}(S)$. If $U \to X$ and $V \to X$ are two strict Nisnevich neighborhoods of S there is at most one X-morphism $U \to V$, by [SGA03, Exposé, I, Corollaire 5.4]. By taking a representative for each isomorphism class the category of strict Nisnevich neighborhoods of S in X with morphisms the X-morphisms, therefore gives rise to a filtered set.

Let $\underline{x} = (x_0, \ldots, x_n)$ be a chain on X. A *strict Nisnevich neighborhood of* $\underline{x}$ is a sequence of maps

$$\mathcal{U} = (U_n \to \ldots \to U_1 \to U_0 \to X =: U_{-1}),$$

such that $U_i \to U_{i-1}$ is a strict Nisnevich neighborhood of U_{i-1,x_i} for all $i = 0, \ldots, n$. A morphism $\mathcal{V} \to \mathcal{U}$ between two strict Nisnevich neighborhoods of $\underline{x}$ consists of a collection of a U_{i-1}-morphism $V_i \to U_i$ for each $i \geq 0$. There is again at most one such morphism. Given two such neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of $\underline{x}$ we can form a new neighborhood $\mathcal{U} \times_X \mathcal{V}$. As above we obtain a filtered set

$$N(\underline{x}) := \{\text{strict Nisnevich neighborhoods of } \underline{x}\}.$$

Assume $\underline{x} \in \mathrm{mc}_r(X)$ and $y \in \mathrm{b}(\underline{x})$. Then we have a map of filtered sets

$$N(\underline{x}) \to N(\underline{x}(y)) \tag{5.2.1}$$

given by

$$(U_{d-1} \xrightarrow{u_{d-1}} \ldots \xrightarrow{u_0} X) \mapsto (U_{d-1} \xrightarrow{u_{d-1}} \ldots \xrightarrow{u_r} U_{r-1} \xrightarrow{\mathrm{id}} U_{r-1} \xrightarrow{u_{r-1}} \ldots \xrightarrow{u_0} X).$$

Let F be a presheaf of abelian groups on X_{Nis} and $\underline{x} = (x_0, \ldots, x_n) \in \mathrm{c}(X)$. The Nisnevich stalk of F at $\underline{x}$ is defined to be

$$F^h_{\underline{x}} := \varinjlim_{\mathcal{U}=(U_n \to \ldots \to X) \in N(\underline{x})} F(U_n).$$

Note that for $\underline{x} \in \mathrm{mc}_r(X)$ and $y \in \mathrm{b}(\underline{x})$ the map (5.2.1) induces a natural map

$$\iota_y : F^h_{\underline{x}} \to F^h_{\underline{x}(y)}. \tag{5.2.2}$$

For $\underline{x} = (x_0, \ldots, x_n) \in \mathrm{c}(X)$, write $\mathcal{O}^h_{X,\underline{x}} = F^h_{\underline{x}}$ with $F = \mathcal{O}$ and $K^h_{X,\underline{x}} = \mathrm{Frac}(\mathcal{O}^h_{X,\underline{x}})$. By Lemma 5.3 below, $\mathcal{O}^h_{X,\underline{x}}$ is a finite product of henselian local rings and $K^h_{X,\underline{x}}$ is a finite product of fields. If $x_n \in X_{(d)}$, then $\mathcal{O}^h_{X,\underline{x}} = K^h_{X,\underline{x}}$. Otherwise, we have

$$K^h_{X,\underline{x}} = \prod_{\eta \in X_{(d)}, \eta > x_n} \mathcal{O}^h_{X,(\underline{x},\eta)}.$$

For $\underline{x} \in \mathrm{mc}_r(X)$ and $y \in \mathrm{b}(\underline{x})$ we have by (5.2.2) natural maps

$$K^h_{X,\underline{x}} \to K^h_{X,\underline{x}(y)}, \text{ for } r < d \quad \text{and} \quad \mathcal{O}^h_{X,\underline{x}} \to K^h_{X,\underline{x}(y)} = \mathrm{Frac}(\mathcal{O}^h_{X,\underline{x}}), \text{ for } r = d.$$

Lemma 5.3. *For $\underline{x} = (x_0, \ldots, x_n) \in \mathrm{c}(X)$ we have $\mathcal{O}^h_{X,\underline{x}} = R_n$, where*

$$R_0 = \mathcal{O}^h_{X,x_0} \quad \textit{and} \quad R_i = \prod_{\mathfrak{p} \in T_i} R^h_{i-1,\mathfrak{p}}, \quad i \geq 1,$$

with $T_i = \operatorname{Spec} R_{i-1} \times_X \{x_i\}$ the finite set of prime ideals in R_{i-1} lying over the prime ideal in $\mathcal{O}_{X,x_0}$ corresponding to x_i.

Proof. The finiteness of the T_i holds by [Gro67, Théorème (18.6.9)]. Set $\underline{x}_i = (x_0, \ldots, x_i)$. As in (5.2.1) we obtain maps $N(\underline{x}_0) \to N(\underline{x}_1) \to \ldots \to N(\underline{x}_n)$. The equality $\mathcal{O}^h_{X,\underline{x}} = R_n$ then follows from

$$\varinjlim_{\mathcal{U} \in N(\underline{x})} \mathcal{O}(U_n) = \varinjlim_{\mathcal{V}_0 \in N(\underline{x}_0)} \varinjlim_{\mathcal{V}_1 \in N(\underline{x}_1)/\mathcal{V}_0} \ldots \varinjlim_{\mathcal{U} \in N(\underline{x}_n)/\mathcal{V}_{n-1}} \mathcal{O}(U_n).$$

□

5.4. Let F be an abelian Nisnevich sheaf on X. For a finite subset $S \subset X$ we write $H^i_S(X_{\mathrm{Nis}}, F) = \varinjlim_{S\subset V\subset X} H^i_{\overline{S}\cap V}(V_{\mathrm{Nis}}, F)$, where the colimit is over all open subsets V of X containing S and $\overline{S}$ denotes the closure of S in X.

Let $\underline{x} = (x_0, \dots, x_n) \in \mathrm{c}(X)$. We set

$$H^i_{\underline{x}}(X, F) := \varinjlim_{\mathcal{U}=(U_n\to\ldots\to X)\in N(\underline{x})} H^i_{U_{n,x_n}}(U_{n,\mathrm{Nis}}, F).$$

Assume $x_{n-1} \in \overline{\{x_n\}}^{(1)}$ and write $\underline{x}' = (x_0, \dots, x_{n-1})$. We define a map

$$\delta_{\underline{x}} : H^i_{\underline{x}}(X, F) \to H^{i+1}_{\underline{x}'}(X, F) \tag{5.4.1}$$

as follows: set $y := x_n$ and $x := x_{n-1}$. For $\mathcal{U} = (U_n \to \ldots \to U_0 \to X) \in N(\underline{x})$, set $V := U_n$, $U := U_{n-1}$, and set $U_{(x)} := U \times_X \operatorname{Spec} \mathcal{O}_{X,x}$ and define $\delta_{\mathcal{U}}$ as the composition (we drop the index "Nis" everywhere)

$$\delta_{\mathcal{U}} : H^i_{V_y}(V, F) \cong H^i_{U_{\overline{\{y\}}}\setminus U_x}(U_{(x)} \setminus U_x, F) \xrightarrow{\delta} H^{i+1}_{U_x}(U_{(x)}, F) \cong H^{i+1}_{U_x}(U, F),$$

where the first isomorphism is Nisnevich excision plus $\overline{\{y\}} = \{x, y\} \subset \operatorname{Spec} \mathcal{O}_{X,x}$, δ is the connecting homomorphism from the long exact localization sequence, and the last isomorphism is again excision. Then $\delta_{\underline{x}} = \varinjlim_{\mathcal{U}\in N(\underline{x})} \delta_{\mathcal{U}}$.

Let $\underline{x} = (x_0, \dots, x_d) \in \mathrm{mc}(X)$. As in [KS86, Definition 1.6.2(5)] we define

$$c_{\underline{x}} := s_{x_0} \circ c_{\underline{x},0} : F(K^h_{X,\underline{x}}) := F^h_{\underline{x}} \to H^d(X_{\mathrm{Nis}}, F), \tag{5.4.2}$$

where $s_{x_0} : H^d_{x_0}(X_{\mathrm{Nis}}, F) \to H^d(X_{\mathrm{Nis}}, F)$ is the forget-support-map and $c_{\underline{x},0}$ is the composition

$$c_{\underline{x},0} : F^h_{\underline{x}} = H^0_{(x_0,\ldots,x_d)}(X_{\mathrm{Nis}}, F) \xrightarrow{\delta_{(x_0,\ldots,x_d)}} H^1_{(x_0,\ldots x_{d-1})}(X_{\mathrm{Nis}}, F) \xrightarrow{\delta_{(x_0,\ldots x_{d-1})}} \ldots \tag{5.4.3}$$
$$\ldots \xrightarrow{\delta_{(x_0,x_1)}} H^d_{x_0}(X_{\mathrm{Nis}}, F).$$

Note that for $\underline{x} \in \mathrm{mc}_d(X)$, the composition

$$F^h_{\underline{x}} \xrightarrow{\iota_y} F^h_{\underline{x}(y)} \xrightarrow{\delta_{\underline{x}(y)}} H^1_{\underline{x}}(X_{\mathrm{Nis}}, F) \tag{5.4.4}$$

is zero, so that $c_{\underline{x}(y)} \circ \iota_y = 0$ for all $y \in \mathrm{b}(\underline{x})$.

Proposition 5.5 ([KS86, Lemma 1.6.3])**.** *Let F be an abelian Nisnevich sheaf on X. Then for any abelian group A the map*

$$\Phi : \operatorname{Hom}(H^d(X_{\mathrm{Nis}}, F), A) \to \prod_{\underline{x}\in\mathrm{mc}(X)} \operatorname{Hom}(F^h_{\underline{x}}, A), \quad \alpha \mapsto (\alpha \circ c_{\underline{x}})_{\underline{x}\in\mathrm{mc}(X)},$$

is injective. Furthermore, the image of Φ consists of those tuples $(\chi_{\underline{x}})_{\underline{x}\in\mathrm{mc}(X)}$ satisfying the following condition: for any $r \in \{0, \dots, d\}$, $\underline{x} \in \mathrm{mc}_r(X)$, $y \in \mathrm{b}(\underline{x})$ and for any $a \in F^h_{\underline{x}}$, we have $\chi_{\underline{x}(y)}(\iota_y(a)) = 0$ for almost all $y \in \mathrm{b}(\underline{x})$, and

$$\sum_{y\in\mathrm{b}(\underline{x})} \chi_{\underline{x}(y)}(\iota_y(a)) = 0.$$

6. Reciprocity Pairings and Zariski-Nagata Purity

In this section k is perfect. We fix $F \in \mathbf{RSC}_{\mathrm{Nis}}$ and set $\widetilde{F} = \tau_! \omega^{\mathbf{CI}} F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, see (2.2.2).

Definition 6.1. Let X be a reduced separated scheme of finite type over a function field K over k and let $D \subset X$ be a closed subscheme, such that $U = X \setminus D$ is regular (in this case U is also pro-smooth over k). We define

$$F_{\mathrm{gen}}(X, D) := \mathrm{Ker}\left(F(U) \to \bigoplus_{x \in \widetilde{X}^{(1)} \cap \nu^{-1}(D)} \frac{F(\widetilde{X}^h_x \setminus x)}{\widetilde{F}(\widetilde{X}^h_x, D^h_x)}\right),$$

where $\nu : \tilde{X} \to X$ is the normalization, $\widetilde{X}^h_x = \mathrm{Spec}\, \mathcal{O}^h_{\widetilde{X},x}$, and $D^h_x = D \times_{\widetilde{X}} \widetilde{X}^h_x$.

The aim of this section is to show that if X is smooth and projective over K and D_{red} is a SNCD on X, then $F_{\mathrm{gen}}(X, D) = \widetilde{F}(X, D)$, i.e., the modulus of an element $a \in F(U)$ is determined by the local modulus (or the motivic conductor in the language of [RS21]) at the 1-codimensional points of X. If the support of D is smooth, this also follows from [Sai20, Corollary 8.6(2)] and if D is reduced, it follows from [Sai, Theorem 2.3]. The proof of the general case given here is completely different.

Lemma 6.2. *Let X and Y be separated and finite type schemes over (maybe different) function fields over k and let $f : Y \to X$ be a* flat *morphism. Let $D \subset X$ be a closed subscheme such that $U = X \setminus D$ and $V = Y \setminus f^{-1}(D)$ are regular. Then the pullback $f^* : F(U) \to F(V)$ induces a morphism*

$$f^* : F_{\mathrm{gen}}(X, D) \to F_{\mathrm{gen}}(Y, f^{-1}(D)).$$

Proof. Let $y \in Y^{(1)}$ and $x = f(y)$. Since f is flat we have $\dim \mathcal{O}_{Y,y} = \dim \mathcal{O}_{X,x} + \dim(\mathcal{O}_{Y,y}/\mathfrak{m}_x \mathcal{O}_{Y,y})$. Thus a 1-codimensional point in Y can only be mapped to a 0- or 1-codimensional point in X. The statement follows. $\square$

6.3. Let X be a k-scheme. We denote by $K^M_{r,X}$ ($r \geq 0$) the Nisnevich sheafification of the improved Milnor K-theory from [Ker10]. Let $D \subset X$ be a closed subscheme and denote by $j : U := X \setminus D \hookrightarrow X$ (resp. $i : D \hookrightarrow X$) the corresponding open (resp. closed) immersion. We consider the following Nisnevich sheaves for $r \geq 1$

$$V_{r,X|D} := \mathrm{Im}(\mathcal{O}^\times_{X|D} \otimes_{\mathbb{Z}} K^M_{r-1,X} \to K^M_{r,X}) \quad \text{with } \mathcal{O}^\times_{X|D} := \mathrm{Ker}(\mathcal{O}^\times_X \to i_* \mathcal{O}^\times_D).$$

Note $V_{1,X|D} = \mathcal{O}^\times_{X|D}$. Let $I \subset \mathcal{O}_X$ denote the ideal sheaf of $D \hookrightarrow X$. In [KS86, (1.3)] the sheaf $K^M_r(\mathcal{O}_X, I) = \mathrm{Ker}(K^M_{r,X} \to K^M_{r,D})$ is considered. The inclusion $V_{r,X|D} \subset K^M_r(\mathcal{O}_X, I)$ is an equality for $r = 1$ and for $r \geq 2$ it induces an equality between the Nisnevich stalks at points $x \in X$ with infinite residue field (this follows from [KS86, Lemma 1.3.1] and [Ker10, Proposition 10(5)]). In particular, if X has dimension d, Grothendieck-Nisnevich vanishing implies that the natural map

$$H^d(X_{\mathrm{Nis}}, V_{d,X|D}) \xrightarrow{\simeq} H^d(X_{\mathrm{Nis}}, K^M_d(\mathcal{O}_X, I)) \tag{6.3.1}$$

is an isomorphism. Now assume X is of finite type and pure dimension d over k. Let $U \subset X$ be a regular dense open subscheme. For $x \in U_{(0)}$ the Gersten resolution ([Ker10, Proposition 10(8)]) yields an isomorphism

$$\theta_x : \mathbb{Z} \xrightarrow{\simeq} H^d_x(U_{\mathrm{Nis}}, K^M_{d,U}) \cong H^d_x(X_{\mathrm{Nis}}, V_{d,X|D}). \tag{6.3.2}$$

By [KS86, Theorem 2.5] and (6.3.1), we obtain a surjective map

$$\theta = \sum_x \theta_x : Z_0(U) = \bigoplus_{x \in U_{(0)}} \mathbb{Z} \twoheadrightarrow H^d(X_{\mathrm{Nis}}, V_{d,X|D}). \tag{6.3.3}$$

Proposition 6.4 ([RS23])**.** *Let the notation be as above and assume additionally that X is smooth of pure dimension d and that the support of D is a simple normal crossing divisor. Then (6.3.3) factors to give a surjective map*

$$\mathrm{CH}_0(X|D) \twoheadrightarrow H^d(X_{\mathrm{Nis}}, V_{d,X|D}).$$

Here, $\mathrm{CH}_0(X|D)$ is the Chow group of zero-cycles with modulus introduced in [KS16], see the introduction.

6.5. Let $U \to S$ be a quasi-projective dominant morphism in $\mathbf{Sm}$, with U and S integral. Set $d = \dim U - \dim S$. We fix a compactification

$$U \xrightarrow{j} X \xrightarrow{f} S$$

with j a dense open immersion, f projective, and X integral. There is a natural map of abelian Nisnevich sheaves on U (see 2.3(3))

$$F_U \otimes_{\mathbb{Z}} K^M_{d,U} \xrightarrow{\text{nat.}} (F \otimes_{\mathbf{NST}} K^M_d)_U \to F\langle d\rangle_U \tag{6.5.1}$$

inducing

$$K^M_{d,U} \to \underline{\mathrm{Hom}}_{\mathrm{Sh}(U_{\mathrm{Nis}})}(F_U, F\langle d\rangle_U) = j^{-1}\,\underline{\mathrm{Hom}}_{\mathrm{Sh}(X_{\mathrm{Nis}})}(j_*F_U, j_!F\langle d\rangle_U).$$

By adjunction we obtain the following morphism in $\mathrm{Sh}(X_{\mathrm{Nis}})$

$$j_*F_U \otimes_{\mathbb{Z}} j_!K^M_{d,U} \to j_!F\langle d\rangle_U. \tag{6.5.2}$$

We define the pairing

$$(-,-)_{U\subset X/S} : F(U) \otimes_{\mathbb{Z}} H^d(X_{\mathrm{Nis}}, j_!K^M_{d,U}) \to F(S) \tag{6.5.3}$$

as the composition:

$$\begin{aligned} F(U) \otimes_{\mathbb{Z}} H^d(X_{\mathrm{Nis}}, j_!K^M_{d,U}) &\to H^d(X_{\mathrm{Nis}}, j_*F_U \otimes_{\mathbb{Z}} j_!K^M_{d,U}) \\ &\xrightarrow{(6.5.2)} H^d(X_{\mathrm{Nis}}, j_!F\langle d\rangle_U) \xrightarrow{(f,j)_*} F(S), \end{aligned}$$

where $(f,j)_*$ is the pushforward from Definition 4.6 and the first map sends $a \in F(U)$ and $\beta \in H^d(X_{\mathrm{Nis}}, j_!K^M_{d,U})$ corresponding to maps $a : \mathbb{Z}_X \to j_*F_U$ and $\beta : \mathbb{Z}_X \to j_!K^M_{d,U}[d]$ in $D(X_{\mathrm{Nis}})$, respectively, to $a \otimes \beta : \mathbb{Z}_X \to (j_*F_U \otimes_{\mathbb{Z}} j_!K^M_{d,U})[d]$.

Lemma 6.6. *Let the assumptions be as above. Let $Z \subset U$ be an integral closed subscheme which is finite and surjective over S. By abuse of notation, we also denote by Z the finite correspondence in $\mathbf{Cor}(S,U)$ defined by the image of the natural map $Z \to S \times U$ and we denote by $[Z]$ the image of $1 \in \mathbb{Z}$ under the following map*

$$\mathbb{Z} \xrightarrow{\theta_Z, \simeq} H^d_Z(U_{\mathrm{Nis}}, K^M_{d,U}) = H^d_Z(X_{\mathrm{Nis}}, j_!K^M_{d,U}) \to H^d(X_{\mathrm{Nis}}, j_!K^M_{d,U}),$$

where the isomorphism θ_Z is induced by the Gersten resolution, noting $\mathrm{codim}_U(Z) = d$. Then for $a \in F(U)$

$$(a, [Z])_{U\subset X/S} = Z^*a \quad \text{in } F(S).$$

If Z is furthermore smooth and we denote by $i : Z \hookrightarrow U$ the closed immersion and $g : Z \to S$ the finite and surjective map, then

$$(a, [Z])_{U\subset X/S} = g_*i^*a, \tag{6.6.1}$$

with g_* as in (4.1.1).

Proof. Since the restriction map $F(S) \to F(V)$ is injective for a dense open $V \subset S$ (see [Sai20, Theorem 3.1]), we can shrink S around its generic point and therefore may assume that Z is smooth. Let $\Gamma_i \in \mathbf{Cor}(Z, U)$ and $\Gamma_g^t \in \mathbf{Cor}(S, Z)$ be the (transpose of the) graph of i and g from the second part of the statement, respectively. We have

$$Z^* a = (\Gamma_i \circ \Gamma_g^t)^* a = (\Gamma_g^t)^* \Gamma_i^* a = g_* i^* a,$$

where for the last equality we use that $g_* : F(Z) \to F(S)$ from (4.1.1) is equal to $(\Gamma_g^t)^*$ by [BRS22, Proposition 8.10(3)]. Thus it remains to show (6.6.1). To this end, consider the diagram

$$\begin{tikzcd}
F(U) \arrow[r, "\cup\theta_Z(1)"] \arrow[d, "i^*"'] & H^d_Z(U_{\mathrm{Nis}}, F\langle d\rangle) \arrow[d] \\
F(Z) \arrow[ur, "i_{Z*}"] \arrow[dr, "g_*"'] & H^d(X_{\mathrm{Nis}}, j_! F\langle d\rangle_U) \arrow[d, "(f,j)_*"] \\
& F(S)
\end{tikzcd}$$

where i_{Z*} is induced by H^d_Z of the map (4.1.1), where $f = i$, $r = -d$, $a = d$, $b = 0$, and $\cup\theta_Z(1)$ is induced by the pairing

$$F(U) \times H^d_Z(U_{\mathrm{Nis}}, K^M_{d,U}) \to H^d_Z(U_{\mathrm{Nis}}, F\langle d\rangle_U).$$

Since by definition i_{Z*} is induced by the Gysin map from [BRS22, 7.4], the top triangle commutes by Theorem 7.11 of *loc. cit.* The lower triangle is commutative by Lemma 4.7(3). The equality (6.6.1) follows from this. □

Proposition 6.7. *Let K be a function field over k. Let X be a reduced and projective K-scheme of pure dimension d and $D \subset X$ a nowhere dense closed subscheme, such that $U = X \setminus D$ is regular. Denote by $j : U \hookrightarrow X$ the open immersion. Then the natural map $H^d(X_{\mathrm{Nis}}, j_! K^M_{d,U}) \to H^d(X_{\mathrm{Nis}}, V_{d,X|D})$ is surjective (see 6.3 for notation) and there is a unique pairing*

$$(-,-)_{(X,D)/K} : F_{\mathrm{gen}}(X, D) \otimes H^d(X_{\mathrm{Nis}}, V_{d,X|D}) \to F(K) \tag{6.7.1}$$

such that the following diagram commutes

$$\begin{tikzcd}
F(U) \arrow[r, "(6.5.3)"] & \mathrm{Hom}(H^d(X_{\mathrm{Nis}}, j_! K^M_{d,U}), F(K)) \\
F_{\mathrm{gen}}(X, D) \arrow[r, "(6.7.1)"] \arrow[u, hook] & \mathrm{Hom}(H^d(X_{\mathrm{Nis}}, V_{d,X|D}), F(K)). \arrow[u, hook]
\end{tikzcd}$$

Proof. Since $V_{d,X|D}/j_! K^M_{d,U}$ has support in $|D|$, we see $H^d(X_{\mathrm{Nis}}, V_{d,X|D}/j_! K^M_{d,U}) = 0$, whence the surjectivity of the natural map

$$H^d(X_{\mathrm{Nis}}, j_! K^M_{d,U}) \twoheadrightarrow H^d(X_{\mathrm{Nis}}, V_{d,X|D}). \tag{6.7.2}$$

To show the existence of the pairing (6.7.1), we first reduce to the case where X is normal. Let $\nu : \widetilde{X} \to X$ be the normalization and denote by $\tilde{j} : U \hookrightarrow \widetilde{X}$ the induced open immersion. We obtain a diagram of solid arrows

$$\begin{tikzcd}
F_{\mathrm{gen}}(X,D) \arrow[r] \arrow[d, equal] & H^d(X_{\mathrm{Nis}}, j_!K^M_{d,U})^\vee \arrow[d, "\simeq"] & H^d(X_{\mathrm{Nis}}, V_{d,X|D})^\vee \arrow[l, hook'] \\
F_{\mathrm{gen}}(\tilde{X},\nu^{-1}D) \arrow[r] \arrow[rr, dashed, bend right] & H^d(\widetilde{X}_{\mathrm{Nis}}, \tilde{j}_!K^M_{d,U})^\vee & H^d(\widetilde{X}_{\mathrm{Nis}}, V_{d,\widetilde{X}|\nu^{-1}D})^\vee \arrow[l, hook'] \arrow[u, "(\nu^*)^\vee"']
\end{tikzcd}$$

where $(-)^\vee = \mathrm{Hom}(-, F(K))$, the top and bottom horizontal maps on the left are induced by the pairing (6.5.3), the horizontal injections on the right by (6.7.2), and the vertical isomorphism in the middle by $j_! \cong \nu_*\tilde{j}_!$, see (4.3.2). It is direct to check that the diagram commutes. Assume the statement is proven for $(\widetilde{X}, \nu^{-1}D)$. Then the dashed arrow in the diagram exists, which induces the desired pairing for (X, D).

From now on we assume that X is normal and integral. Let $a \in F_{\mathrm{gen}}(X,D) \subset F(U)$ and denote by $\chi_a \in \mathrm{Hom}(H^d(X_{\mathrm{Nis}}, j_!K^M_{d,U}), F(K))$ the image of a under the pairing (6.5.3). For $\underline{x} \in \mathrm{mc}(X)$, set (see (5.2) for notation)

$$\chi_{a,\underline{x}} := \chi_a \circ c_{\underline{x}} \in \mathrm{Hom}(K^M_d(K^h_{X,\underline{x}}), F(K)),$$

where $c_{\underline{x}}$ is the map (5.4.2). Note that for all $\underline{x} \in \mathrm{mc}(X) \cup \bigcup_{r=0}^{d-1} \mathrm{mc}_r(X)$, we have

$$(V_{d,X|D})^h_{\underline{x}} = (j_!K^M_{d,U})^h_{\underline{x}} = K^M_d(K^h_{X,\underline{x}}).$$

Thus by Proposition 5.5 the map χ_a lies in the image of

$$\mathrm{Hom}(H^d(X_{\mathrm{Nis}}, V_{d,X|D}), F(K)) \hookrightarrow \mathrm{Hom}(H^d(X_{\mathrm{Nis}}, j_!K^M_{d,U}), F(K))$$

if the following composition vanishes for all $\underline{x} \in \mathrm{mc}_d(X)$:

$$(V_{d,X|D})^h_{\underline{x}} \xrightarrow{\iota_\eta\ (5.2.2)} (V_{d,X|D})^h_{\underline{x}(\eta)} = K^M_d(K^h_{X,\underline{x}(\eta)}) \xrightarrow{\chi_{a,\underline{x}(\eta)}} F(K), \tag{6.7.3}$$

where $\eta \in X$ is the generic point. Let $\underline{x} \in \mathrm{mc}_d(X)$ and set $\underline{y} := \underline{x}(\eta)$. By the normality of X and Lemma 5.3, the ring $\mathcal{O}^h_{X,\underline{x}}$ is a finite product of henselian dvr's of geometric type over k and $K^h_{X,\underline{y}} = \mathrm{Frac}(\mathcal{O}^h_{X,\underline{x}})$. Let $a_{\underline{y}} \in F(K^h_{X,\underline{y}})$ be the restriction of a. By definition the map $\chi_{a,\underline{y}}$ is equal to the composition

$$K^M_d(K^h_{X,\underline{y}}) \xrightarrow{\cup a_{\underline{y}}} F\langle d\rangle(K^h_{X,\underline{y}}) \xrightarrow{c_{\underline{y}}} H^d(X_{\mathrm{Nis}}, j_!F\langle d\rangle_U) \xrightarrow{(f,j)_*} F(K), \tag{6.7.4}$$

where the first map is induced by the restriction $a_{\underline{y}}$ of a via the pairing (6.5.2)

$$F(K^h_{X,\underline{y}}) \otimes K^M_d(K^h_{X,\underline{y}}) \to F\langle d\rangle(K^h_{X,\underline{y}}).$$

In fact in this description of $\chi_{a,\underline{y}}$, we use that the following diagram commutes

$$\begin{tikzcd}
F(U) \otimes K^M_d(K^h_{X,\underline{y}}) \arrow[r, "\mathrm{id}\otimes c_{\underline{y}}"] \arrow[d, "\cup"] & H^0(X_{\mathrm{Nis}}, j_*F_U) \otimes H^d(X_{\mathrm{Nis}}, j_!K^M_{d,U}) \arrow[d, "\cup"] \\
F\langle d\rangle(K^h_{X,\underline{y}}) \arrow[r, "c_{\underline{y}}"] & H^d(X_{\mathrm{Nis}}, j_!F\langle d\rangle_U)
\end{tikzcd} \tag{6.7.5}$$

which follows from the compatibility of the cup product with the boundary maps in long exact sequences, e.g., [Swa99, Corollary 3.7]. Since X is normal, we find a regular open subset $V \subset X$ which contains U and $X^{(1)}$; let $j' : V \to X$ be the open immersion. By Lemma 4.7(2), we have a commutative diagram

$$\begin{tikzcd} F(d)(K^h_{X,\underline{y}}) \arrow[r, "c_{\underline{y}}"] \arrow[dr, "c_{\underline{y}}"'] & H^d(X_{\mathrm{Nis}}, j_!F\langle d\rangle_U) \arrow[r, "(f,j)_*"] \arrow[d] & F(K) \\ & H^d(X_{\mathrm{Nis}}, j'_!F\langle d\rangle_V). \arrow[ur, "(f,j')_*"'] & \end{tikzcd}$$

Since $X^{(1)} \subset V$ we have $(j'_!F\langle d\rangle_V)_{\underline{x}} = F\langle d\rangle(\mathcal{O}^h_{X,\underline{x}})$, and hence the lower $c_{\underline{y}}$ annihilates $F\langle d\rangle(\mathcal{O}^h_{X,\underline{x}})$, see (5.4.4). Thus to show the vanishing of (6.7.3), it suffices to show that the image of the composition

$$F_{\mathrm{gen}}(\mathcal{O}^h_{X,\underline{x}}, I_{\underline{x}}^{-1}) \otimes V_{d,X|D,\underline{x}} \to F(K^h_{X,\underline{y}}) \otimes K^M_d(K^h_{X,\underline{y}}) \xrightarrow{(6.5.1)} F\langle d\rangle(K^h_{X,\underline{y}})$$

lies in $F\langle d\rangle(\mathcal{O}^h_{X,\underline{x}})$, where $I_{\underline{x}} \subset \mathcal{O}^h_{X,\underline{x}}$ denotes the ideal of D around $\underline{x}$ and we use the notation from (2.2.4). Since $F_{\mathrm{gen}}(\mathcal{O}^h_{X,\underline{x}}, I_{\underline{x}}^{-1}) = \widetilde{F}(\mathcal{O}^h_{X,\underline{x}}, I_{\underline{x}}^{-1})$ by definition, the image of the above composition is equal to the image of the following composition

$$(\widetilde{F}(\mathcal{O}^h_{X,\underline{x}}, I_{\underline{x}}^{-1}) \otimes \mathcal{O}^\times_{X|D,\underline{x}}) \otimes K^M_{d-1}(\mathcal{O}^h_{X,\underline{x}}) \to F\langle 1\rangle(K^h_{X,\underline{y}}) \otimes K^M_{d-1}(\mathcal{O}^h_{X,\underline{x}}) \to F\langle d\rangle(K^h_{X,\underline{y}}).$$

Hence the desired assertion follows from Theorem 3.7. □

Theorem 6.8. *Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$. Let X be a smooth projective k-scheme of pure dimension d and D an effective Cartier divisor on X, such that its support $|D|$ is SNCD. Denote by $j : U = X \setminus |D| \hookrightarrow X$ the open immersion. For $a \in F(U)$, the following conditions are equivalent:*

(i) $a \in \widetilde{F}(X, D)$;

(ii) $a \in F_{\mathrm{gen}}(X, D)$;

(iii) for any function field K over k, the map

$$(a_K, -)_{U_K \subset X_K/K} : H^d(X_{K,\mathrm{Nis}}, j_!K^M_{d,U_K}) \to F(K)$$

induced by the pairing (6.5.3) (with $a_K \in F(U_K)$ the pullback of a) factors through $H^d(X_{K,\mathrm{Nis}}, V_{d,X_K|D_K})$.

Proof. The implication (i)⇒(ii) is obvious. If $a \in F_{\mathrm{gen}}(X, D)$, then Lemma 6.2 implies $a_K \in F_{\mathrm{gen}}(X_K, D_K)$. Thus (ii)⇒(iii) follows from Proposition 6.7. Assume (iii) holds. We have to show that this implies (i), i.e., that the Yoneda map $a : \mathbb{Z}_{\mathrm{tr}}(U) \to F$ factors through the quotient map $q : \mathbb{Z}_{\mathrm{tr}}(U) \to h_0(X, D)$ in **PST**, see 2.2. This means that we have to show that $a(S) : \mathbb{Z}_{\mathrm{tr}}(U)(S) \to F(S)$ factors through $q(S)$, for any $S \in \mathbf{Sm}$. If S is connected with function field K, then $F(S) \to F(K)$ is injective by [Sai20, Theorem 3.1]. Hence it suffices to show the claim in case $S = \operatorname{Spec} K$. By [BS19, Theorem 3.3] we have a commutative diagram in which the vertical maps are canonical isomorphisms

(6.8.1)
$$\begin{tikzcd} \mathbb{Z}_{\mathrm{tr}}(U)(K) \arrow[r, "q(K)"] \arrow[d, equal] & h_0(X, D)(K) \arrow[d, "\simeq"] \\ Z_0(U_K) \arrow[r, "\pi"] & \mathrm{CH}_0(X_K|D_K). \end{tikzcd}$$

where $\mathrm{CH}_0(X_K|D_K)$ is the Chow group of zero-cycles with modulus introduced in the introduction. Thus we are reduced to show the following.

Claim 6.8.1. $a(K) : Z_0(U_K) = \mathbb{Z}_{\mathrm{tr}}(U)(K) \to F(K)$ factors via π from (6.8.1).

By Lemma 6.6 we have a commutative diagram

$$\begin{tikzcd} F(U) \arrow[r, "\rho_K"] \arrow[d, "\simeq"'] & \mathrm{Hom}(H^d(X_{K,\mathrm{Nis}}, j_! K^M_{d,U_K})), F(K)) \arrow[d, "\gamma"] \\ \mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(U), F) \arrow[r, "\beta"] & \mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}(U)(K), F(K)) \end{tikzcd}$$

where ρ_K is induced by (6.5.3), β is the evaluation map, and γ is induced by (6.8.2)

$$\mathbb{Z}_{\mathrm{tr}}(U)(K) = \bigoplus_{z\in (U_K)_{(0)}} \mathbb{Z} \underset{\simeq}{\xrightarrow{(6.3.2)}} \bigoplus_{z\in (U_K)_{(0)}} H^d_z(U_{K,\mathrm{Nis}}, K^M_{d,U}) \xrightarrow{\Sigma} H^d(X_{K,\mathrm{Nis}}, j_! K^M_{d,U_K}).$$

By the diagram we have $a(K) = \gamma(\rho_K(a)) = (a_K, -)_{U_K\subset X_K/K} \circ$ (6.8.2). Moreover, by (6.8.1) and Proposition 6.4, the composite of (6.8.2) with the natural map

$$H^d(X_{K,\mathrm{Nis}}, j_! K^M_{d,U_K})) \to H^d(X_{K,\mathrm{Nis}}, V_{d,X_K|D_K})$$

factors as

$$\mathbb{Z}_{\mathrm{tr}}(U)(K) \xrightarrow{q(K)} h_0(X,D)(K) \simeq \mathrm{CH}_0(X_K|D_K) \to H^d(X_{K,\mathrm{Nis}}, V_{d,X_K|D_K}).$$

Thus the condition (iii) implies Claim 6.8.1. This completes the proof. □

Examples and Remarks 6.9.

(1) Let F be the reciprocity sheaf given by

$$F(X) = \mathrm{Hom}_{\mathrm{cts}}(\pi_1^{\text{ét}}(X), \mathbb{Q}/\mathbb{Z}), \quad X \in \mathbf{Sm},$$

see [RS21, 8] for the fact that this is a reciprocity sheaf. Set $\pi_1^{\text{ét,ab}}(X,D) := \mathrm{Hom}(\widetilde{F}(X,D), \mathbb{Q}/\mathbb{Z})$, cf. [KS16, Definition 2.9]. Let (X,D) be as in Theorem 6.8 and assume that k is a finite field. Then $F(k) \cong \mathbb{Q}/\mathbb{Z}$ and the pairing $(-,-)_{(X,D)/K}$ induces a morphism

$$H^d(X_{\mathrm{Nis}}, V_{d,X|D}) \to \pi_1^{\text{ét,ab}}(X,D).$$

Taking the inverse limit over multiples of D we obtain the reciprocity map constructed in [KS86, (3.7)]. (Actually in *loc. cit.* the case where k is a number field is considered, which requires some additional attention to the places at infinity.) Composing with the cycle map from Proposition 6.4 yields the reciprocity map

$$\mathrm{CH}_0(X|D) \to \pi_1^{\text{ét,ab}}(X,D),$$

from [KS16, Proposition 3.2]. (But in *loc. cit.* only $X \setminus |D|$ is assumed to be smooth.) Finally we note that in the special case at hand and in view of [GK22, Theorem 7.16], Proposition 6.7 essentially reproves [GK22, Theorem 1.2]. In fact, in *loc. cit.* X is just assumed to be of finite type instead of being projective. However, if X is at least quasi-projective we can construct a pairing as in Proposition 6.7 by considering projective compactifications of (X,D). Also note that the paring in *loc. cit.* is constructed only over a finite field.

(2) Assume $\mathrm{char}(k) = 0$ and let (X,D) be as in Theorem 6.8. Denote by $\mathrm{Conn}^1_{\mathrm{abs}}(X)$ the group of isomorphism classes of absolute rank one connections on X relative to k. ("Absolute" refers to the fact that the connection

has values in absolute differentials, $\nabla : \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{Z}}$.) This defines a reciprocity sheaf, cf. [RS21, 6.10]. Theorem 6.8.1 yields a pairing

$$\widetilde{\mathrm{Conn}}^1_{\mathrm{abs}}(X, D) \otimes H^d(X_{\mathrm{Nis}}, V_{d,X|D}) \to \mathrm{Conn}^1_{\mathrm{abs}}(k).$$

By construction and (an absolute version of) [RS21, Theorem 6.11], this pairing is a higher-dimensional version of the pairing constructed in [BE01, 4.] in the case X is a curve, i.e., $d = 1$.

(3) Assume $\mathrm{char}(k) = p > 0$. For $j \geq 1$ and $X \in \mathbf{Sm}$ consider

$$H^j_{p^n}(X) := H^0(X, R^j\varepsilon_*\mathbb{Z}/p^n(j-1)) = H^0(X, R^1\varepsilon_* W_n\Omega^{j-1}_{X,\log}),$$

where $\varepsilon : X_{\mathrm{\acute{e}t}} \to X_{\mathrm{Nis}}$ is the natural change of sites morphism, $W_n\Omega^{j-1}_{X,\log}$ denotes the subsheaf of the de Rham-Witt differentials $W_n\Omega^{j-1}_X$ generated locally by log-forms (see [Ill79]), and $\mathbb{Z}/p^n(j-1)$ denotes the mod-p^n-motivic complex of weight $j-1$. The last equality above holds by [GL00, Theorem 8.3]. In fact we have

$$H^j_{p^n} = \mathrm{Coker}(W_n\Omega^{j-1} \xrightarrow{F-R} W_n\Omega^{j-1}/dV^{n-1}\Omega^{j-1}).$$

Since $\mathbf{RSC}_{\mathrm{Nis}}$ is an abelian category by [Sai20, Theorem 0.1] it follows that $H^j_{p^n}$ is a reciprocity sheaf.

Let (X, D) be as in Theorem 6.8. By [Izh91] (see also [GL00, Theorem 8.1]) and the Gersten resolution [Ker10, Proposition 10(8)], $K^M_{d,X}$ is p-torsion free, hence so is $V_{d,X|D}$. Grothendieck-Nisnevich vanishing yields $H^d(X_{\mathrm{Nis}}, V_{d,X|D})/p^n = H^d(X_{\mathrm{Nis}}, V_{d,X|D}/p^n)$. Thus for every function field K/k, Theorem 6.8 yields a pairing

$$\widetilde{H^j_{p^n}}(X_K, D_K) \otimes H^d(X_{K,\mathrm{Nis}}, V_{d,X_K|D_K}/p^n) \to H^j_{p^n}(K),$$

which induces a pairing, for all $j \geq 1$

$$H^j_{p^n}(U_K) \times \varprojlim_r H^d(X_{K,\mathrm{Nis}}, V_{d,X_K|rD_K}/p^n) \to H^j_{p^n}(K).$$

This should be compared to [JSZ18, Theorem 2] and [GK, Theorem 1.1] where similar pairings are constructed in different cohomological degrees. Note that the filtrations on $H^j(U_{\mathrm{\acute{e}t}}, \mathbb{Z}/p^n(j-1))$ used there to define the pairings are more ad hoc and only well-behaved in the colimit.

By [Sai20, Corollary 8.6] we have $\widetilde{F}(X, D) = F_{\mathrm{gen}}(X, D)$, for any $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$ with X and $|D| \in \mathbf{Sm}$. The following corollary of Theorem 6.8 generalizes this result to the case where $|D|$ is only SNCD - at least under a mild extra assumption. Recall from [KMSY21a, Definition 1.8.1], that a *compactification of a modulus pair* $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$ is a proper modulus pair $(\overline{X}, \overline{D} + B) \in \mathbf{MCor}$ with effective Cartier divisors $\overline{D}$ and B, such that there is a dense open immersion $j : X \hookrightarrow \overline{X}$ with $j(X) = \overline{X} \setminus |B|$ and $D = j^*\overline{D}$. A compactification of (X, D) always exists.

Corollary 6.10. *Assume F has level $n \geq 0$ (see Definition 1.3) and resolutions of singularities hold over k in dimension $\leq n$ (see Theorem 1.4). Let $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$ and $U = X \setminus |D|$. Let $a \in F(U)$. The following statements are equivalent:*

(i) $a \in \widetilde{F}(X, D)$;

(ii) $h^*a \in \widetilde{F}(Z, h^*D)$, *for all k-morphisms $h : Z \to X$ with* $\dim(Z) \leq n$, *such that $h^{-1}(U)$ is smooth and non-empty;*

(iii) $h^*a \in F_{\rm gen}(Z, h^*D)$, *for all* k*-morphisms* $h : Z \to X$ *with* Z *smooth quasi-projective, and* $\dim(Z) \leq n$*, such that* $|h^*D|$ *is SNCD.*

Furthermore, if there is a compactification $(\overline{X}, \overline{D} + B)$ *of* (X, D)*, such that* $\overline{X}$ *is smooth projective and* $|\overline{D} + B|$ *is SNCD, the above conditions are equivalent to*

(iv) $a \in F_{\rm gen}(X, D)$.

Proof. We assume $D \neq \emptyset$ else there is nothing to prove. The implication (i)$\Rightarrow$ (ii) is direct. The implication (i)$\Rightarrow$ (iv) always holds, hence so does (ii)$\Rightarrow$ (iii). Assume we have a compactification $(\overline{X}, \overline{D} + B)$ of (X, D) with $\overline{X}$ smooth projective and $|\overline{D}+B|$ SNCD. We may assume that $|\overline{D}|$ and $|B|$ have no common components. For any $N \geq 1$ we have

$$\begin{aligned}(6.10.1)\quad F_{\rm gen}(X, D) \cap \widetilde{F}(\overline{X}, N(\overline{D} + B)) &= F_{\rm gen}(X, D) \cap F_{\rm gen}(\overline{X}, N(\overline{D} + B)) \\ &= F_{\rm gen}(\overline{X}, \overline{D} + NB) = \widetilde{F}(\overline{X}, \overline{D} + NB) \subset \widetilde{F}(X, D),\end{aligned}$$

where the first and the last equality hold by Theorem 6.8 and the equality in the middle holds by definition of $F_{\rm gen}$. Since there exits an $N \geq 0$, such that $a \in \widetilde{F}(\overline{X}, N(\overline{D} + B))$, the equality (6.10.1) proves the implication (iv)$\Rightarrow$ (i) under the extra assumption on (X, D). It remains to show (iii)$\Rightarrow$ (i) (assuming resolution of singularities in dimension $\leq n$). Let $(\overline{X}, \overline{D} + B)$ be any (not necessarily smooth) compactification of (X, D). Take $N \geq 0$, such that $a \in \widetilde{F}(\overline{X}, N(\overline{D}+B))$. We claim that *(iii)* implies

$$a \in \widetilde{F}(\overline{X}, \overline{D} + NB) \subset \widetilde{F}(X, D).$$

Indeed, by [RS21, Corollary 4.18] and since F has level n, it suffices to show

$$(6.10.2)\qquad \rho^*a \in \widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-v_L(\overline{D}+NB)})$$

for any $\rho : \operatorname{Spec} L \to U$, where L is a henselian discrete valuation field of geometric type over k, which has $\operatorname{trdeg}(L/k) \leq n$, and where $v_L(\overline{D} + NH)$ denotes the multiplicity of $\rho^*(\overline{D}+NH)$ on $\operatorname{Spec} \mathcal{O}_L$. (Note that ρ uniquely extends to $\operatorname{Spec} \mathcal{O}_L \to \overline{X}$.) Let $\overline{Z}_1$ be the closure of the image of $\operatorname{Spec} \mathcal{O}_L$ in $\overline{X}$. Using the Chow Lemma we find a proper morphism $\bar{h} : \overline{Z} \to \overline{X}$ which maps birational onto $\overline{Z}_1$ and such that $\overline{Z}$ is a projective k-scheme; using resolutions of singularities in dimension $\leq n$, we can additionally assume $\overline{Z}$ is smooth and $|\bar{h}^*(\overline{D} + B)|$ is SNCD. Note that by the valuative criterion for properness, $\operatorname{Spec} \mathcal{O}_L \to \overline{X}$ factors via $\bar{h}$. Set $Z := \bar{h}^{-1}(X)$ and denote by $h : Z \to X$ the restriction of $\bar{h}$. By (iii) and the above we find

$$\bar{h}^*a \in F_{\rm gen}(Z, h^*D) \cap \widetilde{F}(\overline{Z}, N\bar{h}^*(\overline{D} + B)) = \widetilde{F}(\overline{Z}, \bar{h}^*(\overline{D} + NB)),$$

where the equality follows from (6.10.1) applied to Z instead of X. Pulling $\bar{h}^*a$ further back to $\operatorname{Spec} \mathcal{O}_L$ yields (6.10.2). This completes the proof. $\square$

Examples 6.11. We list some examples where we can apply Corollary 6.10 unconditionally (recall this is the case if $\operatorname{ch}(k) = 0$ or if F has level≤ 3):

(1) Let G be a commutative algebraic group over k. By [RS21, Theorem 5.2] G has level 1. By [RS21, Theorem 7.20] we obtain a cut-by-curves criterion for a conductor defined by Kato-Russell (see [KR10]).

(2) $(\operatorname{char}(k) = p > 0)$ Let F be one of the following two reciprocity sheaves

$$\mathbf{Sm} \ni X \mapsto \operatorname{Hom}_{\rm cts}(\pi_1^{\rm ét}(X), \mathbb{Q}/Z) \quad \text{or} \quad \mathbf{Sm} \ni X \mapsto \operatorname{Lisse}^1_\ell(X),$$

where $\mathrm{Lisse}^1_\ell(X)$ denotes the group of isomorphism classes of lisse $\overline{\mathbb{Q}}_\ell$-sheaves of rank 1 on X. Then F has level 1 and we obtain a cut-by-curves criterion for the Artin conductor of torsion characters or lisse rank 1 sheaves, by [RS21, Theorem 8.8, Corollary 8.10]. This also follows (by a different method) from the main results in [Kat89] and [Mat97], see [KS16, Corollary 2.8]. Also note that the statement for $D = \emptyset$ and $F = \mathrm{Hom}_{\mathrm{cts}}(\pi_1^{\text{ét}}(-), \mathbb{Q}/Z)$ is a direct consequence of the Zariski-Nagata purity theorem.

(3) ($\mathrm{char}(k) = 0$) By [RS21, Theorem 6.11] assigning to $X \in \mathbf{Sm}$ the group of isomorphism classes of integrable rank 1 connections defines a reciprocity sheaf of level 1. We obtain a cut-by-curves criterion for the irregularity. In the tame case (i.e. regular connections at infinity) this was proven by Deligne in [Del70, II, Proposition 4.4] - by a different method and even for higher rank connections. In general this is well-known, but we don't know a reference.

(4) ($\mathrm{char}(k) = p > 0$) Let G be a finite flat algebraic k-group (G infinitesimal unipotent, like α_p, is the most interesting case). By [RS21, Theorem 9.12] the presheaf $X \mapsto H^1_{\mathrm{fppf}}(X, G)$ defines a reciprocity sheaf of level at most 2. Thus we get a cut-by-surfaces criterion for the ramification of G-torsors, which we believe to be new.

(5) ($\mathrm{char}(k) = p > 0$) Let $\varepsilon : \mathbf{Sm}_{\text{ét}} \to \mathbf{Sm}_{\mathrm{Nis}}$ be the change of sites map. We expect the reciprocity sheaves $R^n\varepsilon_*(\mathbb{Q}/\mathbb{Z}(n-1))$ to be of level n, where $\mathbb{Q}/\mathbb{Z}(n-1)$ denotes the étale motivic complex of weight $n-1$ with $\mathbb{Q}/\mathbb{Z}$-coefficients so that for $n = 2$ (resp. $n = 3$), we get a cut-by-surfaces (resp. cut-by-threefolds) criterion for the ramification of $R^n\varepsilon_*(\mathbb{Q}/\mathbb{Z}(n-1))$. We hope to come back to this point somewhere else.

7. An application to rational singularities

In this section we assume $\mathrm{char}(k) = 0$ and give an application of Corollary 6.10 to the theory of singularities. Recall that by Kempf's criterion a separated finite type k-scheme X of pure dimension d has rational singularities if and only if it is normal, Cohen-Macaulay, and for one/any resolution of singularities $f : Y \xrightarrow{\simeq} X$ we have $f_*\omega_{Y/k} = \omega_{X/k}$, where $\omega_{Y/k} = \Omega^d_{Y/k}$ and $\omega_{X/k} = j_*\Omega^d_{X_{\mathrm{sm}}/k}$, with $j : X_{\mathrm{sm}} = (\text{smooth locus}) \hookrightarrow X$, are the relative dualizing sheaves over k. There are various alternative descriptions of rational singularities, all relying on some sort of resolutions (alterations, Macaulifications, etc...), see [Kov] for the state of the art. Corollary 6.10 yields a resolution-free characterization as shown in the next theorem.

Theorem 7.1. *Assume* $\mathrm{char}(k) = 0$.[8] *Let X be an affine k-scheme of finite type of pure dimension d, that is normal and Cohen-Macaulay. Let D be an effective Cartier divisor on X, such that $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$ (i.e., D contains the singular locus of X). The following are equivalent:*

(1) X has rational singularities.

(2) We have $\widetilde{\Omega^d}(X, D) = \Omega^d_{\mathrm{gen}}(X, D)$.

Proof. We find an open immersion $j : V \hookrightarrow X$, such that $V \supset (X \setminus |D|) \cup X^{(1)}$ and such that $D_V = j^*D$ has SNC support. We obtain

$$\Omega^d_{\mathrm{gen}}(X, D) \overset{(1)}{=} \Omega^d_{\mathrm{gen}}(V, D_V) \overset{(2)}{=} \widetilde{\Omega^d}(V, D_V) \overset{(3)}{=} \Gamma(V, \Omega^d_V(\log D_V) \otimes_{\mathcal{O}_V} \mathcal{O}_V(D_V - D_{V,\mathrm{red}}))$$

[8]The statement is extended to the case $\mathrm{ch}(k) > 0$ in [RSb, Cor.7.3].

$$\overset{(4)}{=} \Gamma(V, \Omega^d_V \otimes_{\mathcal{O}_V} j^*\mathcal{O}_X(D)) = \Gamma(X, j_*(\Omega^d_V \otimes_{\mathcal{O}_V} j^*\mathcal{O}_X(D))) \overset{(5)}{=} \Gamma(X, \omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)),$$

where (1) holds by definition, (2) follows from Corollary 6.10 and resolutions of singularities, (3) follows from [RS21, Corollary 6.8], (4) holds since $d = \dim V$, and (5) follows from the projection formula and a well-known formula for the dualizing sheaf $\omega_{X/k}$ of a normal Cohen-Macaulay k-scheme. On the other hand, if $f : Y \to X$ is a resolution of singularities such that f^*D has SNC support, then (X, D) and (Y, f^*D) are isomorphic in $\underline{\mathbf{M}}\mathbf{Cor}$ and we find similarly

$$\widetilde{\Omega^d}(X, D) = \widetilde{\Omega^d}(Y, f^*D) = \Gamma(Y, \omega_{Y/k} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(f^*D)) = \Gamma(X, f_*\omega_{Y/k} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)).$$

Thus

$$\widetilde{\Omega^d}(X, D) = \Omega^d_{\mathrm{gen}}(X, D) \iff \Gamma(X, f_*(\omega_{Y/k}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)) = \Gamma(X, \omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)).$$

Since X is affine, the equality on the right is equivalent to

$$f_*(\omega_{Y/k}) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) = \omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D),$$

which is equivalent to $f_*(\omega_{Y/k}) = \omega_{X/k}$. Thus the statement follows from Kempf's criterion stated in the beginning of this section. $\square$

References

[AGV72] Michael Artin, Alexander Grothendieck, and J. L. Verdier, *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas. (SGA 4). Un séminaire dirigé par M. Artin, A. Grothendieck, J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint-Donat. Tome 1: Théorie des topos. Exposés I à IV. 2e éd.*, vol. 269, Springer, Cham, 1972.

[AS11] Ahmed Abbes and Takeshi Saito, *Ramification and cleanliness*, Tohoku Math. J. (2) **63** (2011), no. 4, 775–853.

[BE01] Spencer Bloch and Hélène Esnault, *Gauss-Manin determinants for rank 1 irregular connections on curves*, Math. Ann. **321** (2001), no. 1, 15–87, With an appendix in French by P. Deligne.

[BRS22] Federico Binda, Kay Rülling, and Shuji Saito, *On the cohomology of reciprocity sheaves*, Forum of Mathematics, Sigma **10**:e72 (2022), 1–111.

[Bry83] Jean-Luc Brylinski, *Théorie du corps de classes de Kato et revêtements abéliens de surfaces*, Ann. Inst. Fourier (Grenoble) **33** (1983), no. 3, 23–38. MR 723946

[BS19] Federico Binda and Shuji Saito, *Relative cycles with moduli and regulator maps*, J. Inst. Math. Jussieu **18** (2019), no. 6, 1233–1293.

[CP09] Vincent Cossart and Olivier Piltant, *Resolution of singularities of threefolds in positive characteristic. II*, J. Algebra **321** (2009), no. 7, 1836–1976.

[Del70] Pierre Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin-New York, 1970.

[GK] Rahul Gupta and Amalendu Krishna, *Ramified class field theory and duality over finite fields*, Preprint 2021, https://arxiv.org/abs/2104.03029.

[GK22] Rahul Gupta and Amalendu Krishna, *Reciprocity for Kato-Saito idele class group with modulus*, J. Algebra **608** (2022), 487–552.

[GL00] Thomas Geisser and Marc Levine, *The K-theory of fields in characteristic p*, Invent. Math. **139** (2000), no. 3, 459–493.

[GLL15] Ofer Gabber, Qing Liu, and Dino Lorenzini, *Hypersurfaces in projective schemes and a moving lemma*, Duke Math. J. **164** (2015), no. 7, 1187–1270.

[Gro67] A. Grothendieck, *éléments de géométrie algébrique. IV. étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361.

[Ill79] Luc Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Sci. École Norm. Sup. (4) **12** (1979), no. 4, 501–661.

[Izh91] O. Izhboldin, *On p-torsion in K_*^M for fields of characteristic p*, Algebraic K-theory, Adv. Soviet Math., vol. 4, Amer. Math. Soc., Providence, RI, 1991, pp. 129–144. MR 1124629

[JSZ18] Uwe Jannsen, Shuji Saito, and Yigeng Zhao, *Duality for relative logarithmic de Rham–Witt sheaves and wildly ramified class field theory over finite fields*, Compos. Math. **154** (2018), no. 6, 1306–1331.

[Kat89] Kazuya Kato, *Swan conductors for characters of degree one in the imperfect residue field case*, Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987), Contemp. Math., vol. 83, Amer. Math. Soc., Providence, RI, 1989, pp. 101–131. MR 991978

[Ker10] Moritz Kerz, *Milnor K-theory of local rings with finite residue fields*, J. Algebraic Geom. **19** (2010), no. 1, 173–191.

[KMSY21a] Bruno Kahn, Hiroyasu Miyazaki, Shuji Saito, and Takao Yamazaki, *Motives with modulus, I: Modulus sheaves with transfers for non-proper modulus pairs*, Épijournal Géom. Algébrique **5** (2021), Art. 1, 46.

[KMSY21b] ———, *Motives with modulus, II: Modulus sheaves with transfers for proper modulus pairs*, Épijournal Géom. Algébrique **5** (2021), Art. 2, 31.

[Kov] Sándor Kovács, *Rational Singularities*, Preprint 2017 https://arxiv.org/abs/1703.02269.

[KR10] Kazuya Kato and Henrik Russell, *Modulus of a rational map into a commutative algebraic group*, Kyoto Journal of Mathematics **50** (2010), no. 3, 607–622.

[KS86] Kazuya Kato and Shuji Saito, *Global class field theory of arithmetic schemes*, Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 255–331. MR 862639

[KS16] Moritz Kerz and Shuji Saito, *Chow group of 0-cycles with modulus and higher-dimensional class field theory*, Duke Math. J. **165** (2016), no. 15, 2811–2897. MR 3557274

[KSY16] Bruno Kahn, Shuji Saito, and Takao Yamazaki, *Reciprocity sheaves*, Compos. Math. **152** (2016), no. 9, 1851–1898, With two appendices by Kay Rülling.

[KSY22] ———, *Reciprocity sheaves, II*, Homol. Homotopy Appl. **24** (2022), no. 1.

[Lev06] Marc Levine, *Chow's moving lemma and the homotopy coniveau tower*, K-Theory **37** (2006), no. 1-2, 129–209.

[Mac71] Saunders MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics, Vol. 5, Springer-Verlag, New York-Berlin, 1971.

[Mat97] Shigeki Matsuda, *On the Swan conductor in positive characteristic*, Amer. J. Math. **119** (1997), no. 4, 705–739.

[MS] Alberto Merici and Shuji Saito, *Cancellation theorems for reciprocity sheaves*, Preprint 2020, to appear in Algebraic Geometry, https://arxiv.org/abs/2001.07902.

[Pop86] Dorin Popescu, *General Néron desingularization and approximation*, Nagoya Math. J. **104** (1986), 85–115.

[RS23] Kay Rülling and Shuji Saito, *Cycle class maps for Chow groups of zero-cycles with modulus*, J. Pure Appl. Algebra **227** (2023), no. 5.

[RSa] ———, *Ramification theory of reciprocity sheaves, II, Higher local symbols*, Preprint 2021, https://arxiv.org/abs/2111.13373.

[RSb] ———, *Ramification theory of reciprocity sheaves, III, Abbes-Saito formula*, Preprint 2022, https://arxiv.org/abs/2204.10637.

[RS21] ———, *Reciprocity sheaves and their ramification filtrations*, Journal of the Institute of Mathematics of Jussieu (2021), 1–74.

[RSY21] Kay Rülling, Rin Sugiyama, and Takao Yamazaki, *Tensor structures in the theory of modulus presheaves with transfers*, Math. Z. (2021), 1–49.

[TSai17] Takeshi Saito, *Wild ramification and the cotangent bundle*, J. Algebraic Geom. **26** (2017), no. 3, 399–473.

[Sai20] Shuji Saito, *Purity of reciprocity sheaves*, Adv. Math. **366** (2020), 107067, 70.

[Sai] Shuji Saito, *Reciprocity sheaves and logarithmic motives*, Preprint 2021, to appear in Compositio Math., https://arxiv.org/abs/2107.00381.

[SGA73] *Théorie des topos et cohomologie étale des schémas. Tome 3*, Lecture Notes in Mathematics, Vol. 305, Springer-Verlag, Berlin-New York, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat.

[SGA03] *Revêtements étales et groupe fondamental (SGA 1)*, Documents Mathématiques (Paris), vol. 3, Société Mathématique de France, Paris, 2003, Séminaire de géométrie algébrique du Bois Marie 1960–61. Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original.

[Swa99] Richard G. Swan, *Cup products in sheaf cohomology, pure injectives, and a substitute for projective resolutions*, J. Pure Appl. Algebra **144** (1999), no. 2, 169–211.

[Yat17] Yuri Yatagawa, *Equality of two non-logarithmic ramification filtrations of abelianized Galois group in positive characteristic*, Doc. Math. **22** (2017), 917–952.

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