The Abbes-Saito formula for motivic ramification filtrations

Shuji Saito (joint work with Kay Rülling)

University of Tokyo

Takeshi 60

Plan of the talk

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- §3 Comparison theorem of motivic and Abbes-Saito filtrations
- §4 Reciprocity pairings (key in the proof)
- §5 Computations of characteristic forms
- §6 Application: Cdarc descent for reciprocity sheaves (Kelly-S)

§1. Review of Abbes-Saito's theory

Abbes-Saito's (non-logarithmic) ramification theory

X smooth separated scheme over a perfect field k of $\mathrm{ch}(k) = p > 0$.

$D \subset X$ a SNC divisor, $U = X - D$.

$R = \sum_{i \in I} n_i D_i$ an effective divisor supported on D $(n_i \in \mathbb{Z}_{>0})$.

$$P_X^{(R)} = \mathbf{Bl}_{\Delta \cap (R \times R)}(X \times X) \backslash \big(\widetilde{X \times R} \cup \widetilde{R \times X}\big).$$

$$p_1, p_2 : P_X^{(R)} \to X \times X \rightrightarrows X, \ p_i^{-1}(U) = U \times U \ (i = 1, 2).$$

For finite group G and G-torsor $\varphi : V \to U$, AS give a condition:

<u>the ramification of φ is bounded by R</u> in terms of $P_X^{(R)}$.

Remark The AS theory works even for case R is a $\mathbb{Q}$-divisor.

Assume G abelian. The Abbes-Saito condition is equivalent to

$(\spadesuit)^{(R)}$ G-torsor $p_1^*(\varphi) - p_2^*(\varphi)$ over $U \times U$ extends to that on $P_X^{(R)}$.

Define

$$H^1_G(U) := \{G\text{-torsors } \varphi : V \to U\} \simeq H^1(U_{\text{ét}}, G)$$

Then

$$H^{1,AS}_G(X,R) := \{\varphi \in H^1_G(U) |\, \varphi \text{ satisfy } (\spadesuit)^{(R)}\ \}$$

give filtration on $H^1_G(U)$ parametrized by (X,R) with $U = X \backslash R$.

Characteristic forms (T. Saito)

Assume G cyclic, and (for simplicity) $p \neq 2$ or multiplicities in $R > 2$.

$$\mathrm{char}^{(R)} : H^{1,AS}_G(X, R) \to \Gamma(D, \Omega^1_X(R)_{|D}).$$

Construction:

- $V_R := P^{(R)}_X \times_X D \simeq \mathbb{V}(\Omega^1_X(R)) \times_X D$.
  $q_1, q_2, \mu : V_R \times V_R \to V_R$: projections and addition.
- $\mathrm{Image}(\chi^{(R)}) \subset H^1_G(V_R)_{ad} := \{\varphi \in H^1_G(V_R) \mid q_1^*\varphi + q_2^*\varphi = \mu^*\varphi\}$,

  $$\chi^{(R)} : H^{1,AS}_G(X, R) \to H^1_G(P^{(R)}_X) \to H^1_G(V_R) \; ; \; \varphi \to (p_1^*\varphi - p_2^*\varphi)_{|V_R}$$
- every $\varphi \in H^1_G(V_R)_{ad}$ is given the Artin-Schreir covering

  $$t^p - t = s \ \text{ for some } s \in \Omega^1_X(R)_{|D}.$$

Theorem 1.1 (Abbes-Saito (Zariski-Nagata purity))

$$H^{1,AS}_G(X,R) = \mathrm{Ker}\left(H^1_G(U) \to \bigoplus_{\eta \in R^{(0)}} \frac{H^1_G(\mathrm{Spec}\,\mathcal{O}^h_{X,\eta}\backslash\eta)}{H^{1,AS}_G(\mathrm{Spec}\,\mathcal{O}^h_{X,\eta}, R_\eta)}\right),$$

where $R_\eta = R \times_X \mathrm{Spec}\,\mathcal{O}^h_{X,\eta}$ *and* $U = X\backslash R$.

Theorem 1.2 (T. Saito)

For $D_i \subset R$ *irreducible component with* $n_i \geq 2$ *and dense open* $V \subset D_i$,

$$\frac{H^{1,AS}_G(X,R)}{H^{1,AS}_G(X,R-D_i)} \overset{\mathrm{char}^{(R)}}{\hookrightarrow} \Gamma(V, \Omega^1_X(R)_{|V}).$$

Computation of characteristic form

Assume $G = \mathbb{Z}/p^s\mathbb{Z}$, $p \neq 2$, D smooth and $R = nD$ with $n \geq 2$,

$X = \mathrm{Spec}\, A$, $D = \{\pi = 0\}$ with $\pi \in A$, $U = \mathrm{Spec}\, A[1/\pi]$.

$H^1_G(U) = W_s(A[1/\pi])/1 - F$ (Artin-Schreier-Witt theory).

Theorem 1.3 (Abbes-Saito, Yatagawa)

$H^{1,AS}_G(X, nD) = \mathrm{Image}\big(\mathrm{fil}'_n W_s(A[1/\pi]) \to H^1_G(U)\big)$.

$\mathrm{fil}'_n W_s(A[1/\pi])$: *Brylinski-Kato-Matsuda's filtration.*

For $\alpha = (a_{s-1}, \ldots, a_0) \in \mathrm{fil}'_n W_s(A[1/\pi])$,

$$\mathrm{char}(\overline{\alpha}) = -F^{s-1} d(\alpha) = -\sum_{i=0}^{s-1} a_i^{p^i - 1} da_i.$$

Brylinski-Kato's filtration on $W_s(A[1/\pi])$:

$$\mathrm{fil}_n W_s(A[1/\pi]) = \{(a_{s-1}, \dots, a_0) \mid p^i \mathrm{ord}_\pi(a_i) \geq -n\}$$

Matsuda's variant:

$$\mathrm{fil}'_n W_s(A[1/\pi]) = \mathrm{fil}_{n-1} W_s(A[1/\pi]) + V^{s-s'} \mathrm{fil}_n W_{s'}(A[1/\pi])$$

where $s' = \min\{\mathrm{ord}_p(n), s\}$.

§2. Motivic ramification filtrations

In the rest of this talk, fix a perfect base field k, and set

$$\begin{aligned}\mathbf{Sch} &:= \{\text{separated schemes over } k\}\\ \mathbf{Sm} &:= \{X \in \mathbf{Sch} \mid X \text{ smooth of finite type over } k\}\\ \widetilde{\mathbf{Sm}} &:= \{X \in \mathbf{Sch} \mid X \text{ essentially smooth over } k\}\end{aligned}$$

We extend $F \in \mathrm{PSh}(\mathbf{Sm})$ to $\widetilde{\mathbf{Sm}}$ by

$$F(X) := \varinjlim_{i\in I} F(X_i) \text{ for } X = \varprojlim_{i\in I} X_i$$

where I filtered, $X_i \in \mathbf{Sm}$, and all transition maps are étale.

e.g. $\mathrm{Spec}(\mathcal{O}^h_{X,x}) \in \widetilde{\mathbf{Sm}}$ for $X \in \mathbf{Sm}$ and $x \in X$.

We also use the following categories:

$$\underline{\mathbf{M}}\mathbf{Sm} := \{(X,R) \mid X \in \mathbf{Sch},\ R \in \mathrm{Div}^+(X),\ X\backslash R \in \mathbf{Sm}\}$$
$$\mathbf{MSm} := \{(X,R) \in \underline{\mathbf{M}}\mathbf{Sm} \mid X \text{ proper over } k\}$$

For fixed $U \in \mathbf{Sm}$,

$$\mathbf{MSm}(U) := \{(X,R) \in \mathbf{MSm} \mid X\backslash R = U\}$$
$$\underline{\mathbf{M}}\mathbf{Sm}(U) := \{(X,R) \in \underline{\mathbf{M}}\mathbf{Sm} \mid X\backslash R = U\}$$

Construction of functors (Kahn-Miyazaki-S-Yamazaki):

$$\underline{\omega}^{\mathbf{CI}} : \mathrm{PSh}^{tr}(\mathbf{Sm}) \xrightarrow{\omega^{\mathbf{CI}}} \mathrm{PSh}^{tr}(\mathbf{MSm}) \xrightarrow{\tau_!} \mathrm{PSh}^{tr}(\underline{\mathbf{M}}\mathbf{Sm}).$$

$\mathrm{PSh}^{tr}(-)$ category of presheaves of abelian groups with transfers.

For $X, Y \in \mathbf{Sm}$, an *elementary correspondence from X to Y* is an irreducible closed subset $Z \subset X \times Y$ finite and surjective over a component of X (e.g., a graph of a morphism).

$\mathbf{Cor}$: The category defined by

- Objects = the same as $\mathbf{Sm}$,
- Morphisms =*finite correspondences* = finite integral sums of elementary correspondences.

"Taking graph" induces a functor $\mathbf{Sm} \to \mathbf{Cor}$; $f \mapsto \Gamma_f$.

$\mathrm{PSh}^{tr}(\mathbf{Sm})$ = category of additive presheaves $\mathbf{Cor}^{\mathrm{op}} \to \mathbf{Ab}$

Representable object for $U \in \mathbf{Sm}$:

$$\mathbb{Z}_{\mathrm{tr}}(U) = \mathbf{Cor}(-, U) \in \mathrm{PSh}^{tr}(\mathbf{Sm})$$

For $F \in \mathrm{PSh}^{tr}(\mathbf{Sm})$ and $U \in \mathbf{Sm}$, Yoneda's lemma gives

$$F(U) = \mathrm{Hom}_{\mathrm{PSh}^{tr}(\mathbf{Sm})}(\mathbb{Z}_{\mathrm{tr}}(U), F).$$

For $(X, R) \in \mathbf{MSm}(U)$, we define

$$\omega^{\mathbf{CI}} F(X, R) := \mathrm{Hom}_{\mathrm{PSh}^{tr}(\mathbf{Sm})}(h_0(X, R), F),$$

$h_0(X, R)$ is a quotient of $\mathbb{Z}_{\mathrm{tr}}(U) = \mathbf{Cor}(-, U)$ defined as follows:
For irreducible $T \in \mathbf{Sm}$ with $\eta \in T$ generic point,

$$\mathbb{Z}_{\mathrm{tr}}(U)(T) = \{\alpha \in Z_0(U_\eta) \mid \text{closure in } U \times_k T \text{ of } |\alpha| \text{ is finite over } T\}$$

$$h_0(X, R)(T) = \mathrm{Image}\big(\mathbb{Z}_{\mathrm{tr}}(U)(T) \hookrightarrow Z_0(U_\eta) \twoheadrightarrow \mathrm{CH}_0(X_\eta | R_\eta)\big)$$

$$(Y_\eta = Y \times_k \eta \text{ for } Y \in \mathbf{Sch}.)$$

$\mathrm{CH}_0(X_\eta|R_\eta)$ is *Chow group with modulus* defined as

$$Z_0(U_\eta)/\langle \mathrm{div}_C(f)|\ C \subset U_\eta \text{ curves}\rangle,$$

with $f \in k(\eta)(C)^\times$ such that $f \equiv 1 \mod R_{|\overline{C}^N}$.

$$(\overline{C} \subset X_\eta \text{ the closure of } C,\ \overline{C}^N \to \overline{C} \text{ the normalization})$$

Warning: The description of $h_0(X,R)(T)$ is valid only locally on T.

We have gotten the first functor in

$$\underline{\omega}^{\mathbf{CI}} : \mathrm{PSh}^{tr}(\mathbf{Sm}) \xrightarrow{\omega^{\mathbf{CI}}} \mathrm{PSh}^{tr}(\mathbf{MSm}) \xrightarrow{\tau_!} \mathrm{PSh}^{tr}(\underline{\mathbf{M}}\mathbf{Sm}).$$

$\tau_!$ is the left Kan extension along $\mathbf{MSm} \hookrightarrow \underline{\mathbf{M}}\mathbf{Sm}$:

For $G \in \mathrm{PSh}^{tr}(\mathbf{MSm})$ and $(Y, S) \in \underline{\mathbf{M}}\mathbf{Sm}$, put

$$\tau_! G(Y,S) = \varinjlim_{(X,R) \in \mathbf{Comp}(Y,S)} G(X,R),$$

where an object of $\mathbf{Comp}(Y,S)$ is a pair $((X,R), j)$ with

- $(X, R) \in \mathbf{MSm}$,
- $j : Y \hookrightarrow X$ open immersion such that $X \backslash R = Y$.

$$\underline{\omega}^{\mathbf{CI}} : \mathrm{PSh}^{tr}(\mathbf{Sm}) \xrightarrow{\omega^{\mathbf{CI}}} \mathrm{PSh}^{tr}(\mathbf{MSm}) \xrightarrow{\tau_!} \mathrm{PSh}^{tr}(\underline{\mathbf{M}}\mathbf{Sm}).$$

Write $\widetilde{F} = \underline{\omega}^{\mathbf{CI}} F \in \mathrm{PSh}^{tr}(\underline{\mathbf{M}}\mathbf{Sm})$ for $F \in \mathrm{PSh}^{tr}(\mathbf{Sm})$.

$$\widetilde{F}(X,R) \subset F(U), \;\; U \in \mathbf{Sm}, \; (X,R) \in \underline{\mathbf{M}}\mathbf{Sm}(U)$$

give a filtration on $F(U)$ equipped with modulus transfers:

$$[Z]^*(\widetilde{F}(X,R)) \subset \widetilde{F}(Y,S) \;\; \text{for } V \in \mathbf{Sm}, \; (Y,S) \in \underline{\mathbf{M}}\mathbf{Sm}(V),$$

where $Z \subset V \times U$ is an elementary correspondence satisfying

- the closure $\overline{Z} \subset Y \times X$ of Z is proper over Y,
- $p_Y^* S \geq p_X^* R \;\; (p_X : \overline{Z}^N \to X, \; p_Y : \overline{Z}^N \to Y)$,

Definition 2.1

$F \in \mathrm{PSh}^{tr}(\mathbf{Sm})$ *is a reciprocity sheaf if*

- *The restriction* $F_{|\mathbf{Sm}} \in \mathrm{PSh}(\mathbf{Sm})$ *is a Nisnevich sheaf,*
- *(reciprocity) For any* $U \in \mathbf{Sm}$*,*

$$F(U) = \bigcup_{(X,R)\in \mathbf{MSm}(U)} \widetilde{F}(X,R).$$

$\mathbf{RSC}_{\mathrm{Nis}} \subset \mathrm{PSh}^{tr}(\mathbf{Sm})$ *the full subcategory of reciprocity sheaves.*

Theorem 2.1 (S.)

$\mathbf{RSC}_{\mathrm{Nis}}$ *is a Grothendieck abelian category.*

Point: The Nisnevich sheafication preserves reciprocity.

Examples of reciprocity sheaves

1. $\mathbf{A}^1$-invariant object in $F \in \mathrm{Shv}^{tr}_{\mathrm{Nis}}(\mathbf{Sm})$ (e.g. $\mathbf{G}_m$, K^M_d).

   $F \in \mathrm{Shv}^{tr}_{\mathrm{Nis}}$ $\mathbf{A}^1$-inv iff $\widetilde{F}(X,R) = F(U)$ for $(X,R) \in \mathbf{MSm}(U)$.
2. The sheaf Ω^i of (absolute or relative) Kähler differentials.
3. The cohomology sheaves $\mathcal{H}^i(\Omega^\bullet)$ of the de Rham complex.
4. The de Rham-Witt sheaves $W_n\Omega^i$ of Bloch-Deligne-Illusie.
5. A smooth commutative algebraic k-group G (e.g. $\mathbf{G}_a$).
6. The group Conn^1 (resp. $\mathrm{Conn}^1_{\mathrm{int}}$) of isomorphism classes of (resp. integrable) rank 1 connections.

1. The group of isomorphism classes of $\overline{\mathbb{Q}}_\ell$-sheaves of rank 1.
2. $H^1_G := H^1_{\mathrm{fppf}}(-, G)$ for a finite flat k-group scheme G.
3. $H^{i,n}_{\text{ét}} := R^i\varepsilon_*\mathbb{Q}/\mathbb{Z}(n)_{\text{ét}}$ $(i, n \in \mathbb{Z}_{\geq 0})$, where

   $\varepsilon : X_{\text{ét}} \to X_{\mathrm{Nis}}$,

   $\mathbb{Q}/\mathbb{Z}(n)_{\text{ét}} = \mathbb{Z}(n)^{\text{ét}} \otimes \mathbb{Q}/\mathbb{Z} = \mu_\infty^{\otimes n} \oplus W_\infty\Omega^n_{\log}[-n]$

   is the étale motivic complex with $\mathbb{Q}/\mathbb{Z}$-coefficient.

   $H^{1,0}_{\text{ét}} = H^1_{\text{ét}}(-, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(\pi_1(-)^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z})$,

   $H^{2,1}_{\text{ét}} = \mathrm{Br}(-)$ cohomological Brauer group.

   $H^{n+1,n}_{\text{ét}}$ *unramified cohomology group*.

Remark 2.2

Out of the above examples, many more examples can be manufactured by taking (co)kernels (since $\mathbf{RSC}_{\mathrm{Nis}}$ is abelian), and tensor products and internal homs.

Fact(Rülling-Sugiyama-Yamazaki): $\mathbf{RSC}_{\mathrm{Nis}}$ carries a monoidal structure $\otimes^{\mathbf{RSC}}$.

Ex: $\mathbf{G}_m \otimes^{\mathbf{RSC}} \mathbf{G}_a \simeq \Omega^1$ and $\mathbf{G}_a \otimes^{\mathbf{RSC}} \mathbf{G}_a \simeq \mathcal{P}^1$ if $\mathrm{ch}(k) \neq 2$.

Fact(Merici-S): For $F \in \mathbf{RSC}_{\mathrm{Nis}}$ and $G \in \mathrm{PSh}^{tr}(\mathbf{Sm})$ which is a quotient of a finite sum of representable sheaves, we have

$$\underline{\mathrm{Hom}}_{\mathrm{PSh}^{tr}(\mathbf{Sm})}(G, F) \in \mathbf{RSC}_{\mathrm{Nis}}.$$

Ex: $\underline{\mathrm{Hom}}_{\mathrm{PSh}^{tr}(\mathbf{Sm})}(\Omega^n, \Omega^m) \simeq \Omega^{m-n} \oplus \Omega^{m-n-1}$ if $\mathrm{ch}(k) = 0$.

Summary of §2: Construction of

$$\mathrm{PSh}^{tr}(\mathbf{Sm}) \to \mathrm{PSh}^{tr}(\underline{\mathbf{M}}\mathbf{Sm})\ ;\ F \to \widetilde{F}$$

giving the *motivic filtration* for every $U \in \mathbf{Sm}$:

$$(*) \qquad \{\widetilde{F}(X,R) \subset F(U)\}_{(X,R)\in \underline{\mathbf{M}}\mathbf{Sm}(U)}$$

satisfying a functoriality called *modulus transfer*.

F is *reciprocity sheaf* $\overset{\text{def}}{\Leftrightarrow}$ F is a sheaf on $\mathbf{Sm}_{\mathrm{Nis}}$ and $(*)$ is exhaustive.

Remark: $\mathrm{CH}_0(X|R)$ is very hard to compute, so it seems hopeless to compute $\widetilde{F}(X,R) \subset F(U)$ just by definition.

It's desirable to invent effective tools to compute it.

§3. Comparison theorem of motivic and Abbes-Saito filtrations

Fix $(X, R) \in \underline{\mathbf{M}}\mathbf{Sm}$ with $X \in \mathbf{Sm}$ and $D = |R|$ SNCD on X.

We say (X, R) admits a smooth comactification if $\exists X \hookrightarrow \overline{X}$ such that $\overline{X} \in \mathbf{Sm}$ proper over k and $\overline{X} \backslash (X \backslash R)$ is SNCD on $\overline{X}$.

This is the case if X is already proper.

Theorem 3.1 (Rülling-S (Zariski-Nagata purity))

Assume (X, R) admits a smooth comactification. For $F \in \mathbf{RSC}_{\mathrm{Nis}}$,

$$\widetilde{F}(X, R) = \mathrm{Ker}\left(F(U) \to \bigoplus_{\eta \in R^{(0)}} \frac{F(\mathrm{Spec}\, \mathcal{O}^h_{X,\eta} \backslash \eta)}{\widetilde{F}(\mathrm{Spec}\, \mathcal{O}^h_{X,\eta}, R_\eta)}\right),$$

where $U = X \backslash R$ and $R_\eta = R \times_X \mathrm{Spec}\, \mathcal{O}^h_{X,\eta}$.

Recall Abbes-Saito's construction for (X,R) with $U = X \backslash R$:

$$p_1, p_2 : P_X^{(R)} \to X \times X \rightrightarrows X,\ p_i^{-1}(U) = U \times U\ (i = 1, 2).$$

Definition 3.1

Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$. *Define* $F^{AS}(X,R) \subset F(U)$ *as*

$\{\varphi \in F(U) |\ p_1^*(\varphi) - p_2^*(\varphi) \in F(U \times U)$ *extends to* $F(P_X^{(R)}).\}$

Theorem 3.2 (Rülling-S)

(i) $\widetilde{F}(X,R) \subset F^{AS}(X,R)$.

(ii) $\widetilde{F}(X,R) = F^{AS}(X,R)$ *if* (X,R) *admits smooth comactification.*

Corollary: $F^{AS}(X,R)$ admits modulus transfers assuming RS.

Characteristic forms of reciprocity sheaves

Fix $(X, R) \in \underline{\mathbf{M}}\mathbf{Sm}$ with $X \in \mathbf{Sm}$ and $D = |R|$ SNCD on X.

$\mathbf{Sm}_D$: category of separated smooth schemes of finite type over D.

For $F \in \mathbf{RSC}_{\mathrm{Nis}}$, we can construct the *characteristic form*

$$\mathrm{char}_F^{(R)} : \widetilde{F}(X, R) \to \Gamma\big(D, \Omega^1_X(R)_{|D} \otimes_{\mathcal{O}_D} \underline{\mathrm{Hom}}_{\mathrm{PSh}(\mathbf{Sm}_D)}(\mathcal{O}, F_D)\big)$$

$$F_D = \mathrm{Ker}\left(\bigoplus_{1\le i\le r} (h_i)_* F_{|\mathbf{Sm}_{D_i}} \xrightarrow{\partial} \bigoplus_{1\le i<j\le r} (h_{ij})_* F_{|\mathbf{Sm}_{D_i\cap D_j}}\right)$$

$$h_i : D_i \hookrightarrow D, \quad h_{ij} : D_i \cap D_j \hookrightarrow D$$

For $F = H^1_G$ (G finite abelian group), $\mathrm{char}_F^{(R)}$ agrees with T. Saito's.

The construction done in the same manner as T. Saito's :

- $V_R := P_X^{(R)} \times_X D \simeq \mathbb{V}(\Omega^1_X(R)) \times_X D$.

  $q_1, q_2, \mu : V_R \times V_R \to V_R$: projections and addition.

- $\mathrm{Image}(\chi^{(R)}) \subset F(V_R)_{ad} := \{\alpha \in F(V_R) \mid q_1^*\alpha + q_2^*\alpha = \mu^*\alpha\}$,

$$\chi^{(R)} : \widetilde{F}(X,R) \hookrightarrow F^{AS}(X,R) \to F(P_X^{(R)}) \to F(V_R)$$

$$\alpha \longrightarrow (p_1^*\alpha - p_2^*\alpha)_{|V_R}$$

- $F(V_R)_{ad} \hookrightarrow \Gamma\big(D, \Omega^1_X(R)_{|D} \otimes_{\mathcal{O}_D} \underline{\mathrm{Hom}}_{\mathrm{PSh}(\mathbf{Sm}_D)}(\mathcal{O}, F_D)\big)$.

For $F \in \mathbf{RSC}_{\mathrm{Nis}}$, we have *the characteristic form at* (X, R):

$$\mathrm{char}_F^{(R)} : \widetilde{F}(X,R) \to \Gamma\big(D, \Omega^1_X(R)_{|D} \otimes_{\mathcal{O}_D} \underline{\mathrm{Hom}}_{\mathrm{PSh}(\mathbf{Sm}_D)}(\mathcal{O}, F_D)\big)$$

Theorem 3.3 (Rülling-S)

Assume (X, R) *admits smooth comactification.*

For D_i *irreducible component of* R *with multiplicity*≥ 2 *and dense open* $V \subset D_i$,

$$\frac{\widetilde{F}(X,R)}{\widetilde{F}(X,R-D_i)} \overset{\mathrm{char}_F^{(R)}}{\hookrightarrow} \Gamma(V, \Omega^1_X(R)_{|V} \otimes_{\mathcal{O}_V} \underline{\mathrm{Hom}}_{\mathrm{PSh}(\mathbf{Sm}_V)}(\mathcal{O}, F_V))).$$

§5. Reciprocity pairings (key in the proof)

Fix $(X, R) \in \mathbf{MSm}$ with $U = X \backslash R \hookrightarrow X$.

For K function field over k, write $X_K = X \times_k \operatorname{Spec} K$.

For coherent ideal $I \subset \mathcal{O}_X$, define $K_d^M(\mathcal{O}_{X_K}, I) \subset K_d^M(\mathcal{O}_{X_K})$ as

$$\left\langle \{\alpha_1, \ldots, \alpha_d\} \middle| \; \alpha_1 \in 1 + I\mathcal{O}_{X_K},\; \alpha_i \in \mathcal{O}_{X_K}^\times \; (i = 2, \ldots, d) \right\rangle$$

Fact: If $I\mathcal{O}_U = \mathcal{O}_U$, there exists a surjective map

$$\theta : Z_0(U_K) = \bigoplus_{z \in U_K} \mathbb{Z} \xrightarrow{\theta_z} H^d_{\mathrm{Nis}}(X_K, K_d^M(\mathcal{O}_{X_K}, I)),$$

induced by the maps for closed points $z \in U_K$

$$\theta_z : \mathbb{Z} \xrightarrow{\simeq} H^d_z(X_K, K_d^M(\mathcal{O}_{X_K}, I)) \to H^d_{\mathrm{Nis}}(X_K, K_d^M(\mathcal{O}_{X_K}, I)).$$

Theorem 4.1 (Rülling-S)

Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$*. There exists a bilinear pairing*

$$(-,-)_{X/K} : F(U) \times \varprojlim_{\substack{I \subset \mathcal{O}_X \\ I\mathcal{O}_U = \mathcal{O}_U}} H^d_{\mathrm{Nis}}(X_K, K^M_d(\mathcal{O}_{X_K}, I)) \to F(K),$$

satisfying the following conditions:

(i) *Let* $I_R \subset \mathcal{O}_X$ *be the ideal sheaf of* R*. The pairing induces*

$$(-,-)_{X/K} : \widetilde{F}(X,R) \times H^d_{\mathrm{Nis}}(X_K, K^M_d(\mathcal{O}_{X_K}, I_R)) \to F(K).$$

(ii) *For* $a \in F(U)$ *and a closed point* $z \in U_K$*,*

$$(a, \theta_z(1))_{X/K} = (g_z)_* i_z^*(a \otimes_k K) \in F(K),$$

$$(i_z : z \hookrightarrow U_K,\ g_z : z \to \operatorname{Spec} K)$$

Examples of reciprocity pairings

$$(-,-)_{X/K} : F(U) \times \varprojlim_{I\mathcal{O}_U=\mathcal{O}_U} H^d_{\mathrm{Nis}}(X_K, K^M_d(\mathcal{O}_{X_K}, I)) \to F(K).$$

<u>Case $F = H^1_{\text{ét}}(-,\mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{ab}}(-),\mathbb{Q}/\mathbb{Z})$ and k finite:</u>

$F(k) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\overline{k}/k),\mathbb{Q}/\mathbb{Z}) \xrightarrow{\simeq} \mathbb{Q}/\mathbb{Z}\ ;\ \chi \to \chi(Frob_k)$.

Then $(-,-)_{X/K}$ with $K = k$ induces a map

$$\varprojlim_{I\mathcal{O}_U=\mathcal{O}_U} H^d(X, K^M_d(\mathcal{O}_X, I)) \to \pi_1^{\mathrm{ab}}(U) = \mathrm{Hom}(H^1_{\text{ét}}(U,\mathbb{Q}/\mathbb{Z}),\mathbb{Q}/\mathbb{Z}).$$

This coincides with the reciprocity map of Kato-S.

<u>Case $F = \mathrm{Conn}^1$ and $\dim(X) = 1$:</u>

$(-,-)_{X/K}$ coincides with Bloch-Esnault's pairing in:

Gauss-Manin determinants for rank 1 connections on curves.

Construction of $(-,-)_{X/K}$ uses pushforward maps for $F \in \mathbf{RSC}_{\mathrm{Nis}}$:

For proper $f : Y \to X$ in $\mathbf{Sm}$, Binda-Rülling-S construct

$$f_* : Rf_*F(d)_Y[d] \to F_X \ (d = \dim(Y) - \dim(X)),$$

$$F(d) = F \otimes^{\mathbf{RSC}} \overbrace{\mathbf{G}_m \otimes^{\mathbf{RSC}} \cdots \otimes^{\mathbf{RSC}} \mathbf{G}_m}^{d \text{ times}}, \ F_X = F_{|X_{\mathrm{Nis}}}.$$

The construction recovers Grothendieck's pushforward maps for $F = \Omega^i$ and Gros' pushforward maps for $F = W\Omega^i$ without using general machinery of Grothendieck and Ekedahl dualities.

Recall $(X,R) \in \mathbf{MSm}$, $U = X \backslash R \in \mathbf{Sm}$, K/k function field.

$(-,-)_{X/K}$ induces the following local versions (Higher local symbols)

$$\Big\{(-,-)_{X/K,x} : F(U) \otimes K_d^M(Q(\mathcal{O}^h_{X_K,x})) \to F(K)\Big\}_{x \in \mathrm{mc}(X_K)},$$

$$\mathrm{mc}(X_K) := \{x = (x_1,\dots,x_d) |\ x_i \in X_K^{(i)},\ \overline{\{x_i\}} \supset \overline{\{x_{i+1}\}}\}.$$

Theorem 4.2 (Rülling-S)

Assume $X \in \mathbf{Sm}$ and $|R|$ is a SNCD on X. For $a \in F(U)$, the following conditions are equivalent:

(i) $a \in \widetilde{F}(X,R)$.

(ii) $(a,\beta)_{X/K,x} = 0$ *for all* K/k *and* $x \in \mathrm{mc}(X_K)$ *and* $\beta \in U^R K_d^M(Q(\mathcal{O}^h_{X_K,x}))$.

Point of proof of $(ii) \Rightarrow (i)$ is a factorization of θ:

$$Z_0(U_K) = \mathbb{Z}_{\mathrm{tr}}(U)(K) \to h_0(X,R)(K) = \mathrm{CH}_0(X_K|R_K) \longrightarrow H^d_{\mathrm{Nis}}(X_K, K^M_d(\mathcal{O}_{X_K}, I_R)),$$

where $h_0(X,R)$ is a quotient of $\mathbb{Z}_{\mathrm{tr}}(U)$ used to define $\widetilde{F}(X,R)$.

Idea of proof of $\widetilde{F}(X,R) = F^{AS}(X,R)$:

For $a \in F(U)$, apply *the projection formula of higher local symbols* to

$$p_1, p_2 : P_X^{(R)} \to X \times X \rightrightarrows X,$$

and use the above theorem to compare ramification of a along $X - U$ and ramification of $p_1^*(a) - p_2^*(a) \in F(U \times U)$ along $P_X^{(R)} - U \times U$.

§4. Computations of characteristic forms

Fix $(X, nD) \in \underline{\mathbf{M}}\mathbf{Sm}$ with $X, D \in \mathbf{Sm}$ and $n \in \mathbb{Z}_{>0}$. Assume

$$X = \operatorname{Spec} A \text{ and } D = \{\pi = 0\} \text{ with } \pi \in A.$$

Witt vectors

Assume $\mathrm{ch}(k) = p > 0$. Consider $W_s \in \mathbf{RSC}_{\mathrm{Nis}}$. We have

$$\widetilde{W_s}(X, nD) = \sum_{r \geq 0} F^r \left(\mathrm{fil}'_n W_s(A[1/\pi]) \right),$$

$$(F^r : W_s \to W_s \text{ the iterative Frobenius})$$

This filtration has been studied by Kato-Russell.

$$\mathrm{char}^{(R)}_{W_s} : \widetilde{W_s}(X, nD) \to \Omega^1_X(R)_{|D} \otimes_{\mathcal{O}_D} \underline{\mathrm{Hom}}_{\mathrm{PSh}(\mathbf{Sm}_D)}(\mathcal{O}, W_s)$$

For $\alpha = (a_{s-1}, \ldots, a_1, a_0) \in \mathrm{fil}'_n W_s(A[1/\pi])$

$$\mathrm{char}^{(R)}_{W_s}(\alpha) = \begin{cases} -F^{s-1} d(\alpha) \otimes V^{s-1} & \text{if } (p, n) \neq (2, 2), \\ -F^{s-1} d(\alpha) \otimes V^{s-1} + \dfrac{d\pi}{\pi^2} \otimes V^{s-1} \varphi_{a_0} & \text{if } (p, n) = (2, 2), \end{cases}$$

$F^{s-1}d(\alpha) = \sum_{i=0}^{s-1} a_i^{p^i-1} da_i$,

$V^{s-1} : \mathcal{O} \to W_s$ the iterative Verschiebung,

$\varphi_{a_0} : \mathcal{O} \overset{\alpha \to \overline{\pi^2 a_0} \cdot \alpha}{\longrightarrow} \mathcal{O} \overset{\alpha \to \alpha^2}{\longrightarrow} \mathcal{O}$,

where $\overline{\pi^2 a_0} \in \Gamma(D, \mathcal{O})$ is the residue class of $\pi^2 a_0$.

The computation is due to Y. Yatagawa.

H^1_G with $G = \mathbb{Z}/p^s\mathbb{Z}$ $(\mathrm{ch}(k) = p > 0)$

$$H^1_G(U) := \{G\text{-torsors } \varphi : V \to U\} \simeq H^1(U_{\text{ét}}, \mathbb{Z}/p^s\mathbb{Z})$$

By Artin-Schreier-Witt theory, we have an isomorphism in $\mathbf{RSC}_{\mathrm{Nis}}$:

$$\delta : H^1_G \simeq W_s/1 - F.$$

We have a commutative diagram

$$\begin{array}{ccc} \widetilde{W_s}(X, nD) & \xrightarrow{\mathrm{char}^{(R)}_{W_s}} & \Omega^1_X(R)_{|D} \otimes_{\mathcal{O}} \underline{\mathrm{Hom}}_{\mathrm{PSh}(\mathbf{Sm}_D)}(\mathcal{O}, W_s) \\ \Big\downarrow \widetilde{\delta} & & \Big\downarrow id_{\Omega^1_X} \otimes \underline{\mathrm{Hom}}(\mathcal{O},\delta) \\ \widetilde{H^1_G}(X, nD) & \xrightarrow{\mathrm{char}^{(R)}_{H^1_G}} & \Omega^1_X(nD)_{|D} \otimes_{\mathcal{O}} \underline{\mathrm{Hom}}_{\mathrm{PSh}(\mathbf{Sm}_D)}(\mathcal{O}, H^1_G) \end{array}$$

Point: $\widetilde{\delta}$ is surjective (non-trivial!).

This enables us to compute $\mathrm{char}^{(R)}_{H^1_G}$ from $\mathrm{char}^{(R)}_{W_s}$.

Kähler differentials

Let $\mathrm{ch}(k) = p \geq 0$. Consider $\Omega^i \in \mathbf{RSC}_{\mathrm{Nis}}$. For $R = nD$, we have

$$\mathrm{char}^{(R)}_{\Omega^i} : \widetilde{\Omega}^i(X, nD) \to \Omega^1_X(nD)_{|D} \otimes_{\mathcal{O}} \underline{\mathrm{Hom}}_{\mathrm{PSh}(\mathbf{Sm}_D)}(\mathcal{O}, \Omega^i)$$

We can show

$$\widetilde{\Omega}^i(X, nD) = \begin{cases} \Omega^i_X(\log D)((n-1)D) & \text{if } p = 0 \text{ or } 0 < p \not| n, \\ \Omega^i_X(nD) & \text{if } 0 < p | n. \end{cases}$$

Furthermore, $\mathrm{char}^{(R)}_{\Omega^i}$ factors via maps

$$\frac{\widetilde{\Omega}^i(X, nD)}{\widetilde{\Omega}^i(X, (n-1)D)} \underset{\hookrightarrow}{\xrightarrow{\overline{\mathrm{char}}^{(R)}_{\Omega^i}}} \Omega^1_X(nD)_{|D} \otimes_{\mathcal{O}_D} (\Omega^i_D \oplus \Omega^{i-1}_D)$$

$$\underset{\hookrightarrow}{\xrightarrow{id_{\Omega^1} \otimes \xi}} \Omega^1_X(nD)_{|D} \otimes_{\mathcal{O}_D} \underline{\mathrm{Hom}}_{\mathrm{PSh}(\mathbf{Sm}_D)}(\mathcal{O}, \Omega^i)$$

where

$$\xi : \Omega^i_D \oplus \Omega^{i-1}_D \hookrightarrow \underline{\mathrm{Hom}}_{\mathrm{PSh}(\mathbf{Sm}_D)}(\mathcal{O}, \Omega^i)$$

is $\mathcal{O}_D$-hom given for $\beta \in \Omega^i_D$, $\alpha \in \Omega^{i-1}_D$, $\lambda \in \mathcal{O}_D$

$$\xi(\beta, \alpha)(\lambda) = (-1)^{i-1}\lambda(\beta - d\alpha) - \alpha \wedge d\lambda$$

Define (independent of choice of π with $D = \{\pi = 0\} \subset X$)

$$L_X(nD) := \mathcal{O}_D \cdot \frac{d\pi}{\pi^n} \subset \Omega^1_X(nD)_{|D}$$

$$\mathrm{Res}_{D,n} : \Omega^j_X(\log D)((n-1)D) \to L_X(nD) \otimes_{\mathcal{O}} \Omega^{j-1}_D$$

$$\beta \to \frac{d\pi}{\pi^n} \otimes \mathrm{Res}_D(\pi^{n-1}\beta)$$

where $\mathrm{Res}_D : \Omega^j_X(\log D) \to \Omega^{j-1}_D$ is the residue map.

$$\overline{\mathrm{char}}^{(R)}_{\Omega^i} : \frac{\widetilde{\Omega}^i(X, nD)}{\widetilde{\Omega}^i(X,(n-1)D)} \hookrightarrow \Omega^1_X(nD)_{|D} \otimes_{\mathcal{O}_D} (\Omega^i_D \oplus \Omega^{i-1}_D)$$

is computed as follows:

Case $p = 0$ *or* $0 < p \not| n(n-1)$*:*

$$\frac{\Omega^i_X(\log D)((n-1)D)}{\Omega^i_X(\log D)((n-2)D)} \xrightarrow[\simeq]{(\mathrm{Res}_{D,n}\circ d,\, \mathrm{Res}_{D,n})} L_X(nD) \otimes (\Omega^i_D \oplus \Omega^{i-1}_D)$$

Case $0 < p | n-1$*:*

$$\frac{\Omega^i_X(\log D)((n-1)D)}{\Omega^i_X((n-1)D)} \xrightarrow[\simeq]{\mathrm{Res}_{D,n}} L_X(nD) \otimes \Omega^{i-1}_D$$

Case $0 < p|n$:

$$\overline{\mathrm{char}}^{(R)}_{\Omega^i} : \frac{\Omega^i_X(nD)}{\Omega^i_X(\log D)((n-2)D)} \hookrightarrow \Omega^1_X(nD)_{|D} \otimes_{\mathcal{O}_D} (\Omega^i_D \oplus \Omega^{i-1}_D)$$

For simplicity, assume $X = D \times \operatorname{Spec} k[\pi]$, $D = \operatorname{Spec} k[z_1, \ldots, z_d]$.

For $\omega = \dfrac{f dz_1 \wedge \cdots \wedge dz_i}{\pi^n}$ with $f \in k[z_1, \ldots, z_d]$,

$$\overline{\mathrm{char}}^{(R)}_{\Omega^i}(\omega) = \sum_{\nu=1}^{d} \frac{dz_\nu}{\pi^n} \otimes ((-1)^{i-1}\delta_\nu, 0) - \sum_{\nu=1}^{i} \frac{dz_\nu}{\pi^n} \otimes (d\gamma_\nu, \gamma_\nu),$$

$$\delta_\nu = \frac{\partial f}{\partial z_\nu} dz_1 \wedge \cdots \wedge dz_i, \quad \gamma_\nu = (-1)^{\nu+1} f dz_1 \wedge \cdots \wedge \overset{\vee}{dz_\nu} \wedge \cdots \wedge dz_i.$$

Cdarc descent for reciprocity sheaves

$\mathrm{LK} : \mathrm{PSh}(\mathbf{Sm}) \to \mathrm{PSh}(\mathbf{Sch})$ left Kan extension along $\mathbf{Sm} \to \mathbf{Sch}$.

Theorem 5.1 (Kelly-S)

Assume $\mathrm{ch}(k) = 0$ *or embedded resolution of singularities over* k.

For $F \in \mathbf{RSC}_{\mathrm{Nis}}$, $\mathrm{LK}(F)_{\mathrm{cdh}} \in \mathrm{Shv}_{\mathrm{cdh}}(\mathbf{Sch})$ *satisfies cdarc descent.*

Recall (Bhatt-Mathew, Elmanto-Hoyois-Iwasa-Kelly)

A qcqs morphism $Y \to X$ in $\mathbf{Sch}$ is a cdarc cover if

$$\mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}\, R, Y) \to \mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}\, R, X)$$

is surjective for all henselian valuation ring of rank≤ 1.

The above theorem follows from the following, which follows from the Abbes-Saito formula for motivic filtrations.

Theorem 5.2 (Brylinski-Kato formula for motivic conductors)

K: function field over k, X: regular separated of finite type over K, K-rational point $0 \in X$, $D \subset X$ regular divisor containing 0.
$C, C' \subset X$ regular curves intersecting transversally with D at 0.
Put $M = \mathrm{Frac}(\mathcal{O}^h_{X,D})$, $L = \mathrm{Frac}(\mathcal{O}^h_{C,0})$, $L' = \mathrm{Frac}(\mathcal{O}^h_{C',0})$.
Take $F \in \mathbf{RSC}_{\mathrm{Nis}}$ and $a \in F(X - D)$ and consider $a_{|C} \in F(C - 0)$, $a_{|C'} \in F(C' - 0)$ the restrictions of a. Then

$$(C, C')_0 \geq c^F_M(a) \;\Rightarrow\; c^F_L(a_{|C}) = c^F_{L'}(a_{|C'}).$$

Motivic conductors

$\Phi = \{\mathrm{Frac}(\mathcal{O}^h_{X,x}) \mid X \in \mathbf{Sm},\ x \in X^{(1)}\}$

For each $F \in \mathbf{RSC}_{\mathrm{Nis}}$, we get a collection of maps

$$\{c_L^F : F(L) \to \mathbb{N}\}_{L\in\Phi}$$

$$c_L^F(a) = \min\{n \in \mathbb{N} |\ a \in \widetilde{F}(\mathrm{Spec}\,\mathcal{O}_L, \mathfrak{m}_L^n)\}\ (a \in F(L)).$$

It recovers known conductors such as

1. Kato-Matsuda's Swan conductors for $F = H^1_G$ with $G = \mathbb{Z}/p^s\mathbb{Z}$,
2. irregularities for $F = \mathrm{Conn}^1$,
3. Rosenlicht-Serre conductor for F represented by a commutative algebraic group.

Thank you for attention!