# MOTIVES WITH MODULUS

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Abstract. We construct and study a triangulated category of motives with modulus $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ over a field $k$ that extends Voevodsky's category $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ in such a way as to encompass non-homotopy invariant phenomena. In a similar way as $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ is constructed out of smooth $k$-varieties, $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ is constructed out of *proper modulus pairs*, that is, pairs of a proper $k$-variety $X$ and an effective divisor $D$ on $X$ such that $X \setminus |D|$ is smooth. To a modulus pair $(X, D)$ we associate its motive $M(X, D) \in \mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$. In some cases the Hom group in $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ between the motives of two modulus pairs can be described in terms of Bloch's higher Chow groups.

Contents



In this paper, we construct triangulated categories of "motives with modulus" over a field $k$, in parallel with Voevodsky's construction of triangulated categories of motives in [49]. Our motivation comes from

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the reciprocity sheaves studied in [18]; the link between the present theory and [18] will be established in [19].

Voevodsky's construction is based on $\mathbf{A}^1$-invariance. It captures many important invariants such as Bloch's higher Chow groups, but not their natural generalisations like additive Chow groups [5, 40] or higher Chow groups with modulus [7]. Our basic motivation is to build a theory that captures such non $\mathbf{A}^1$-invariant phenomena, as an extension of [18].

Let $\mathbf{Sm}$ be the category of smooth separated $k$-schemes of finite type. Voevodsky's construction starts from an additive category $\mathbf{Cor}$, whose objects are those of $\mathbf{Sm}$ and morphisms are finite correspondences. The category of effective geometric motives $\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ is then defined to be the pseudo-abelian envelope of the localisation of the homotopy category $K^b(\mathbf{Cor})$ of bounded complexes by two types of "relations":

**(HI):** $[X \times \mathbf{A}^1] \to [X]$, $X \in \mathbf{Cor}$;
**(MV):** $[U \cap V] \to [U] \oplus [V] \to [X]$, $X, U, V \in \mathbf{Cor}$

where in the latter $U \sqcup V \to X$ ranges over all open cover of $X$. This makes $\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ a tensor triangulated category. We denote by $M^V$ the canonical functor $\mathbf{Sm} \to \mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}$. The following fundamental result computes some Hom groups in concrete terms:

**Theorem 1** ([6, 6.7.3] and [50, Cor. 2])**.** *Assume that $k$ is perfect. For $X, Y \in \mathbf{Sm}$, with $X$ proper of dimension $d$ and $j \in \mathbb{Z}$, there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}}(M^V(Y)[j], M^V(X)) \simeq CH^d(Y \times X, j)$$

*where the right hand side is Bloch's higher Chow groups. In particular, this group is $0$ for $j < 0$ and isomorphic to $CH^d(Y \times X)$ for $j = 0$.*

In the present work, we construct a tensor triangulated category $\mathbf{MDM}_{\mathrm{gm}}^{\mathrm{eff}}$ in a parallel way. The category $\mathbf{Cor}$ is replaced by a category $\mathbf{MCor}$ whose objects are *modulus pairs*, which only played an auxiliary rôle in [18]. A modulus pair $M = (\overline{M}, M^\infty)$ consists of a proper $k$-variety $\overline{M}$ and an effective Cartier divisor $M^\infty$ such that $\overline{M} \setminus |M^\infty| \in \mathbf{Sm}$. A morphism from $(\overline{M}, M^\infty)$ to $(\overline{N}, N^\infty)$ is a finite correspondences from $\overline{M} - M^\infty$ to $\overline{N} - N^\infty$ which satisfies a certain inequality on Cartier divisors (Definition 1.3.1)

The category $\mathbf{MCor}$ enjoys a symmetric monoidal structure (Definition 1.4.1). The right object replacing $\mathbf{A}^1$ in this context turns out to be

$$\overline{\square} = (\mathbf{P}^1, \infty), \tag{0.1}$$

the compactification of $\mathbf{A}^1 \simeq \mathbf{P}^1 - \{\infty\}$ with reduced divisor at infinity. This provides an analogue of (HI):

**(CI):** $[M \otimes \overline{\square}] \to [M]$, $M \in \mathbf{MCor}$.

Unfortunately, the properness of $\overline{M}$ in the definition of $\mathbf{MCor}$ prevents a direct analogue of (MV). To overcome this difficulty, we are led to relax this condition and introduce a larger category $\underline{\mathbf{M}}\mathbf{Cor}$ of not necessarily proper modulus pairs (Definition 1.3.1). This yields a Mayer-Vietoris condition (MV1) in $K^b(\underline{\mathbf{M}}\mathbf{Cor})$ (§6.2). We may then define a tensor triangulated category $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ in the same fashion as Voevodsky (Definition 6.2.1), with a "motive" functor $\underline{M} : \underline{\mathbf{M}}\mathbf{Cor} \to \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$, and get a full subcategory $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}} \subset \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ which contains the essential image of $K^b(\mathbf{MCor})$ (Theorem 6.9.4). We have the following partial analogue of Theorem 1:

**Theorem 2** (See Cor. 7.3.3)**.** *Suppose $k$ is perfect. Let $X$ be a smooth proper $k$-variety of dimension $d$. We have a canonical isomorphism:*

$$\mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}}(\underline{M}(\mathcal{Y})[j], \underline{M}(X, \emptyset)) \simeq CH^d((\overline{\mathcal{Y}} - \mathcal{Y}^\infty) \times X, j)$$

*for any modulus pair $\mathcal{Y} = (\overline{\mathcal{Y}}, \mathcal{Y}^\infty)$ and $j \in \mathbb{Z}$. The same formula holds in $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ if $\mathcal{Y} \in \mathbf{MCor}$.*

Remarkably, non-trivial Mayer-Vietoris relations do arise in $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$, see Theorem 7.5.2 (2); this has been considerably amplified in [17].

An important step in the proof of Theorem 2 is the following analogue of [49, Cor. 3.2.7], which requires no hypothesis on $k$:

**Theorem 3** (see Theorem 7.1.1)**.** *For any $\mathcal{X} = (\overline{\mathcal{X}}, \mathcal{X}^\infty), \mathcal{Y} \in \underline{\mathbf{M}}\mathbf{Cor}$ and $i \in \mathbb{Z}$, we have an isomorphism*

$$\mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}}(\underline{M}(\mathcal{X}), \underline{M}(\mathcal{Y})[i]) \simeq \mathbb{H}^i_{\mathrm{Nis}}(\overline{\mathcal{X}}, RC^{\overline{\square}}_*(\mathcal{Y})_{\mathcal{X}}). \tag{0.2}$$

*The same formula holds in $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ if $\mathcal{X}, \mathcal{Y} \in \mathbf{MCor}$.*

Here, $RC^{\overline{\square}}_*(\mathcal{Y})$ is the *derived Suslin complex* of the modulus pair $\mathcal{Y}$ (see Subsection 6.8), and $RC^{\overline{\square}}_*(\mathcal{Y})_{\mathcal{X}}$ denotes its restriction to $\overline{\mathcal{X}}_{\mathrm{Nis}}$ (see Notation 3.6.1). Briefly, it is defined like the Suslin complex of a smooth variety $X$, with 3 differences: a) we use $\overline{\square}$ instead of $\mathbf{A}^1$; b) we use a cubical version instead of Suslin-Voevodsky's simplicial version (see Remarks 5.2.6 and 6.8.3 for an important comment on this point); c) we use derived internal Homs instead of classical internal Homs. In [8], (0.2) is refined to an isomorphism

$$\mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}}(\underline{M}(\mathcal{X}), \underline{M}(\mathcal{Y})[i]) \simeq \mathbb{H}^i_{\mathrm{Nis}}(\overline{\mathcal{X}}, C^{\overline{\square}}_*(\mathcal{Y})_{\mathcal{X}}), \tag{0.3}$$

assuming $\mathcal{Y} \in \mathbf{MCor}$ and $\overline{\mathcal{X}} \in \mathbf{Sm}$ and $|\mathcal{X}^\infty|$ is a normal crossing divisor on $\overline{\mathcal{X}}$. Here $C_*^{\overline{\square}}(\mathcal{Y})$ is a "naïve" Suslin complex, defined using classical internal Homs.

Recall that a key technical tool of Voevodsky for proving Theorem 1 is to embed $\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ into a larger triangulated category $\mathbf{DM}^{\mathrm{eff}}$ of *motivic complexes*. The situation is similar here: $\underline{\mathbf{M}}\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ and $\mathbf{MDM}_{\mathrm{gm}}^{\mathrm{eff}}$ are respectively embedded into categories $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ and $\mathbf{MDM}^{\mathrm{eff}}$. This is how the derived Suslin complex $RC_*^{\overline{\square}}(\mathcal{Y})$ arises.

On the other hand, there is a canonical "forgetful" functor $\underline{\omega} : \underline{\mathbf{M}}\mathbf{Cor} \to \mathbf{Cor}$ sending $(\overline{X}, X^\infty)$ to $\overline{X} - |X^\infty|$, whence a comparison between our theory and Voevodsky's. This is summarised in the following diagram, assuming $k$ perfect:

$$\begin{array}{ccccc}
\mathbf{MCor} & \xrightarrow{M} & \mathbf{MDM}_{\mathrm{gm}}^{\mathrm{eff}} & \xrightarrow{\iota} & \mathbf{MDM}^{\mathrm{eff}} \\
\downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau_{\mathrm{eff,gm}}} & & \downarrow{\scriptstyle \tau_{\mathrm{eff}}} \\
\underline{\mathbf{M}}\mathbf{Cor} & \xrightarrow{M} & \underline{\mathbf{M}}\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}} & \xrightarrow{\iota} & \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}} \\
\downarrow{\scriptstyle \underline{\omega}} & & \downarrow{\scriptstyle \underline{\omega}_{\mathrm{eff,gm}}} & & \downarrow{\scriptstyle \underline{\omega}_{\mathrm{eff}}} \\
\mathbf{Cor} & \xrightarrow{M^V} & \mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}} & \xrightarrow{\iota} & \mathbf{DM}^{\mathrm{eff}},
\end{array} \tag{0.4}$$

in which the functors denoted $\iota$ are fully faithful. The yoga of Thomason-Neeman of compactly generated categories [35] shows that the functors $\tau_{\mathrm{eff}}$ and $\underline{\omega}_{\mathrm{eff}}$ have right adjoints $\tau^{\mathrm{eff}}, \underline{\omega}^{\mathrm{eff}}$. Hence their composition $\omega_{\mathrm{eff}} = \underline{\omega}_{\mathrm{eff}}\tau_{\mathrm{eff}}$ also has a right adjoint $\omega^{\mathrm{eff}}$, and we have:

**Theorem 4** (See Th. 7.3.1 and Cor. 7.3.2)**.** *a) Let $X$ be a smooth proper $k$-variety. Then $\underline{\omega}^{\mathrm{eff}} M^V(X) = \underline{M}(X, \emptyset)$ and $\omega^{\mathrm{eff}} M^V(X) = M(X, \emptyset)$.*
*b) If $p$ is the exponential characteristic of $k$, then $\underline{\omega}^{\mathrm{eff}}(\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}[1/p]) \subset \underline{\mathbf{M}}\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}[1/p]$ and $\omega^{\mathrm{eff}}(\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}[1/p]) \subset \mathbf{MDM}_{\mathrm{gm}}^{\mathrm{eff}}[1/p]$.*

Part a) of Theorem 4 is the second important step in the proof of Theorem 2. Note that $\underline{\omega}^{\mathrm{eff}}$ and $\omega^{\mathrm{eff}}$ are fully faithful (Propositions 6.10.2 and 6.10.3).

We should point out that, while the category $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ is relevant and plays an essential rôle in the proofs of [43], most results of [19], notably the existence of a $\overline{\square}$-homotopy $t$-structure, apply to $\mathbf{MDM}^{\mathrm{eff}}$ and do not seem to extend to $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$.

This is a revised version of the initial version of this paper. There are two main differences. The first one: we develop of good theory of the categories $\mathbf{MDM}^{\mathrm{eff}}$ and $\mathbf{MDM}_{\mathrm{gm}}^{\mathrm{eff}}$ (rather than just $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ and

$\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$), which was initially out of reach. The second: we decided to promote Section 7 of the initial version to [19].

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Independently of our work, Moritz Kerz conjectured in [27] the existence of a category of motives with modulus and gave a list of expected properties. His conjectures are closely related to our construction. We thank him for communicating his ideas, which were very helpful to us.

The first author thanks Joseph Ayoub for explaining him two easy but crucial results on unbounded derived categories (Lemmas A.13.8 and A.13.12), as well as for Remark 6.8.3 a). We also thank Ofer Gabber and Michel Raynaud for their help with Lemma 1.9.1, Joseph Oesterlé for his help in the proof of Lemma 5.3.2, Kay Rülling for pointing out an error and correcting Definition 1.8.1, and Hiroyasu Miyazaki for several helpful observations and corrections.

Finally, the influence of Voevodsky's ideas is all-pervasive, as will be evident when reading this paper.

**Notation and conventions.** In the whole paper we fix a base field $k$. Let **Sm** be the category of separated smooth schemes of finite type over $k$, and let **Sch** be the category of separated schemes of finite type over $k$. We write **Cor** for Voevodsky's category of finite correspondences [49].

## 1. Modulus pairs and admissible correspondences

### 1.1. Admissible correspondences.

**Definition 1.1.1.**

(1) A *modulus pair* $M$ consists of $\overline{M} \in \mathbf{Sch}$ and an effective Cartier divisor $M^\infty \subset \overline{M}$ such that the dense open subset $M^o := \overline{M} - |M^\infty|$ is smooth over $k$. (The case $|M^\infty| = \emptyset$ is allowed.) We say that $M$ is *proper* if $\overline{M}$ is.

We write $M = (\overline{M}, M^\infty)$, since $M$ is completely determined by the pair, although we regard $M^o$ as the main part of $M$.

(2) Let $M_1, M_2$ be modulus pairs. Let $Z \in \mathbf{Cor}(M_1^o, M_2^o)$ be an elementary correspondence. We write $\bar{Z}^N$ for the normalization of the closure of $Z$ in $\overline{M}_1 \times \overline{M}_2$ and $p_i : \bar{Z}^N \to \overline{M}_i$ for the canonical morphisms for $i = 1, 2$. We say $Z$ is *admissible* for $(M_1, M_2)$ if

$p_1^* M_1^\infty \geq p_2^* M_2^\infty$. An element of $\mathbf{Cor}(M_1^o, M_2^o)$ is called admissible if all of its irreducible components are admissible. We write $\mathbf{Cor}_{\mathrm{adm}}(M_1, M_2)$ for the subgroup of $\mathbf{Cor}(M_1^o, M_2^o)$ consisting of all admissible correspondences.

*Remarks* 1.1.2.

(1) In [18, Def. 2.1.1], we used a different notion of modulus pair, where $\overline{M}$ is supposed proper, $M^o$ smooth quasi-affine and $M^\infty$ is any closed subscheme of $\overline{M}$. Definition 1.1.1 (1) is the right one for the present work. Definition 1.1.1 (2) is the same as [18, Def. 2.6.1], mutatis mutandis.

(2) In the first version of this paper, we imposed the condition that $\overline{M}$ is locally integral; it is now removed. The main reason for this change is that this condition is not stable under products or extension of the base field. The next remark shows that this removal is reasonable.

(3) Let $M$ be a modulus pair. Then $M^o$ is dense in $\overline{M}$, since the Cartier divisor $M^\infty$ is everywhere of codimension 1. Moreover, $\overline{M}$ is reduced. (In particular, $\overline{M}$ has no embedded component.) Indeed, take $x \in \overline{M}$ and let $f \in \mathcal{O}_{\overline{M},x}$ be a local equation for $M^\infty$. Then $f$ is not a zero-divisor (since $M^\infty$ is Cartier), and hence $\mathcal{O}_{\overline{M},x} \to \mathcal{O}_{\overline{M},x}[1/f]$ is injective, but $\mathcal{O}_{\overline{M},x}[1/f]$ is reduced as $M^o$ is smooth. In particular, $\overline{M}$ is integral if $M^o$ is.

The following lemma will play a key rôle:

**Lemma 1.1.3.** *Let $X \in \mathbf{Sm}$ and let $\overline{X}$ be a variety containing $X$ as a dense open subset. Assume that $\overline{X} - X$ is the support of a Cartier divisor. Then we have*

$$\bigcup_{\{M \text{ modulus pair } | \overline{M} = \overline{X} \text{ and } M^o = X\}} \mathbf{Cor}_{\mathrm{adm}}(M, N) = \mathbf{Cor}(X, N^o)$$

*for any modulus pair $N$.*

*Proof.* This is proven in [18, Lemma 2.6.2]. In loc. cit. $X$ and $N^o$ are assumed to be quasi-affine, and $\overline{X}$ and $\overline{N}$ proper and normal (see Remark 1.1.2). But these assumptions are not used in the proof. (Nor is the assumption on Cartier divisors, but the latter is essential for the proof of Proposition 1.2.3 below.) ☐

1.2. **Composition.** To discuss composability of admissible correspondences, we need the following lemma of Krishna and Park [25, Lemma 2.2].

**Lemma 1.2.1.** *Let $f : X \to Y$ be a surjective morphism of normal integral schemes, and let $D, D'$ be two Cartier divisors on $Y$. If $f^*D' \leq f^*D$, then $D' \leq D$.*

**Definition 1.2.2.** Let $M_1, M_2, M_3$ be three modulus pairs, and let $\alpha \in \mathbf{Cor}_{\mathrm{adm}}(M_1, M_2), \beta \in \mathbf{Cor}_{\mathrm{adm}}(M_2, M_3)$. We say that $\alpha$ and $\beta$ are *composable* if their composition $\beta\alpha$ is admissible.

**Proposition 1.2.3.** *With the above notation, assume $\alpha$ and $\beta$ integral and let $\bar{\alpha}$ and $\bar{\beta}$ be their closures in $\overline{M}_1 \times \overline{M}_2$ and $\overline{M}_2 \times \overline{M}_3$ respectively. Then $\alpha$ and $\beta$ are composable provided the projection $\bar{\alpha} \times_{\overline{M}_2} \bar{\beta} \to \overline{M}_1 \times \overline{M}_3$ is proper. This happens in the following cases:*

(i) $\bar{\alpha} \to \overline{M}_1$ *is proper.*
(ii) $\bar{\beta} \to \overline{M}_3$ *is proper.*

*In particular, both (i) and (ii) hold if $\overline{M}_2$ is proper.*

*Proof.* Note that $\alpha \times_{M_2^o} \beta$ is a closed subscheme of $(M_1^o \times M_2^o) \times_{M_2^o} (M_2^o \times M_3^o) = M_1^o \times M_2^o \times M_3^o$; we have $|\beta\alpha| = |p_{13*}(\alpha \times_{M_2^o} \beta)|$ where $p_{13} : M_1^o \times M_2^o \times M_3^o \to M_1^o \times M_3^o$ is the projection. Let $\gamma$ be a component of $\alpha \times_{M_2^o} \beta$. We have a commutative diagram

$$\begin{tikzcd}
\gamma \arrow[r, hook] \arrow[d, hook] & \alpha \times_{M_2^o} \beta \arrow[r, hook] \arrow[d, hook] & M_1^o \times M_2^o \times M_3^o \arrow[r, "p_{13}"] \arrow[d, hook] & M_1^o \times M_3^o \arrow[d, hook] & \delta \arrow[l, hook'] \arrow[d, hook] \\
\bar{\gamma} \arrow[r, hook] & \bar{\alpha} \times_{\overline{M}_2} \bar{\beta} \arrow[r, hook] & \overline{M}_1 \times \overline{M}_2 \times \overline{M}_3 \arrow[r, "\bar{p}_{13}"] & \overline{M}_1 \times \overline{M}_3 & \bar{\delta} \arrow[l, hook']
\end{tikzcd}$$

where $\overline{p}_{ij} : \overline{M}_1 \times \overline{M}_2 \times \overline{M}_3 \to \overline{M}_i \times \overline{M}_j$ denotes the projection, $\delta = p_{13}(\gamma)$, and $^-$ denotes closure. The hypothesis implies that $\bar{\gamma} \to \bar{\delta}$ is proper surjective. The same holds for $\pi^N_{\gamma\delta}$ appearing in the second of the two other commutative diagrams:

$$\begin{tikzcd}
\bar{\alpha} & \bar{\alpha}^N \arrow[l] \arrow[r, "\varphi_\alpha"] & \overline{M}_1 \times \overline{M}_2 \\
\bar{\gamma} \arrow[u, "\pi_{\gamma\alpha}"] \arrow[d, "\pi_{\gamma\beta}"] & \bar{\gamma}^N \arrow[l] \arrow[r, "\varphi_\gamma"] & \overline{M}_1 \times \overline{M}_2 \times \overline{M}_3 \arrow[u, "\bar{p}_{12}"] \arrow[d, "\bar{p}_{23}"] \\
\bar{\beta} & \bar{\beta}^N \arrow[l] \arrow[r, "\varphi_\beta"] & \overline{M}_2 \times \overline{M}_3
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{\gamma}^N \arrow[r, "\varphi_\gamma"] \arrow[d, "\pi^N_{\gamma\delta}"] & \overline{M}_1 \times \overline{M}_2 \times \overline{M}_3 \arrow[d, "\bar{p}_{13}"] \\
\bar{\delta}^N \arrow[r, "\varphi_\delta"] & \overline{M}_1 \times \overline{M}_3
\end{tikzcd}$$

where $^N$ means normalisation. (Note that $\pi_{\gamma\alpha}$ and $\pi_{\gamma\beta}$ need not extend to the normalisations, as they need not be dominant.) We have the admissibility conditions for $\alpha$ and $\beta$:

$$\varphi_\alpha^*(\overline{M}_1 \times M_2^\infty) \leq \varphi_\alpha^*(M_1^\infty \times \overline{M}_2)$$
$$\varphi_\beta^*(\overline{M}_2 \times M_3^\infty) \leq \varphi_\beta^*(M_2^\infty \times \overline{M}_3).$$

Applying [34, Lemma 2.4][1], we get an inequality

$$\varphi_\gamma^*(\overline{M}_1 \times \overline{M}_2 \times M_3^\infty) \leq \varphi_\gamma^*(\overline{M}_1 \times M_2^\infty \times \overline{M}_3) \leq \varphi_\gamma^*(M_1^\infty \times \overline{M}_2 \times \overline{M}_3),$$

which implies by the right half of the above diagram

$$(\pi_{\gamma\delta}^N)^* \varphi_\delta^*(\overline{M}_1 \times M_3^\infty) \leq (\pi_{\gamma\delta}^N)^* \varphi_\delta^*(M_1^\infty \times \overline{M}_3)$$

hence $\varphi_\delta^*(\overline{M}_1 \times M_3^\infty) \leq \varphi_\delta^*(M_1^\infty \times \overline{M}_3)$ by Lemma 1.2.1.

Finally, cases (i) and (ii) are trivially checked. □

*Example* 1.2.4. Let $\overline{M}_1 = \overline{M}_3 = \mathbf{P}^1$, $\overline{M}_2 = \mathbf{A}^1$, $\overline{M}_i^{\mathrm{o}} = \mathbf{A}^1$, $M_1^\infty = \infty$, $M_2^\infty = \emptyset$, $M_3^\infty = 2\infty$, $\alpha = \beta =$ graph of the identity. Then $\alpha$ and $\beta$ are admissible but $\beta \circ \alpha$ is not admissible.

1.3. **The categories $\mathbf{MSm}, \underline{\mathbf{M}}\mathbf{Sm}, \mathbf{MCor}$ and $\underline{\mathbf{M}}\mathbf{Cor}$.** We now define 4 categories:

**Definition 1.3.1.** An admissible correspondence $\alpha \in \mathbf{Cor}_{\mathrm{adm}}(M, N)$ is *left proper* if the closures of all components of $\alpha$ are proper over $\overline{M}$; this is automatic if $\overline{N}$ is proper. Modulus pairs and left proper admissible correspondences define an additive category $\underline{\mathbf{M}}\mathbf{Cor}$ by Proposition 1.2.3. We write $\mathbf{MCor}$ for the full subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$ whose objects are proper modulus pairs (see Definition 1.1.1(1)).

Let us check that these are indeed categories. By Case (i) of Proposition 1.2.3, we are left to show that if $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}(M_1, M_2)$ and $\beta \in \underline{\mathbf{M}}\mathbf{Cor}(M_2, M_3)$, their composition $\beta\alpha$ in $\mathbf{Cor}(M_1^{\mathrm{o}}, M_3^{\mathrm{o}})$ is left proper. We may assume $\alpha$ and $\beta$ irreducible. Since $\beta$ is left proper, $\bar{\alpha} \times_{\overline{M}_2} \bar{\beta} \to \bar{\alpha}$ is proper, which implies that $\bar{\alpha} \times_{\overline{M}_2} \bar{\beta}$ is proper over $\overline{M}_1$ by the left properness of $\alpha$, hence so is its image in $\overline{M}_1 \times \overline{M}_3$.

In the context of modulus pairs, the category $\mathbf{Sm}$ and the graph functor $\mathbf{Sm} \to \mathbf{Cor}$ are replaced by the following:

**Definition 1.3.2.** We write $\underline{\mathbf{M}}\mathbf{Sm}$ for the category with same objects as $\underline{\mathbf{M}}\mathbf{Cor}$ a morphism of $\underline{\mathbf{M}}\mathbf{Sm}(M_1, M_2)$ being a (scheme-theoretic) $k$-morphism $f : M_1^{\mathrm{o}} \to M_2^{\mathrm{o}}$ whose graph belongs to $\underline{\mathbf{M}}\mathbf{Cor}(M_1, M_2)$. We write $\mathbf{MSm}$ for the full subcategory of $\underline{\mathbf{M}}\mathbf{Sm}$ whose objects are proper modulus pairs.

[1] To apply this lemma, factor $\pi_{\gamma\alpha}$ and $\pi_{\gamma\beta}$ into dominant morphisms followed by closed immersions.

## 1.4. **Tensor structure.**

**Definition 1.4.1.** For $M, N \in \underline{\mathbf{M}}\mathbf{Cor}$, we define $L = M \otimes N$ by

$$\overline{L} = \overline{M} \times \overline{N}, \quad L^\infty = M^\infty \times \overline{N} + \overline{M} \times N^\infty.$$

This gives the categories $\mathbf{MSm}$, $\underline{\mathbf{M}}\mathbf{Sm}$, $\mathbf{MCor}$ and $\underline{\mathbf{M}}\mathbf{Cor}$ of Definitions 1.3.1 and 1.3.2 symmetric monoidal structures with unit $(\operatorname{Spec} k, \emptyset)$. To see this, we have to check:

**Lemma 1.4.2.** *Let $f \in \underline{\mathbf{M}}\mathbf{Cor}(M_1, N_1)$ and $g \in \underline{\mathbf{M}}\mathbf{Cor}(M_2, N_2)$. Consider the tensor product correspondence $f \otimes g \in \mathbf{Cor}(M_1^o \times M_2^o, N_1^o \times N_2^o)$. Then $f \otimes g \in \underline{\mathbf{M}}\mathbf{Cor}(M_1 \otimes M_2, N_1 \otimes N_2)$.*

*Proof.* We may assume that $f$ and $g$ are given by integral cycles $Z \subset M_1^o \times N_1^o$ and $T \subset M_2^o \times N_2^o$. Then $f \otimes g$ is given by the product cycle $Z \times T$. Let $\overline{Z}^N \to \overline{Z}$ be the normalizations of the closures $\overline{Z}$ of $Z$, and similarly for $\overline{T}^N \to \overline{T}$. By hypothesis, we have

$$(p_1^Z)^* M_1^\infty \geq (p_2^Z)^* N_1^\infty, \quad (p_1^T)^* M_2^\infty \geq (p_2^T)^* N_2^\infty,$$

where $p_1^Z$ is the composition $\overline{Z}^N \to \overline{Z} \subset \overline{M}_1 \times \overline{N}_1 \to \overline{M}_1$, and likewise for $p_2^Z, p_1^T, p_2^T$. Hence:

$$(p_1^Z \times p_1^T)^*(M_1^\infty \times \overline{M}_2 + \overline{M}_1 \times M_2^\infty) = (p_1^Z)^* M_1^\infty \times \overline{T}^N + \overline{Z}^N \times (p_1^T)^* M_2^\infty$$
$$\geq (p_2^Z)^* N_1^\infty \times \overline{T}^N + \overline{Z}^N \times (p_2^T)^* N_2^\infty = (p_2^Z \times p_2^T)^*(N_1^\infty \times \overline{N}_2 + \overline{N}_1 \times N_2^\infty)$$

hence $Z \times T \in \mathbf{Cor}_{\mathrm{adm}}(M_1 \otimes M_2, N_1 \otimes N_2)$, because the projection $(\overline{Z} \times \overline{T})^N \to \overline{Z} \times \overline{T}$ factors through $\overline{Z}^N \times \overline{T}^N$. Finally, $\overline{Z} \times \overline{T}$ is obviously proper over $\overline{M}_1 \times \overline{M}_2$. $\square$

*Remark* 1.4.3. At the end of this proof, $\overline{Z} \times \overline{T}$ is finite over $\overline{M}_1 \times \overline{M}_2$ if $\overline{Z}$ and $\overline{T}$ are finite over $\overline{M}_1$ and $\overline{M}_2$ respectively. This remark will be used in Definition 1.10.1.

*Warning* 1.4.4. Definition 1.4.1 does not have the universal property of products, even when restricted to $\underline{\mathbf{M}}\mathbf{Sm}$. Indeed, take $M = N \in \underline{\mathbf{M}}\mathbf{Sm}$. If $M \otimes M$ represented the self-product of $M$ in $\underline{\mathbf{M}}\mathbf{Sm}$, the diagonal $M \to M \otimes M$ would have to be admissible; this happens if and only if $M^\infty = \emptyset$.

Nevertheless, there do exist finite products in $\underline{\mathbf{M}}\mathbf{Sm}$ and $\mathbf{MSm}$, that yield another symmetric monoidal structure on $\underline{\mathbf{M}}\mathbf{Cor}$ and $\mathbf{MCor}$. This involves blow-ups and is more difficult to handle. In the present paper, we will stick to our naive monoidal structure $\otimes$.

### 1.5. The functors $(-)^{(n)}$.

**Definition 1.5.1.** Let $n \geq 1$ and $M = (\overline{M}, M^\infty) \in \underline{\mathbf{M}}\mathbf{Cor}$. We write
$$M^{(n)} = (\overline{M}, nM^\infty).$$
This defines an endofunctor of $\underline{\mathbf{M}}\mathbf{Cor}$. Those come with natural transformations
$$M^{(n)} \to M^{(m)} \quad \text{if } m \leq n. \tag{1.1}$$

**Lemma 1.5.2.** *The functor $(-)^{(n)}$ is monoidal and fully faithful.* □

### 1.6. Changes of categories.

We now have a basic diagram of additive categories and functors
$$\begin{tikzcd} \mathbf{MCor} \arrow[rr, "\tau"] \arrow[dr, "\omega"'] & & \underline{\mathbf{M}}\mathbf{Cor} \arrow[dl, shift left, "\underline{\omega}"] \\ & \mathbf{Cor} \arrow[ur, shift left, "\lambda"] & \end{tikzcd} \tag{1.2}$$
with
$$\tau(M) = M; \quad \omega(M) = M^o; \quad \underline{\omega}(M) = M^o; \quad \lambda(X) = (X, \emptyset).$$

All these functors are monoidal and faithful, and $\tau$ is fully faithful; they "restrict" to analogous functors between $\mathbf{MSm}$, $\underline{\mathbf{M}}\mathbf{Sm}$ and $\mathbf{Sm}$. Note that $\underline{\omega} \circ (-)^{(n)} = \underline{\omega}$ for any $n$. Moreover:

**Lemma 1.6.1.** *We have $\underline{\omega}\tau = \omega$, and $\lambda$ is left adjoint to $\underline{\omega}$. Moreover, the restriction of $\lambda$ to $\mathbf{Cor}^{\mathrm{prop}}$ (finite correspondences on smooth proper varieties) is "right adjoint" to $\underline{\omega}$. (That is, $\mathbf{Cor}(\underline{\omega}(M), X) = \underline{\mathbf{M}}\mathbf{Cor}(M, \lambda(X))$ for $M \in \underline{\mathbf{M}}\mathbf{Cor}$ and $X \in \mathbf{Cor}^{\mathrm{prop}}$).*

*Proof.* The first identity is obvious. For the adjointness, let $X \in \mathbf{Cor}$, $M \in \underline{\mathbf{M}}\mathbf{Cor}$ and $\alpha \in \mathbf{Cor}(X, M^o)$ be an integral finite correspondence. Then $\alpha$ is closed in $X \times \overline{M}$, since it is finite on $X$ and $\overline{M}$ is separated; it is evidently proper over $X$ and $q^*M^\infty = 0$ where $q$ is the composition $\alpha^N \to \alpha \to M^o \to \overline{M}$. Therefore $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}(\lambda(X), M)$.

For the second statement, assume $X$ proper and let $\beta \in \mathbf{Cor}(M^o, X)$ be an integral finite correspondence. Then $\beta$ is trivially admissible, and its closure in $\overline{M} \times X$ is proper over $\overline{M}$, so $\beta \in \underline{\mathbf{M}}\mathbf{Cor}(M, \lambda(X))$. □

The following theorem is an important refinement of Lemma 1.6.1: it will be proven in the next subsections.

**Theorem 1.6.2.** *The functors $\omega$ and $\tau$ have monoidal pro-left adjoints $\omega^!$ and $\tau^!$ (see §A.2).*

General definitions and results on pro-objects and pro-adjoints are gathered in the Appendix. We shall freely use results from there.

1.7. **Proof of Theorem 1.6.2: case of $\omega$.** We need a definition:

**Definition 1.7.1.** Let $\Sigma$ (resp. $\underline{\Sigma}$) be the class of all morphisms $\sigma : M_1 \to M_2$ in $\mathbf{MCor}$ (resp. $\underline{\mathbf{M}}\mathbf{Cor}$) restricting to the identity on $M_1^o = M_2^o$.

In view of Proposition A.6.2, the existence of the pro-left adjoint of $\omega$ is a consequence of the following more precise result:

**Proposition 1.7.2.** *a) The class $\Sigma$ enjoys a calculus of right fractions.*
*b) The functor $\omega$ induces an equivalence of categories*

$$\Sigma^{-1}\,\mathbf{MCor} \xrightarrow{\sim} \mathbf{Cor}\,.$$

*Proof.* a) We check the axioms of Definition A.5.1:

(1) Identities, stability under composition: obvious.
(2) Given a diagram in $\mathbf{MCor}$

$$\begin{tikzcd} & M_2' \arrow[d] \\ M_1 \arrow[r, "\alpha"] & M_2 \end{tikzcd}$$

with $M_2^o = M_2'^o$, Lemma 1.1.3 provides a $M_1'' \in \mathbf{MCor}$ such that $M_1''^o = M_1^o$ and $\alpha \in \mathbf{MCor}(M_1'', M_2')$. We may choose $M_1''$ such that $\overline{M_1''} = \overline{M_1}$. Then $M_1' = (\overline{M_1}, M_1'^\infty)$ with any $M_1'^\infty$ such that $M_1'^\infty \geq M_1^\infty$, $M_1'^\infty \geq M_1''^\infty$ allows us to complete the square.
(3) Given a diagram

$$M_1 \underset{g}{\overset{f}{\rightrightarrows}} M_2 \xrightarrow{s} M_2'$$

with $M_1, M_2, M_2'$ as in (2) and such that $sf = sg$, the underlying correspondences to $f$ and $g$ are equal since the one underlying $s$ is $1_{M_2^o}$. Hence $f = g$.

b) now follows from a), Lemma 1.1.3 and Corollary A.5.5, noting that $\omega$ is essentially surjective. $\square$

Let $\omega^! : \mathbf{Cor} \to \text{pro}-\mathbf{MCor}$ be the pro-left adjoint of $\omega$. It remains to show that $\omega^!$ is monoidal (for the monoidal structure on pro$-\mathbf{MCor}$ induced by the one on $\mathbf{MCor}$, given by Definition 1.4.1). By Proposition A.6.2, we have for $X \in \mathbf{Cor}$:

$$\omega^! X = \text{“}\varprojlim_{M \in \Sigma \downarrow X}\text{”} M.$$

Let us spell out the indexing set $\mathbf{MSm}(X)$ of this pro-object, and refine it:

**Definition 1.7.3.**

(1) For $X \in \mathbf{Sm}$, we define a subcategory $\mathbf{MSm}(X)$ of $\mathbf{MSm}$ as follows. The objects of $\mathbf{MSm}(X)$ are those $M \in \mathbf{MSm}$ such that $M^o = X$. Given $M_1, M_2 \in \mathbf{MSm}(X)$, we define $\mathbf{MSm}(X)(M_1, M_2)$ to be $\{1_X\}$ if $1_X$ is admissible for $(M_1, M_2)$, and $\emptyset$ otherwise.

(2) Let $X \in \mathbf{Sm}$ and fix a compactification $\overline{X}$ such that $\overline{X} - X$ is the support of a Cartier divisor (for short, a *Cartier compactification*). Define $\mathbf{MSm}(\overline{X}, X)$ to be the full subcategory of $\mathbf{MSm}(X)$ consisting of objects $M \in \mathbf{MSm}(X)$ such that $\overline{M} = \overline{X}$.

**Lemma 1.7.4.** *a) For any $X \in \mathbf{Sm}$ and any Cartier compactification $\overline{X}$, $\mathbf{MSm}(X)$ is a cofiltered ordered set, and $\mathbf{MSm}(\overline{X}, X)$ is cofinal in $\mathbf{MSm}(X)$.*
*b) Let $X \in \mathbf{Cor}$, and let $M \in \mathbf{MSm}(X)$. Then $(M^{(n)})_{n\geq 1}$ defines a cofinal subcategory of $\mathbf{MSm}(X)$.*

*Proof.* a) "Ordered" is obvious and "cofiltered" follows from Propositions 1.7.2 and A.5.2 a); the cofinality follows again from Lemma 1.1.3.

b) Let $M = (\overline{X}, X^\infty)$. By a) it suffices to show that $(M^{(n)})_{n\geq 1}$ defines a cofinal subcategory of $\mathbf{MSm}(\overline{X}, X)$. If $(\overline{X}, Y) \in \mathbf{MSm}(\overline{X}, X)$, $Y$ and $X^\infty$ both have support $\overline{X} - X$, so there exists $n > 0$ such that $nX^\infty \geq Y$. $\square$

Let $X, Y \in \mathbf{Cor}$ and let $M \in \mathbf{MSm}(X)$, $N \in \mathbf{MSm}(Y)$. Then $M \otimes N \in \mathbf{MSm}(X \times Y)$. By Lemma 1.7.4, we have

$$\omega^! X = \text{“}\varprojlim_{n\geq 1}\text{”} M^{(n)}, \omega^! Y = \text{“}\varprojlim_{n\geq 1}\text{”} N^{(n)}, \omega^!(X \times Y) = \text{“}\varprojlim_{n\geq 1}\text{”} (M \otimes N)^{(n)}.$$

This shows the monoidality of $\omega^!$, since $(M \otimes N)^{(n)} = M^{(n)} \otimes N^{(n)}$. We also note:

**Proposition 1.7.5.** *Proposition 1.7.2 extends to $\underline{\mathbf{M}}\mathbf{Cor}$, $\underline{\Sigma}$ and $\underline{\omega}$ (see Definition 1.7.1 for $\underline{\Sigma}$).*

*Proof.* Same as for Proposition 1.7.2, except for b): we must show for $M, N \in \underline{\mathbf{M}}\mathbf{Cor}$, the injection

$$\varinjlim_{M' \in \underline{\Sigma}\downarrow M} \underline{\mathbf{M}}\mathbf{Cor}(M', N) \hookrightarrow \mathbf{Cor}(M^o, N^o)$$

is surjective (Lemma 1.1.3 is not sufficient because of the properness condition). For this, we simply note that $\underline{\Sigma} \downarrow M$ contains the initial object $(M^o, \emptyset) = \lambda\underline{\omega}(M)$, and apply Lemma 1.6.1. $\square$

1.8. **Proof of Theorem 1.6.2: case of $\tau$.** We need a definition:

**Definition 1.8.1.** Take $M = (\overline{M}, M^\infty) \in \underline{\mathbf{M}}\mathbf{Cor}$. Let $\mathbf{Comp}(M)$ be the category whose objects are pairs $(N, j)$ consisting of a modulus pair $N = (\overline{N}, N^\infty) \in \mathbf{MCor}$ equipped with a dense open immersion $j : \overline{M} \hookrightarrow \overline{N}$ such that $N^\infty = M^\infty_N + C$ for some effective Cartier divisors $M^\infty_N, C$ on $\overline{N}$ satisfying $\overline{N} \setminus |C| = j(\overline{M})$ and $j^*(M^\infty_N) = M^\infty$. Note that for $N \in \mathbf{Comp}(M)$ we have $j(M^o) = N^o$ and $N$ is equipped with $j_N \in \underline{\mathbf{M}}\mathbf{Cor}(M, N)$ which is the graph of $j|_{M^o} : M^o \cong N^o$. For $N_1, N_2 \in \mathbf{Comp}(M)$ we define

$$\mathbf{Comp}(M)(N_1, N_2) = \{\gamma \in \mathbf{MCor}(N_1, N_2) \mid \gamma \circ j_{N_1} = j_{N_2}\}.$$

**Lemma 1.8.2.** *The category* $\mathbf{Comp}(M)$ *is a cofiltered ordered set.*

*Proof.* Ordered is obvious. For cofiltering, we first show that $\mathbf{Comp}(M)$ is nonempty. For this, choose a compactification $j_0 : \overline{M} \hookrightarrow \overline{N}_0$, with $\overline{N}_0 \in \mathbf{Sch}$ proper. Let $\overline{N}_1 = \mathbf{Bl}_{(\overline{N}_0 - \overline{M})_{\mathrm{red}}}(\overline{N_0})$; then $j_0$ lifts to $j_1 : \overline{M} \hookrightarrow \overline{N}_1$, and $\overline{N}_1 - \overline{M}$ is the support of an effective Cartier divisor $C_1$. Consider now the scheme-theoretic closure $N_1^\infty$ of $M^\infty$ in $\overline{N}_1$, and define $\overline{N} = \mathbf{Bl}_{N_1^\infty}(\overline{N}_1)$, $N^\infty =$ pull-back of $N_1^\infty$, $C =$pull-back of $C_1$ and $N = (\overline{N}, N^\infty)$: then $j_1$ lifts to $j : \overline{M} \hookrightarrow \overline{N}$ which defines an object of $\mathbf{Comp}(M)$.

Let now $N_i = (\overline{N}_i, N_i^\infty)$ with $i = 1, 2$ be in $\mathbf{Comp}(M)$. Let $\overline{N}$ be the blowup of the closure $\overline{N}'$ of the image $\overline{M}'$ of $(j_1, j_2) : \overline{M} \to \overline{N}_1 \times \overline{N}_2$ along $(\overline{N}' - \overline{M}')_{\mathrm{red}}$, and $M^\infty_N \subset \overline{N}$ be the pullback of $M^\infty_{N_1} \times \overline{N_2} + \overline{N_1} \times M^\infty_{N_2}$. Then $(\overline{N}, M^\infty_N + C)$ with a suitable divisor $C$ supported on $\overline{N} - \overline{M}'$ is an element of $\mathbf{Comp}(M)$ which dominates both $N_1$ and $N_2$. $\square$

For $M \in \underline{\mathbf{M}}\mathbf{Cor}$ and $L \in \mathbf{MCor}$ we have a natural map

$$\Phi : \varinjlim_{N \in \mathbf{Comp}(M)} \mathbf{MCor}(N, L) \to \underline{\mathbf{M}}\mathbf{Cor}(M, \tau L),$$

which maps a representative $\alpha_N \in \mathbf{MCor}(N, L)$ to $\alpha_N \circ j_N$ which is independent of $N$ by definition. We also have a natural map

$$\Psi : \underline{\mathbf{M}}\mathbf{Cor}(\tau L, M) \to \varprojlim_{N \in \mathbf{Comp}(M)} \mathbf{MCor}(L, N),$$

which maps a morphism $\beta$ to $(j_N \circ \beta)_N$.

The following is an analogue to Lemma 1.1.3:

**Lemma 1.8.3.** $\Phi$ *and* $\Psi$ *are isomorphisms. In particular, the formula*

$$\tau^! M = \text{“}\varprojlim_{N \in \mathbf{Comp}(M)}\text{”} N$$

*defines a pro-left adjoint to $\tau$.*

*Proof.* We start with $\Phi$. Injectivity is obvious since both sides are subgroups of $\mathbf{Cor}(M^o, L^o)$. We prove surjectivity. Choose a dense open immersion $j_1 : \overline{M} \hookrightarrow \overline{N}_1$ with $\overline{N}_1$ proper such that $\overline{N}_1 - \overline{M}$ is the support of an effective Cartier divisor $C_1$. Let $M_1^\infty$ be the scheme-theoretic closure of $M^\infty$ in $\overline{N}_1$. (This may not be Cartier.) Let $\pi : \overline{N}_2 \to \overline{N}_1$ be the blowup with center in $M_1^\infty$ and put $M_2^\infty = \pi^{-1}(M_1^\infty)$ and $C_2 = \pi^{-1}(C_1)$. Note that $M_2^\infty$ and $C_2$ are effective Cartier divisors on $\overline{N}_2$. By the universal property of the blowup [16, Ch. II, Prop. 7.14], $j_1$ extends to an open immersion $j_2 : \overline{M} \to \overline{N}_2$ so that $j_1 = \pi j_2$. Then $\overline{N}_2 - M^o$ is the support of the Cartier divisor $N_2^\infty := M_2^\infty + C_2$ so that

$$((\overline{N}_2, N_2^\infty), j_2) \in \mathbf{Comp}(M).$$

It suffices to show the following:

*Claim* 1.8.4. For any $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}(M, L)$, there exists an integer $n > 0$ such that $\alpha \in \mathbf{MCor}((\overline{N}_2, M_2^\infty + nC_2), L)$.

Indeed we may assume $\alpha$ is an integral closed subscheme of $M^o \times L^o$. We have a commutative diagram

$$\begin{tikzcd}
\overline{\alpha}^N \arrow[r, "j_1"] \arrow[d, "\varphi_\alpha"'] & \overline{\alpha}_1^N \arrow[d, "\varphi_{\alpha_1}"'] & \overline{\alpha}_2^N \arrow[l, "\pi"'] \arrow[d, "\varphi_{\alpha_2}"'] \\
\overline{M} \times \overline{L} \arrow[r, "j_1"] & \overline{N}_1 \times \overline{L} & \overline{N}_2 \times \overline{L} \arrow[l, "\pi"']
\end{tikzcd}$$

where $\overline{\alpha}^N$ (resp. $\overline{\alpha}_1^N$, resp. $\overline{\alpha}_2^N$) is the normalization of the closure of $\alpha \subset M^o \times L^0$ in $\overline{M} \times \overline{L}$ (resp. $\overline{N}_1 \times \overline{L}$, resp. $\overline{N}_2 \times \overline{L}$), and $j_1$ and $\pi$ are iduced by $j_1 : \overline{M} \to \overline{N}_1$ and $\pi : \overline{N}_2 \to \overline{N}_1$ respectively. Now the admissibility of $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}(M, L)$ implies

$$\varphi_\alpha^*(\overline{M} \times L^\infty) \leq \varphi_\alpha^*(M^\infty \times \overline{L}).$$

Since $\overline{\alpha}_1^N - j_1(\overline{\alpha}^N)$ is supported on $\varphi_{\alpha_1}^{-1}(C_1 \times \overline{L})$, this yields an inclusion of closed subschemes

$$\varphi_{\alpha_1}^*(\overline{N}_1 \times L^\infty) \subseteq \varphi_{\alpha_1}^*((M_1^\infty + nC_1) \times \overline{L})$$

for a sufficiently large $n > 0$. Appllying $\pi^*$ to this inclusion, we get an inequality of Cartier divisors

$$\varphi_{\alpha_2}^*(\overline{N}_2 \times L^\infty) \leq \varphi_{\alpha_2}^*((M_2^\infty + nC_2) \times \overline{L})$$

which proves the claim.

Next we prove that $\Psi$ is an isomorphism. Injectivity is obvious since both sides are subgroups of $\mathbf{Cor}(L^o, M^o)$. We prove surjectivity. Take

$\gamma \in \varprojlim_{N \in \mathbf{Comp}(M)} \mathbf{MCor}(L, N)$. Then $\gamma \in \mathbf{Cor}(L^o, M^o)$ is such that any component $\delta \subset L^o \times M^o$ of $\gamma$ satisfies the following condition: Take any $(N, j) \in \mathbf{Comp}(M)$ and write $N^\infty = M_N^\infty + C$ as in Definition 1.8.1. Let $\overline{\delta}^N$ be the normalization of the closure of $\delta$ in $\overline{L} \times \overline{N}$ with the natural map $\varphi_\delta : \overline{\delta}^N \to \overline{L} \times \overline{N}$. Then we have

$$\varphi_\delta^*(\overline{L} \times (M_N^\infty + nC)) \leq \varphi_\delta^*(L^\infty \times \overline{N})$$

for any integer $n > 0$. Clearly this implies that $|\overline{\delta}|$ does not intersect with $|C|$ so that $\overline{\delta} \subset \overline{L} \times \overline{M}$. Noting $\overline{\delta}$ is proper over $\overline{L}$ since $\overline{N}$ is proper, this implies $\delta \in \underline{\mathbf{M}}\mathbf{Cor}(L, M)$ which proves the surjectivity of $\Psi$ as desired. $\square$

We now show the monoidality of $\tau^!$, arguing as in the case of $\omega^!$ (although we cannot quite use the functors $(-)^{(n)}$ here). Let $M \in \underline{\mathbf{M}}\mathbf{Cor}$. Take $N \in \mathbf{Comp}(M)$ and write $N^\infty = M_N^\infty + C$ as in Definition 1.8.1. Define $\mathbf{Comp}(N, M)$ as the full subcategory of $\mathbf{Comp}(M)$ consisting of those $P$ such that $\overline{P} = \overline{N}$ (compatibly with the open immersions $\overline{M} \hookrightarrow \overline{N}$, $\overline{M} \hookrightarrow \overline{P}$) and $P^\infty = M_N^\infty + nC$ for some $n > 0$. (Strictly speaking, $\mathbf{Comp}(N, M)$ depends on the choice of the decomposition $N^\infty = M_N^\infty + C$.) The proof of Claim 1.8.4 shows that $\mathbf{Comp}(N, M)$ is cofinal in $\mathbf{Comp}(M)$. If $M' \in \underline{\mathbf{M}}\mathbf{Cor}$ is another object and $N' \in \mathbf{Comp}(M')$ with a decomposition $N'^\infty = M'^\infty_{N'} + C'$, then $N \otimes N' \in \mathbf{Comp}(M \otimes M')$ as $(N \otimes N')^\infty = (M_N^\infty \times \overline{N'} + \overline{N} \times M'^\infty_{N'}) + (C \times \overline{N'} + \overline{N} \times C')$, and it is easy to see that the obvious functor

$$\mathbf{Comp}(N, M) \times \mathbf{Comp}(N', M') \to \mathbf{Comp}(N \otimes N', M \otimes M')$$

is cofinal.

This concludes the proof of Theorem 1.6.2.

## 1.9. **The closure of a finite correspondence.**

**Lemma 1.9.1.** *Let $X$ be a Noetherian scheme, $(\pi_i : Z_i \to X)_{1 \leq i \leq n}$ a finite set of proper surjective morphisms with $Z_i$ integral, and let $U \subseteq X$ be a normal open subset. Suppose that $\pi_i : \pi_i^{-1}(U) \to U$ is finite for every $i$. Then there exists a proper birational morphism $X' \to X$ which is an isomorphism over $U$, such that the closure of $\pi_i^{-1}(U)$ in $Z_i \times_X X'$ is finite over $X'$ for every $i$.*

*Proof.* By induction, we reduce to $n = 1$; then this follows from [41, Cor. 5.7.10] applied with $(S, X, U) \equiv (X, Z_1, U)$ and $n = 0$ (note that quasi-finite + proper $\iff$ finite, and that an admissible blow-up of an algebraic space is a scheme if the algebraic space happens to be a scheme). $\square$

**Theorem 1.9.2.** *Let $X, Y \in$ **Sch**. Let $U$ be a normal dense open subscheme of $X$, and let $\alpha$ be a finite correspondence from $U$ to $Y$. Suppose that the closure $\overline{Z}$ of $Z$ in $X \times Y$ is proper over $X$ for any component $Z$ of $\alpha$. Then there is a proper birational morphism $X' \to X$ which is an isomorphism over $U$, such that $\alpha$ extends to a finite correspondence from $X'$ to $Y$.*

*Proof.* Apply Lemma 1.9.1, noting that $Z = \overline{Z} \times_X U$ by [18, Lemma 2.6.3]. □

## 1.10. The categories $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ and $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$.

**Definition 1.10.1.** We write $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$ for the subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$ with the same objects and the following condition on morphisms: $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}(M, N)$ belongs to $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(M, N)$ if and only if, for any component $Z$ of $\alpha$, the projection $\overline{Z} \to \overline{M}$ is *finite*. We write $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ for the subcategory of $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$ with the same objects and such that a morphism $f : M \to N$ belongs to $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ if and only if $f : \overline{M} \to \overline{N}$ is a morphism. Definition 1.4.1 yields symmetric monoidal structures with unit $(\operatorname{Spec} k, \emptyset)$ on $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$ and $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ by Remark 1.4.3.

(The same argument as after Definition 1.3.1 shows that $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$ is indeed a subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$.)

*Remark* 1.10.2. If $\overline{M}$ is normal, then $\underline{\mathbf{M}}\mathbf{Sm}(M, N) \cap \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(M, N) = \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}(M, N)$ for any $N$ by Zariski's connectedness theorem.

We shall also need the following definition:

**Definition 1.10.3.** a) A morphism $f : M \to N$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ is *minimal* if $f^* N^\infty = M^\infty$.
b) A morphism $f : M \to N$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ is in $\Sigma^{\mathrm{fin}}$ if it is minimal, $f : \overline{M} \to \overline{N}$ is a proper morphism and $f^o$ is the identity.

In particular, $\Sigma^{\mathrm{fin}} \subset \underline{\Sigma}$ (see Definition 1.7.1).

**Proposition 1.10.4.** *a) The class $\Sigma^{\mathrm{fin}}$ enjoys a calculus of right fractions within $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ and $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$.*
*b) Any morphism in $\Sigma^{\mathrm{fin}}$ is invertible in $\underline{\mathbf{M}}\mathbf{Sm}$ (hence in $\underline{\mathbf{M}}\mathbf{Cor}$).*
*c) The induced functors $A : (\Sigma^{\mathrm{fin}})^{-1}\, \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{Sm}$ and $A' : (\Sigma^{\mathrm{fin}})^{-1}\, \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{Cor}$ are isomorphisms of categories.*

*Proof.* a) Same as the proof of Proposition 1.7.2 a), except for (2): consider a diagram in $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$

$$\begin{tikzcd} & M_2' \arrow[d, "f"] \\ M_1 \arrow[r, "\alpha"] & M_2 \end{tikzcd}$$

with $f \in \Sigma^{\mathrm{fin}}$ (in particular $M_2'^{\mathrm{o}} = M_2^{\mathrm{o}}$). By the properness of $f$, the finite correspondence $\alpha^{\mathrm{o}} : M_1^{\mathrm{o}} \to M_2'^{\mathrm{o}}$ satisfies the hypothesis of Theorem 1.9.2. Applying this theorem, we find a proper birational morphism $f' : \overline{M}_1' \to \overline{M}_1$ which is the identity on $\overline{M}_1^{\mathrm{o}}$ and such that $\alpha^{\mathrm{o}}$ defines a finite correspondence $\alpha' : \overline{M}_1' \to \overline{M}_2'$. If we define $M_1'^{\infty} = f'^* M_1^{\infty}$, then $f' \in \Sigma^{\mathrm{fin}}$ and $\alpha' \in \mathbf{MCor}^{\mathrm{fin}}(M_1', M_2')$.

If $\alpha \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}(M_1, M_2)$, then $\alpha'$ is not in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}(M_1', M_2')$ in general (unless $\overline{M_1'}$ is normal, see Remark 1.10.2). However, write $\overline{M_1''}$ for the closure of the graph of the rational map $\alpha' : \overline{M_1'} \dashrightarrow \overline{M_2'}$, and $\pi$ for the projection $\overline{M_1''} \to \overline{M_1'}$: by hypothesis, $\pi$ is finite birational. Define a modulus pair $M_1'' = (\overline{M_1''}, M_1''^{\infty})$ by putting $M_1''^{\infty} := \pi^* M_1'^{\infty}$. Then $\pi$ defines a minimal morphism $M_1'' \to M_1'$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, hence the morphism $\alpha'' : M_1'' \to M_2'$ determined by $\alpha'$ is in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$.

b) is clear. To prove c), it suffices as in Corollary A.5.5 to show that for any $M, N \in \underline{\mathbf{M}}\mathbf{Cor}$, the obvious map

$$\varinjlim_{M' \in \Sigma^{\mathrm{fin}} \downarrow M} \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(M', N) \to \underline{\mathbf{M}}\mathbf{Cor}(M, N)$$

is an isomorphism. This map is clearly injective, and its surjectivity follows again from Theorem 1.9.2. □

**Corollary 1.10.5.** *Let $\mathcal{C}$ be a category and let $F : \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}} \to \mathcal{C}$, $G : \underline{\mathbf{M}}\mathbf{Sm} \to \mathcal{C}$ be two functors whose restrictions to the common subcategory $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ are equal. Then $(F, G)$ extends (uniquely) to a functor $H : \underline{\mathbf{M}}\mathbf{Cor} \to \mathcal{C}$.*

*Proof.* The hypothesis implies that $F$ inverts the morphisms in $\Sigma^{\mathrm{fin}}$; the conclusion now follows from Proposition 1.10.4 c). □

**Corollary 1.10.6.** *Any modulus pair in $\underline{\mathbf{M}}\mathbf{Sm}$ is isomorphic to a modulus pair $M$ in which $\overline{M}$ is normal. Under resolution of singularities, we may even choose $\overline{M}$ smooth and the support of $M^{\infty}$ to be a divisor with normal crossings.*

*Proof.* Let $M_0 \in \underline{\mathbf{M}}\mathbf{Sm}$. Consider a proper morphism $\pi : \overline{M} \to \overline{M_0}$ which is an isomorphism over $M_0^{\mathrm{o}}$. Define $M^{\infty} := \pi^* M_0^{\infty}$. Then the induced morphism $\pi : M \to M_0$ of $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ is in $\Sigma^{\mathrm{fin}}$, hence invertible in $\underline{\mathbf{M}}\mathbf{Sm}$. The corollary readily follows. □

We also have the following important lemma:

**Lemma 1.10.7.** *Let $M, L, N \in \underline{\mathbf{M}}\mathbf{Sm}$. Let $f : L \to N$ be a minimal morphism in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ such that $\overline{f} : \overline{L} \to \overline{N}$ is faithfully flat. Then the* *diagram*

$$\begin{tikzcd}
\underline{\mathbf{M}}\mathbf{Cor}(N,M) \arrow[r, "f^*"] \arrow[d, "\cap"'] & \underline{\mathbf{M}}\mathbf{Cor}(L,M) \arrow[d, "\cap"'] \\
\mathbf{Cor}(N^{\mathrm{o}},M^{\mathrm{o}}) \arrow[r, "(f^{\mathrm{o}})^*"] & \mathbf{Cor}(L^{\mathrm{o}},M^{\mathrm{o}})
\end{tikzcd}$$

*is cartesian. The same holds when* $\underline{\mathbf{M}}\mathbf{Cor}$ *is replaced by* $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$*.*

*Proof.* As the second statement is proven in a completely parallel way, we only prove the first one. Take $\alpha \in \mathbf{Cor}(N^{\mathrm{o}},M^{\mathrm{o}})$ such that $(f^{\mathrm{o}})^*(\alpha) \in \underline{\mathbf{M}}\mathbf{Cor}(L,M)$. We need to show $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}(N,M)$.

We first reduce to the case where $\alpha$ is integral. To do this, it suffices to show that for two distinct integral finite correspondences $V, V' \in \mathbf{Cor}(N^{\mathrm{o}},M^{\mathrm{o}})$, $(f^{\mathrm{o}})^*(V)$ and $(f^{\mathrm{o}})^*(V')$ have no common component. For this, we may assume $M^{\mathrm{o}}$ and $N^{\mathrm{o}}$ integral. By the injectivity of $\mathbf{Cor}(N^{\mathrm{o}},M^{\mathrm{o}}) \to \mathbf{Cor}(k(N^{\mathrm{o}}),M^{\mathrm{o}})$, this can be reduced to the case where $N^{\mathrm{o}}$ and $L^{\mathrm{o}}$ are fields, and then the claim is obvious.

Now assume $\alpha$ is integral and put $\beta := (f^{\mathrm{o}})^*(\alpha)$. We have a commutative diagram

$$\begin{tikzcd}
\overline{\beta}^N \arrow[r] \arrow[d, "f^N"'] \arrow[rr, bend left, "\varphi_\beta"] & \overline{\beta} \arrow[r] \arrow[d] \arrow[rr, bend left, "a'"] & \overline{L}\times\overline{M} \arrow[r] \arrow[d] & \overline{L} \arrow[d, "\overline{f}"'] \\
\overline{\alpha}^N \arrow[r] \arrow[rr, bend right, "\varphi_\alpha"'] & \overline{\alpha} \arrow[r] \arrow[rr, bend right, "a"'] & \overline{N}\times\overline{M} \arrow[r] & \overline{N}.
\end{tikzcd}$$

Here $\overline{\alpha}$ (resp. $\overline{\beta}$) is the closure of $\alpha$ (resp. $\beta$) in $\overline{N}\times\overline{M}$ (resp. $\overline{L}\times\overline{M}$) and $\overline{\alpha}^N$ (resp. $\overline{\beta}^N$) is the normalization of $\overline{\alpha}$ (resp. $\overline{\beta}$). By hypothesis $a'$ is proper and $\overline{f}$ is faithfully flat. This implies that $a$ is proper [SGA1, Exp. VIII, Cor. 4.8]. We also have

$$\begin{aligned}
(f^N)^*(\varphi_\alpha^*(N^\infty\times\overline{M})) &= \varphi_\beta^*(\overline{f}^*(N^\infty)\times\overline{M})\\
&= \varphi_\beta^*(L^\infty\times\overline{M}) \geq \varphi_\beta^*(\overline{L}\times M^\infty) = (f^N)^*(\varphi_\alpha^*(\overline{N}\times M^\infty))
\end{aligned}$$

(the second equality by the minimality of $f$). Note that $f^N$ is surjective since $\overline{f}$ is. Hence Lemma 1.2.1 shows that $\varphi_\alpha^*(N^\infty\times\overline{M}) \geq \varphi_\alpha^*(\overline{N}\times M^\infty)$, and we are done. $\square$

1.11. **Quarrable morphisms.** Recall [SGA3, IV.1.4.0] that a morphism $f : M \to N$ in a category $\mathcal{C}$ is *quarrable* if, for any $g : N' \to N$, the fibred product $N' \times_N M$ is representable in $\mathcal{C}$. We have:

**Proposition 1.11.1.** *Let $f : M \to N$ be a minimal morphism in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ (Definition 1.10.3). If $f^{\mathrm{o}}$ is smooth, $f$ is quarrable.*

*Proof.* Let $g : N' \to N$ be a morphism in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$; let $\overline{M'} = \overline{N'} \times_{\overline{N}} \overline{M}$, and let $f' : \overline{M'} \to \overline{N'}$, $g' : \overline{M'} \to \overline{M}$ be the two projections. Since $f^{\mathrm{o}}$ is smooth, $f'^{-1}(N'^{\mathrm{o}})$ is smooth, and defines $M'^{\mathrm{o}}$. We define $M'^{\infty}$ as $f'^* N'^{\infty}$, yielding a modulus pair $M'$ and morphisms $f' : M' \to N'$, $g' : M' \to M$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ with $f'$ minimal:

$$\begin{array}{ccc} M' & \xrightarrow{g'} & M \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ N' & \xrightarrow{g} & N. \end{array}$$

Indeed, $g'^* M^{\infty} = g'^* f^* N^{\infty} = f'^* g^* N^{\infty}$. Since $f' : \overline{M'} \to \overline{N'}$ is dominant, it lifts to a morphism of normalisations $f'^N : \overline{M'}^N \to \overline{N'}^N$. If $q : \overline{M'}^N \to \overline{M'}$ and $r : \overline{N'}^N \to \overline{N'}$ denote the projections, we have

$$q^* f'^* g^* N^{\infty} = f'^{N*} r^* g^* N^{\infty} \leq f'^{N*} r^* N'^{\infty} = q^* f'^* N^{\infty} = q^* M'^{\infty}$$

which shows that $g'$ is admissible.

Let $P$ be another modulus pair and $a : P \to M$, $b : P \to N'$ be morphisms in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ such that $fa = gb$. Let $c_0 : \overline{P} \to \overline{N'} \times_{\overline{N}} \overline{M}$ be the morphism such that $f' c_0 = b$ and $g' c_0 = a$. Then $c_0$ lifts uniquely to a morphism $c : \overline{P} \to \overline{M'}$. Indeed, it suffices to check this component by component, so we may assume $\overline{P}$ and hence $P^{\mathrm{o}}$ irreducible. But $c_0(P^{\mathrm{o}})$ is contained in one component of $M'^{\mathrm{o}}$, hence $c_0(\overline{P})$ is contained in a single irreducible component of $\overline{N'} \times_{\overline{N}} \overline{M}$.

Now

$$c^* M'^{\infty} = c^* f'^* N'^{\infty} = b^* N'^{\infty}$$

so if $p : \overline{P}^N \to \overline{P}$ is the normalisation of $\overline{P}$, we have $p^* c^* M'^{\infty} = p^* b^* N'^{\infty} \leq p^* P^{\infty}$, and $c$ is admissible. □

**Corollary 1.11.2.** *Let $f : M \to N$ be a morphism in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. If $f^{\mathrm{o}}$ is étale and $f$ is minimal in the sense of Definition 1.10.3 a), then so is the diagonal $\Delta_f : M \to M \times_N M$.*

*Proof.* Thanks to Proposition 1.11.1, $\Delta_f$ is really a morphism in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, and it is clearly minimal. Finally, $\Delta_f^{\mathrm{o}}$ is a closed and open immersion as the diagonal of a separated étale morphism. □

As another application, here is a lemma which shows how one can relax the conditions defining the categories **Comp** of Definition 1.8.1, using the notion of minimality:

**Lemma 1.11.3.**

(1) *Let* $\theta : M \to N$ *be a morphism in* $\underline{\mathbf{M}}\mathbf{Cor}$*, such that*
- $\theta \in \underline{\Sigma}$ *(see Definition 1.7.1);*
- $N \in \mathbf{MCor}$*;*
- $\theta : \overline{M} \to \overline{N}$ *is an open immersion;*
- $\theta$ *is minimal (see Definition 1.10.3).*

*Then there exists a morphism* $\varphi : N_1 \to N$ *in* $\Sigma^{\mathrm{fin}}$ *(see Definition 1.10.3) and a lift* $\theta_1 : M \to N_1$ *of* $\theta$ *such that* $(N_1, \theta_1) \in \mathbf{Comp}(M)$*.*

(2) *For* $M \in \underline{\mathbf{M}}\mathbf{Cor}$*, let* $\mathbf{Comp}_1(M)$ *be the category whose objects are morphisms* $\theta : M \to N$ *verifying the conditions of (1), morphisms being given as in Definition 1.8.1. Then* $\mathbf{Comp}_1(M)$ *is a cofiltering ordered set containing* $\mathbf{Comp}(M)$*, and* $\mathbf{Comp}(M)$ *is cofinal in* $\mathbf{Comp}_1(M)$*. In particular,* $\tau^! M$ *in Theorem 1.6.2 may be computed using* $\mathbf{Comp}_1(M)$*.*

*Proof.* (1) Let $M_N^\infty$ be the scheme-theoretic closure of $M^\infty$ in $\overline{N}$, $\pi : \overline{N_1} \to \overline{N}$ the blowing-up along $M_N^\infty$, $N_1^\infty = \pi^* N^\infty$, and $N_1 = (\overline{N_1}, N_1^\infty)$. The lift $\theta_1$ of $\theta$ exists by the universal property of blowing-up, and $\varphi$ is given by $\pi$. By construction, we have $M_{N_1}^\infty := \pi^*(M_N^\infty) \leq N_1^\infty$ and both $M_{N_1}^\infty$ and $N_1^\infty$ are effective Cartier divisors. Then the effective Cartier divisor $C = N_1^\infty - M_{N_1}^\infty$ has support contained in $\overline{N_1} - \overline{M}$; but we have equality since $M^o = N^o = N_1^o$. Hence $N_1 \in \mathbf{Comp}(M)$.

(2) Any morphism between $(N_1, \theta_1)$ and $(N_2, \theta_2)$ in $\mathbf{Comp}_1(M)$ is in $\Sigma^{\mathrm{fin}}$, in particular induces the identity: $N_1^o \to N_2^o$; hence $\mathbf{Comp}_1(M)$ is ordered. Proposition 1.11.1 implies that it is cofiltering, and (1) implies that $\mathbf{Comp}(M)$ is cofinal in $\mathbf{Comp}_1(M)$. □

## 2. Presheaf theory

### 2.1. Modulus presheaves with transfers.

**Definition 2.1.1.** By a presheaf we mean that a contravariant functor to the category of abelian groups.

(1) The category of presheaves on $\mathbf{MSm}$ (resp. $\underline{\mathbf{M}}\mathbf{Sm}, \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$) is denoted by $\mathbf{MPS}$ (resp. $\underline{\mathbf{M}}\mathbf{PS}, \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}$).

(2) The category of additive presheaves on $\mathbf{MCor}$ (resp. on $\underline{\mathbf{M}}\mathbf{Cor}$, on $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$) is denoted by $\mathbf{MPST}$ (resp. $\underline{\mathbf{M}}\mathbf{PST}, \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$.)

Applying Theorem A.13.2, we get:

**Proposition 2.1.2.** *The categories* $\mathbf{MPST}$*,* $\underline{\mathbf{M}}\mathbf{PST}$ *and* $\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ *all have closed tensor structures that extend the tensor structures of* $\mathbf{MCor}$*,* $\underline{\mathbf{M}}\mathbf{Cor}$ *and* $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$ *via the additive Yoneda functors.*

*Remark* 2.1.3. There are no natural maps
$$F(M)\times G(M)\to (F\otimes_{\underline{\mathbf{M}}\mathbf{PST}} G)(M)$$
for $F,G\in \underline{\mathbf{M}}\mathbf{PST}$ and $M\in \underline{\mathbf{M}}\mathbf{Cor}$. This is because the diagonal map $M\to M\otimes M$ is usually not a morphism in $\underline{\mathbf{M}}\mathbf{Cor}$ (see Remark 1.4.4). The same remark applies to $\mathbf{MPST}$ and $\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ as well.

**Notation 2.1.4.** We write
$$\begin{aligned}
\mathbb{Z}^{\mathrm{fin}} &: \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}\to \underline{\mathbf{M}}\mathbf{PS}\\
\mathbb{Z}^{\mathrm{fin}}_{\mathrm{tr}} &: \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}\to \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}\\
\mathbb{Z}_{\mathrm{tr}} &: \underline{\mathbf{M}}\mathbf{Cor}\to \underline{\mathbf{M}}\mathbf{PST}\\
\mathbb{Z}_{\mathrm{tr}} &: \mathbf{MCor}\to \mathbf{MPST}\\
\mathbb{Z}^{V}_{\mathrm{tr}} &: \mathbf{Cor}\to \mathbf{PST}
\end{aligned}$$
for the associated representable additive presheaf functors.

We now briefly describe the main properties of the functors induced by those of the previous section.

## 2.2. MPST and PST.

**Proposition 2.2.1.** *The functor $\omega:\mathbf{MCor}\to\mathbf{Cor}$ of §1.6 yields a string of 3 adjoint functors $(\omega_!,\omega^*,\omega_*)$:*
$$\mathbf{MPST}\ \substack{\xrightarrow{\omega_!}\\ \xleftarrow{\omega^*}\\ \xrightarrow{\omega_*}}\ \mathbf{PST}$$
*where $\omega^*$ is fully faithful and $\omega_!,\omega_*$ are localisations; $\omega_!$ is monoidal and has a pro-left adjoint, hence is exact. The pro-left adjoint $\omega^!$ of $\omega_!$ is monoidal.*

*Proof.* This follows from Theorems 1.6.2, A.8.1 and A.13.2. □

Let $X\in\mathbf{Sm}$ and let $M\in\mathbf{MSm}(X)$. Lemma 1.7.4 and Proposition A.4.2 show that the inclusions $\{M^{(n)}\mid n>0\}\subset \mathbf{MSm}(\overline{M},X)\subset \mathbf{MSm}(X)$ induce isomorphisms (see Def. 1.7.3)

$$\omega_!(F)(X)\simeq \varinjlim_{N\in\mathbf{MSm}(X)} F(N)\simeq \varinjlim_{N\in\mathbf{MSm}(\overline{M},X)} F(N)\simeq \varinjlim_{n>0} F(M^{(n)}). \tag{2.1}$$

**Proposition 2.2.2.** *Let $M,N\in\mathbf{MCor}$ and let $X\in\mathbf{Sm}$. Then*
$$\omega_!(\underline{\mathrm{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\mathrm{tr}}(N),\mathbb{Z}_{\mathrm{tr}}(M))(X)$$
*is the subgroup of $\mathbf{Cor}(N^{\mathrm{o}}\times X,M^{\mathrm{o}})$ generated by all elementary correspondences $Z\in\mathbf{Cor}(N^{\mathrm{o}}\times X,M^{\mathrm{o}})$ such that*
$$\varphi_Z^*(M^\infty\times\overline{N}\times X)\le\varphi_Z^*(\overline{M}\times N^\infty\times X),$$

*where* $\varphi_Z : \overline{Z}^N \to \overline{M} \times \overline{N} \times X$ *denotes the normalization of the closure of* $Z$*.*

*Proof.* (2.1) shows that $\omega_!(\underline{\mathrm{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\mathrm{tr}}(N), \mathbb{Z}_{\mathrm{tr}}(M))(X)$ agrees with

$$\bigcup_{L \in \mathbf{MSm}(X)} \mathbf{MCor}(N \otimes L, M) \quad \Big( \subset \mathbf{Cor}(N^0 \times X, M^o) \Big),$$

from which the proposition follows. □

2.3. $\underline{\mathbf{M}}$**PST and PST.**

**Proposition 2.3.1.** *The adjoint functors* $(\lambda, \underline{\omega})$ *of Lemma 1.6.1 induce a string of* 4 *adjoint functors* $(\lambda_! = \underline{\omega}^!, \lambda^* = \underline{\omega}_!, \lambda_* = \underline{\omega}^*, \underline{\omega}_*)$*:*

$$\underline{\mathbf{M}}\mathbf{PST} \underset{\overset{\underline{\omega}^*}{\longleftarrow}\atop \overset{\underline{\omega}_*}{\longrightarrow}}{\overset{\overset{\underline{\omega}^!}{\longleftarrow}}{\overset{\underline{\omega}_!}{\longrightarrow}}} \mathbf{PST}$$

*where* $\underline{\omega}_!, \underline{\omega}_*$ *are localisations while* $\underline{\omega}^!$ *and* $\underline{\omega}^*$ *are fully faithful. The functors* $\underline{\omega}^!$ *and* $\underline{\omega}_!$ *are monoidal. Moreover, if* $X \in \mathbf{Cor}$ *is proper, we have a canonical isomorphism* $\underline{\omega}^*\mathbb{Z}_{\mathrm{tr}}(X) \simeq \mathbb{Z}_{\mathrm{tr}}(X, \emptyset)$*.*

*Proof.* The only non obvious fact is the last claim, which follows from Lemma 1.6.1. □

2.4. **MPST and** $\underline{\mathbf{M}}$**PST.**

**Proposition 2.4.1.** *The functor* $\tau : \mathbf{MCor} \to \underline{\mathbf{M}}\mathbf{Cor}$ *of* (1.2) *yields a string of* 3 *adjoint functors* $(\tau_!, \tau^*, \tau_*)$*:*

$$\mathbf{MPST} \overset{\overset{\tau_!}{\longrightarrow}}{\underset{\overset{\tau_*}{\longrightarrow}}{\overset{\tau^*}{\longleftarrow}}} \underline{\mathbf{M}}\mathbf{PST}$$

*where* $\tau_!, \tau_*$ *are fully faithful and* $\tau^*$ *is a localisation;* $\tau_!$ *is monoidal and has a pro-left adjoint* $\tau^!$*, hence is exact; moreover,* $\tau^!$ *is monoidal. There are natural isomorphisms*

$$\omega_! \simeq \underline{\omega}_!\tau_!, \quad \omega_* \sim \underline{\omega}_*\tau_*, \quad \omega^! \sim \tau^!\underline{\omega}^!.$$

*Proof.* This follows from Theorem 1.6.2 and Proposition A.4.2. □

**Lemma 2.4.2.**

(1) *For* $G \in \mathbf{MPST}$ *and* $M \in \underline{\mathbf{M}}\mathbf{Sm}$*, we have*

$$\varinjlim_{N \in \mathbf{Comp}(M)} G(N) \simeq \tau_! G(M).$$

(2) *The adjunction map* $\mathrm{Id} \to \tau^*\tau_!$ *is an isomorphism.*

(3) *There is a natural isomorphism* $\tau_! \omega^* \simeq \underline{\omega}^*$.

*Proof.* (1) This follows from Lemma 1.8.3 Proposition A.4.2.

(2) This follows from (1) since $\mathbf{Comp}(M) = \{M\}$ for $M \in \mathbf{MSm}$.

(3) For $F \in \mathbf{PST}$ and $M \in \mathbf{MCor}$, we compute

$$\begin{aligned}\tau_! \omega^* F(M) &= \varinjlim_{N \in \mathbf{Comp}(M)} \omega^* F(N) \\ &= \varinjlim_{N \in \mathbf{Comp}(M)} F(N^\circ) = F(M^\circ) = \underline{\omega}^* F(M).\end{aligned}$$

□

*Remark* 2.4.3. By Lemma 1.8.3 we have the formulas

$$\tau^! \mathbb{Z}_{\mathrm{tr}}(M) = \text{“}\varprojlim\text{”}_{N \in \mathbf{Comp}(M)} \mathbb{Z}_{\mathrm{tr}}(N), \quad \tau^* \mathbb{Z}_{\mathrm{tr}}(M) = \varprojlim_{N \in \mathbf{Comp}(M)} \mathbb{Z}_{\mathrm{tr}}(N)$$

the latter an inverse limit computed in **MPST**.

*Question* 2.4.4. Is $\tau^!$ exact?

### 2.5. $\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ and $\underline{\mathbf{M}}\mathbf{PST}$.

**Proposition 2.5.1.** *Let* $b : \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{Cor}$ *be the inclusion functor from Definition 1.10.1. Then* $b$ *is monoidal, is a localisation and has a pro-left adjoint; it yields a string of* 3 *adjoint functors* $(b_!, b^*, b_*)$*:*

$$\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}} \begin{array}{c} \xrightarrow{b_!} \\ \xleftarrow{b^*} \\ \xrightarrow{b_*} \end{array} \underline{\mathbf{M}}\mathbf{PST}$$

*where* $b_!, b_*$ *are localisations and* $b^*$ *is fully faithful;* $b_!$ *is monoidal and has a pro-left adjoint, hence is exact.*

*Proof.* The monoidality of $b$ is obvious; the rest follows from the usual yoga applied with Proposition 1.10.4. □

### 2.6. The functors $n_!$ and $n^*$.

We write $(n_!, n^*)$ for the pair of adjoint endofunctors of **MPST** induced by the functor $(-)^{(n)}$ of Definition 1.5.1 ($n_!$ is left adjoint to $n^*$ and extends $(-)^{(n)}$ via the Yoneda embedding).

**Lemma 2.6.1.** *The functor* $n_!$ *is fully faithful and monoidal.*

*Proof.* This follows formally from the same properties of $(-)^{(n)}$. □

**Proposition 2.6.2.** *For any* $F \in \mathbf{MPST}$*, there is a natural isomorphism*

$$\omega^* \omega_! F \simeq \infty^* F$$

*where* $\infty^* F(M) := \varinjlim_n F(M^{(n)})$ *(for the natural transformations (1.1)).*

*Proof.* Let $M \in \mathbf{MCor}$ and $X = \omega M$. Then

$$\omega^*\omega_! F(M) = \varinjlim_{M' \in \mathbf{MSm}(X)} F(M')$$

and the claim follows from Lemma 1.7.4. $\square$

**Proposition 2.6.3.** *For all $n \geq 1$, the natural transformation $\omega_! \to \omega_! n^*$ stemming from (1.1) is an isomorphism.*

*Proof.* Let $F \in \mathbf{MPST}$. For $X \in \mathbf{Cor}$, we have

$$\omega_! n^* F(X) = \varinjlim_{M \in \mathbf{MSm}(X)} n^* F(M) = \varinjlim_{M \in \mathbf{MSm}(X)} F(M^{(n)}) = \varinjlim_{M \in \mathbf{MSm}(X)} F(M)$$

where the last isomorphism follows from Lemma 1.7.4. $\square$

## 3. Sheaf theory

### 3.1. Grothendieck topologies on $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$.

**Definition 3.1.1.** Let $\sigma \in \{\text{ét}, \text{Nis}, \text{Zar}\}$. We call a morphism $p : U \to M$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ a $\sigma$-cover if

(i) $\overline{p} : \overline{U} \to \overline{M}$ is a $\sigma$-cover of $\overline{M}$ in the usual sense;
(ii) $p$ is minimal (that is, $U^\infty = p^*(M^\infty)$).

Since the morphisms appearing in the $\sigma$-cover are quarrable by Proposition 1.11.1, we obtain a Grothendieck topology on $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. The category $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ endowed with this topology will be called the big $\sigma$-site of $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ and denoted by $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}_\sigma$.

The following lemma is easily checked:

**Lemma 3.1.2.** *If $p : U \to M$ is a $\sigma$-cover in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, so is the induced morphism $p^o : U^o \to M^o$ in* **Sm**. $\square$

**Definition 3.1.3.** Let us fix $M \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. We define three (small) sites:

(1) Let $M_{\text{ét}}$ be the category of morphisms $f : N \to M$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ such that $\overline{f}$ is étale, endowed with the topology induced by $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}_{\text{ét}}$.
(2) Let $M_{\mathrm{Nis}}$ be the same category as $M_{\text{ét}}$, but endowed with the topology induced by $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}_{\mathrm{Nis}}$.
(3) Let $M_{\mathrm{Zar}}$ be the category of morphisms $f : N \to M$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ such that $\overline{f}$ is open immersion, endowed with the topology induced by $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}_{\mathrm{Zar}}$.

The following lemma is obvious from the definitions:

**Lemma 3.1.4.** *Let $\sigma \in \{\text{ét}, \text{Nis}, \text{Zar}\}$ and $M \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. Let $(\overline{M})_\sigma$ be the (usual) small $\sigma$-site on $\overline{M}$. Then we have an isomorphism of sites*

$$M_\sigma \to (\overline{M})_\sigma, \qquad N \mapsto \overline{N},$$

*whose inverse is given by $(p : X \to \overline{M}) \mapsto (X, p^*(M^\infty))$. (This isomorphism of sites depends on the choice of $M^\infty$.)* □

**Lemma 3.1.5.** *Let $\sigma \in \{\text{ét}, \text{Nis}\}$. Let $\alpha : M \to N$ be a morphism in $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$ and let $p : U \to N$ be a $\sigma$-cover of $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. Then there is a commutative diagram*

$$\begin{tikzcd} V \arrow[r, "\alpha'"] \arrow[d, "p'"'] & U \arrow[d, "p"] \\ M \arrow[r, "\alpha"'] & N, \end{tikzcd}$$

*where $\alpha' : V \to U$ is a morphism in $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$ and $p' : V \to M$ is a $\sigma$-cover of $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$.*

*Proof.* We may assume $\alpha$ is integral. Let $\overline{\alpha}$ be the closure of $\alpha$ in $\overline{M} \times \overline{N}$. Since $\overline{\alpha}$ is finite over $\overline{M}$, we may find a $\sigma$-cover $p' : \overline{V} \to \overline{M}$ such that $\tilde{p}$ in the diagram (all squares being cartesian)

$$\begin{tikzcd} \overline{V} \times_{\overline{M}} \times (\overline{\alpha} \times \overline{U}) \arrow[r] \arrow[d, "\tilde{p}"'] & \overline{\alpha} \times_{\overline{N}} \overline{U} \arrow[r] \arrow[d] & \overline{U} \arrow[d, "\overline{p}"] \\ \overline{V} \times_{\overline{M}} \times \overline{\alpha} \arrow[r] \arrow[d] & \overline{\alpha} \arrow[r] \arrow[d] & \overline{N} \\ \overline{V} \arrow[r, "p'"'] & \overline{M} & \end{tikzcd}$$

has a splitting $s$. Put $V := (\overline{V}, p'^*(M^\infty)) \in \underline{\mathbf{M}}\mathbf{Sm}$. The image of $s$ gives us a desired correspondence $\alpha'$. □

## 3.2. Sheaves on $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$.

**Definition 3.2.1.** For $\sigma \in \{\text{ét}, \text{Nis}, \text{Zar}\}$, we define $\underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}_\sigma$ to be the full subcategory of $\underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}$ consisting of $\sigma$-sheaves.

**Theorem 3.2.2.** *Let $\sigma \in \{\text{ét}, \text{Nis}, \text{Zar}\}$, and let $F \in \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}_\sigma$. Then*

- *If $\sigma \in \{\text{Nis}, \text{Zar}\}$, then $H^i_\sigma(X, F) = 0$ for any $X \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ and $i > \dim X (= \dim X^o = \dim \overline{X})$.*
- *If $\sigma = \text{ét}$ and $cd(k) = d < +\infty$, then $H^i_\sigma(X, F) = 0$ for $i > 2 \dim X + d$.*

*Proof.* This is clear from Lemma 3.1.4 and the known properties of $\sigma$. □

For the sequel, the following terminology is helpful:

**Definition 3.2.3.** An additive functor $F$ between additive categories is *strongly additive* if it commutes with infinite direct sums.

**Lemma 3.2.4.** *The category* $\underline{\mathbf{M}}\mathbf{PS}_\sigma^{\mathrm{fin}}$ *is closed under infinite direct sums and the inclusion functor* $\underline{i}_\sigma^{\mathrm{fin}} : \underline{\mathbf{M}}\mathbf{PS}_\sigma^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}$ *is strongly additive.*

*Proof.* Indeed, the sheaf condition is tested on finite diagrams, hence the presheaf given by a direct sum of sheaves is a sheaf. □

**Proposition 3.2.5.** *For any* $M \in \underline{\mathbf{M}}\mathbf{Sm}$ *and* $\sigma \in \{\text{ét}, \mathrm{Nis}, \mathrm{Zar}\}$*, we have*

$$c^*\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M), \quad c^*b^*\mathbb{Z}_{\mathrm{tr}}(M) \in \underline{\mathbf{M}}\mathbf{PS}_\sigma^{\mathrm{fin}},$$

*where* $\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}, \mathbb{Z}_{\mathrm{tr}}$ *are the representable presheaf functors (notation 2.1.4) and the functors*

$$b^* : \underline{\mathbf{M}}\mathbf{PST} \to \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}, \quad c^* : \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}. \tag{3.1}$$

*are induced by* $b$ *from Proposition 2.5.1 and* $c : \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$*.*

*Proof.* It suffices to show the case $\sigma = $ ét. Let $U \to N$ be an étale cover. We have a commutative diagram

$$\begin{array}{ccccc}
0 \to \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(N, M) & \longrightarrow & \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(U, M) & \longrightarrow & \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(U \times_N U, M) \\
\cap\big\downarrow & & \cap\big\downarrow & & \cap\big\downarrow \\
0 \to \underline{\mathbf{M}}\mathbf{Cor}(N, M) & \longrightarrow & \underline{\mathbf{M}}\mathbf{Cor}(U, M) & \longrightarrow & \underline{\mathbf{M}}\mathbf{Cor}(U \times_N U, M) \\
\cap\big\downarrow & & \cap\big\downarrow & & \cap\big\downarrow \\
0 \to \mathbf{Cor}(N^o, M^o) & \longrightarrow & \mathbf{Cor}(U^o, M^o) & \longrightarrow & \mathbf{Cor}(U^o \times_{N^o} U^o, M^o).
\end{array}$$

The bottom row is exact by [30, Lemma 6.2]. The exactness of the top and middle row now follows from Lemma 1.10.7. □

## 3.3. Čech complex.

**Theorem 3.3.1.** *Let* $\sigma \in \{\text{ét}, \mathrm{Nis}\}$*. If* $p : U \to M$ *is a* $\sigma$*-cover in* $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$*, then the Čech complex*

$$\cdots \to c^*\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U \otimes_M U) \to c^*\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U) \to c^*\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M) \to 0 \tag{3.2}$$

*is exact in* $\underline{\mathbf{M}}\mathbf{PS}_\sigma^{\mathrm{fin}}$*. Here we write* $U \otimes_M U$*, etc., for the modulus pair corresponding to* $\overline{U} \times_{\overline{M}} \overline{U}$ *under the isomorphism of sites from Lemma 3.1.4. Note that it is a fibre product in* $\underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}$ *thanks to Proposition 1.11.1, hence in* $\underline{\mathbf{M}}\mathbf{PS}$ *by Propositions 1.10.4 a) and A.5.6 a).*

*Remark* 3.3.2. This result will be refined several times, see Corollary 3.4.6 and Theorems 3.5.7, 3.9.3.

*Proof.* It is adapted from [49, Prop. 3.1.3]. As both proofs go parallel, we only write down the one for $\sigma = \mathrm{Nis}$. In view of Lemma 3.1.4, it suffices to show the exactness of (3.2) evaluated at $S = (\overline{S}, D)$

$$\cdots \to \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(S, U \otimes_M U) \to \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(S, U) \to \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(S, M) \to 0 \tag{3.3}$$

for a henselian local $\overline{S}$ and an effective Cartier divisor $D$ on $\overline{S}$. To a diagram $\overline{S} \xleftarrow{f} Z \xrightarrow{g} \overline{M}$ of $k$-schemes with $f$ quasi-finite, we associate the free abelian group $L(Z)$ on the set of irreducible components $V$ of $Z$ such that $f|_V$ is finite and surjective over an irreducible component of $\overline{S}$ and such that $g(V) \not\subset M^\infty$ and $(fi_V v)^*(D) \geq (gi_V v)^*(M^\infty)$, where $v : V^N \to V$ is the normalization and $i_V : V \hookrightarrow Z$ is the inclusion. Note that $L$ is covariantly functorial in $Z$. Then (3.3) is obtained as the inductive limit of

$$\cdots \to L(Z \times_{\overline{M}} (\overline{U} \times_{\overline{M}} \overline{U})) \to L(Z \times_{\overline{M}} \overline{U}) \to L(Z) \to 0 \tag{3.4}$$

where $Z$ ranges over all closed subschemes of $\overline{S} \times \overline{M}$ that are finite surjective over an irreducible component of $\overline{S}$. It suffices to show the exactness of (3.4).

Since $Z$ is finite over a henselian local scheme $\overline{S}$, $Z$ is a disjoint union of henselian local schemes. Thus the Nisnevich cover $Z \times_{\overline{M}} \overline{U} \to Z$ admits a section $s_0 : Z \to Z \times_{\overline{M}} \overline{U}$. Define for $k \geq 1$

$$s_k := s_0 \times_{\overline{M}} \mathrm{Id}_{\overline{U}^k} : Z \times_{\overline{M}} \overline{U}^k \to Z \times_{\overline{M}} \overline{U} \times_{\overline{M}} \overline{U}^k = Z \times_{\overline{M}} \overline{U}^{k+1}$$

where $\overline{U}^k = \overline{U} \times_{\overline{M}} \cdots \times_{\overline{M}} \overline{U}$. Then the maps

$$L(Z \times_{\overline{M}} \overline{U}^k) \to L(Z \times_{\overline{M}} \overline{U}^{k+1})$$

induced by $s_k$ give us a homotopy from the identity to zero. □

3.4. **Sheafification preserves transfers.** For $\sigma \in \{\text{ét}, \mathrm{Nis}, \mathrm{Zar}\}$, let $\underline{a}^{\mathrm{fin}}_{s,\sigma} : \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}_\sigma$ be the sheafification functor, that is, the left adjoint of the inclusion functor $\underline{i}^{\mathrm{fin}}_{s,\sigma} : \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}_\sigma \hookrightarrow \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}$. It exists for general reasons and is exact [SGA4-I, II.3.4].

**Definition 3.4.1.** Let $\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}_\sigma$ be the full subcategory of $\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ consisting of all objects $F \in \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ such that $c^*F \in \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}_\sigma$ (see (3.1) for the functor $c^*$).

**Lemma 3.4.2.** *The category $\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}_\sigma$ is closed under infinite direct sums in $\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$, and the inclusion functor $\underline{i}^{\mathrm{fin}}_\sigma : \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}_\sigma \to \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ is strongly additive.*

*Proof.* This follows from Lemma 3.2.4, because $c^*$ is strongly additive as a left adjoint. $\square$

Recall that there is a functor $\check{c}_\sigma^{\mathrm{fin}} : \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}$ given by

$$\check{c}_\sigma^{\mathrm{fin}}(G)(M) = \varinjlim_{U\to M} \check{H}^0(U,G) \quad (G \in \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}},\ M \in \underline{\mathbf{M}}\mathbf{Sm}), \tag{3.5}$$

where $U \to M$ ranges over all $\sigma$-cover [SGA4-I, II.3.0.5]. It comes equipped with a canonical morphism $u : G \to \check{c}_\sigma^{\mathrm{fin}}(G)$. The presheaf $\check{c}_\sigma^{\mathrm{fin}}(G)$ is always $\sigma$-separated, and if $G \in \underline{\mathbf{M}}\mathbf{PS}_\sigma^{\mathrm{fin}}$ then we have $\check{c}_\sigma^{\mathrm{fin}}(G) = G$. We also have $\underline{i}_{s,\sigma}^{\mathrm{fin}}\underline{a}_{s,\sigma}^{\mathrm{fin}} = \check{c}_\sigma^{\mathrm{fin}}\check{c}_\sigma^{\mathrm{fin}}$.

**Lemma 3.4.3.** *Let $\sigma \in \{\text{ét}, \mathrm{Nis}\}$ and $F \in \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$. There exists an unique object $G \in \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ such that $c^*(G) = \check{c}_\sigma^{\mathrm{fin}}(c^*(F))$ and such that the canonical morphism $u : c^*(F) \to \check{c}_\sigma^{\mathrm{fin}}(c^*(F)) = c^*(G)$ extends to a morphism in $\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$.*

*Proof.* This can be shown by a rather trivial modification of [49, Th. 3.1.4], but for the sake of completeness we include a proof. To ease the notation, put $F' := \check{c}_\sigma^{\mathrm{fin}} c^* F \in \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}$. First we construct a homomorphism

$$\Phi_M : F'(M) \to \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}(c^*\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M), F')$$

for any $M \in \underline{\mathbf{M}}\mathbf{Sm}$. Take $f \in F'(M)$. There exists a $\sigma$-cover $p : U \to M$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ and $g \in c^*F(U) = F(U)$ such that $f|_U = u(g)$ in $F'(U)$ and that $g|_{U\otimes_M U} = 0$ in $F(U \otimes_M U)$. We have $\check{c}_\sigma^{\mathrm{fin}} c^*\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M) = c^*\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M)$ because $c^*\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M) \in \underline{\mathbf{M}}\mathbf{PS}_\sigma^{\mathrm{fin}}$ by Prop. 3.2.5. Thus we get a commutative diagram in which the horizontal maps are induced by $\check{c}_\sigma^{\mathrm{fin}} c^*$

$$\begin{tikzcd}
0 \arrow[d] & \\
\underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}(c^*\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M), F') \arrow[d, "s"] & \\
\underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}(c^*\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U), F') \arrow[d] & \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}(\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U), F) \arrow[l, "s'"'] \arrow[d, "s''"] \\
\underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}(c^*\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U\otimes_M U), F') & \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}(\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U\otimes_M U), F) \arrow[l]
\end{tikzcd}$$

The left vertical column is exact by Theorem 3.3.1. Since $g \in F(U) = \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}(\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U), F)$ satisfies $s''(g) = g|_{U\otimes_M U} = 0$, there exists a unique $h \in \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}(c^*\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M), F')$ such that $s(h) = s'(g)$. One checks that $h$ does not depend on the choices we made. We define $\Phi_M(f) := h$.

Now we define $G$. On objects we put $G(M) = F'(M)$ for $M \in \underline{\mathbf{M}}\mathbf{Sm}$. For $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(M,N)$, we define $\alpha^* : F'(N) \to F'(M)$ as the composition of

$$F'(N) \xrightarrow{\Phi_N} \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}(c^*\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(N), F') \longrightarrow \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}(c^*\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M), F') \to F'(M),$$

where the middle map is induced by $c^*(\alpha) : c^*\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M) \to c^*\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(N)$, and the last map is given by $f \mapsto f_M(\mathrm{Id}_M)$. One checks that, with this definition, $G$ becomes an object of $\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$.

To prove uniqueness, take $G, G' \in \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ which enjoy the stated properties. We have $G(M) = G'(M) = F'(M)$ for any $M \in \underline{\mathbf{M}}\mathbf{Sm}$. (Recall that $F' := \check{c}_\sigma^{\mathrm{fin}} c^* F \in \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}$.) We also have $G(c(q)) = G'(c(q)) = F'(q)$ for any morphism $q$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. Let $\alpha : M \to N$ be a morphism in $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$ and let $f \in F'(N)$. Take a $\sigma$-cover $p : U \to N$ of $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ and $g \in c^*F(U) = F(U)$ such that $f|_U = u(g)$ in $F'(U)$. Apply Lemma 3.1.5 to get a morphism $\alpha' : V \to U$ in $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$ and a $\sigma$-cover $p' : V \to M$ of $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ such that $\alpha p' = p\alpha'$. Then we have

$$\begin{aligned} G(p')G(\alpha)(f) &= G(\alpha')G(p)(f) = G(\alpha')(u(g)) = u(F(\alpha')(g)) \\ &= G'(\alpha')(u(g)) = G'(\alpha')G'(p)(f) = G'(p')G'(\alpha)(f) = G(p')G'(\alpha)(f). \end{aligned}$$

Since $p' : V \to M$ is a $\sigma$-cover and $G$ is $\sigma$-separated, this implies $G(\alpha)(f) = G'(\alpha)(f)$. This completes the proof. $\square$

**Theorem 3.4.4.** *Let $\sigma \in \{\text{ét}, \mathrm{Nis}\}$.*

(1) *Let $F \in \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$. There exists a unique object $F_\sigma \in \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ such that $c^*(F_\sigma) = \underline{a}_{s,\sigma}^{\mathrm{fin}}(c^*(F))$ and such that the canonical morphism $u : c^*(F) \to \underline{a}_{s,\sigma}^{\mathrm{fin}}(c^*(F)) = c^*(F_\sigma)$ extends to a morphism in $\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$.*

(2) *$\underline{i}_\sigma^{\mathrm{fin}}$ has a left adjoint $\underline{a}_\sigma^{\mathrm{fin}} : \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}}$, which is exact; in particular the category $\underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}}$ is Grothendieck (§A.12). The following diagram commutes*

$$\begin{tikzcd} \underline{\mathbf{M}}\mathbf{PS}_\sigma^{\mathrm{fin}} \arrow[r, "\underline{i}_{s,\sigma}^{\mathrm{fin}}"'] \arrow[rr, bend left, "\mathrm{Id}"] & \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}} \arrow[r, "\underline{a}_{s,\sigma}^{\mathrm{fin}}"'] & \underline{\mathbf{M}}\mathbf{PS}_\sigma^{\mathrm{fin}} \\ \underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}} \arrow[u, "c^\sigma"'] \arrow[r, "\underline{i}_\sigma^{\mathrm{fin}}"] \arrow[rr, bend right, "\mathrm{Id}"'] & \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}} \arrow[u, "c^*"'] \arrow[r, "\underline{a}_\sigma^{\mathrm{fin}}"] & \underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}} \arrow[u, "c^\sigma"'] \end{tikzcd}$$

*where $c^\sigma$ is the restriction of $c^*$ to $\underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}}$. Moreover, $c^\sigma$ is exact, strongly additive and has the left adjoint $c_\sigma = \underline{a}_\sigma^{\mathrm{fin}} c_! \underline{i}_\sigma^{\mathrm{fin}}$.*

*Proof.* (1) is deduced by applying the previous lemma twice. (2) is a consequence of (1) and the fact that $\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ is Grothendieck as a category of modules (see Theorem A.12.1 d)). Note that we cannot apply Lemma A.9.1, because $c^*$ is faithful but not full. □

**Definition 3.4.5.** An additive functor $\varphi$ between abelian categories is *faithfully exact* if a complex $F' \to F \to F''$ is exact if and only if $\varphi F' \to \varphi F \to \varphi F''$ is.

This happens if $\varphi$ is exact and either faithful or conservative.

By Theorems 3.4.4 and 3.3.1 we get:

**Corollary 3.4.6.** *Let $\sigma \in \{\text{ét}, \mathrm{Nis}\}$. The functor $c^\sigma$ is faithfully exact. In particular, if $p : U \to M$ is a $\sigma$-cover in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, then the Čech complex*

$$\cdots \to \mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U \otimes_M U) \to \mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U) \to \mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M) \to 0 \tag{3.6}$$

*is exact in $\underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}}$.*

**Notation 3.4.7.** Take $\sigma \in \{\text{ét}, \mathrm{Nis}, \mathrm{Zar}\}$. Let $M \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ and let $K$ be an object or a complex in $\underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}}$. We write $K_M$ for the sheaf or the complex of sheaves on $(\overline{M})_\sigma$ deduced from $c^\sigma K$ via the isomorphism of sites from Lemma 3.1.4. (Note that $K_M$ depends not only on $\overline{M}$, but also on $M^\infty$.) We thus have a canonical isomorphism

$$H^i_\sigma(M, c^\sigma K) \simeq H^i_\sigma(\overline{M}, K_M) \tag{3.7}$$

where the right hand side denotes the cohomology of the (usual) small site $(\overline{M})_\sigma$.

Let $S$ be a scheme and consider the small site $S_\sigma$ with $\sigma \in \{\text{ét}, \mathrm{Nis}, \mathrm{Zar}\}$. A sheaf $F$ on $S_\sigma$ is called *flabby* if $H^i_\sigma(U, F) = 0$ for any $U \in S_\sigma$ and $i > 0$. When $\sigma \in \{\mathrm{Nis}, \mathrm{Zar}\}$, we say $F$ is *flasque* if $F(V) \to F(U)$ is surjective for any open dense immersion $U \to V$. Flasque sheaves are flabby (see [16, II 2.5], [42, lemme 1.40]).

Let us now take $F \in \underline{\mathbf{M}}\mathbf{PS}_\sigma^{\mathrm{fin}}$. We say $F$ flabby (resp. flasque) if $F_M$ is for any $M \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ (see Notation 3.4.7).

**Lemma 3.4.8.** *Suppose that $\sigma \in \{\text{ét}, \mathrm{Nis}, \mathrm{Zar}\}$, and let $I \in \underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}}$ be an injective object. Then $c^\sigma(I) \in \underline{\mathbf{M}}\mathbf{PS}_\sigma^{\mathrm{fin}}$ is flabby, and is flasque if $\sigma \in \{\mathrm{Nis}, \mathrm{Zar}\}$.*

*Proof.* Suppose first that $\sigma \in \{\mathrm{Nis}, \mathrm{Zar}\}$. Let $j : U \hookrightarrow M$ be a minimal open immersion of modulus pairs in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. The morphism of sheaves $\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(j)$ is a monomorphism, hence $j^* : I(M) \to I(U)$ is surjective.

We now assume $\sigma = \text{ét}$ and show that $c^\sigma(I)$ is flabby. (This proof, adapted from [49, 3.1.7], also works for $\sigma = \mathrm{Nis}$.) We take a $\sigma$-cover

$p : U \to M$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. By [33, III 2.12], it suffices to show that its Čech cohomology $\check{H}^i(U/M, c^\sigma I)$ vanishes for all $i > 0$. Denote by $U_M^n \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ the $n$-fold fiber product of $U$ over $M$. Then $\check{H}^i(U/M, c^\sigma I)$ is computed as the cohomology of the complex $c^\sigma I(U_M^{\bullet+1}) = I(U_M^{\bullet+1})$. Thus an element of $\check{H}^i(U/M, c^\sigma I)$ is represented by $a \in \ker(I(U_M^{i+1}) \to I(U_M^{i+2}))$. By Yoneda, this yields a morphism $\overline{a}$ appearing in a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U_M^{i+2}) \arrow[r, "d"] \arrow[dr, "0"'] & \mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U_M^{i+1}) \arrow[r] \arrow[d, "a"'] & \mathrm{Coker}(d) \arrow[r, hook, "d'"] \arrow[dl, "\overline{a}"] & \mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U_M^{i}) \arrow[dll, dotted, "\tilde{a}"] \\
 & I. & &
\end{tikzcd}$$

We know $d'$ is injective by (3.6). Since $I$ is injective, there is an extension $\tilde{a}$ of $\overline{a}$. This shows the vanishing of the class of $a$ in $\check{H}^i(U/M, c^\sigma I)$. □

**Proposition 3.4.9.** *Let $\sigma \in \{\text{ét}, \text{Nis}, \text{Zar}\}$.*

(1) *Let $F \in \underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}}$, and let $M \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$. Then there is a canonical isomorphism for any $i \geq 0$:*

$$\mathrm{Ext}^i_{\underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}}}(\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M), F) \simeq H^i_\sigma(\overline{M}, F_M)$$

*where $F_M$ is as in Notation 3.4.7.*

(2) *Assume that $cd(k) < +\infty$ if $\sigma = \text{ét}$. Let $K$ be a chain complex in $C(\underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}})$, and let $M \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$. Then there are canonical isomorphisms*

$$\mathrm{Hom}_{D(\underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}})}(\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M), K[i]) \simeq H^i_\sigma(\overline{M}, K_M), \quad i \in \mathbb{Z}$$

*where $K_M$ is as in Notation 3.4.7.*

(3) *Assume that $cd(k) < +\infty$ if $\sigma = \text{ét}$. Then the functor $D(c^\sigma) : D(\underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}}) \to D(\underline{\mathbf{M}}\mathbf{PS}_\sigma^{\mathrm{fin}})$ obtained from Theorem 3.4.4 (2) has a left adjoint $Lc_\sigma$, such that*

$$Lc_\sigma \mathbb{Z}^{\mathrm{fin}}(M)[0] = \mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M)[0]$$

*for any $M \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$.*

*Proof.* For $M \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$, write $\Gamma_M$ (resp. $\tilde{\Gamma}_M$) for the functor $F \mapsto F(M)$ (resp. $G \mapsto \mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M), G)$) from $\underline{\mathbf{M}}\mathbf{PS}_\sigma^{\mathrm{fin}}$ (resp. $\underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}}$) to abelian groups (these are iterated Yoneda constructions). Since $c_\sigma \mathbb{Z}^{\mathrm{fin}}(M) = \mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M)$, we have a natural isomorphism

$$\tilde{\Gamma}_M \simeq \Gamma_M \circ c^\sigma.$$

Then (1) follows from (3.7), Lemma 3.4.8 and Theorem A.10.1.

Similarly, (2) will follow if we show that the natural transformation (A.4)

$$R\tilde{\Gamma}_M \Rightarrow R\Gamma_M \circ D(c^\sigma)$$

is invertible. For this, we apply Lemma A.13.8. Its first condition is given as above by Lemma 3.4.8, and we are left to show that $R\tilde{\Gamma}_M$, $R\Gamma_M$ and $D(c^\sigma)$ are strongly additive. For $D(c^\sigma)$, this follows from Theorem 3.4.4 (2) and Proposition A.13.9 a). For $R\Gamma_M$ and $R\tilde{\Gamma}_M$, we check that the conditions of Proposition A.13.9 b) are verified; by (1), it suffices to do it for $R\Gamma_M$. Then Condition (ii) follows from Theorem 3.2.2 (use the compact projective generator $\mathbb{Z}$ of **Ab**), and Condition (i) follows similarly from the known commutation of $\sigma$-cohomology with filtering colimits of sheaves.

(3) Since $D(\underline{\mathbf{M}}\mathbf{PS}_\sigma^{\mathrm{fin}})$ is generated by the $\mathbb{Z}^{\mathrm{fin}}(M)[0]$'s, it suffices to show the given formula, which formally follows from [the proof of] (2). □

*Remark* 3.4.10. It is likely that $Lc_\sigma$ is the total left derived functor of $c_\sigma$. We shall not need this in the sequel.

*Remark* 3.4.11. The proof of Proposition 3.4.9 (1) is adapted from [49, 3.1.8], for which there is another proof in [30, Ex. 6.20, 13.3]. One could adapt this other proof here, by constructing transfers on the Godement resolution of $F \in \underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}}$.

## 3.5. From $\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ to $\underline{\mathbf{M}}\mathbf{PST}$.

**Definition 3.5.1.** Let $\sigma \in \{\text{ét}, \mathrm{Nis}, \mathrm{Zar}\}$. We define $\underline{\mathbf{M}}\mathbf{PST}_\sigma$ to be the full subcategory of $\underline{\mathbf{M}}\mathbf{PST}$ consisting of those $F$ such that $b^*F \in \underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}}$. We denote by $\underline{i}_\sigma : \underline{\mathbf{M}}\mathbf{PST}_\sigma \to \underline{\mathbf{M}}\mathbf{PST}$ the inclusion functor.

**Lemma 3.5.2.** *The category $\underline{\mathbf{M}}\mathbf{PST}_\sigma$ is closed under infinite direct sums in $\underline{\mathbf{M}}\mathbf{PST}$, and $\underline{i}_\sigma$ is strongly additive.*

*Proof.* This follows from Lemma 3.4.2, because $b^*$ is strongly additive as a left adjoint. □

**Proposition 3.5.3.** *For $\sigma \in \{\text{ét}, \mathrm{Nis}\}$, the functor $\underline{i}_\sigma$ has an exact left adjoint $\underline{a}_\sigma$. The category $\underline{\mathbf{M}}\mathbf{PST}_\sigma$ is Grothendieck and the functor $\mathbb{Z}_{\mathrm{tr}}$ takes values in $\underline{\mathbf{M}}\mathbf{PST}_\sigma$. The fully faithful functor $b^\sigma : \underline{\mathbf{M}}\mathbf{PST}_\sigma \to \underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}}$ induced by $b^*$ is exact, strongly additive and has an exact left adjoint $b_\sigma = \underline{a}_\sigma b_! \underline{i}_\sigma^{\mathrm{fin}}$.*

*Proof.* We apply Lemma A.9.1 with $\mathcal{D} = \underline{\mathbf{M}}\mathbf{PST}$, $\mathcal{D}' = \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$, $\mathcal{C}' = \underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}}$, $f = b^*$ and $i' = \underline{i}_\sigma^{\mathrm{fin}}$. Let us check that its hypotheses are verified: $b^*$ is fully faithful, has a right adjoint and is exact by

Proposition 2.5.1. The functor $\underline{i}_\sigma^{\mathrm{fin}}$ has an exact left adjoint by Theorem 3.4.4. Finally, $b^*\mathbb{Z}_{\mathrm{tr}}(M) \in \underline{\mathbf{M}}\mathbf{PST}_\sigma^{\mathrm{fin}}$ for any $M \in \underline{\mathbf{M}}\mathbf{Cor}$ by Proposition 3.2.5, so that the $\mathbb{Z}_{\mathrm{tr}}(M)$ form a set of generators of $\underline{\mathbf{M}}\mathbf{PST}$ by strict epimorphisms which belong to $\underline{\mathbf{M}}\mathbf{PST}_\sigma$. The second claim again follows from Theorem A.12.1. The last one follows from Lemma A.9.1 c) and d) (strongly additive follows from Theorem 3.4.4 (2)). □

**Corollary 3.5.4.** *For $\sigma \in \{\text{ét}, \text{Nis}\}$ and for any $p \geq 0$, we have a natural isomorphism*

$$c^*b^*R^p\underline{i}_\sigma = R^p\underline{i}_{s,\sigma}^{\mathrm{fin}}c^\sigma b^\sigma.$$

*Proof.* Since $c^*$ and $b^*$ are exact, we have

$$c^*b^*R^p\underline{i}_\sigma = R^p(c^*b^*\underline{i}_\sigma) = R^p(\underline{i}_{s,\sigma}^{\mathrm{fin}}c^\sigma b^\sigma).$$

By Lemma 3.4.8, $c^\sigma$ sends injectives to $\underline{i}_{s,\sigma}^{\mathrm{fin}}$-acyclics, and by Proposition 3.5.3 $b^\sigma$ preserves injectives. Hence the conclusion. □

Similarly to Corollary 3.4.6, we get:

**Corollary 3.5.5.** *For $\sigma \in \{\text{ét}, \text{Nis}\}$, the functors $c^\sigma$ and $c^\sigma b^\sigma$ are faithfully exact.*

**Definition 3.5.6.** (1) A Cartesian square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "e"] & X \end{tikzcd} \tag{3.8}$$

in **Sch** is called an *elementary Nisnevich square* if $e$ is an open embedding, $p$ is étale and $p^{-1}(X \setminus U)_{\mathrm{red}} \to (X \setminus U)_{\mathrm{red}}$ is an isomorphism. In this situation, we say $U \sqcup V \to X$ is an *elementary Nisnevich cover*.

(2) A diagram (3.8) in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ is called an *elementary Nisnevich square* if it becomes so (in **Sch**) after replacing $X, U, V, W$ by $\overline{X}, \overline{U}, \overline{V}, \overline{W}$, all morphisms are minimal, and it is cartesian. (Note that such fibre products exist in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ thanks to Proposition 1.11.1.)

**Theorem 3.5.7.** *Let $\sigma \in \{\text{ét}, \text{Nis}\}$. If $p : U \to M$ is a $\sigma$-cover in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, then the Čech complex*

$$\cdots \to \mathbb{Z}_{\mathrm{tr}}(U \otimes_M U) \to \mathbb{Z}_{\mathrm{tr}}(U) \to \mathbb{Z}_{\mathrm{tr}}(M) \to 0 \tag{3.9}$$

*is exact in $\underline{\mathbf{M}}\mathbf{PST}_\sigma$. If $\sigma = \text{Nis}$, the sequence*

$$0 \to \mathbb{Z}_{\mathrm{tr}}(W) \to \mathbb{Z}_{\mathrm{tr}}(U) \oplus \mathbb{Z}_{\mathrm{tr}}(V) \to \mathbb{Z}_{\mathrm{tr}}(X) \to 0$$

*is exact in* $\underline{\mathbf{M}}\mathbf{PST}_{\mathrm{Nis}}$ *for any elementary Nisnevich square* (3.8) *in* $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$.

*Proof.* By (3.6), the complex

$$\cdots \to \mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U \otimes_M U) \to \mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U) \to \mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M) \to 0$$

is exact in $\underline{\mathbf{M}}\mathbf{PST}_{\sigma}^{\mathrm{fin}}$. Applying the exact functor $b_\sigma$, we get (3.9) thanks to Proposition 3.5.3. The second statement follows from the first and a small computation (cf. [30, Prop. 6.14]). □

**Corollary 3.5.8.** *Let* $\sigma \in \{\text{ét}, \mathrm{Nis}\}$. *Let* $F \in \underline{\mathbf{M}}\mathbf{PST}$ *be such that* $\underline{a}_\sigma F = 0$. *Then* $F$ *may be written as a quotient of a direct sum of presheaves of the form* $\mathbb{Z}_{\mathrm{tr}}(M/U) := \mathrm{Coker}(\mathbb{Z}_{\mathrm{tr}}(U) \to \mathbb{Z}_{\mathrm{tr}}(M))$, *where* $M \in \underline{\mathbf{M}}\mathbf{Sm}$ *and* $U \to M$ *is a* $\sigma$*-cover. Moreover,* $\underline{a}_\sigma \mathbb{Z}_{\mathrm{tr}}(M/U) = 0$ *for any such* $\sigma$*-cover.*

*Proof.* Let $G = c^* b^* F$. We have $\underline{a}_{s,\sigma}^{\mathrm{fin}} G = c^\sigma b^\sigma \underline{a}_\sigma F = 0$. This implies that $\check{c}_\sigma^{\mathrm{fin}} G = 0$ where $\check{c}_\sigma^{\mathrm{fin}}$ is taken from (3.5), because $\check{c}_\sigma^{\mathrm{fin}} G \to \underline{a}_{s,\sigma}^{\mathrm{fin}} G$ is mono as $\check{c}_\sigma^{\mathrm{fin}} G$ is $\sigma$-separated. Let $M \in \underline{\mathbf{M}}\mathbf{Sm}$ and let $f \in F(M) = G(M)$. We may find a $\sigma$-cover $U \xrightarrow{\varphi} M$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ such that $\varphi^* f = 0$. Thus the Yoneda map $\mathbb{Z}^{\mathrm{fin}}(M) \to G$ given by $f$ factors through $\mathbb{Z}^{\mathrm{fin}}(M/U) := \mathrm{Coker}(\mathbb{Z}^{\mathrm{fin}}(U) \to \mathbb{Z}^{\mathrm{fin}}(M))$. Here $\mathbb{Z}^{\mathrm{fin}} : \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}$ is the representable presheaf functor, so that we have $b_! c_! \mathbb{Z}^{\mathrm{fin}}(N) = \mathbb{Z}_{\mathrm{tr}}(N)$ for any $N \in \underline{\mathbf{M}}\mathbf{Sm}$. By adjunction, the Yoneda map $\mathbb{Z}_{\mathrm{tr}}(M) \to F$ given by $f$ factors through $\mathbb{Z}_{\mathrm{tr}}(M/U)$. Collecting over all pairs $(M, f)$ as usual, we get what we want. Finally, the last statement is a consequence of Theorem 3.5.7. □

## 3.6. Cohomology in $\underline{\mathbf{M}}\mathbf{PST}_\sigma$.

**Notation 3.6.1.** Take $\sigma \in \{\text{ét}, \mathrm{Nis}, \mathrm{Zar}\}$. Let $M \in \underline{\mathbf{M}}\mathbf{Sm}$ and let $K$ be an object or a complex in $\underline{\mathbf{M}}\mathbf{PST}_\sigma$. We write

$$K_M = (b^\sigma K)_M$$

(see Notation 3.4.7).

**Proposition 3.6.2.** *Let* $\sigma \in \{\text{ét}, \mathrm{Nis}\}$, $M \in \underline{\mathbf{M}}\mathbf{Sm}$ *and let* $K$ *be a complex of* $\underline{\mathbf{M}}\mathbf{PST}_\sigma$. *If* $\sigma = \text{ét}$, *assume that* $cd(k) < +\infty$. *Then we have a canonical isomorphism for any* $i \in \mathbb{Z}$

$$\mathrm{Hom}_{D(\underline{\mathbf{M}}\mathbf{PST}_\sigma)}(\mathbb{Z}_{\mathrm{tr}}(M), K[i]) \simeq \mathbb{H}_\sigma^i(\overline{M}, K_M). \tag{3.10}$$

*Proof.* Since $b^\sigma$ has the exact left adjoint $b_\sigma$ (Proposition 3.5.3), this follows formally from Proposition 3.4.9 (2). □

### 3.7. From $\underline{\mathbf{M}}\mathbf{PST}$ to $\mathbf{MPST}$.

**Definition 3.7.1.** Let $\sigma \in \{\text{ét}, \text{Nis}, \text{Zar}\}$. We define $\mathbf{MPST}_\sigma$ to be the full subcategory of $\mathbf{MPST}$ consisting of those $F$ such that $\tau_! F \in \underline{\mathbf{M}}\mathbf{PST}_\sigma$. We denote by $i_\sigma : \mathbf{MPST}_\sigma \to \mathbf{MPST}$ the inclusion functor.

**Lemma 3.7.2.** *The category $\mathbf{MPST}_\sigma$ is closed under infinite direct sums in $\mathbf{MPST}$, and $i_\sigma$ is strongly additive.*

*Proof.* This follows from Lemma 3.5.2, because $\tau_!$ is strongly additive as a left adjoint. □

**Proposition 3.7.3.** *For $\sigma \in \{\text{ét}, \text{Nis}\}$, the functor $i_\sigma$ has an exact left adjoint $a_\sigma$. The category $\mathbf{MPST}_\sigma$ is Grothendieck and the functor $\mathbb{Z}_{\mathrm{tr}}$ takes values in $\mathbf{MPST}_\sigma$. The fully faithful functor $\tau_\sigma : \mathbf{MPST}_\sigma \to \underline{\mathbf{M}}\mathbf{PST}_\sigma$ induced by $\tau_!$ is exact, strongly additive, has a right adjoint $\tau^\sigma$ and has a pro-left adjoint $\tau^{!\sigma} = \mathrm{pro}_\Sigma\text{–}a_\sigma \tau^! \underline{i}_\sigma$. Moreover, there is a natural isomorphism $\underline{a}_\sigma \tau_! \simeq \tau_\sigma a_\sigma$.*

*Proof.* We apply Lemma A.9.1 this time with $\mathcal{D} = \mathbf{MPST}$, $\mathcal{D}' = \underline{\mathbf{M}}\mathbf{PST}$, $\mathcal{C}' = \underline{\mathbf{M}}\mathbf{PST}_\sigma$, $f = \tau_!$ and $i' = \underline{i}_\sigma$. Let us check that its hypotheses are verified: $\tau_!$ is fully faithful, has a right adjoint and is exact by Proposition 2.4.1. The functor $\underline{i}_\sigma$ has an exact left adjoint by Proposition 3.5.3. Finally, $\tau_! \mathbb{Z}_{\mathrm{tr}}(M) = \mathbb{Z}_{\mathrm{tr}}(\tau(M)) \in \underline{\mathbf{M}}\mathbf{PST}_\sigma$ for any $M \in \mathbf{MCor}$ by Proposition 3.2.5, so that the $\mathbb{Z}_{\mathrm{tr}}(M)$ form a set of generators of $\mathbf{MPST}$ by strict epimorphisms which belong to $\mathbf{MPST}_\sigma$. The second and third claim follow as above. The strong additivity of $\tau_\sigma$ follows from that of $\tau_!$, which is a left adjoint, and those of $\underline{i}_\sigma$ (Lemma 3.5.2) and $i_\sigma$ (Lemma 3.7.2); its exactness then implies that it is cocontinuous. The existence of $\tau^\sigma$ now follows from Theorem A.12.1 b). The last claim follows from (A.3) in Lemma A.9.1. □

*Remark* 3.7.4.

(1) We will show later that $\tau^\sigma$ is exact (see Corollary 3.9.6). Note also that $\tau^{!\sigma}$ is exact provided the answer to Question 2.4.4 is yes.

(2) The existence of $\tau^\sigma$ means that $\tau^*(\underline{i}_\sigma \underline{\mathbf{M}}\mathbf{PST}_\sigma) \subseteq i_\sigma \mathbf{MPST}_\sigma$, and it is computed by restricting $\tau^*$ to $\underline{i}_\sigma \underline{\mathbf{M}}\mathbf{PST}_\sigma$, cf. Lemma A.9.1 b). (Checking this compatibility directly seems non-trivial.) This yields an isomorphism $\tau^* \underline{i}_\sigma \simeq i_\sigma \tau^\sigma$ and thus a base change morphism $a_\sigma \tau^* \Rightarrow \tau^\sigma \underline{a}_\sigma$, which will be shown to be an isomorphism in Corollary 3.9.6.

**Corollary 3.7.5.** *For $\sigma \in \{\text{ét}, \text{Nis}\}$, the functor $\tau_\sigma$ is faithfully exact.*

**Proposition 3.7.6.** *Suppose that $\sigma \in \{\text{Nis}, \text{Zar}\}$, and let $I \in \mathbf{MPST}_\sigma$ be an injective object. Then $c^\sigma b^\sigma \tau_\sigma(I) \in \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}_\sigma$ is flasque.*

*Proof.* It is a more sophisticated version of that of Lemma 3.4.8. Let $j : U \hookrightarrow X$ be a minimal open immersion of modulus pairs in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. We must show that

$$\varinjlim_{M \in \mathbf{Comp}(X)} I(M) \to \varinjlim_{N \in \mathbf{Comp}(U)} I(N)$$

is surjective. Let $N \in \mathbf{Comp}(U)$ and $x \in I(N)$. The functoriality of $\tau^!$ defined using $\mathbf{Comp}$ (Lemma 1.8.3) shows that, for any $M \in \mathbf{Comp}(X)$ we may find $N' \in \mathbf{Comp}(U)$ and morphisms

$$\begin{tikzcd} & N \\ N' \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\ & M \end{tikzcd}$$

with $\alpha$ in $\mathbf{Comp}(U)$ and $\beta$ compatible with $j$. Since $j$ is an open immersion, the morphism of sheaves $\mathbb{Z}_{\mathrm{tr}}(\beta)$ is a monomorphism. Hence $\beta^* : I(M) \to I(N')$ is surjective and $\alpha^* x \in \operatorname{Im} \beta^*$. $\square$

*Remark* 3.7.7. We do not know if $c^{\text{ét}} b^{\text{ét}} \tau_{\text{ét}}(I) \in \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}_{\text{ét}}$ is flabby for any injective $I \in \mathbf{MPST}_{\text{ét}}$. An affirmative answer to this question would make all results in the rest of this section valid for $\sigma = \text{ét}$.

**Corollary 3.7.8.** *Suppose $\sigma = \mathrm{Nis}$. Let $I \in \mathbf{MPST}_\sigma$ be an injective object. Then $\tau_\sigma(I)$ is $\underline{i}_\sigma$-acyclic.*

*Proof.* Let $p > 0$. We must show that $R^p \underline{i}_\sigma \tau_\sigma(I) = 0$. Since $b^*$ and $c^*$ are faithful, it suffices to show that $c^* b^* R^p \underline{i}_\sigma \tau_\sigma(I) = 0$. By Corollary 3.5.4 this is also $R^p \underline{i}^{\mathrm{fin}}_{s,\sigma}(c^\sigma b^\sigma \tau_\sigma)(I)$, which vanishes by Prop. 3.7.6. $\square$

**Corollary 3.7.9.** *Suppose $\sigma = \mathrm{Nis}$. We have a natural isomorphism*

$$R^p(\underline{i}_\sigma \tau_\sigma) \simeq (R^p \underline{i}_\sigma)\tau_\sigma \tag{3.11}$$

*for all $p \geq 0$, and an isomorphism*

$$\operatorname{Ext}^p_{\mathbf{MPST}_\sigma}(\mathbb{Z}_{\mathrm{tr}}(M), F) \simeq \operatorname{Ext}^p_{\underline{\mathbf{M}}\mathbf{PST}_\sigma}(\mathbb{Z}_{\mathrm{tr}}(M), \tau_\sigma F) \tag{3.12}$$

*for any $M \in \mathbf{MCor}$ and any $F \in \mathbf{MPST}_\sigma$.*

*Proof.* The isomorphism (3.11) follows from Theorem A.10.1, Corollary 3.7.8 and the exactness of $\tau_\sigma$ (Proposition 3.7.3). To get (3.12), we apply Proposition A.10.3 and the projectivity of $\mathbb{Z}_{\mathrm{tr}}(M)$ in $\mathbf{MPST}$

and $\underline{\mathbf{M}}\mathbf{PST}$ to get isomorphisms

$$\operatorname{Ext}^p_{\mathbf{MPST}_\sigma}(\mathbb{Z}_{\mathrm{tr}}(M), F) \simeq \mathbf{MPST}(\mathbb{Z}_{\mathrm{tr}}(M), R^p i_\sigma F) \tag{3.13}$$

$$\operatorname{Ext}^p_{\underline{\mathbf{M}}\mathbf{PST}_\sigma}(\mathbb{Z}_{\mathrm{tr}}(M), \tau_\sigma F) \simeq \underline{\mathbf{M}}\mathbf{PST}(\mathbb{Z}_{\mathrm{tr}}(M), R^p \underline{i}_\sigma \tau_\sigma F). \tag{3.14}$$

The last term may be rewritten

$$\begin{aligned}\underline{\mathbf{M}}\mathbf{PST}(\mathbb{Z}_{\mathrm{tr}}(M), R^p \underline{i}_\sigma \tau_\sigma F) &\simeq \underline{\mathbf{M}}\mathbf{PST}(\tau_!\mathbb{Z}_{\mathrm{tr}}(M), R^p (\underline{i}_\sigma \tau_\sigma) F)\\ &\simeq \underline{\mathbf{M}}\mathbf{PST}(\tau_!\mathbb{Z}_{\mathrm{tr}}(M), R^p (\tau_! i_\sigma) F) \simeq \underline{\mathbf{M}}\mathbf{PST}(\tau_!\mathbb{Z}_{\mathrm{tr}}(M), \tau_! R^p i_\sigma F)\end{aligned}$$

where we used (3.11) for the first isomorphism and the exactness of $\tau_!$ for the last one (Proposition 2.4.1). Now (3.12) follows from the full faithfulness of $\tau_!$ (ibid.). □

**Lemma 3.7.10.** *Suppose $\sigma = \mathrm{Nis}$. In $\underline{\mathbf{M}}\mathbf{PST}_\sigma$ (resp. $\mathbf{MPST}_\sigma$), infinite direct sums of injectives are $\underline{i}_\sigma$-acyclic (resp. $i_\sigma$-acyclic).*

*Proof.* Let us start with the first case: by the same reasoning as in the proof of Corollary 3.7.8 and the strong additivity of $b^\sigma$ and $c^\sigma$ (Theorem 3.4.4 and Proposition 3.5.3), the claim follows from the fact that in $\underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}_\sigma$, infinite direct sums of flasque sheaves are flasque.

Let now $(I_\alpha)_{\alpha\in A}$ be a family of injectives in $\mathbf{MPST}_\sigma$, and $p > 0$. To show that $R^p i_\sigma(\bigoplus I_\alpha) = 0$, it suffices to show that

$$\begin{aligned}\tau_! R^p i_\sigma(\bigoplus I_\alpha) &= R^p(\tau_! i_\sigma)(\bigoplus I_\alpha)\\ &= R^p(\underline{i}_\sigma \tau_\sigma)(\bigoplus I_\alpha) = (R^p \underline{i}_\sigma)\tau_\sigma(\bigoplus I_\alpha) = (R^p \underline{i}_\sigma)(\bigoplus \tau_\sigma I_\alpha)\end{aligned}$$

is 0. Here the first equality uses the exactness of $\tau_!$ (Proposition 2.4.1), the third one follows from Corollary 3.7.8 and the exactness of $\tau_\sigma$ (Proposition 3.7.3) and the last one follows from the strong additivity of $\tau_\sigma$ (Proposition 3.7.3 again). Reasoning again as in the proof of Corollary 3.7.8, it remains to show that

$$R^p \underline{i}^{\mathrm{fin}}_{s,\sigma} c^\sigma b^\sigma(\bigoplus \tau_\sigma I_\alpha) = R^p \underline{i}^{\mathrm{fin}}_{s,\sigma}(\bigoplus c^\sigma b^\sigma \tau_\sigma I_\alpha)$$

vanishes, where we used the strong additivity of $b^\sigma$ and $c^\sigma$ once again. We conclude as in the first case, using Proposition 3.7.6. □

**Theorem 3.7.11.** *Suppose $\sigma = \mathrm{Nis}$. There is a canonical isomorphism of functors*

$$D(\tau_!) R i_\sigma \simeq R\underline{i}_\sigma D(\tau_\sigma) : D(\mathbf{MPST}_\sigma) \to D(\underline{\mathbf{M}}\mathbf{PST}).$$

(If $F$ is an exact functor between abelian categories, we write $D(F)$ for the functor it induces on the derived categories [trivial derivation].)

*Proof.* We have a tautological isomorphism of functors
$$\tau_! i_\sigma \simeq \underline{i}_\sigma \tau_\sigma : \mathbf{MPST}_\sigma \to \mathbf{\underline{M}PST}$$
hence an isomorphism of total derived functors
$$R(\tau_! i_\sigma) \simeq R(\underline{i}_\sigma \tau_\sigma). \tag{3.15}$$

Since $\tau_!$ is exact, we have by Lemma A.13.5
$$R(\tau_! i_\sigma) \xrightarrow{\sim} R\tau_! \circ Ri_\sigma = D(\tau_!) Ri_\sigma. \tag{3.16}$$

To conclude it suffices to prove that, similarly, the natural transformation (A.4)
$$R(\underline{i}_\sigma \tau_\sigma) \Longrightarrow R\underline{i}_\sigma \circ R\tau_\sigma = R\underline{i}_\sigma D(\tau_\sigma)$$
is invertible. By Corollary 3.7.8 and Lemma A.13.8, it suffices to show that $D(\tau_\sigma)$, $R\underline{i}_\sigma$ and $R(\underline{i}_\sigma \tau_\sigma)$ are strongly additive. Thanks to (3.15) and (3.16), the last one is equivalent to that of $D(\tau_!)Ri_\sigma$. Since $\tau_!$ and $\tau_\sigma$ are strongly additive (see Proposition 3.7.3 for the latter), so are $D(\tau_!)$ and $D(\tau_\sigma)$ and we are left to show that $Ri_\sigma$ and $R\underline{i}_\sigma$ are strongly additive. For this, we shall apply Proposition A.13.9.

We first check that its Condition (i) is verified, namely, that $R^p i_\sigma$ and $R^p \underline{i}_\sigma$ are strongly additive for all $p \geq 0$. This is true for $p = 0$ by Lemmas 3.5.2 and 3.7.2, hence it follows from Lemmas 3.7.10 and A.13.10 for $p > 0$.

We now check Condition (ii) of Proposition A.13.9. In the case of $\underline{i}_\sigma$, we take for $\mathcal{E}$ the $\mathbb{Z}_{\mathrm{tr}}(M)$'s for $M$ running through a set of representatives of the isomorphism classes of $\mathbf{\underline{M}Cor}$: they are compact and projective generators by [1, Prop. 1.3.6 f)]. The vanishing condition now follows from (3.10), (3.14) and Theorem 3.2.2. In the case of $i_\sigma$, we take the same $\mathcal{E}$ with respect to $\mathbf{MCor}$, and get the same vanishing using (3.12) and (3.13). □

**Proposition 3.7.12.** *Suppose* $\sigma = \mathrm{Nis}$. *The categories* $D(\mathbf{MPST}_\sigma)$ *and* $D(\mathbf{\underline{M}PST}_\sigma)$ *are left complete (see Definition A.13.11).*

*Proof.* Indeed, $\mathbf{MPST}_\sigma$ (resp. $\mathbf{\underline{M}PST}_\sigma$) is generated by the $\mathbb{Z}_{\mathrm{tr}}(M)$ for $M \in \mathbf{MCor}$ (resp. $M \in \mathbf{\underline{M}Cor}$) because the same fact holds in the categories of presheaves (apply the exact functors $a_\sigma$ and $\underline{a}_\sigma$ from Propositions 3.5.3 and 3.7.3), and we saw that these objects have finite Ext-dimension. The conclusion then follows from Lemma A.13.12 and Remark A.13.13. □

**Lemma 3.7.13.** *Suppose* $\sigma = \mathrm{Nis}$. *Let* $\cdots \to C_{n+1} \to C_n \to \cdots$ *be a tower of objects of* $D^+(\mathbf{MPST})$ *such that* $n \mapsto H^i(C_n)$ *is stationary for any* $i \in \mathbb{Z}$. *Then the map*
$$D(\tau_!)(\operatorname{holim} C_n) \to \operatorname{holim} D(\tau_!)(C_n)$$

*is an isomorphism. The same holds when replacing* $\tau_! : \mathbf{MPST} \to \underline{\mathbf{M}}\mathbf{PST}$ *by* $\tau_\sigma : \mathbf{MPST}_\sigma \to \underline{\mathbf{M}}\mathbf{PST}_\sigma$.

*Proof.* Let $\mathcal{A} = \mathbf{MPST}$ or $\underline{\mathbf{M}}\mathbf{PST}$. Since $\mathcal{A}$ is a category of presheaves, infinite products are exact and we have a Milnor exact sequence

$$0 \to \varprojlim_n^1 H^{i-1}(C_n) \to H^i(\operatorname{holim}_n C_n) \to \varprojlim_n H^i(C_n) \to 0$$

for any tower in $D(\mathcal{A})$. For $(C_n)$ as in the lemma, the exactness of $\tau_!$ thus reduces us to showing that the maps

$$D(\tau_!) \varprojlim H^i(C_n) \to \varprojlim D(\tau_!) H^i(C_n)$$
$$D(\tau_!) \varprojlim^1 H^i(C_n) \to \varprojlim^1 D(\tau_!) H^i(C_n)$$

are isomorphisms for all $i \in \mathbb{Z}$. But this is obvious by the hypothesis (which implies that both $\varprojlim^1$ vanish).

In the case of $\tau_\sigma$, it suffices by full faithfulness to show the isomorphism after applying $R\underline{i}_\sigma$. Since this functor and $Ri_\sigma$ commute with infinite products as right adjoints, using Theorem 3.7.11 again we reduce to showing that the map

$$D(\tau_!)(\operatorname{holim} C'_n) \to \operatorname{holim} D(\tau_!)(C'_n)$$

is an isomorphism, where $C'_n = Ri_\sigma C_n$. Since $C_n$ is bounded below, the hypercohomology spectral sequence

$$R^p i_\sigma H^q(C_n) \Rightarrow H^{p+q}(C'_n) \tag{3.17}$$

is strongly convergent for all $n$, which shows that the $H^*(C'_n)$ are stationary and we conclude. □

*Remark* 3.7.14. The hypothesis that the $C_n$'s are bounded below is not necessary: in the case of $\tau_!$ this is clear, and in the case of $\tau_\sigma$ the spectral sequence (3.17) is always convergent since $i_\sigma$ is locally of finite cohomological dimension (see end of proof of Theorem 3.7.11).

**Theorem 3.7.15.** *Suppose* $\sigma = \mathrm{Nis}$.
*a)* $\tau_\sigma(\mathbf{MPST}_\sigma)$ *is thick in* $\underline{\mathbf{M}}\mathbf{PST}_\sigma$, *hence*

$$D_{\mathbf{MPST}_\sigma}(\underline{\mathbf{M}}\mathbf{PST}_\sigma) = \{C \in D(\underline{\mathbf{M}}\mathbf{PST}_\sigma) \mid \forall i \in \mathbb{Z}, H^i(C) \in \tau_\sigma(\mathbf{MPST}_\sigma)\}$$

*is a (full) triangulated subcategory of* $D(\underline{\mathbf{M}}\mathbf{PST}_\sigma)$.
*b) The functor* $D(\tau_\sigma) : D(\mathbf{MPST}_\sigma) \to D(\underline{\mathbf{M}}\mathbf{PST}_\sigma)$ *is fully faithful and strongly additive, with essential image* $D_{\mathbf{MPST}_\sigma}(\underline{\mathbf{M}}\mathbf{PST}_\sigma)$.

*Proof.* a) Let $F \in \underline{\mathbf{M}}\mathbf{PST}_\sigma$. If $F \in \tau_\sigma(\mathbf{MPST}_\sigma)$, then $R\underline{i}_\sigma F \in D(\tau_!)D(\mathbf{MPST}_\sigma)$ by Theorem 3.7.11; equivalently, the counit map $D(\tau_!)D(\tau^*)R\underline{i}_\sigma F \to R\underline{i}_\sigma F$ is an isomorphism. The converse is true by applying $H^0$. Let now $0 \to F' \to F \to F'' \to 0$ be a short exact sequence in $\underline{\mathbf{M}}\mathbf{PST}_\sigma$. Consider the associated exact triangle in $D(\underline{\mathbf{M}}\mathbf{PST})$:

$$R\underline{i}_\sigma F' \to R\underline{i}_\sigma F \to R\underline{i}_\sigma F'' \xrightarrow{+1}.$$

Applying the exact functor $\tau_!\tau^*$, we get a commutative diagram of exact triangles:

$$\begin{tikzcd}
D(\tau_!\tau^*)R\underline{i}_\sigma F' \arrow[r] \arrow[d, "f'"'] & D(\tau_!\tau^*)R\underline{i}_\sigma F \arrow[r] \arrow[d, "f"'] & D(\tau_!\tau^*)R\underline{i}_\sigma F'' \arrow[r, "+1"] \arrow[d, "f''"'] & {} \\
R\underline{i}_\sigma F' \arrow[r] & R\underline{i}_\sigma F \arrow[r] & R\underline{i}_\sigma F'' \arrow[r, "+1"] & {}.
\end{tikzcd}$$

Among $f, f', f''$ if two are isomorphisms, so is the third. Whence the conclusion.

b) Theorem 3.7.11 gives a naturally commutative diagram of categories

$$\begin{tikzcd}
D(\mathbf{MPST}) \arrow[r, "D(\tau_!)"] & D(\underline{\mathbf{M}}\mathbf{PST}) \\
D(\mathbf{MPST}_\sigma) \arrow[r, "D(\tau_\sigma)"] \arrow[u, "Ri_\sigma"] & D(\underline{\mathbf{M}}\mathbf{PST}_\sigma) \arrow[u, "R\underline{i}_\sigma"']
\end{tikzcd}$$

where the vertical functors are fully faithful by Lemma A.13.5; so is $D(\tau_!)$, which has the (exact) right adjoint/left inverse $D(\tau^*)$. The first claim follows; the strong additivity of $\tau_\sigma$ (Proposition 3.7.3) implies that of $D(\tau_\sigma)$ via Proposition A.13.9 a).

For the essential image, the inclusion $\subseteq$ is obvious. Conversely, we have the inclusion $D^b_{\mathbf{MPST}_\sigma}(\underline{\mathbf{M}}\mathbf{PST}_\sigma) \subseteq D(\tau_\sigma)D(\mathbf{MPST}_\sigma)$ by induction on the length of a complex and the full faithfulness of $D(\tau_\sigma)$, and we extend it to $D^+_{\mathbf{MPST}_\sigma}(\underline{\mathbf{M}}\mathbf{PST}_\sigma)$ by writing a complex $C \in D^+_{\mathbf{MPST}_\sigma}(\underline{\mathbf{M}}\mathbf{PST}_\sigma)$ as $\operatorname{hocolim}_n \tau_{\leq n} C$.

Let now $C \in D_{\mathbf{MPST}_\sigma}(\underline{\mathbf{M}}\mathbf{PST}_\sigma)$. Then $\tau_{\geq -n}C \simeq D(\tau_\sigma)(C_n)$ for some $C_n \in D^{\geq -n}(\mathbf{MPST}_\sigma)$, for all $n \in \mathbb{Z}$, and the $C_n$'s form a tower by full faithfulness. By Proposition 3.7.12, we get an isomorphism $C \simeq \operatorname{holim} D(\tau_\sigma)(C_n)$. Since $D(\tau_\sigma)$ is $t$-exact and fully faithful, the $H^*(C_n)$ are stationary and the map

$$D(\tau_\sigma)(\operatorname{holim} C_n) \to \operatorname{holim} D(\tau_\sigma)(C_n)$$

is an isomorphism by Lemma 3.7.13. ☐

We finally arrive at:

**Theorem 3.7.16.** *Suppose* $\sigma = \mathrm{Nis}$. *Let* $M \in \mathbf{MSm}$ *and* $K \in D(\mathbf{MPST}_\sigma)$. *Then we have a canonical isomorphism for any* $i \in \mathbb{Z}$

$$\mathrm{Hom}_{D(\mathbf{MPST}_\sigma)}(\mathbb{Z}_{\mathrm{tr}}(M), K[i]) \simeq \mathbb{H}^i_\sigma(\overline{M}, \tau_\sigma K_M).$$

*Proof.* By Proposition 3.6.2, it suffices to prove an isomorphism

$$\mathrm{Hom}_{D(\mathbf{MPST}_\sigma)}(\mathbb{Z}_{\mathrm{tr}}(M), K[i]) \xrightarrow{\sim} \mathrm{Hom}_{D(\underline{\mathbf{M}}\mathbf{PST}_\sigma)}(\mathbb{Z}_{\mathrm{tr}}(M), \tau_\sigma K[i])$$

which follows from Theorem 3.7.15 b). □

*Remark* 3.7.17. This proof is easier than that of (3.12) in Corollary 3.7.9. But Corollary 3.7.9 was used in the proof of Theorem 3.7.15.

3.8. **From MPST and $\underline{\mathbf{M}}$PST to PST.** Let $\mathbf{PST}_\sigma$ be the category of abelian $\sigma$-sheaves with transfers. The inclusion $i^V_\sigma : \mathbf{PST}_\sigma \hookrightarrow \mathbf{PST}$ has a left adjoint $a^V_\sigma$ (i.e. sheafification) by [49, Thm. 3.1.4].

**Proposition 3.8.1.** *a) Let* $\sigma \in \{\text{ét}, \mathrm{Nis}, \mathrm{Zar}\}$. *Then, for* $F \in \mathbf{PST}$, $\omega^* F \in \mathbf{MPST}_\sigma \iff \underline{\omega}^* F \in \underline{\mathbf{M}}\mathbf{PST}_\sigma \iff F \in \mathbf{PST}_\sigma$.
*b) Let* $\sigma \in \{\text{ét}, \mathrm{Nis}\}$. *The functor* $\omega^\sigma : \mathbf{PST}_\sigma \to \mathbf{MPST}_\sigma$ *from a) is exact, has an exact left adjoint* $\omega_\sigma$, *given by* $a^V_\sigma \omega_! i_\sigma$. *Both functors commute with* $i$ *and* $a$. *Similarly for* $\underline{\omega}^\sigma$ *and* $\underline{\omega}_\sigma$.

*Proof.* The first equivalence of a) follows from Lemma 2.4.2 (3), and the second is trivial. Therefore we may apply Lemma A.9.1 to the cartesian squares involving $\omega^*, i_\sigma, i^V_\sigma$ and $\underline{\omega}^*, \underline{i}_\sigma, i^V_\sigma$. This yields

(i) $\underline{a}_\sigma \underline{\omega}^* \simeq \underline{\omega}^\sigma a^V_\sigma$, $a_\sigma \omega^* \simeq \omega^\sigma a^V_\sigma$ by a) (use the existence of $\underline{\omega}_*$ and $\omega_*$).
(ii) $\underline{\omega}^\sigma$ and $\omega^\sigma$ are exact by c).
(iii) $\underline{\omega}_\sigma = a^V_\sigma \underline{\omega}_! \underline{i}_\sigma$, $\omega_\sigma = a^V_\sigma \omega_! i_\sigma$ and the exactness of these functors by d) and the exactness of $\underline{\omega}_!$ and $\omega_!$.
(iv) $\underline{\omega}_\sigma \underline{a}_\sigma \simeq a^V_\sigma \underline{\omega}_!$ and $\omega_\sigma a_\sigma \simeq a^V_\sigma \omega_!$, by adjunction from $\underline{i}_\sigma \underline{\omega}^\sigma \simeq \underline{\omega}^* i^V_\sigma$ and $i_\sigma \omega^\sigma \simeq \omega^* i^V_\sigma$.

It remains to prove the commutation of $\omega_\sigma$ and $\underline{\omega}_\sigma$ with $i$. Since $\tau_!$ preserves sheaves, we only have to show that the base change morphism $\underline{\omega}_! \underline{i}_\sigma \Rightarrow i^V_\sigma \underline{\omega}_\sigma$ induced by the isomorphism (iv) is an isomorphism. Using (iv) again, it suffices to show that the induced map $\underline{\omega}_! \underline{i}_\sigma \underline{a}_\sigma(F) \to i^V_\sigma a^V_\sigma \underline{\omega}_!(F)$ is an isomorphism in **PST** for any $F \in \underline{\mathbf{M}}\mathbf{PST}$.

We use $\underline{\omega}_! = \lambda^*$ (see Prop. 2.3.1). Let **PS** be the category of abelian presheaves on **Sm**, and let $\chi^* : \mathbf{PST} \to \mathbf{PS}$ be the functor induced by $\chi : \mathbf{Sm} \to \mathbf{Cor}$. Since $\chi^*$ is conservative, it suffices to show that for all $F \in \underline{\mathbf{M}}\mathbf{PST}$

$$\chi^* \lambda^* \underline{i}_\sigma \underline{a}_\sigma(F) \xrightarrow{\sim} \chi^* i^V_\sigma a^V_\sigma \lambda^*(F). \tag{3.18}$$

Now consider the commutative diagram

$$\begin{tikzcd}
\underline{\mathbf{M}}\mathbf{Cor} & \mathbf{Cor} \arrow[l, "\lambda"'] \\
\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}} \arrow[u, "bc"] & \mathbf{Sm} \arrow[l, "\lambda_s"] \arrow[u, "\chi"'] .
\end{tikzcd}$$

where $\lambda_s(X) = (X, \emptyset)$ (and $\lambda$ is from (1.2)). This induces the rightmost square in

$$\begin{tikzcd}
\underline{\mathbf{M}}\mathbf{PST} \arrow[r, "\underline{a}_\sigma"] \arrow[d, "c^*b^*"'] & \underline{\mathbf{M}}\mathbf{PST}_\sigma \arrow[r, "\underline{i}_\sigma"] \arrow[d, "c^\sigma b^\sigma"'] & \underline{\mathbf{M}}\mathbf{PST} \arrow[r, "\lambda^*"] \arrow[d, "c^*b^*"'] & \mathbf{PST} \arrow[d, "\chi^*"'] \\
\underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}} \arrow[r, "\underline{a}^{\mathrm{fin}}_{s,\sigma}"'] & \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}_\sigma \arrow[r, "\underline{i}^{\mathrm{fin}}_{s,\sigma}"'] & \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}} \arrow[r, "\lambda_s^*"'] & \mathbf{PS}
\end{tikzcd}$$

which is also the leftmost square in

$$\begin{tikzcd}
\underline{\mathbf{M}}\mathbf{PST} \arrow[r, "\lambda^*"] \arrow[d, "c^*b^*"'] & \mathbf{PST} \arrow[r, "a^V_\sigma"] \arrow[d, "\chi^*"'] & \mathbf{PST}_\sigma \arrow[r, "i^V_\sigma"] \arrow[d] & \mathbf{PST} \arrow[d, "\chi^*"'] \\
\underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}} \arrow[r, "\lambda_s^*"'] & \mathbf{PS} \arrow[r, "a^V_{s,\sigma}"'] & \mathbf{PS}_\sigma \arrow[r, "i^V_{s,\sigma}"'] & \mathbf{PS},
\end{tikzcd}$$

where $\mathbf{PS}_\sigma$ is the category of abelian $\sigma$-sheaves on $\mathbf{Sm}$, and $a^V_{s,\sigma}$ : $\mathbf{PS}_\sigma \to \mathbf{PS}$ is the left adjoint of the inclusion $i^V_{s,\sigma} : \mathbf{PS}_\sigma \to \mathbf{PS}$ (sheafification). By the commutativity of the two diagrams, we rewrite (3.18) as

$$\lambda_s^* \underline{i}^{\mathrm{fin}}_\sigma \underline{a}^{\mathrm{fin}}_\sigma G \to i^V_{s,\sigma} a^V_{s,\sigma} \lambda_s^* G, \quad G = c^*b^*(F) \in \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}} .$$

We now use the functor $\check{c}^{\mathrm{fin}}_\sigma : \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}$ from (3.5). We also use a similar functor $\check{c}_\sigma : \mathbf{PS} \to \mathbf{PS}$ given by

$$\check{c}_\sigma(G)(X) = \varinjlim_{V \to X} \check{H}^0(V, G) \quad (G \in \mathbf{PS},\ X \in \mathbf{Sm}),$$

where $V \to X$ ranges over all $\sigma$-covers, which satisfies $i^V_{s,\sigma} a^V_{s,\sigma} = \check{c}_\sigma \check{c}_\sigma$. We are reduced to showing for each $F \in \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}$, $X \in \mathbf{Sm}$

$$\lambda_s^* \check{c}^{\mathrm{fin}}_\sigma(F)(X) \simeq \check{c}_\sigma \lambda_s^*(F)(X)$$

The left and right hand sides can be rewriten as

$$\varinjlim_{U \to \lambda(X)} \check{H}^0(U, F), \qquad \varinjlim_{V \to X} \check{H}^0(V, \lambda_s^* F).$$

Note that $\sigma$-covers of $\lambda(X)$ are in one-to-one correspondence with $\sigma$-covers of $X$ under $((V, \emptyset) \to \lambda(X)) \leftrightarrow (V \to X)$. Moreover, for any

$\sigma$-cover $V \to X$, the two Čech complexes $\check{C}((V,\emptyset),F)$ and $\check{C}(V,\lambda_s^*F)$ are canonically isomorphic by definition. The proposition is proved. □

We say $F \in \mathbf{PST}_\sigma$ is flasque if so is its restriction $F_X$ to $X_\sigma$ for any $X \in \mathbf{Sm}$ (compare Lemma 3.1.4).

**Proposition 3.8.2.** *Suppose $\sigma = \mathrm{Nis}$. If $I \in \mathbf{MPST}_\sigma$ be an injective object, then $\omega_\sigma I \in \mathbf{PST}_\sigma$ is flasque.*

*Proof.* Since flasqueness is a presheaf condition and $i_\sigma F$ is injective, it suffices to show that $\omega_! F$ is flasque if $F \in \mathbf{MPST}$ is injective. This is similar to the proofs of Lemma 3.4.8 and Proposition 3.7.6: let $j : U \hookrightarrow X$ be an open immersion in $\mathbf{Sm}$. Then $\omega^!\mathbb{Z}_{\mathrm{tr}}^V(j) : \omega^!\mathbb{Z}_{\mathrm{tr}}^V(U) \to \omega^!\mathbb{Z}_{\mathrm{tr}}^V(X)$ is a monomorphism. Indeed, if $M \in \mathbf{MSm}(U)$, $N \in \mathbf{MSm}(X)$ and $\tilde{j} : M \to N$ is a morphism above $j$, then the map

$$\mathbf{MCor}(P,M) \xrightarrow{\tilde{j}_*} \mathbf{MCor}(P,N)$$

is injective for any $P \in \mathbf{MCor}$, because its domain and range are respectively subgroups of $\mathbf{Cor}(P^o,U)$ and $\mathbf{Cor}(P^o,X)$. □

**Proposition 3.8.3.** *Suppose $\sigma = \mathrm{Nis}$. The diagram*

$$\begin{array}{ccc} D(\mathbf{MPST}_\sigma) & \xrightarrow{Ri_\sigma} & D(\mathbf{MPST}) \\ {\scriptstyle D(\omega_\sigma)}\downarrow & & \downarrow{\scriptstyle D(\omega_!)} \\ D(\mathbf{PST}_\sigma) & \xrightarrow{Ri_\sigma^V} & D(\mathbf{PST}) \end{array}$$

*is naturally commutative.*

*Proof.* As in the proof of Theorem 3.7.11, we start from the natural isomorphism $i_\sigma^V\omega_\sigma \simeq \omega_! i_\sigma$ from Proposition 3.8.1 b). We need to show that the natural transformations

$$R(\omega_! i_\sigma) \Rightarrow D(\omega_!)Ri_\sigma, \qquad R(i_\sigma^V\omega_\sigma) \Rightarrow Ri_\sigma^V D(\omega_\sigma)$$

are invertible. This is clear for the first one by Lemma A.13.5. For the second one, we use Lemma A.13.8: we need to show that

(i) $\omega_\sigma$ carries injectives to $i_\sigma^V$-acyclics;
(ii) the 3 functors $R(i_\sigma^V\omega_\sigma)$, $Ri_\sigma^V$ and $D(\omega_\sigma)$ are strongly additive.

(i) is shown in Proposition 3.8.2. The proof of (ii) is completely parallel to the relevant part of that of Theorem 3.7.11, and we skip it. □

**Corollary 3.8.4.** *Suppose $\sigma = \mathrm{Nis}$. For any $C \in D(\mathbf{MPST}_\sigma)$ and any $X \in \mathbf{Sm}$, we have a canonical isomorphism*

$$D(\mathbf{PST}_\sigma)(\mathbb{Z}_{\mathrm{tr}}^V(X), D(\omega_\sigma)(C)) \simeq \varinjlim_{M \in \mathbf{MSm}(X)} D(\mathbf{MPST}_\sigma)(\mathbb{Z}_{\mathrm{tr}}(M), C).$$

*In particular, for any $F \in \mathbf{MPST}_\sigma$, any $X \in \mathbf{Sm}$ and any $q \geq 0$ we have a canonical isomorphism*

$$H^q_\sigma(X, \omega_\sigma F) \simeq \varinjlim_{M \in \mathbf{MSm}(X)} H^q_\sigma(\overline{M}, F_M)$$

*with $M = (\overline{M}, M^\infty)$.*

*Proof.* For the first statement, we have

$$D(\mathbf{PST}_\sigma)(\mathbb{Z}^V_{\mathrm{tr}}(X), D(\omega_\sigma)(C)) \simeq D(\mathbf{PST})(\mathbb{Z}^V_{\mathrm{tr}}(X), Ri^V_\sigma D(\omega_\sigma)(C))$$

because $\mathbb{Z}^V_{\mathrm{tr}}(X)$ is a sheaf, and a similar isomorphism for the groups $D(\mathbf{MPST}_\sigma)(\mathbb{Z}_{\mathrm{tr}}(M), C)$. Now apply Proposition 3.8.3. For the second one, take $C = F[q]$ and use [49, Prop. 3.1.8], [30, Ex. 6.25] and Theorem 3.7.16. $\square$

## 3.9. **A refinement of Theorem 3.3.1.**

**Definition 3.9.1.** Let $M, N \in \underline{\mathbf{M}}\mathbf{Cor}$.

(1) We put

$$\mathbb{Z}_{\mathrm{tr}}(M)^\tau = \tau_! \tau^* \mathbb{Z}_{\mathrm{tr}}(M) \in \underline{\mathbf{M}}\mathbf{PST}$$

and $\mathbb{Z}^{\mathrm{fin}}_{\mathrm{tr}}(M)^\tau = b^* \mathbb{Z}_{\mathrm{tr}}(M)^\tau \in \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$. Note $\mathbb{Z}_{\mathrm{tr}}(M)^\tau \in \underline{\mathbf{M}}\mathbf{PST}_{\text{ét}}$ by Remark 3.7.4.

(2) Let $\underline{\mathbf{M}}\mathbf{Cor}^\tau(N, M)$ (resp. $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin},\tau}(N, M)$) be the subgroup of $\underline{\mathbf{M}}\mathbf{Cor}(N, M)$ (resp. $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(N, M)$) generated by elementary correspondences $Z$ in $\mathbf{Cor}(N^\circ, M^\circ)$ which lie in $\underline{\mathbf{M}}\mathbf{Cor}(N, M)$ (resp. $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(N, M)$) and satisfy the condition:

($\spadesuit$) There exists a dense open immersion $j : \overline{N} \hookrightarrow \overline{L}$ with $\overline{L}$ proper such that the closure $\overline{Z}$ of $Z$ in $\overline{L} \times \overline{M}$ is proper over $\overline{L}$.

**Lemma 3.9.2.** *For $N, M$ as above,*

$$\mathbb{Z}_{\mathrm{tr}}(M)^\tau(N) = \underline{\mathbf{M}}\mathbf{Cor}^\tau(N, M), \quad \mathbb{Z}^{\mathrm{fin}}_{\mathrm{tr}}(M)^\tau(N) = \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin},\tau}(N, M).$$

*Proof.* The second equality follows immediately from the first so we prove the first. We first note that ($\spadesuit$) is independent of the choice of $j : \overline{N} \hookrightarrow \overline{L}$. Indeed, if $j' : \overline{N} \hookrightarrow \overline{L}'$ is another choice equipped with proper surjective $f : \overline{L} \to \overline{L}'$ such that $j' = fj$, writing $\overline{Z}' \subset \overline{L}' \times \overline{M}$ for the closure of $Z$, $f$ induces a proper surjective map $\overline{Z}' \to \overline{Z}$. Then it is easy to see that $\overline{Z}'$ is proper over $\overline{L}'$ if and only if so is $\overline{Z}$ over $\overline{L}$. Now the first equality follows by the same argument as the proof of Lemma 1.8.3. $\square$

**Theorem 3.9.3.** *Let $\sigma \in \{\text{ét}, \mathrm{Nis}\}$. If $p : U \to M$ is a $\sigma$-cover in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, then the Čech complex*

$$\cdots \to \mathbb{Z}_{\mathrm{tr}}(U \otimes_M U)^\tau \to \mathbb{Z}_{\mathrm{tr}}(U)^\tau \to \mathbb{Z}_{\mathrm{tr}}(M)^\tau \to 0 \tag{3.19}$$

*is exact in $\underline{\mathbf{M}}\mathbf{PST}_\sigma$.*

Below we give only a proof for $\sigma = \mathrm{Nis}$, as the same proof works for $\sigma = \text{ét}$. We need a preliminary for the proof. Take $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$ and a point $x \in X$. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be the filtered system of connected affine étale neighborhoods of $x \in X$ and $\overline{S} = \varprojlim_{\lambda\in\Lambda} X_\lambda$ be the henselization of $X$ at $x$. Take $M \in \underline{\mathbf{M}}\mathbf{Cor}$ and let $\mathcal{D}$ be the category of diagrams

$$\overline{S} \xleftarrow{f} Z \xrightarrow{g} \overline{M} \tag{3.20}$$

of $k$-schemes with $f$ quasi-finite and such that $g(V) \not\subset M^\infty$ for any irreducible component $V$ of $Z$. We denote (3.20) by $(Z, f, g)$. A morphism from $(Z, f, g)$ to $(Z', f', g')$ is given by a quasi-finite morphism $\varphi : Z \to Z'$ which fits into a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\varphi"] & \\ \overline{S} & & \overline{M} \\ & Z'. \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd} \tag{3.21}$$

For $(Z, f, g) \in \mathcal{D}$ let $E(Z) = E(Z, f, g)$ be the set of irreducible components $V$ of $Z$ such that $f|_V$ is finite and surjective over an irreducible component of $\overline{S}$ and satisfies the admissibility condition:

$$(fi_V v)^*(D \times_X \overline{S}) \geq (gi_V v)^*(M^\infty), \tag{3.22}$$

where $v : V^N \to V$ is the normalization and $i_V : V \hookrightarrow Z$ is the inclusion. Let $E^\tau(Z) \subset E(Z)$ be the subset of those $V$ satsifying the following condition: There exists $\lambda \in \Lambda$ such that $(Z, f, g)$ (resp. $V \hookrightarrow Z$) is the base change via $\overline{S} \to X_\lambda$ of

$$X_\lambda \xleftarrow{f_\lambda} Z_\lambda \xrightarrow{g_\lambda} \overline{M} \quad (\text{resp. } V_\lambda \hookrightarrow Z_\lambda), \tag{3.23}$$

where $V_\lambda$ is an irreducible component of $Z_\lambda$ satisfying the condition:

$(\clubsuit)_\lambda$ $V_\lambda$ is finite over $X_\lambda$ and satisfies the admissibility condition

$$(f_\lambda i_{V_\lambda} v_\lambda)^*(D \times_X X_\lambda) \geq (g_\lambda i_{V_\lambda} v_\lambda)^*(M^\infty), \tag{3.24}$$

similar to (3.22). Moreover, letting $\tilde{V_\lambda} = h_\lambda(V_\lambda)$ with $h_\lambda = (f_\lambda, g_\lambda) : Z_\lambda \to X_\lambda \times \overline{M}$ ($\tilde{V_\lambda}$ is finite over $X_\lambda$ by the finiteness of $V_\lambda \to X_\lambda$), there exists a dense open immersion $X_\lambda \hookrightarrow \overline{X_\lambda}$ with

$\overline{X_\lambda}$ proper such that the closure $\overline{\tilde{V_\lambda}}$ of $\tilde{V_\lambda}$ in $\overline{X_\lambda} \times \overline{M}$ is proper over $\overline{X_\lambda}$.

Let $L^\tau(Z)$ be the free abelian group on the set $E^\tau(Z)$.

**Lemma 3.9.4.** *Let $V_\lambda$ be as in $(\clubsuit)_\lambda$ and $X_\mu \to X_\lambda$ $(\lambda, \mu \in \Lambda)$ be a map in the system of étale neighborhoods of $x \in X$. Let*

$$X_\mu \xleftarrow{f_\mu} Z_\mu \xrightarrow{g_\mu} \overline{M} \qquad (\textit{resp. } V_\mu \hookrightarrow Z_\mu) \tag{3.25}$$

*be the base change of* (3.23) *(resp. $V_\lambda \hookrightarrow Z_\lambda$). If $V_\lambda \subset Z_\lambda$ satisfies $(\clubsuit)_\lambda$, then any component of $V_\mu$ satisfies $(\clubsuit)_\mu$.*

*Proof.* The finiteness over $X_\mu$ and the admissibility condition of $(\clubsuit)_\mu$ are clear. To check the last condition of $(\clubsuit)_\mu$, let $X_\mu \hookrightarrow \overline{X_\mu}$ be the normalization in $X_\mu$ of $\overline{X_\lambda}$ from $(\clubsuit)_\lambda$ and let $\tilde{V_\mu} = h_\mu(V_\mu)$ with $h_\mu = (f_\mu, g_\mu) : Z_\mu \to X_\mu \times \overline{M}$ ($\tilde{V_\mu}$ is finite over $X_\mu$ by the finiteness of $V_\mu \to X_\mu$). Then $\tilde{V_\mu} \subset \tilde{V_\lambda} \times_{X_\lambda} X_\mu$ so that the closure $\overline{\tilde{V_\mu}}$ of $\tilde{V_\mu}$ in $\overline{X_\mu} \times \overline{M}$ is contained in $\overline{\tilde{V_\lambda}} \times_{\overline{X_\lambda}} \overline{X_\mu}$, which is proper over $\overline{X_\mu}$ by the assumption. Hence $\overline{\tilde{V_\mu}}$ is also proper over $\overline{X_\mu}$, which implies the desired condition. □

**Lemma 3.9.5.** *For a commutative diagram* (3.21)*, there is a natural induced map*

$$\varphi_* : E^\tau(Z) \to E^\tau(Z')$$

*which makes $E^\tau$ a covariant functor on $\mathcal{D}$.*

*Proof.* Take $V \in E(Z)$ and let $V' = \varphi(V)$. By the finiteness of $V \to \overline{S}$, $V'$ finite over $\overline{S}$ and closed in $Z'$. The admissibility condition (3.22) for $V$ implies that for $V'$ by Lemma 1.2.1. Hence $V' \in E(Z')$. To show $V' \in E^\tau(Z')$, take $\lambda \in \Lambda$ and $V_\lambda$ as in $(\clubsuit)_\lambda$. By Lemma 3.9.4 we may assume that the diagram (3.21) is the base chage via $\overline{S} \to X_\lambda$ of

$$\begin{tikzcd}
 & Z_\lambda \arrow[dl, "f_\lambda"'] \arrow[dr, "g_\lambda"] \arrow[dd, "\varphi_\lambda"] & \\
X_\lambda & & \overline{M} \\
 & Z'_\lambda \arrow[ul, "f'_\lambda"] \arrow[ur, "g'_\lambda"'] &
\end{tikzcd}$$

and $V' = V'_\lambda \times_{X_\lambda} \overline{S}$ with $V'_\lambda = \varphi_\lambda(V_\lambda)$. By the finiteness of $V_\lambda \to X_\lambda$, $V'_\lambda$ is finite over $X_\lambda$ and closed in $Z'_\lambda$ so that it is an irreducible component of $Z'_\lambda$. The admissibility condition (3.24) for $V_\lambda$ implies that for $V'_\lambda$ by

Lemma 1.2.1. Letting $h'_\lambda = (f'_\lambda, g'_\lambda) : Z'_\lambda \to X_\lambda \times \overline{M}$, we have $h_\lambda = h'_\lambda \varphi_\lambda$ so that $h'_\lambda(V'_\lambda) = h_\lambda(V_\lambda)$. Hence $V'_\lambda$ satisfies the last condition of $(\clubsuit)_\lambda$ since $V_\lambda$ does. This implies $V' \in E^\tau(Z')$. □

*Proof of Theorem 3.9.3.* It is adapted from that of Theorem 3.3.1. By Lemma 3.9.2 and Corollary 3.5.5, it suffices to show the exactness of

$$(3.26)\quad \cdots \to \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin},\tau}(S, U \otimes_M U) \to \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin},\tau}(S, U) \\ \to \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin},\tau}(S, M) \to 0$$

where $S = (\overline{S}, D \times_X \overline{S})$ with $(X, D)$ and $\overline{S}$ as above. We first note that for a closed subscheme $Z \subset \overline{S} \times \overline{U} \times_{\overline{M}} \cdots \times_{\overline{M}} \overline{U}$ finite and surjective over an irreducible component of $\overline{S}$, the image of $Z$ in $\overline{S} \times \overline{M}$ is finite over $\overline{S}$. From this fact we see that (3.26) is obtained as the inductive limit of

$$(3.27)\quad \cdots \to L^\tau(Z \times_{\overline{M}} (\overline{U} \times_{\overline{M}} \overline{U})) \to L^\tau(Z \times_{\overline{M}} \overline{U}) \to L^\tau(Z) \to 0$$

where $Z$ ranges over all closed subschemes of $\overline{S} \times \overline{M}$ that is finite surjective over an irreducible component of $\overline{S}$. It suffices to show the exactness of (3.27).

Since $Z$ is finite over a henselian local scheme $\overline{S}$, $Z$ is a disjoint union of henselian local schemes. Thus the Nisnevich cover $Z \times_{\overline{M}} \overline{U} \to Z$ admits a section $s_0 : Z \to Z \times_{\overline{M}} \overline{U}$. Define for $k \geq 0$

$$s_k := s_0 \times_{\overline{M}} \mathrm{Id}_{\overline{U}^k} : Z \times_{\overline{M}} \overline{U}^k \to Z \times_{\overline{M}} \overline{U} \times_{\overline{M}} \overline{U}^k = Z \times_{\overline{M}} \overline{U}^{k+1},$$

where $\overline{U}^k$ is the $k$-fold fiber product of $\overline{U}$ over $\overline{M}$. Then the maps

$$(s_k)_* : L^\tau(Z \times_{\overline{M}} \overline{U}^k) \to L^\tau(Z \times_{\overline{M}} \overline{U}^{k+1})$$

give us a homotopy from the identity to zero. □

**Corollary 3.9.6.** *Let $\sigma \in \{\text{ét}, \mathrm{Nis}\}$.*

1. *Let $G \in \underline{\mathbf{M}}\mathbf{PST}$. If $\underline{a}_\sigma G = 0$, then $\underline{a}_\sigma \tau_! \tau^* G = 0$.*
2. *The base change morphism $a_\sigma \tau^* \Rightarrow \tau^\sigma \underline{a}_\sigma$ from Remark 3.7.4 is an isomorphism.*
3. *The functor $\tau^\sigma$ of Proposition 3.7.3 is exact.*

*Proof.* (1) Since $\underline{a}_\sigma, \tau_!$ and $\tau^*$ are all cocontinuous as left adjoints, we reduce by Corollary 3.5.8 to $G$ of the form $\mathbb{Z}_{\mathrm{tr}}(M/U)$. Then the claim follows from Theorem 3.9.3.

(2) Let $F \in \underline{\mathbf{M}}\mathbf{PST}_\sigma$. The base change morphism $a_\sigma \tau^* F \to \tau^\sigma \underline{a}_\sigma F$ is defined as the composition

$$a_\sigma \tau^* F \xrightarrow{a_\sigma \tau^*(\eta_F)} a_\sigma \tau^* \underline{i}_\sigma \underline{a}_\sigma F \simeq a_\sigma i_\sigma \tau^\sigma \underline{a}_\sigma F \xrightarrow{\varepsilon_{\tau^\sigma \underline{a}_\sigma F}} \tau^\sigma \underline{a}_\sigma F$$

where $\eta$ (resp. $\varepsilon$) is the unit (resp. counit) of the adjunction $(\underline{a}_\sigma, \underline{i}_\sigma)$ (resp. $(a_\sigma, i_\sigma)$). Since the latter is an isomorphism, it remains to show that the former is an isomorphism. By the full faithfulness of $\tau_\sigma$, it suffices to show it after applying this functor. But $\underline{a}_\sigma\tau_! \simeq \tau_\sigma a_\sigma$ by Proposition 3.7.3, so we are left to showing that the map

$$\underline{a}_\sigma\tau_!\tau^*F \xrightarrow{\underline{a}_\sigma\tau_!\tau^*(\eta_F)} \underline{a}_\sigma\tau_!\tau^*\underline{i}_\sigma\underline{a}_\sigma F$$

is an isomorphism. This follows from (1), since $\underline{a}_\sigma\tau_!\tau^*$ is exact and $\operatorname{Ker}\eta_F, \operatorname{Coker}\eta_F$ are killed by $\underline{a}_\sigma$.

(3) By (2), $\tau^\sigma\underline{a}_\sigma$ is exact. Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\underline{\mathbf{M}}\mathbf{PST}_\sigma$. Applying $\underline{i}_\sigma$, we get an exact sequence

$$0 \to \underline{i}_\sigma F' \to \underline{i}_\sigma F \to \underline{i}_\sigma F'' \to G \to 0$$

with $\underline{a}_\sigma G = 0$. Applying now $\tau^\sigma\underline{a}_\sigma$ and using the isomorphism $\underline{a}_\sigma\underline{i}_\sigma \overset{\sim}{\Rightarrow} \mathrm{Id}$, we get an exact sequence

$$0 \to \tau^\sigma F' \to \tau^\sigma F \to \tau^\sigma F'' \to 0$$

as desired. □

3.10. **Tensor structures.** We now show that the closed tensor structures on presheaves from Proposition 2.1.2 carry over to sheaves.

**Proposition 3.10.1.** *Let $\sigma \in \{\text{ét}, \text{Nis}\}$. The sheafification functors $a_\sigma$ and $\underline{a}_\sigma$ induce closed monoidal structures on $\mathbf{MPST}_\sigma$ and $\underline{\mathbf{M}}\mathbf{PST}_\sigma$ such that $\tau_\sigma$, $\omega_\sigma$ and $\underline{\omega}_\sigma$ are monoidal.*

*Proof.* By Lemma A.5.7, to get the monoidal structures it suffices to show that $\operatorname{Ker} a_\sigma$ and $\operatorname{Ker}\underline{a}_\sigma$ are $\otimes$-ideals, which follows immediately from Corollary 3.5.8. The monoidality of $\tau_\sigma$, $\omega_\sigma$ and $\underline{\omega}_\sigma$ then follows from that of $\tau_!$, $\omega_!$ and $\underline{\omega}_!$ via Proposition 3.7.3 and Proposition 3.8.1 b).

Let $F, G \in \mathbf{MPST}_\sigma$ and $H \in \mathbf{MPST}$. By adjunction and the monoidality of $a_\sigma$, we have an isomorphism

$$\mathbf{MPST}(H, \underline{\mathrm{Hom}}_{\mathbf{MPST}}(i_\sigma F, i_\sigma G)) \simeq \mathbf{MPST}_\sigma(a_\sigma H \otimes F, G). \tag{3.28}$$

By Proposition A.4.2 d), we have $\underline{\mathrm{Hom}}_{\mathbf{MPST}}(i_\sigma F, i_\sigma G) \simeq i_\sigma M$ for some $M \in \mathbf{MPST}_\sigma$ which verifies the identity

$$\mathbf{MPST}_\sigma(K, M) \simeq \mathbf{MPST}_\sigma(K \otimes F, G), \quad K \in \mathbf{MPST}_\sigma$$

so $M$ represents $\underline{\mathrm{Hom}}_{\mathbf{MPST}_\sigma}(F, G)$. Same reasoning for $\underline{\mathbf{M}}\mathbf{PST}_\sigma$. □

## 4. Mayer-Vietoris sequences in MNST

In this short section, we study analogues of Theorem 3.5.7 in **MNST** $:= \mathbf{MPST}_{\mathrm{Nis}}$. They are of 3 kinds.

We give ourselves an elementary Nisnevich square in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow \\ \mathcal{U} & \longrightarrow & \mathcal{X} \end{array} \tag{4.1}$$

where $\overline{\mathcal{U}} \to \overline{\mathcal{X}}$ is open embedding and $\overline{\mathcal{V}} \to \overline{\mathcal{X}}$ is étale (see Definition 3.5.6).

### 4.1. Using $\tau^{\mathrm{Nis}}$.

**Theorem 4.1.1.** *The sequence*

$$0 \to \tau^{\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(\mathcal{W}) \to \tau^{\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(\mathcal{U}) \oplus \tau^{\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(\mathcal{V}) \to \tau^{\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) \to 0$$

*is exact in* **MNST**.

*Proof.* Apply Theorem 3.5.7 and Corollary 3.9.6 (3). □

### 4.2. Using $\tau^{!,\mathrm{Nis}}$.

**Theorem 4.2.1.** *The sequence*

$$0 \to \tau^{!,\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(\mathcal{W}) \to \tau^{!,\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(\mathcal{U}) \oplus \tau^{!,\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(\mathcal{V}) \to \tau^{!,\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) \to 0$$

*is exact in* pro$-$**MNST**.

*Proof.* Since $\tau^{!,\mathrm{Nis}}$ is right exact as a pro-left adjoint, we have to show that the first map $\varphi$ of the sequence is a monomorphism.

Let $F \in$ pro$-$**MNST**, and let $\psi : F \to \tau^{!,\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(\mathcal{W})$ be such that $\varphi \circ \psi = 0$. We must show that $\psi = 0$. By formal arguments, we reduce to $F$ constant and then to $F$ representable, say $F = \mathbb{Z}_{\mathrm{tr}}(M)$. Then, by Yoneda's lemma and Lemma 1.8.3, we have

$$\begin{aligned} \text{pro}-\mathbf{MNST}(\mathbb{Z}_{\mathrm{tr}}(M), \tau^{!,\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(\mathcal{W})) &= \varprojlim_{N \in \mathbf{Comp}(\mathcal{W})} \mathbf{MCor}(M, N) \\ &= \underline{\mathbf{M}}\mathbf{Cor}(\tau M, \mathcal{W}) = \underline{\mathbf{M}}\mathbf{NST}(\mathbb{Z}_{\mathrm{tr}}(\tau M), \mathbb{Z}_{\mathrm{tr}}(\mathcal{W})) \end{aligned}$$

which concludes the proof. □

4.3. **Compactifying** (4.1)**.** For a development of these ideas, see [17]. Here we consider commutative diagrams in **MCor**

$$\begin{tikzcd} & \mathcal{V}_c \arrow[d, "d"'] \\ \mathcal{U}_c \arrow[r, "b"] & \mathcal{X}_c \end{tikzcd} \tag{4.2}$$

$$\begin{tikzcd} \mathcal{W}_c \arrow[r, "a"] \arrow[d, "c"'] & \mathcal{V}_c \arrow[d, "d"'] \\ \mathcal{U}_c \arrow[r, "b"] & \mathcal{X}_c \end{tikzcd} \tag{4.3}$$

such that

(i) $\mathcal{X}_c \in \mathbf{Comp}_1(\mathcal{X}), \dots, \mathcal{W}_c \in \mathbf{Comp}_1(\mathcal{W})$ (see Lemma 1.11.3);
(ii) morphisms extend those of (4.1) via the embeddings $\mathcal{X} \to \mathcal{X}_c$, etc.

We call (4.2) (resp. (4.3)) a *partial completion* (resp. a *completion*) of (4.1); we say that (4.3) *extends* (4.2) if it contains (4.2) as a subdiagram.

**Lemma 4.3.1.** *a) Given $\mathcal{X}_c$, there exist partial completions of* (4.1)*.*
*b) Given* (4.2)*, there exist completions of* (4.1) *extending it.*

*Proof.* Both statements follow from Lemma 1.11.3 (2). □

We are looking for sufficient conditions on a completion (4.3) implying that the sequence

$$0 \to \mathbb{Z}_{\mathrm{tr}}(\mathcal{W}_c) \to \mathbb{Z}_{\mathrm{tr}}(\mathcal{U}_c) \oplus \mathbb{Z}_{\mathrm{tr}}(\mathcal{V}_c) \to \mathbb{Z}_{\mathrm{tr}}(\mathcal{X}_c) \to 0 \tag{4.4}$$

is exact in **MNST**.

**Lemma 4.3.2.** (4.4) *is always exact at $\mathbb{Z}_{\mathrm{tr}}(\mathcal{W}_c)$.*

This is obvious, since $0 \to \mathbb{Z}_{\mathrm{tr}}^V(\mathcal{W}^o) \to \mathbb{Z}_{\mathrm{tr}}^V(\mathcal{U}^o) \oplus \mathbb{Z}_{\mathrm{tr}}^V(\mathcal{V}^o)$ is exact in **NST**. □

**Lemma 4.3.3.** *If $\mathcal{X}$ is proper (hence $\mathcal{X}_c = \mathcal{X}$),* (4.4) *is exact at $\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}_c)$.*

Indeed, $\mathbb{Z}_{\mathrm{tr}}(\mathcal{U}) \oplus \mathbb{Z}_{\mathrm{tr}}(\mathcal{V}) \to \mathbb{Z}_{\mathrm{tr}}(\mathcal{X})$ is epi in $\underline{\mathbf{M}}\mathbf{NST}$ by Theorem 3.5.7, hence a fortiori so is $\mathbb{Z}_{\mathrm{tr}}(\mathcal{U}_c) \oplus \mathbb{Z}_{\mathrm{tr}}(\mathcal{V}_c) \to \mathbb{Z}_{\mathrm{tr}}(\mathcal{X})$, since $\tau_{\mathrm{Nis}}$ is faithfully exact (Proposition 3.7.3). □

To pass from the proper to the non-proper case in Lemma 4.3.3, we need to extend elementary Nisnevich squares to compactifications. This is at least possible for Zariski covers:

**Definition 4.3.4.** We say that an elementary Nisnevich square (4.1) is *Zariski* if $\overline{\mathcal{V}} \to \overline{\mathcal{X}}$ is an open immersion.

**Lemma 4.3.5.** *If* (4.1) *is Zariski, there exists a cofinal set of* $\mathcal{X} \to \mathcal{X}_c \in \mathbf{Comp}_1(\mathcal{X})$ *such that* (4.1) *extends to an elementary Nisnevich square covering* $\mathcal{X}_c$ *(in* $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$*).*

*Proof.* Put $Z_{\mathcal{U}} = \overline{\mathcal{X}} - \overline{\mathcal{U}}$, $Z_{\mathcal{V}} = \overline{\mathcal{X}} - \overline{\mathcal{V}}$. Choose $\mathcal{X} \to \mathcal{X}_{c,0} \in \mathbf{Comp}_1(\mathcal{X})$; let $\overline{Z_{\mathcal{U}}}, \overline{Z_{\mathcal{V}}}$ be the scheme-theoretic closures of $Z_{\mathcal{U}}$ and $Z_{\mathcal{V}}$ in $\overline{\mathcal{X}_{c,0}}$, and let $F = \overline{Z_{\mathcal{U}}} \times_{\overline{\mathcal{X}_{c,0}}} \overline{Z_{\mathcal{V}}}$. Define

$$\overline{\mathcal{X}_c} = \mathbf{Bl}_F(\overline{\mathcal{X}_{c,0}}), \quad \pi : \overline{\mathcal{X}_c} \to \overline{\mathcal{X}_{c,0}}, \quad \mathcal{X}_c^\infty = \pi^* \mathcal{X}_{c,0}^\infty$$

and $\mathcal{X}_c = (\overline{\mathcal{X}_c}, \mathcal{X}_c^\infty)$. The morphism $\mathcal{X} \to \mathcal{X}_{c,0}$ lifts to a morphism $\mathcal{X} \to \mathcal{X}_c$ in $\mathbf{Comp}_1(\mathcal{X}_c)$. Write $\overline{Z_{\mathcal{U}}}', \overline{Z_{\mathcal{V}}}' \subset \overline{\mathcal{X}_c}$ for the strict transforms of $\overline{Z_{\mathcal{U}}}, \overline{Z_{\mathcal{V}}}$ under $\pi$: then $\overline{Z_{\mathcal{U}}}' \cap \overline{Z_{\mathcal{V}}}' = \emptyset$, hence $\overline{\mathcal{U}'} = \overline{\mathcal{X}_c} - \overline{Z_{\mathcal{U}}}'$ and $\overline{\mathcal{V}'} = \overline{\mathcal{X}_c} - \overline{Z_{\mathcal{V}}}'$ yield the desired cover (apply Lemma 3.1.4). □

*Remark* 4.3.6. When (4.1) is not Zariski, some extra conditions seem necessary for the conclusion of Lemma 4.3.5 to hold.

**Definition 4.3.7.** We say that a partial completion (4.2) of (4.1) is MV if $F = \mathrm{Ker}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{U}_c) \oplus \mathbb{Z}_{\mathrm{tr}}(\mathcal{V}_c) \to \mathbb{Z}_{\mathrm{tr}}(\mathcal{X}_c))$ is representable (in $\mathbf{MCor}$).

Suppose that the condition of Definition 4.3.7 is satisfied. Then $F \simeq \mathbb{Z}_{\mathrm{tr}}(\mathcal{W}_c)$, where $\mathcal{W}_c \in \mathbf{MCor}$ is unique up to unique isomorphism by Yoneda's lemma. The exactness of

$$0 \to \mathbb{Z}_{\mathrm{tr}}(\mathcal{W}) \to \mathbb{Z}_{\mathrm{tr}}(\mathcal{U}) \oplus \mathbb{Z}_{\mathrm{tr}}(\mathcal{V}) \to \mathbb{Z}_{\mathrm{tr}}(\mathcal{X})$$

in $\underline{\mathbf{M}}\mathbf{NST}$ and that of $\tau_{\mathrm{Nis}}$ yield a monomorphism

$$\mathbb{Z}_{\mathrm{tr}}(\mathcal{W}) \hookrightarrow \tau_{\mathrm{Nis}} F \simeq \mathbb{Z}_{\mathrm{tr}}(\tau \mathcal{W}_c)$$

hence, still by Yoneda, we get a morphism $\theta : \mathcal{W} \to \tau \mathcal{W}_c$ in $\mathbf{MCor}$, yielding a square (4.3) compatible with (4.1).

**Proposition 4.3.8.** *In this situation, we may choose* $(\mathcal{W}_c, \theta)$ *such that* $\theta \in \mathbf{Comp}_1(\mathcal{X})$*: in other words, such that* (4.3) *is a completion of* (4.2)*. We call it an* associated completion*.*

*Proof.* We need to achieve the conditions of Lemma 1.11.3 (1). The first two are automatic. Up to blowing up, we may assume that $\overline{\mathcal{W}_c} \to \overline{\mathcal{V}_c}$ is a morphism: this forces $\theta$ to be an open immersion. Because $\mathcal{W} \to \mathcal{V}$ and $\mathcal{V} \to \tau \mathcal{V}_c$ are minimal, $\theta$ is minimal. □

**Definition 4.3.9.** We say that a completion (4.3) of (4.1) is *minimal* if $a, c$ define morphisms $\overline{\mathcal{U}_c} \xleftarrow{c} \overline{\mathcal{W}_c} \xrightarrow{a} \overline{\mathcal{V}_c}$ and

$$\mathcal{W}_c^\infty = \sup(c^* \mathcal{U}_c^\infty, a^* \mathcal{V}_c^\infty)$$

in the ordered group of Cartier divisors. We say that a completion is *universally minimal* if it is minimal and if, moreover,

$$f^*\mathcal{W}_c^\infty = \sup(f^*c^*\mathcal{U}_c^\infty, f^*a^*\mathcal{V}_c^\infty)$$

for any $f : X \to \overline{\mathcal{W}_c}$ such that $f^*\mathcal{W}_c^\infty$, $f^*c^*\mathcal{U}_c^\infty$ and $f^*a^*\mathcal{V}_c^\infty$ are well-defined as Cartier divisors.

**Proposition 4.3.10.** *If the underlying partial completion* (4.2) *is MV, then* (4.3) *is minimal; conversely, if* (4.1) *is Zariski and* (4.3) *is universally minimal, then* (4.2) *is MV and* (4.3) *is an associated completion of* (4.2).

(Said more plainly: minimality is a necessary condition for the exactness of (4.4) at the middle term, and universal minimality is sufficient when (4.1) is Zariski.)

*Proof.* By Lemma 4.3.2 and the exactness of $a_{\mathrm{Nis}}$, exactness of (4.4) at the middle term may be checked in **MPST**. Let $M \in \mathbf{MPST}$. In the commutative diagram

$$\begin{tikzcd}
\mathbf{MCor}(M,\mathcal{W}_c) \arrow[r] \arrow[d, "r"'] & \mathbf{MCor}(M,\mathcal{U}_c) \oplus \mathbf{MCor}(M,\mathcal{V}_c) \arrow[r] \arrow[d, "s"'] & \mathbf{MCor}(M,\mathcal{X}_c) \arrow[d, "t"'] \\
\mathbf{Cor}(M^o,\mathcal{W}^o) \arrow[r, "\varphi"] & \mathbf{Cor}(M^o,\mathcal{U}^o) \oplus \mathbf{Cor}(M^o,\mathcal{V}^o) \arrow[r] & \mathbf{Cor}(M^o,\mathcal{X}^o)
\end{tikzcd}$$

the bottom row is easily checked to be exact. Since $\varphi$ and $t$ are injective, the top row is exact if and only if the left square is cartesian.

Suppose the left square is cartesian for any $M$. Take $\overline{M} = \overline{\mathcal{W}_c}$ and $\alpha = 1_{\mathcal{W}^o} \in \mathbf{Cor}(M^o,\mathcal{W}^o)$. First, $\varphi(\alpha) \in \operatorname{Im} s$ if and only if

$$M^\infty \geq c^*\mathcal{U}_c^\infty, M^\infty \geq a^*\mathcal{V}_c^\infty.$$

Next, $\alpha \in \operatorname{Im} r$ if and only if $M^\infty \geq \mathcal{W}_c^\infty$. By hypothesis, the first condition implies the second for any such $M$: this means that (4.3) is minimal.

Assume now (4.1) Zariski, and let $x = \sum n_\alpha \alpha \in \mathbf{Cor}(M^o,\mathcal{W}^o)$ ($n_\alpha \neq 0$), where the $\alpha$'s are pairwise distinct elementary correspondences. Then the two components of the $\varphi(\alpha)$ are still elementary and pairwise distinct. Hence $\varphi(x)$ is in the image of $s$ if and only if $\varphi(\alpha)$ is in the image of $s$ for each $\alpha$. Thus we are reduced to showing that, if $x$ is elementary and $\varphi(x)$ is in the image of $s$, then $x$ is in the image of $r$.

Let $Z \subset M^o \times \mathcal{W}^o$ be the support of $x$; write $Z_\mathcal{U}, Z_\mathcal{V}, Z_\mathcal{W}$ for the closures of $Z$ respectively in $\overline{M} \times \overline{\mathcal{U}_c}$, $\overline{M} \times \overline{\mathcal{V}_c}$, $\overline{M} \times \overline{\mathcal{W}_c}$, and $Z_\mathcal{U}^N$; etc. for their normalisations. Write $p_\mathcal{U}, q_\mathcal{U} \dots$ for the projections $Z_\mathcal{U}^N \to \overline{M}$, $Z_\mathcal{U}^N \to \overline{\mathcal{U}_c} \dots$ By definition, $\varphi(x) \in \operatorname{Im} s$ means that

$$q_\mathcal{U}^*\mathcal{U}_c^\infty \leq p_\mathcal{U}^* M^\infty, \quad q_\mathcal{V}^*\mathcal{V}_c^\infty \leq p_\mathcal{V}^* M^\infty.$$

Using the projections $Z^N_{\mathcal{U}} \leftarrow Z^N_{\mathcal{W}} \to Z^N_{\mathcal{V}}$, we deduce

$$q_{\mathcal{W}}^* c^* \mathcal{U}_c^\infty \leq p_{\mathcal{W}}^* M^\infty, \quad q_{\mathcal{W}}^* a^* \mathcal{V}_c^\infty \leq p_{\mathcal{W}}^* M^\infty$$

hence $q_{\mathcal{W}}^* \mathcal{W}_c^\infty \leq p_{\mathcal{W}}^* M^\infty$ by universal minimality. □

**Corollary 4.3.11.** *Let $X$ be proper and let $D, D_1, D_2, D'$ be effective Cartier divisors on $X$ such that*

$$X - D \text{ is smooth} \tag{4.5}$$

$$D \leq D_i \leq D' \tag{4.6}$$

$$|D_1 - D| \cap |D_2 - D| = \emptyset \tag{4.7}$$

$$D' - D_2 = D_1 - D. \tag{4.8}$$

*Then we have a short exact sequence in* **MNST***:*

$$0 \to \mathbb{Z}_{\mathrm{tr}}(X, D') \to \mathbb{Z}_{\mathrm{tr}}(X, D_1) \oplus \mathbb{Z}_{\mathrm{tr}}(X, D_2) \to \mathbb{Z}_{\mathrm{tr}}(X, D) \to 0.$$

*Proof.* Let $\mathcal{X} = (X, D)$. Write $E_i = D_i - D$ as a Cartier divisor of $X$. By (4.7), $\mathcal{U} = (X - E_1, D\,|_{X-E_1})$ and $\mathcal{V} = (X - E_2, D\,|_{X-E_2}))$ define a Zariski cover of $\mathcal{X}$, with $\mathcal{W} = (X - E_1 - E_2, D\,|_{X-E_1-E_2})$. Then $(X, D_1)$, $(X, D_2)$, $(X, D')$ yield a completion of the corresponding square. Since (4.7) and (4.8) imply that this completion is universally minimal, the conclusion follows from Proposition 4.3.10. □

**Corollary 4.3.12.** *Any partial completion* (4.2) *of a Zariski square* (4.1) *is MV when* $\dim \mathcal{X} \leq 1$ *and* $\mathcal{W}$ *is regular.*

*Proof.* We may find a regular projective $\overline{\mathcal{W}_c}$ and morphisms $\overline{\mathcal{U}_c} \xleftarrow{c} \overline{\mathcal{W}_c} \xrightarrow{a} \overline{\mathcal{V}_c}$ extending the morphisms $\overline{\mathcal{U}} \leftarrow \overline{\mathcal{W}} \to \overline{\mathcal{V}}$ of Diagram (4.1). Then we may define $\mathcal{W}_c$ by setting $\mathcal{W}_c^\infty = \sup(c^*\mathcal{U}_c^\infty, a^*\mathcal{V}_c^\infty)$, since the Weil divisor on the right hand side is a Cartier divisor. We may also define

$$D = \inf(c^*\mathcal{U}_c^\infty, a^*\mathcal{V}_c^\infty).$$

Then $D_1 = c^*\mathcal{U}_c^\infty$, $D_2 = a^*\mathcal{V}_c^\infty$ and $D' = \mathcal{W}_c^\infty$ verify the conditions of Corollary 4.3.11: the only nontrivial one is (4.7), which is true because $\inf(D_1 - D, D_2 - D) = 0$ and $\dim \overline{\mathcal{W}_c} \leq 1$. Thus the completion is universally minimal. □

*Remark* 4.3.13. Condition (4.7) cannot be avoided in Corollary 4.3.11: consider $X = \mathbf{P}^1$, $D = (\infty)$, $D_1 = D_2 = 2(\infty)$, $D' = 3(\infty)$. The following more subtle example, inspired by Hiroyasu Miyazaki, shows that Proposition 4.3.10 may fail if (4.3) is minimal but not universally minimal:

We assume that $k$ has more than 2 elements. Consider $\mathbf{A}^2$ with homogeneous coordinates $(t_1, t_2)$. Let $D_1 = (t_1 = 0)$, $D_2 = (t_1 = 1)$

be two parallel lines. Take for (4.1) the elementary Zariski square with $\mathcal{X} = (\mathbf{A}^2, \emptyset)$, $\mathcal{U} = \mathcal{X} - D_1$, $\mathcal{V} = \mathcal{X} - D_2$ and $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$. Now consider $\mathbf{P}^2$ with homogeneous coordinates $(T_0 : T_1 : T_2)$, and let $E = (T_0 = 0)$ be the line at infinity. Take for (4.3) the square with $\mathcal{X}_c = (\mathbf{P}^2, E)$, $\mathcal{U}_c = (\mathbf{P}^2, E + \bar{D}_1)$, $\mathcal{V}_c = (\mathbf{P}^2, E + \bar{D}_2)$, $\mathcal{W}_c = (\mathbf{P}^2, E + \bar{D}_1 + \bar{D}_2)$, where $\bar{D}_i$ is the closure of $D_i$ in $\mathbf{P}^2$. Then (4.3) is minimal but not universally minimal, and (4.2) is not MV; more precisely, the sequence

$$\mathbf{MCor}(\overline{\square}^{(2)}, \mathcal{W}_c) \to \mathbf{MCor}(\overline{\square}^{(2)}, \mathcal{U}_c) \oplus \mathbf{MCor}(\overline{\square}^{(2)}, \mathcal{V}_c) \\ \to \mathbf{MCor}(\overline{\square}^{(2)}, \mathcal{X}_c)$$

is not exact. Indeed, consider the morphism $\varphi : \mathbf{A}^1 \to \mathcal{W}$ given by $t \mapsto (\alpha, t)$, where $\alpha \in k - \{0, 1\}$. It yields elements of $\mathbf{MCor}(\overline{\square}^{(2)}, \mathcal{U}_c)$ and $\mathbf{MCor}(\overline{\square}^{(2)}, \mathcal{V}_c)$ which match in $\mathbf{MCor}(\overline{\square}^{(2)}, \mathcal{X}_c)$ but don't lift to $\mathbf{MCor}(\overline{\square}^{(2)}, \mathcal{W}_c)$, exactly as in the initial example of this remark.

Miyazaki has proven that Condition (4.7) is equivalent to the equality

$$D = D_1 \times_X D_2,$$

which allows one to construct many universally minimal completions. This will appear in [17].

*Remark* 4.3.14. Here is an example showing that minimality is not sufficient in general for the conclusion of Proposition 4.3.10 when (4.1) is not Zariski, even if $\overline{\mathcal{W}_c}$ is smooth of dimension 1. Take $\mathcal{X} = \overline{\square} = (\mathbf{P}^1, \infty)$, $\mathcal{U} = (\mathbf{P}^1 - \{1\}, (\infty))$ and $\mathcal{V} = (\mathbf{A}^1 - \{0, -1\}, \emptyset)$, where $\mathcal{U} \to \mathcal{X}$ is given by the inclusion and $\mathcal{V} \to \mathcal{X}$ is given by $t \mapsto t^2$; thus, $\mathcal{W} = (\mathbf{A}^1 - \{0, 1, -1\}, \emptyset)$. Choices for $\mathcal{U}_c, \mathcal{V}_c$ and $\mathcal{W}_c$ are $(\mathbf{P}^1, D_{\mathcal{U}})$, $(\mathbf{P}^1, D_{\mathcal{V}})$, $(\mathbf{P}^1, D_{\mathcal{W}})$, with

$$D_{\mathcal{U}} = a_{\mathcal{U}}(\infty) + c_{\mathcal{U}}(1), \quad D_{\mathcal{V}} = a_{\mathcal{V}}(\infty) + b_{\mathcal{V}}(0) + d_{\mathcal{V}}(-1), \\ D_{\mathcal{W}} = a_{\mathcal{W}}(\infty) + b_{\mathcal{W}}(0) + c_{\mathcal{W}}(1) + d_{\mathcal{W}}(-1)$$

subject to

$$a_{\mathcal{U}} = 1, \quad a_{\mathcal{V}} \geq 2,$$
$$a_{\mathcal{W}} \geq \sup(2a_{\mathcal{U}}, a_{\mathcal{V}}), \quad b_{\mathcal{W}} \geq b_{\mathcal{V}}, \quad c_{\mathcal{W}} \geq c_{\mathcal{U}}, \quad d_{\mathcal{W}} \geq \sup(c_{\mathcal{U}}, d_{\mathcal{V}}). \tag{4.9}$$

By Proposition 4.3.10, (4.2) must be minimal. Let now $X = \mathbf{A}^1 - \{0, 1, -1\}$, and consider the finite correspondence $\gamma = \gamma_+ - \gamma_- \in \mathbf{Cor}(X, \mathcal{W}^o)$ where $\gamma_+$ (resp. $\gamma_-$) is the graph of the identity (resp. of the map $t \mapsto -t$). Note that $\gamma \mapsto 0 \in \mathbf{Cor}(X, \mathcal{U}^o)$.

Let $M = (\mathbf{P}^1, D)$ with $D = a_M(\infty) + b_M(0) + c_M(1) + d_M(-1)$. Then $\gamma \in \mathbf{MCor}(M, \mathcal{W}_c)$ if and only if

$$a_M \geq a_{\mathcal{W}}, \quad b_M \geq b_{\mathcal{W}}, \quad c_M \geq c_{\mathcal{W}}, \quad d_M \geq d_{\mathcal{W}} \tag{4.10}$$

and

$$a_M \geq a_{\mathcal{W}}, \quad b_M \geq b_{\mathcal{W}}, \quad c_M \geq d_{\mathcal{W}}, \quad d_M \geq c_{\mathcal{W}}. \tag{4.11}$$

On the other hand, the image of $\gamma$ in $\mathbf{MCor}(M^o, \mathcal{U}^o) \oplus \mathbf{MCor}(M^o, \mathcal{V}^o)$ is in $\mathbf{MCor}(M, \mathcal{U}_c) \oplus \mathbf{MCor}(M, \mathcal{V}_c)$ if and only if

$$a_M \geq a_{\mathcal{V}}, \quad b_M \geq b_{\mathcal{V}}, \quad c_M \geq d_{\mathcal{V}}, \quad d_M \geq d_{\mathcal{V}}. \tag{4.12}$$

We want to find $\mathcal{W}_c$ such that (in particular) (4.12) implies (4.10) and (4.11) for any $(a_M, b_M, c_M, d_M)$. Taking equality in (4.12) and comparing with (4.9), we find

$$a_{\mathcal{W}} = a_{\mathcal{V}} \geq 2a_{\mathcal{U}} = 2, \quad b_{\mathcal{W}} = b_{\mathcal{V}}, \quad d_{\mathcal{W}} = d_{\mathcal{V}}, \quad d_{\mathcal{V}} \geq c_{\mathcal{W}} \geq c_{\mathcal{U}}$$

and in particular

$$a_{\mathcal{V}} \geq 2a_{\mathcal{U}}, \quad d_{\mathcal{V}} \geq c_{\mathcal{U}}. \tag{4.13}$$

Conversely, assume that the minimality condition and (4.13) hold. Let $M \in \mathbf{MCor}$. For $x \in \mathbf{Cor}(M^o, \mathcal{W}^o)$, write $x_- = \gamma_- \circ x$ where, as above, $\gamma_-$ is the graph of the map $t \mapsto -t$. Write $x = \sum_\alpha n_\alpha \alpha$, where $\alpha$ runs through distinct elementary correspondences and $n_\alpha \neq 0$ for all $\alpha$. Let

$$S = \{\alpha \mid n_\alpha + n_{\alpha_-} = 0\}$$

whence

$$x = \sum_{\alpha \notin S} n_\alpha \alpha + \sum_{\alpha \in S'} n_\alpha (\alpha - \alpha_-)$$

where $S'$ is set of representatives of elements of $S$ modulo the relation $\alpha \sim \alpha_-$. Then, with the notation in the proof of Proposition 4.3.10, $\varphi(x) \in \operatorname{Im} s$ if and only if

$$\varphi(\alpha) \in \operatorname{Im} s \quad \forall \alpha \notin S \text{ and } \varphi(\alpha)_{\mathcal{V}_c}, \varphi(\alpha_-)_{\mathcal{V}_c} \in (\operatorname{Im} s)_{\mathcal{V}_c} \quad \forall \alpha \in S'.$$

As in the proof of Proposition 4.3.10, the first condition is assured by minimality and the second one is assured by the latter and (4.13). In particular, (4.2) is MV if and only if (4.13) holds.

## 5. Cubical objects and intervals

This technical section is preparatory to the next one, and may be skipped at first reading.

5.1. **Cubical objects and associated complexes.** We follow [29] but we omit the use of permutations and involutions. Let **Cube** be the subcategory of **Sets** which has as objects $\underline{n} = \{0,1\}^n$ for $n \in \mathbb{Z}_{\geq 0}$ $\underline{0} = *$ the terminal object of **Sets**) and whose morphisms are generated by

$$\begin{aligned} &p_i^n : \underline{n} \to \underline{n-1} \qquad (n \in \mathbb{Z}_{>0},\ i \in \{1,\dots,n\}), \\ &\delta_{i,\varepsilon}^n : \underline{n} \to \underline{n+1} \qquad (n \in \mathbb{Z}_{\geq 0},\ i \in \{1,\dots,n+1\},\ \varepsilon \in \{0,1\}), \end{aligned}$$

where $p_i^n$ omits the $i$-th component and $\delta_{i,\varepsilon}^n$ inserts $\varepsilon$ at the $i$-th component.

**Definition 5.1.1.** Let $\mathcal{A}$ be a category. A covariant (resp. contravariant) functor $A : \mathbf{Cube} \to \mathcal{A}$ is called a *co-cubical* (resp. *cubical*) object in $\mathcal{A}$;

*Remark* 5.1.2. The definition of **Cube** in [29] is different from ours. (It also contains other morphisms called permutations and involutions.) However, concerning the following lemma, the same proof as in loc. cit. works in our more basic setting.

**Lemma 5.1.3.** *Let $A : \mathbf{Cube}^{\mathrm{op}} \to \mathcal{A}$ be a cubical object in a pseudo-abelian category $\mathcal{A}$. Put $A_n := A(\underline{n})$.*

(1) *We have well-defined objects*

$$A_n^{\deg} := \mathrm{Im}\left( \oplus p_i^{n*} : \bigoplus_{i=1}^n A_{n-1} \to A_n \right) \xrightarrow{\sim} \mathrm{Im}\left( \oplus \delta_{i,1}^{(n-1)*} : A_n \to \bigoplus_{i=1}^n A_{n-1} \right),$$

$$A_n^{\nu} := \ker\left( \oplus \delta_{i,1}^{(n-1)*} : A_n \to \bigoplus_{i=1}^n A_{n-1} \right) \xrightarrow{\sim} \mathrm{Coker}\left( \oplus p_i^{n*} : \bigoplus_{i=1}^n A_{n-1} \to A_n \right)$$

*in $\mathcal{A}$, and $A_n^{\nu} \oplus A_n^{\deg} \xrightarrow{\sim} A_n$ holds.*

(2) *Let $d_n := \sum_{i=1}^{n+1} (-1)^i (\delta_{i,1}^{n*} - \delta_{i,0}^{n*}) : A_{n+1} \to A_n$. This makes $A_\bullet$ a complex, of which $A_\bullet^{\nu}$ and $A_\bullet^{\deg}$ are subcomplexes. The two complexes $A_\bullet / A_\bullet^{\deg}$ and $A_\bullet^{\nu}$ are isomorphic.*

*Proof.* See [29, Lemmas 1.3, 1.6]. □

*Remark* 5.1.4. We have obvious dual statements of Lemma 5.1.3 for co-cubical objects. Here we state it for later use. Let $A : \mathbf{Cube} \to \mathcal{A}$ be a co-cubical object in a pseudo-abelian category $\mathcal{A}$. Put $A^n := A(\underline{n})$.

(1) We have well-defined objects

$$A^n_{\deg} := \operatorname{Im}\Big(\oplus\, \delta^{(n-1)}_{i,1*} : \bigoplus_{i=1}^{n} A^{n-1} \to A^n\Big) \xrightarrow{\sim} \operatorname{Im}\Big(\oplus\, p^n_{i*} : A^n \to \bigoplus_{i=1}^{n} A^{n-1}\Big),$$

$$A^n_{\nu} := \operatorname{Ker}\Big(\oplus\, p^n_{i*} : A^n \to \bigoplus_{i=1}^{n} A^{n-1}\Big) \xrightarrow{\sim} \operatorname{Coker}\Big(\oplus\, \delta^{(n-1)}_{i,1*} : \bigoplus_{i=1}^{n} A^{n-1} \to A^n\Big),$$

in $\mathcal{A}$, and $A^n \xrightarrow{\sim} A^n_\nu \oplus A^n_{\deg}$ holds.

(2) Let $d^n := \sum_{i=1}^{n+1}(-1)^i(\delta^n_{i,1*} - \delta^n_{i,0*}) : A^n \to A^{n+1}$. This makes $A^\bullet$ a complex, of which $A^\bullet_\nu$ and $A^\bullet_{\deg}$ are subcomplexes. The two complexes $A^\bullet/A^\bullet_{\deg}$ and $A^\bullet_\nu$ are isomorphic.

*Remarks* 5.1.5. Let $\mathcal{A}$ be a unital symmetric monoidal category (briefly: a $\otimes$-category). Let $A : \mathbf{Cube} \to \mathcal{A}$ be a co-cubical object in a tensor pseudo-abelian category $\mathcal{A}$ and suppose that $A$ is strict monoidal (i.e. $A(\underline{m} \times \underline{n}) = A(\underline{m}) \otimes A(\underline{n})$). Then:
1) $A^0 = A^0_\nu = \mathbf{1}$ is the unit object of $\mathcal{A}$, and $A^0_{\deg} = 0$. For $n > 0$, combining $A^1 = A^1_\nu \oplus A^1_{\deg}$ and $A^n = A^1 \otimes \cdots \otimes A^1$, we get a decomposition

$$A^n_\nu = A^1_\nu \otimes \cdots \otimes A^1_\nu,$$
$$A^n_{\deg} = \bigoplus_{\sigma \not\equiv \nu} A^1_{\sigma(1)} \otimes \cdots \otimes A^1_{\sigma(n)},$$

where $\sigma$ ranges over all maps $\{1, \dots, n\} \to \{\nu, \deg\}$ except for the constant map $\nu$.
2) $A^\bullet$ has a canonical comonoid structure where the counit and comultiplication are respectively given by

$$\pi^\bullet : A^\bullet \to A^0[0] = \mathbf{1}, \quad \pi^n = 0 \ (n > 0) \text{ and } \pi^0 = \mathrm{Id}_{A^0}, \tag{5.1}$$

$$\Delta^\bullet : A^\bullet \to \operatorname{Tot}(A^\bullet \otimes A^\bullet) \tag{5.2}$$

where $\Delta^n = \sum_{p+q=n} \Delta^{p,q}$ with $\Delta^{p,q} : A^{p+q} \xrightarrow{=} A^p \otimes A^q$. In view of 1), we see that $A^\bullet_\nu$ inherits the same structure:

$$\pi^\bullet_\nu : A^\bullet_\nu \to \mathbf{1}, \quad \Delta^\bullet_\nu : A^\bullet_\nu \to \operatorname{Tot}(A^\bullet_\nu \otimes A^\bullet_\nu).$$

5.2. **Interval structure.** Recall from Voevodsky [47] the notion of interval:

**Definition 5.2.1.** Let $\mathbf{1}$ be the unit object of $\mathcal{A}$. An *interval* in $\mathcal{A}$ is a quintuple $(I, p, i_0, i_1, \mu)$, with $I \in \mathcal{A}$, $p : I \to \mathbf{1}$, $i_0, i_1 : \mathbf{1} \to I$, $\mu : I \otimes I \to I$, verifying the identitites

$$pi_0 = pi_1 = 1_{\mathbf{1}}, \quad \mu \circ (1_I \otimes i_0) = i_0 p, \quad \mu \circ (1_I \otimes i_1) = 1_I.$$

**Definition 5.2.2.** Given a interval $(I, p, i_0, i_1, \mu)$ in $\mathcal{A}$, we define a strict monoidal co-cubical object $A : \mathbf{Cube} \to \mathcal{A}$ by

$$A^n = I^{\otimes n},\ p^n_{i*} = 1_I^{\otimes(i-1)} \otimes p \otimes 1_I^{\otimes(n-i)},\ \delta^n_{i\varepsilon *} = 1_I^{\otimes(i-1)} \otimes i_\varepsilon \otimes 1_I^{\otimes(n-i)}$$

(this does not use the morphism $\mu$). When $\mathcal{A}$ is pseudo-abelian, we write $I^\bullet$, $I^\bullet_\nu$, $I^\bullet_{\deg}$ for the associated complexes introduced in Remark 5.1.4.

By definition and Remark 5.1.5, we have

$$I^n_\nu = I_\nu \otimes \cdots \otimes I_\nu \quad \text{with } I_\nu = \operatorname{Ker}(I \xrightarrow{p} \mathbf{1}).$$

*Remark* 5.2.3. Conversely, Levine introduced in [29] a notion of *extended co-cubical object* $A : \mathbf{ECube} \to \mathcal{A}$, where $\mathbf{ECube}$ is the smallest symmetric monoidal subcategory of $\mathbf{Sets}$ that contains $\mathbf{Cube}$ and the morphism

$$\tilde{\mu} : \underline{2} \to \underline{1}; \quad (a, b) \mapsto ab.$$

Given such a (strict monoidal) extended co-cubical object $A$, we may define an interval $(I, p, i_0, i_1, \mu)$ in $\mathcal{A}$ by

$$I = A(\underline{1}),\ p = p^1_{1*},\ i_0 = \delta^0_{1,0*},\ i_1 = \delta^0_{1,1*},\ \mu = \tilde{\mu}_*.$$

Such intervals are not arbitrary, as $\mu$ makes $I$ into a commutative monoid (because so does $\tilde{\mu}$ with $\underline{1}$). However, all intervals encountered in practice are commutative monoids, including in [47, 49] and here (Lemma 6.1.1).

**Definition 5.2.4.** a) An object $X \in \mathcal{A}$ is *$I$-local at $Y \in \mathcal{A}$* [2] if $p$ induces an isomorphism $\mathcal{A}(Y, X) \xrightarrow{\sim} \mathcal{A}(Y \otimes I, X)$; $X$ is *$I$-local* if it is $I$-local at $Y$ for any $Y \in \mathcal{A}$. If $\mathcal{A}$ is closed, it is equivalent to ask for the morphism

$$X \xrightarrow{p^*} \underline{\mathrm{Hom}}(I, X)$$

to be an isomorphism.

b) A morphism $f : Y \to Z$ in $\mathcal{A}$ is called an *$I$-equivalence* if $\mathcal{A}(Z, X) \xrightarrow{f^*} \mathcal{A}(Y, X)$ is an isomorphism for any $I$-local $X$.

**Lemma 5.2.5.** *Let $X, Y \in \mathcal{A}$. Then*

(1) *If $X$ is $I$-local at $Y$, the maps $1_Y \otimes i_0^*, 1_Y \otimes i_1^* : \mathcal{A}(Y \otimes I, X) \to \mathcal{A}(Y, X)$ are equal.*
(2) *If the maps $1_{Y\otimes I} \otimes i_0^*, 1_{Y\otimes I} \otimes i_1^* : \mathcal{A}(Y \otimes I \otimes I, X) \to \mathcal{A}(Y \otimes I, X)$ are equal, then $X$ is $I$-local at $Y$.*

[2] This notion will be useful in [43].

(3) *$X$ is $I$-local if and only if the maps $i_0^*, i_1^* : \mathcal{A}(Y \otimes I, X) \to \mathcal{A}(Y, X)$ are equal for all $Y \in \mathcal{A}$ (equivalently when $\mathcal{A}$ is closed: if and only if the maps $i_0^*, i_1^* : \underline{\mathrm{Hom}}(I, X) \to X$ are equal).*

*Proof.* For (2), the last two identities of Definition 5.2.1 imply that $p^* i_0^* : \mathcal{A}(Y \otimes I, X) \to \mathcal{A}(Y \otimes I, X)$ is the identity, hence the claim since $i_0^* p^*$ is also the identity. (3) now follows from (1) and (2). $\square$

*Remark* 5.2.6. Actually, Definition 5.2.1 is more general than Voevodsky's definition in [47, 2.2] or (with Morel) [31, 2.2.3]. There, the $\otimes$-category $\mathcal{A}$ is a site with products (in [47]) or the category of sheaves on a site (in [31]), and the tensor structure is the one given by products of objects or of sheaves. Voevodsky constructs in [47, loc. cit.] a universal cosimplicial object, whose general term is $I^n$. Unfortunately, the formulas of loc. cit. implicitly use diagonal morphisms which are not available in general $\otimes$-categories, in particular in the ones we use here (see warning 1.4.4). So, while one can develop a cubical theory out of Definition 5.2.1, we do not know if this definition is sufficient to develop a simplicial theory.

## 5.3. **Homotopy equivalences.**

**Proposition 5.3.1.** *Let $\mathcal{A}$ be a pseudo-abelian $\otimes$-category, provided with an interval $I$. Let $I^\bullet$ be as in Definition 5.2.2. Then the morphisms*

$$1 \otimes p^1_{1*} : I^\bullet \otimes I^1[0] \to I^\bullet, \tag{5.3}$$

$$1 \otimes p^1_{1*} : I^\bullet_\nu \otimes I^1[0] \to I^\bullet_\nu, \tag{5.4}$$

$$\Delta^\bullet_\nu : I^\bullet_\nu \to \mathrm{Tot}(I^\bullet_\nu \otimes I^\bullet_\nu) \tag{5.5}$$

*are homotopy equivalences.*

*Proof.* For (5.3), since $p^1_1 \delta^0_{1,0} = 1_{\underline{0}}$, the composition $(1 \otimes p^1_{1*})(1 \otimes \delta^0_{1,0*}) : I^\bullet \to I^\bullet$ is the identity. Let $s^n : I^{n+1} \xrightarrow{\sim} I^n \otimes I^1$ be the tautological isomorphism. The identities

$$s^n \delta^n_{j,\varepsilon *} = \begin{cases} ((\delta^{n-1}_{j,\varepsilon *} \otimes 1) s^{n-1} & \text{if } j < n+1 \\ 1_{I^n} \otimes i_\varepsilon & \text{if } j = n+1 \end{cases}$$

yield

$$s^n d^n - (d^{n-1} \otimes 1) s^{n-1} = 1 \otimes i_1 - 1 \otimes i_0.$$

Then the composition

$$\sigma^{n+1} : I^{n+1} \otimes I^1 \xrightarrow{s^n \otimes 1} I^n \otimes I^1 \otimes I^1 \xrightarrow{1 \otimes \mu} I^n \otimes I^1$$

yields a chain homotopy from $1\otimes(\delta^0_{1,0*}p^1_{1*})$ to $1\otimes 1$, which concludes the proof. Now (5.4) is also homotopy equivalence as a direct summand of (5.3).

Consider (5.5). By induction and the homotopy equivalence (5.3), we find that for any $q>0$

$$\mathrm{Id}\otimes(p^1_1p^2_1\dots p^q_1)_*: I^\bullet\otimes I^q[0]\to I^\bullet \tag{5.6}$$

is a homotopy equivalence. Since $I^q_\nu$ is a direct summand of $I^q$ contained in $\mathrm{Ker}((p^1_1p^2_1\dots p^q_1)_*)$ by Remark 5.1.4, we find that $I^\bullet\otimes I^q_\nu[0]$ is contractible for $q>0$. The same is true of $I^\bullet_\nu\otimes I^q_\nu[0]$ because it is a direct summand of $I^\bullet\otimes I^q_\nu[0]$. Lemma 5.3.2 (2) below then shows that $\mathrm{Tot}(1\otimes\pi^\bullet):\mathrm{Tot}(I^\bullet_\nu\otimes I^\bullet_\nu)\to I^\bullet_\nu$ is a homotopy equivalence, where $\pi^\bullet$ is as in (5.1). Since $\mathrm{Tot}(1\otimes\pi^\bullet)$ is left inverse to $\Delta^\bullet_\nu$, this shows that $\Delta^\bullet_\nu$ is a homotopy equivalence. □

**Lemma 5.3.2.** *Let $\mathcal{A}$ be an additive category. Let us call a double complex $S^{\bullet,\bullet}$ in $\mathcal{A}$ locally finite if $\{p\in\mathbb{Z}\mid S^{p,n-p}\neq 0\}$ is a finite set for each $n\in\mathbb{Z}$.*

1. *Let $S^{\bullet,\bullet}$ be a locally finite double complex in $\mathcal{A}$. Suppose that the single complex $S^{\bullet,q}$ is contractible for each $q\in\mathbb{Z}$. Then $\mathrm{Tot}(S^{\bullet,\bullet})$ is contractible.*
2. *Let $f^{\bullet,\bullet}:S^{\bullet,\bullet}\to T^{\bullet,\bullet}$ be a morphism of locally finite double complexes in $\mathcal{A}$. If $f^{\bullet,q}$ is a homotpy equivalence for each $q\in\mathbb{Z}$, then so is $\mathrm{Tot}(f^{\bullet,\bullet}):S^{\bullet,\bullet}\to T^{\bullet,\bullet}$.*

*Proof.* (1)[3] Let us write $d^S_1:S^{\bullet,\bullet}\to S^{\bullet+1,\bullet}$, $d^S_2:S^{\bullet,\bullet}\to S^{\bullet,\bullet+1}$ for the differentials of $S^{\bullet,\bullet}$, and set $d^S=d^S_1+d^S_2$. By assumption we have a map $s:S^{\bullet,\bullet}\to S^{\bullet,\bullet}$ of bidegree $(-1,0)$ such that $d^S_1s+sd^S_1=\mathrm{Id}_{S^{\bullet,\bullet}}$. Thus $d^Ss+sd^S-\mathrm{Id}_{S^{\bullet,\bullet}}$ is an endomorphism of $S^{\bullet,\bullet}$ of bidegree $(-1,1)$, which defines an endomorphism $u$ of $\mathrm{Tot}(S^{\bullet,\bullet})$ of degree 0. By assumption, $u$ restricted to each degree is nilpotent. Hence $\mathrm{Id}+u$ is an isomorphism, which implies that $\mathrm{Tot}(S)$ is contractible.

(2) We shall use the following fact:

> (*) A morphism $g$ of (single) complexes is a homotopy equivalence if and only if $\mathrm{Cone}(g)$ is contractible.

Let $U^{\bullet,\bullet}$ be a cone of $f$, that is, $U^{p,q}=T^{p,q}\oplus S^{p+1,q}$ equipped with $d^U_1=\begin{pmatrix} d^T_1 & f\\ 0 & d^S_1\end{pmatrix}:U^{p,q}\to U^{p+1,q}$ and $d^U_2=\begin{pmatrix} d^T_2 & 0\\ 0 & d^S_2\end{pmatrix}:U^{p,q}\to U^{p,q+1}$. For each $q\in\mathbb{Z}$, we have $U^{\bullet,q}=\mathrm{Cone}(f^{\bullet,q})$. as (single) complexes. By assumption and (*), they are contractible. Then (1) shows that $\mathrm{Tot}(U)$

[3] We learned this proof from J. Œsterlé. We thank him.

is contractible. Since we have $\mathrm{Cone}(\mathrm{Tot}(f)) = \mathrm{Tot}(U)$ by definition, this implies that $\mathrm{Tot}(f)$ is contractible by (*). □

5.4. **An adjunction.** Let $\mathcal{T}$ be a tensor triangulated category, compactly generated (Definition A.11.4) and equipped with an interval $(I, p, i_0, i_1, \mu)$. We assume that the tensor structure of $\mathcal{T}$ is strongly biadditive (Definition 3.2.3) and that $-\otimes I$ preserves the full subcategory $\mathcal{T}^c$ of compact objects.

By Theorem A.11.8, $\mathcal{T}$ enjoys the Brown representability property of Definition A.11.1. By Lemma A.11.2, $\otimes$ therefore has a right adjoint $\underline{\mathrm{Hom}}$.

**Definition 5.4.1.** Let $\mathcal{R}_I \subset \mathcal{T}$ be the localising subcategory generated by objects of the form $\mathrm{Cone}(X \otimes I \xrightarrow{1\otimes p} X)$ for $X \in \mathcal{T}$. We write $\mathcal{T}_I$ for the Verdier quotient $\mathcal{T}/\mathcal{R}_I$.

**Proposition 5.4.2.**

(1) *The functor* $\underline{\mathrm{Hom}}_{\mathcal{T}}(I, -)$ *is strongly additive.*
(2) *The category* $\mathcal{T}_I$ *is compactly generated, hence has the Brown representability property.*
(3) *The localisation functor* $L^I : \mathcal{T} \to \mathcal{T}_I$ *has a (fully faithful) right adjoint* $j^I$*, which also has a right adjoint* $R^I$*.*
(4) *The essential image of* $j^I$ *consists of the* $I$*-local objects (Definition 5.2.4 a)).*
(5) *The tensor structure on* $\mathcal{T}$ *induces a tensor structure on* $\mathcal{T}_I$*.*

*Proof.* For $(X_j)_{j\in J}$ a family of objects of $\mathcal{T}$, the invertibility of the map

$$\bigoplus \underline{\mathrm{Hom}}_{\mathcal{T}}(I, X_j) \to \underline{\mathrm{Hom}}_{\mathcal{T}}(I, \bigoplus X_j)$$

can be tested on a set of compact generators; it then follows from the hypothesis that $\mathcal{T}^c$ is preserved under $-\otimes I$. This also implies that $\mathcal{R}_I$ is generated by a set of compact objects of $\mathcal{T}$, hence (2) follows from Theorem A.11.9. Then (3) follows from Corollary A.11.10. (4) is obvious by adjunction, and (5) follows from the fact that if $A \in \mathcal{R}_I$ and $B \in \mathcal{T}$, then $A \otimes B \in \mathcal{R}_I$. □

*Remark* 5.4.3. The functor $j^I R^I$ can be described by a double adjunction as follows: for $X, Y \in \mathcal{T}$, we have

$$\mathcal{T}(X, j^I R^I Y) = \mathcal{T}(L^I X, R^I Y) = \mathcal{T}(j^I L^I X, Y).$$

Our main theorem in this section is a computation of the localisation functor $j^I L^I$ in terms of $I_\nu^\bullet$ (see Definition 5.2.2). Ideally it should be expressed in the above framework. Unfortunately, we do not know how to totalise $I_\nu^\bullet$ into an object of $\mathcal{T}$ in general (compare [9, §3]). So we take refuge in the situation where $\mathcal{T}$ is of the form $D(\mathcal{A})$ for an abelian category $\mathcal{A}$, and where $I \in \mathcal{A}$.

The proof of the following theorem will occupy the next two subsections (see Theorem 5.6.3).

**Theorem 5.4.4.** *Under suitable additional hypotheses (5.6.1 below), there is a canonical isomorphism*

$$j^I L^I(K) \simeq \underline{\mathrm{Hom}}_{D(\mathcal{A})}(I_\nu^\bullet, K)$$

*for any $K \in D(\mathcal{A})$.*

5.5. **Monadic intermezzo.** Let $\mathcal{C}$ be a category and $(C, \eta, \mu)$ be a monad in $\mathcal{C}$ in the sense of [32, Ch. VI]. Recall what this means:

- $C$ is an endofunctor of $\mathcal{C}$.
- $\eta : \mathrm{Id} \to C$ is a natural transformation (unit).
- $\mu : C^2 \to C$ is a natural transformation (multiplication).
- For any $X \in \mathcal{C}$, we have the identities

$$\mu_X \circ C(\mu_X) = \mu_X \circ \mu_{C(X)} \tag{5.7}$$

$$\mu_X \circ C(\eta_X) = \mu_X \circ \eta_{C(X)} = 1_{C(X)}. \tag{5.8}$$

We shall not use (5.7) in the sequel.

Let $C(\mathcal{C})$ be the strictly full subcategory of $\mathcal{C}$ generated by the image of $C$: an object of $\mathcal{C}$ is in $C(\mathcal{C})$ if and only if it is isomorphic to $C(X)$ for some $X \in \mathcal{C}$; the morphisms of $C(\mathcal{C})$ are the morphisms of $\mathcal{C}$.

**Proposition 5.5.1.** *a) If $\mu$ is a natural isomorphism, the full embedding $j : C(\mathcal{C}) \hookrightarrow \mathcal{C}$ has the left adjoint $C$.*
*b) Let $C_*$ be a second monad in $\mathcal{C}$. Assume that the condition of a) holds for $C$ and $C_*$, and that*

(i) $C_*(\mathcal{C}) \subseteq C(\mathcal{C})$.
(ii) *For any $X \in C(\mathcal{C})$, the unit map $X \to C_*(X)$ is an isomorphism.*

*Then there is a natural isomorphism $C \simeq C_*$.*

*Proof.* a) Let $Y \in C(\mathcal{C})$ and choose an isomorphism $u : Y \xrightarrow{\sim} C(X)$ with $X \in \mathcal{C}$. By assumption, $\eta_Y : Y \to C(Y)$ is an isomorphism, thus the second equality of (5.8) and the naturality of $\eta$ imply that the composite

$$\varepsilon_Y : C(Y) \xrightarrow{C(u)} C^2(X) \xrightarrow{\mu_X} C(X) \xrightarrow{u^{-1}} Y$$

is the inverse of $\eta_Y$, hence does not depend on the choice of $u, X$. One then easily checks that $\varepsilon_Y$ for $Y \in C(\mathcal{C})$ defines a natural transformation $\varepsilon : Cj \to \mathrm{Id}$ and that $(\eta, \varepsilon)$ provides the unit and counit of the desired adjunction.

In b), (i) implies that for any $X \in \mathcal{C}$, the unit $X \to C_*(X)$ factors through the unit $X \to C(X)$ (use a)). On the other hand, (ii) implies that $C(\mathcal{C}) \subseteq C_*(\mathcal{C})$, so the same reasoning shows that the converse is true. □

*Remark* 5.5.2. The converse of a) is certainly false in general. The point is that a given endofunctor $C$ on $\mathcal{C}$ might have two completely different monad structures. However, if $(\eta, \mu)$ yields an adjunction between $j$ and $C$, then $\mu$ must be a natural isomorphism because $j$ is fully faithful. In particular, if we start from an adjunction $(j, C)$ with $j$ fully faithful, then the multiplication of the monad $jC$ is a natural isomorphism.

5.6. **A formula for $j^I L^I$.** Let $\mathcal{A}$ be a closed tensor Grothendieck category equipped with an interval $(I, p, i_0, i_1, \mu)$. We use the notation introduced in Definition 5.4.1. We assume that $\mathcal{A}$ has a set of compact generators which is preserved by tensor product with $I$[4]. We also assume that the tensor structure of $\mathcal{A}$ is right exact and that $D(\mathcal{A})$ is provided with a tensor structure $\otimes_{D(\mathcal{A})}$ with the following properties:

**Hypothesis 5.6.1.**

(i) $\otimes_{D(\mathcal{A})}$ is right $t$-exact and strongly biadditive.

(ii) Let $\otimes_{K(\mathcal{A})}$ be the canonical extension of $\otimes_{\mathcal{A}}$ to $K(\mathcal{A})$. Then the localisation functor $\lambda : K(\mathcal{A}) \to D(\mathcal{A})$ is lax monoidal, i.e., there is a collection of morphisms

$$\lambda C \otimes_{D(\mathcal{A})} \lambda D \to \lambda(C \otimes_{K(\mathcal{A})} D)$$

binatural in $(C, D) \in K(\mathcal{A}) \times K(\mathcal{A})$ and commuting with the associativity and commutativity constraints.

(iii) The object $\lambda \mathbf{1}_{\mathcal{A}}[0]$ is a unit of $\otimes_{D(\mathcal{A})}$.

(iv) The map $(\lambda I[0])^{\otimes_{D(\mathcal{A})} n} \to \lambda(I^{\otimes_{\mathcal{A}} n}[0])$ induced by (ii) is an isomorphism for all $n \geq 0$.

Then $\mathcal{T} = D(\mathcal{A})$ equipped with the interval $\lambda I[0]$ verifies the hypotheses of §5.4; we abbreviate $\lambda I[0]$ to $I$. Hence the localisation functor $L^I : D(\mathcal{A}) \to D(\mathcal{A})_I = D(\mathcal{A})/\mathcal{R}_I$ has a right adjoint $j^I$ (see Proposition 5.4.2). By adjunction, the composed functor $j^I L^I$ has a canonical monad structure. Note that its multiplication is an isomorphism because $j^I$ is fully faithful (compare Remark 5.5.2).

**Definition 5.6.2.** For $K \in D(\mathcal{A})$, we let

$$RC_*^I(K) = \underline{\mathrm{Hom}}_{D(\mathcal{A})}(I_\nu^\bullet, K) \in D(\mathcal{A}).$$

[4]This condition is convenient but not essential, see Remark A.11.11.

Here we view the complex $I_\nu^\bullet$ as an object of $D(\mathcal{A})$. We call $RC_*^I(K)$ the *derived cubical Suslin complex of* $K$ (relative to $I$).

The comonoidal structure on $I_\nu^\bullet$ given by (5.1), (5.2) induces a monad structure on $RC_*^I$. The comonoidal structure on $I_\nu^\bullet$:

$$\pi^\bullet : I_\nu^\bullet \to \mathbf{1}, \quad \Delta^\bullet : I_\nu^\bullet \to \mathrm{Tot}(I_\nu^\bullet \otimes I_\nu^\bullet)$$

given by (5.1), (5.2) induces a monad structure on $RC_*^I$. For example the multiplication is given by

$$\begin{aligned} RC_*^I(RC_*^I(K)) &= \underline{\mathrm{Hom}}_{D(\mathcal{A})}(I_\nu^\bullet, \underline{\mathrm{Hom}}_{D(\mathcal{A})}(I_\nu^\bullet, K)) \\ &\simeq \underline{\mathrm{Hom}}_{D(\mathcal{A})}(I_\nu^\bullet \otimes I_\nu^\bullet, K)) \\ &\xrightarrow{(\Delta^\bullet)^*} \underline{\mathrm{Hom}}_{D(\mathcal{A})}(I_\nu^\bullet, K)) = RC_*^I(K) \end{aligned}$$

Note that the last map is an isomorphism by Proposition 5.3.1. The following theorem completes the proof of Theorem 5.4.4.

**Theorem 5.6.3.** *The two monads $j^I L^I$ and $RC_*^I$ are naturally isomorphic.*

For any $K \in D(\mathcal{A})$, the monad structure on $RC_*^I$ provides us with a natural morphism in $D(\mathcal{A})$:

$$\eta_K : K \to RC_*^I(K). \tag{5.9}$$

We prove the following result concurrently with Theorem 5.6.3.

**Theorem 5.6.4.** *Let $K \in D(\mathcal{A})$.*
*a) The complex $RC_*^I(K)$ is $I$-local (Definition 5.2.4 a)).*
*b) The morphism (5.9) is an isomorphism if and only if $K$ is $I$-local.*
*c) The morphism (5.9) is an $I$-equivalence (Definition 5.2.4 b)).*

*Proof of Theorems 5.6.3 and 5.6.4.* (Compare [49, proof of Lemma 3.2.2] or [30, proof of Lemma 9.14].) We first prove Theorem 5.6.4 a) and b). In view of Definition 5.2.4 and Hypothesis 5.6.1 (iv), a) follows from Proposition 5.3.1 by adjunction. In b), if $K$ is $I$-local, we have $\underline{\mathrm{Hom}}(I_\nu, K) = 0$ and hence

$$\begin{aligned} \underline{\mathrm{Hom}}_{D(\mathcal{A})}(I_\nu^n, K) &\simeq \underline{\mathrm{Hom}}_{D(\mathcal{A})}(I_\nu^{n-1} \otimes I_\nu, K) \\ &\simeq \underline{\mathrm{Hom}}_{D(\mathcal{A})}(I_\nu^{n-1}, \underline{\mathrm{Hom}}_{D(\mathcal{A})}(I_\nu, K)) = 0 \ \text{ for } n > 0, \end{aligned}$$

which implies that (5.9) is an isomorphism. Converselly, if (5.9) is an isomorphism, then $K$ is $I$-local by a).

Next we show Theorem 5.6.3. As mentioned before Definition 5.6.2, the multiplication of the monad $j^I L^I$ is an isomorphism, and the same is true for $RC_*^I$ as proven above. Theorem 5.6.3 now follows from Theorem 5.6.4 a), b) and Proposition 5.5.1 b).

Finally, Theorem 5.6.4 c) follows from Theorem 5.6.3. □

**Corollary 5.6.5.** *a) For any $K \in \mathcal{R}_I$, $RC_*^I(K) = 0$ in $D(\mathcal{A})$.*
*b) The functor $RC_*^I$ is strongly additive.*
*c) The localising subcategory $\mathcal{R}_I \subset D(\mathcal{A})$ is generated by the cones of the $X \to RC_*^I(X)$ for $X \in D(\mathcal{A})$. In particular, $K \in D(\mathcal{A})$ is $I$-local if and only if the natural map*

$$\operatorname{Hom}_{D(\mathcal{A})}(RC_*^I(X), K[i]) \to \operatorname{Hom}_{D(\mathcal{A})}(X, K[i])$$

*is an isomorphism for any $X \in D(\mathcal{A})$ and any $i \in \mathbb{Z}$.*

*Proof.* a) This is obvious from Theorem 5.6.3 since $\mathcal{R}_I \cap jD(\mathcal{A})_I = 0$, the two categories being mutually orthogonal.

b) This follows from Theorem 5.6.3 and the strong additivity of $j^I$ and $L^I$ (Example A.11.3).

c) By Theorem 5.6.4 c), for any $X \in D(\mathcal{A})$ the cone of $X \to RC_*^I(X)$ vanishes in $D(\mathcal{A})_I$, hence it is in $\mathcal{R}_I$. Conversely, let $\mathcal{R}_I' \subset D(\mathcal{A})$ be the localising subcategory generated by these cones. In the commutative diagram

$$\begin{tikzcd}
I[0] \otimes_{D(\mathcal{A})} X \arrow[r, "p"] \arrow[d] & X \arrow[d] \\
RC_*^I(I[0] \otimes_{D(\mathcal{A})} X) \arrow[r, "p'"] & RC_*^I(X)
\end{tikzcd}$$

$p'$ is an isomorphism by a), hence the cone of $p$ belongs to $\mathcal{R}_I'$. The last statement follows. □

*Remark* 5.6.6. Suppose that $\mathcal{A}$ is generated by a class of objects $\mathcal{E}$ of finite Ext-dimension (see Lemma A.13.12). By Remark A.13.13, $\coprod_{n\in\mathbb{Z}} \mathcal{E}[n]$ then generates $D(\mathcal{A})$. By Corollary 5.6.5 b), the condition in Corollary 5.6.5 c) may then be restricted to $X$ of the form $A[0]$ for $A \in \mathcal{E}$.

5.7. **Comparison of intervals.** Let $(\mathcal{A}, I), (\mathcal{A}', I')$ be as in §5.6. We give ourselves a right exact cocontinuous $\otimes$-functor $T : \mathcal{A} \to \mathcal{A}'$ sending $I$ to $I'$ and respecting the constants of structure of $I$ and $I'$. By Theorem A.12.1 b), $T$ has a right adjoint $S$. We assume that $T$ has a total left derived functor $LT : D(\mathcal{A}) \to D(\mathcal{A}')$, which is strongly additive, a $\otimes$-functor and sends $I[0]$ to $I'[0]$ (this is automatic if $T$ is exact). By Brown representability (Lemma A.11.2 and Theorem A.13.1 a)), $LT$ has a right adjoint $RS$, which is the total right derived functor of $S$. Then $LT$ induces a triangulated $\otimes$-functor $\overline{LT} : D(\mathcal{A})_I \to D(\mathcal{A}')_{I'}$ via $L^I$ and $L^{I'}$.

**Lemma 5.7.1.** *Let $j^I$ and $j^{I'}$ be the right adjoints of the localisation functors $L^I : D(\mathcal{A}) \to D(\mathcal{A})_I$ and $L^{I'} : D(\mathcal{A}') \to D(\mathcal{A}')_{I'}$. Then $RS$ sends $j^{I'} D(\mathcal{A}')_{I'}$ into $j^I D(\mathcal{A})_I$, and the induced functor $\overline{RS} : D(\mathcal{A}')_{I'} \to D(\mathcal{A})_I$ is right adjoint to $\overline{LT}$.* □

By construction, we have a natural isomorphism

$$RSj^{I'} \simeq j^I \overline{RS} \tag{5.10}$$

from which we deduce two "base change morphisms"

$$L^I \circ RS \Rightarrow \overline{RS} \circ L^{I'} \tag{5.11}$$

$$LT \circ j^I \Rightarrow j^{I'} \circ \overline{LT}. \tag{5.12}$$

**Theorem 5.7.2.** (5.11) *is an isomorphism.*

*Proof.* The monoidality of $LT$ yields the following identity, for $(X, K) \in D(\mathcal{A}) \times D(\mathcal{A}')$ (Lemma A.7.1):

$$\underline{\mathrm{Hom}}_{D(\mathcal{A})}(X, RSK) \simeq RS\, \underline{\mathrm{Hom}}_{D(\mathcal{A}')}(LTX, K). \tag{5.13}$$

Apply (5.13) to $X = I_\nu^\bullet$: we get an isomorphism

$$RC_*^I(RSK) \simeq RSRC'_*(K).$$

In view of Theorem 5.6.3, this converts to an isomorphism

$$j^I L^I RS(K) \simeq RSj^{I'} L^{I'}(K)$$

hence to an isomorphism $L^I RS(K) \simeq \overline{RS} L^{I'}(K)$ in view of (5.10) and the full faithfulness of $j^I$. One checks that this isomorphism coincides with (5.11). □

**Definition 5.7.3.** We say that $T$ *verifies Condition (V)* if (5.12) is an isomorphism.

**Lemma 5.7.4.** *$T$ verifies Condition (V) if and only if $LT(j^I D(\mathcal{A})_I) \subseteq j^{I'} D(\mathcal{A}')_{I'}$.*

*Proof.* "Only if" is obvious. Conversely, let $X \in D(\mathcal{A})_I$ be such that $LTj^I(X) \simeq j^{I'} Y$ for some $Y \in D(\mathcal{A}')_{I'}$. Applying $L^{I'}$, we get

$$Y \simeq L^{I'} j^{I'} Y \simeq L^{I'} LTj^I(X) \simeq \overline{LT} L^I j^I(X) \simeq \overline{LT}(X).$$

Applying $j^{I'}$, it gives an isomorphism

$$LTj^I(X) \simeq j^{I'} Y \simeq j^{I'} \overline{LT}(X)$$

and one checks that this is induced by (5.12). □

## 6. Motives with modulus

6.1. $\overline{\square}$**-invariance.** We start with:

**Lemma 6.1.1.** *Let* $\overline{\square} = (\mathbf{P}^1, \infty) \in \mathbf{MSm}$. *The interval structure of* $\mathbf{A}^1 \simeq \mathbf{P}^1 - \{\infty\} \in \mathbf{Sm}$ *from* [47] *induces an interval structure on* $\overline{\square} \in \mathbf{MSm}$.

*Proof.* We need to check that the structure maps $p, i_0, i_1, \mu$ are morphisms in $\mathbf{MCor}$. The unit object is $(\operatorname{Spec} k, \emptyset)$, so $i_0, i_1$ and $p$ are clearly admissible. As for $\mu$, its points of indeterminacy in $\mathbf{P}^1 \times \mathbf{P}^1$ are $(0, \infty)$ and $(\infty, 0)$; the closure $\Gamma$ of its graph in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ is isomorphic to $Bl_{(0,\infty),(\infty,0)}(\mathbf{P}^1 \times \mathbf{P}^1)$, where the two exceptional divisors are given by $0 \times \infty \times \mathbf{P}^1$ and $\infty \times 0 \times \mathbf{P}^1$. In particular, $\Gamma$ is smooth. Then

$$p_2^* \infty = \mathbf{P}^1 \times \infty \times \infty + \infty \times \mathbf{P}^1 \times \infty$$

while

$$p_1^*(\mathbf{P}^1 \times \infty + \infty \times \mathbf{P}^1) = \mathbf{P}^1 \times \infty \times \infty + 0 \times \infty \times \mathbf{P}^1 + \infty \times \mathbf{P}^1 \times \infty + \infty \times 0 \times \mathbf{P}^1$$

which completes the proof. $\square$

**Definition 6.1.2.** We say $F \in \underline{\mathbf{M}}\mathbf{PST}$ (resp. $F \in \mathbf{MPST}$) is $\overline{\square}$-*invariant* if the projection map $p : M \otimes \overline{\square} \to M$ induces an isomorphism $p^* : F(M) \simeq F(M \otimes \overline{\square})$ for any $M \in \underline{\mathbf{M}}\mathbf{Sm}$ (resp. $M \in \mathbf{MSm}$). Equivalently, $F \xrightarrow{\sim} \underline{\mathrm{Hom}}(\mathbb{Z}_{\mathrm{tr}}(\overline{\square}), F)$.

We collect basic properties of $\overline{\square}$-invariance.

**Lemma 6.1.3.**

(1) *Let* $F \in \underline{\mathbf{M}}\mathbf{PST}$. *Then* $F$ *is* $\overline{\square}$-*invariant* $\Rightarrow \tau^* F \in \mathbf{MPST}$ *is* $\overline{\square}$-*invariant.*

(2) *Let* $G \in \mathbf{MPST}$. *Then* $G$ *is* $\overline{\square}$-*invariant* $\iff \tau_! G \in \underline{\mathbf{M}}\mathbf{PST}$ *is* $\overline{\square}$-*invariant.*

(3) *Let* $H \in \mathbf{PST}$. *Then* $H$ *is* $\mathbf{A}^1$-*invariant* $\iff \omega^* H \in \mathbf{MPST}$ *is* $\overline{\square}$-*invariant.*

*Proof.* Using Lemma A.7.1, $\Rightarrow$ in (1), (2) and (3) respectively follows from the monoidality of $\tau_!$, $\tau^!$ and $\omega_!$ (Propositions 2.2.1 and 2.4.1) and the identities $\tau\overline{\square} = \overline{\square}$, $\tau^!\overline{\square} = \overline{\square}$, $\omega\overline{\square} = \mathbf{A}^1$, while $\Leftarrow$ in (2) and (3) follows from the full faithfulness of $\tau_!$ and $\omega^*$ (same references). $\square$

6.2. **The category** $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$**.** Inside the homotopy category $K^b(\underline{\mathbf{M}}\mathbf{Cor})$ of bounded complexes on $\underline{\mathbf{M}}\mathbf{Cor}$, we consider two classes of objects of the following type:

($\underline{\mathrm{CI}}1$) for $M \in \underline{\mathbf{M}}\mathbf{Sm}$,

$$[M \otimes \overline{\square}] \to [M];$$

(MV1) for $M \in \underline{\mathbf{M}}\mathbf{Sm}$ and an elementary Nisnevich cover $(U, V)$ of $\overline{M}$,

$$[M_{U \times_{\overline{M}} V}] \to [M_U] \oplus [M_V] \to [M].$$

(For an étale morphism $f : U \to \overline{M}$, we put $M_U = (U, f^*(M^\infty))$.)

**Definition 6.2.1.** We define $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ to be the pseudo-abelian envelope of the localisation of $K^b(\underline{\mathbf{M}}\mathbf{Cor})$ by the smallest thick subcategory containing all objects of the form ($\underline{\mathrm{CI}}1$) and (MV1) above. (One may replace 'thick' by 'triangulated'; the resulting category will be equivalent.) We have an obvious functor $\underline{M} : \underline{\mathbf{M}}\mathbf{Cor} \to \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$.

6.3. **The category $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$.** Inside the derived category $D(\underline{\mathbf{M}}\mathbf{PST})$ of unbounded complexes on $\underline{\mathbf{M}}\mathbf{PST}$, we consider two classes of objects of the following type:

($\underline{\mathrm{CI}}2$) for $M \in \underline{\mathbf{M}}\mathbf{Sm}$,

$$\mathbb{Z}_{\mathrm{tr}}(M \otimes \overline{\square}) \to \mathbb{Z}_{\mathrm{tr}}(M);$$

(MV2) for $M \in \underline{\mathbf{M}}\mathbf{Sm}$ and an elementary Nisnevich cover $(U, V)$ of $\overline{M}$,

$$\mathbb{Z}_{\mathrm{tr}}(M_{U \times_{\overline{M}} V}) \to \mathbb{Z}_{\mathrm{tr}}(M_U) \oplus \mathbb{Z}_{\mathrm{tr}}(M_V) \to \mathbb{Z}_{\mathrm{tr}}(M).$$

**Definition 6.3.1.** We define $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ to be the localisation of $D(\underline{\mathbf{M}}\mathbf{PST})$ by the smallest localising subcategory containing all objects of the form ($\underline{\mathrm{CI}}2$) and (MV2) above. (Then $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ is already pseudo-abelian.)

We put $\underline{\mathbf{M}}\mathbf{NST} := \underline{\mathbf{M}}\mathbf{PST}_{\mathrm{Nis}}$ and $\underline{\mathbf{M}}\mathbf{NS}^{\mathrm{fin}} := \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}_{\mathrm{Nis}}$.

**Proposition 6.3.2.** *The functor $\underline{a}_{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{PST} \to \underline{\mathbf{M}}\mathbf{NST}$ induces a localisation functor $D(\underline{a}_{\mathrm{Nis}}) : D(\underline{\mathbf{M}}\mathbf{PST}) \to D(\underline{\mathbf{M}}\mathbf{NST})$ whose kernel is generated by (MV2) as a localising subcategory. In particular, the localisation functor $D(\underline{\mathbf{M}}\mathbf{PST}) \to \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ factors through $\underline{L}^{\overline{\square}} : D(\underline{\mathbf{M}}\mathbf{NST}) \to \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$, which realises $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ as the localization of $D(\underline{\mathbf{M}}\mathbf{NST})$ with respect to the localising subcategory generated by ($\underline{CI}2$). The categories $D(\underline{\mathbf{M}}\mathbf{NST})$ and $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ are compactly generated.*

*Proof.* For the first statement, we observe that the hypotheses of Proposition A.13.6 are verified with $\mathcal{A} = \underline{\mathbf{M}}\mathbf{NST}$, $\mathcal{B} = \underline{\mathbf{M}}\mathbf{PST}$, $F = \underline{i}_{\mathrm{Nis}}$ and $G = \underline{a}_{\mathrm{Nis}}$, by Proposition 3.5.3. By Example A.11.6, $D(\underline{\mathbf{M}}\mathbf{PST})$ is compactly generated and the objects of the form ($\underline{\mathrm{CI}}2$) and (MV2) are compact. Let $\mathcal{D}$ be the kernel of $D(\underline{\mathbf{M}}\mathbf{PST}) \to D(\underline{\mathbf{M}}\mathbf{NST})$ and $\mathcal{I}$ the localising subcategory of $D(\underline{\mathbf{M}}\mathbf{PST})$ generated by (MV2). We must show $\mathcal{D} = \mathcal{I}$. By Theorem 3.5.7, we have $\mathcal{D} \supset \mathcal{I}$. By Theorem A.11.7, it suffices to show that $\mathcal{I}^\perp = 0$, where $\mathcal{I}^\perp$ is computed inside $\mathcal{D}$.

Let $\mathcal{D}'$ be the kernel of $D(\underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}) \to D(\underline{\mathbf{M}}\mathbf{NS}^{\mathrm{fin}})$ and $\mathcal{I}'$ the localising subcategory of $D(\underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}})$ generated by (MV2). Since the functor $D(\underline{\mathbf{M}}\mathbf{PST}) \to D(\underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}})$ is clearly conservative, we are reduced to showing $\mathcal{D}' = \mathcal{I}'$, or $(\mathcal{I}')^\perp = 0$: the proof is identical to that of [6, Proposition in §4.2.1]. The last statements now follow from Theorem A.11.9. $\square$

*Remark* 6.3.3. Let us give more details on the proof of $(\mathcal{I}')^\perp = 0$ in the above argument: it is really a version of the theorem of Brown and Gersten as abstracted by Voevodsky in [51]. More precisely, Voevodsky defines in loc. cit. a notion of cd-structure on a category $\mathcal{C}$, given by a collection of commutative squares satisfying simple axioms, as well as notions of completeness and boundedness. A cd-strcture canonically defines a topology on $\mathcal{C}$. Theorem 3.2 of loc. cit. says that a complete and bounded cd-structure verifies the analogue of [10, Th. 1]. On the other hand, Voevodsky proves in [52, Th. 2.2] that the elementary Nisnevich squares define a complete bounded cd-strcture on the category of schemes. Let now $M \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. By transport of structure, Lemma 3.1.4 yields a complete and bounded cd-structure on the category $M_{\mathrm{Nis}}$, with associated topology the Nisnevich topology of Definition 3.1.3 (2). This justifies the equality $(\mathcal{I}')^\perp = 0$.

6.4. **Tensor structures.** By Theorem A.13.2, $K^b(\underline{\mathbf{M}}\mathbf{Cor})$ and $D(\underline{\mathbf{M}}\mathbf{PST})$ inherit tensor structures induced by the one of $\underline{\mathbf{M}}\mathbf{Cor}$ (Definition 1.4.1), which are strongly biadditive.

**Proposition 6.4.1.**

(1) *The tensor structure on* $K^b(\underline{\mathbf{M}}\mathbf{Cor})$ *induces a tensor structure on* $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ *via the localisation functor* $K^b(\underline{\mathbf{M}}\mathbf{Cor}) \to \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$.

(2) *The tensor structure on* $D(\underline{\mathbf{M}}\mathbf{PST})$ *induces a tensor structure on* $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ *via the localisation functor* $D(\underline{\mathbf{M}}\mathbf{PST}) \to \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$.

(3) *The tensor structure on* $D(\underline{\mathbf{M}}\mathbf{PST})$ *induces a tensor structure on* $D(\underline{\mathbf{M}}\mathbf{NST})$ *via the localisation functor* $D(\underline{\mathbf{M}}\mathbf{PST}) \to D(\underline{\mathbf{M}}\mathbf{NST})$ *from Proposition 6.3.2, and* $\underline{L}^{\overline{\square}} : D(\underline{\mathbf{M}}\mathbf{NST}) \to \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ *is monoidal.*

*Proof.* As all statements are proven in the same manner, we only provide a proof of (1). Let $\mathcal{R}$ be the thick subcategory of $K^b(\underline{\mathbf{M}}\mathbf{Cor})$ generated by objects of the form ($\underline{\mathrm{CI}}$1) and (MV1). We need to show that if $A \in \mathcal{R}$ and $K^b(\underline{\mathbf{M}}\mathbf{Cor})$, then $A \otimes B \in \mathcal{R}$. For this we may reduce to the case where $A$ is as in ($\underline{\mathrm{CI}}$1) or (MV1), and $B = \mathbb{Z}_{\mathrm{tr}}(N)$ for $M, N \in \underline{\mathbf{M}}\mathbf{Sm}$. For ($\underline{\mathrm{CI}}$1) this is obvious, and for (MV1) this follows by noting that $(U \times \overline{N}, V \times \overline{N})$ is an elementary Nisnevich cover of $\overline{M} \times \overline{N}$. □

*Remark* 6.4.2. One could also obtain a tensor structure on $D(\underline{\mathbf{M}}\mathbf{PST}_{\text{ét}})$ by mimicking the arguments in [30, Ch. 6], but it would be tedious. Since we only need to deal with $D(\underline{\mathbf{M}}\mathbf{NST})$ in this paper, we chose to use the above shortcut which was inspired by the same method in [6].

6.5. **The functor $\underline{L}^{\overline{\square}}$.** As a special case of Definition 5.2.4, we have:

**Definition 6.5.1.** An object $K \in D(\underline{\mathbf{M}}\mathbf{NST})$ is called $\overline{\square}$-local if one of the following equivalent conditions is satisfied:

(a) $H^i_{\mathrm{Nis}}(\overline{M}, K) \simeq H^i_{\mathrm{Nis}}(\overline{M} \otimes \overline{\square}, K)$ for all $M \in \underline{\mathbf{M}}\mathbf{Sm}$ and $i \in \mathbb{Z}$,
(b) $\mathrm{Hom}_{D(\underline{\mathbf{M}}\mathbf{NST})}(L, K[i]) = 0$ for all $L$ of the form ($\underline{\mathrm{CI}}$2) and $i \in \mathbb{Z}$,
(c) $K \to \underline{\mathrm{Hom}}_{D(\underline{\mathbf{M}}\mathbf{NST})}(\mathbb{Z}_{\mathrm{tr}}(\overline{\square}), K)$ is an isomorphism.

(See §6.6 for $\underline{\mathrm{Hom}}_{D(\underline{\mathbf{M}}\mathbf{NST})}$.)

The equivalence follows from Proposition 3.6.2 and

$$\begin{aligned}&\mathrm{Hom}_{D(\underline{\mathbf{M}}\mathbf{NST})}(\mathbb{Z}_{\mathrm{tr}}(M \otimes \overline{\square}),\ K[i])\\ &= \mathrm{Hom}_{D(\underline{\mathbf{M}}\mathbf{NST})}(\mathbb{Z}_{\mathrm{tr}}(M),\ \underline{\mathrm{Hom}}_{D(\underline{\mathbf{M}}\mathbf{NST})}(\mathbb{Z}_{\mathrm{tr}}(\overline{\square}),\ K)[i]).\end{aligned}$$

**Theorem 6.5.2.** *The localisation functor $\underline{L}^{\overline{\square}} : D(\underline{\mathbf{M}}\mathbf{NST}) \to \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ of Proposition 6.3.2 has a fully faithful right adjoint $\underline{j}^{\overline{\square}}$ which itself has a right adjoint $\underline{R}^{\overline{\square}}$. Its essential image consists of $\overline{\square}$-local objects in $D(\underline{\mathbf{M}}\mathbf{NST})$.*

*Proof.* This is a special case of Proposition 5.4.2. □

6.6. **Internal Hom.** Since $D(\underline{\mathbf{M}}\mathbf{NST})$ and $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ are compactly generated (Proposition 6.3.2), Brown's representability theorem applied to their tensor structures provides them with internal Homs. The following is an application of Lemma A.7.1:

**Proposition 6.6.1.** *Let $K \in D(\underline{\mathbf{M}}\mathbf{NST})$ and $L \in \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$. Then we have a natural isomorphism*

$$\underline{j}^{\overline{\square}}\,\underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}}(\underline{L}^{\overline{\square}}(K), L) \simeq \underline{\mathrm{Hom}}_{D(\underline{\mathbf{M}}\mathbf{NST})}(K, \underline{j}^{\overline{\square}}L)$$

*hence, for $K', L \in \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$, a natural isomorphism*

$$\underline{j}^{\overline{\square}}\,\underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}}(K', L) \simeq \underline{\mathrm{Hom}}_{D(\underline{\mathbf{M}}\mathbf{NST})}(\underline{j}^{\overline{\square}}K', \underline{j}^{\overline{\square}}L).$$

6.7. **The Yoneda embedding.** The Yoneda functor $\underline{\mathbf{M}}\mathbf{Cor} \to \underline{\mathbf{M}}\mathbf{PST}$, $M \mapsto \mathbb{Z}_{\mathrm{tr}}(M)$ induces a tensor functor

$$K^b(\underline{\mathbf{M}}\mathbf{Cor}) \to D(\underline{\mathbf{M}}\mathbf{PST}). \tag{6.1}$$

This sends ($\underline{\mathrm{CI}}$1) to ($\underline{\mathrm{CI}}$2) and (MV1) to (MV2), yielding a $\otimes$-functor

$$\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}. \tag{6.2}$$

**Theorem 6.7.1.** *Both functors* (6.1) *and* (6.2) *are fully faithful with dense image. Their essential images consist of the compact objects of* $D(\underline{\mathbf{M}}\mathbf{PST})$ *and of* $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$*.*

*Proof.* Apply Theorem A.11.8 and Example A.11.6. □

By Proposition 6.3.2 and Theorem 6.7.1, we get a commutative diagram

$$\begin{tikzcd} K^b(\underline{\mathbf{M}}\mathbf{Cor}) \arrow[r] \arrow[d] & D(\underline{\mathbf{M}}\mathbf{NST}) \arrow[d, "\underline{L}^{\overline{\square}}"] \\ \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}} \arrow[r] & \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}. \end{tikzcd} \tag{6.3}$$

6.8. **The derived Suslin complex.** We shall need:

**Proposition 6.8.1.** *The interval structure on* $\overline{\square} \in \mathbf{MSm}$ *from Lemma 6.1.1 yields a category with interval* $(\underline{\mathbf{M}}\mathbf{NST}, \mathbb{Z}_{\mathrm{tr}}(\overline{\square}))$ *which verifies the hypotheses 5.6.1.*

*Proof.* Since $\otimes_{D(\underline{\mathbf{M}}\mathbf{PST})}$ is the total derived functor of $\otimes_{\underline{\mathbf{M}}\mathbf{PST}}$ by Theorem A.13.2 d), there is a canonical natural transformation

$$\underline{\lambda}_P C \otimes_{D(\underline{\mathbf{M}}\mathbf{PST})} \underline{\lambda}_P D \Rightarrow \underline{\lambda}_P (C \otimes_{K(\underline{\mathbf{M}}\mathbf{PST})} D)$$

for $(C, D) \in K(\underline{\mathbf{M}}\mathbf{PST}) \times K(\underline{\mathbf{M}}\mathbf{PST})$, where $\underline{\lambda}_P : K(\underline{\mathbf{M}}\mathbf{PST}) \to D(\underline{\mathbf{M}}\mathbf{PST})$ is the localisation functor. Applying $D(\underline{a}_{\mathrm{Nis}})$ to it, we get a natural transformation

$$\begin{aligned} &\underline{\lambda}_N K(\underline{a}_{\mathrm{Nis}})C \otimes_{D(\underline{\mathbf{M}}\mathbf{NST})} \underline{\lambda}_N K(\underline{a}_{\mathrm{Nis}})D \\ &\simeq D(\underline{a}_{\mathrm{Nis}})\underline{\lambda}_P C \otimes_{D(\underline{\mathbf{M}}\mathbf{NST})} D(\underline{a}_{\mathrm{Nis}})\underline{\lambda}_P D \simeq D(\underline{a}_{\mathrm{Nis}})(\underline{\lambda}_P C \otimes_{D(\underline{\mathbf{M}}\mathbf{PST})} \underline{\lambda}_P D) \\ &\quad \Rightarrow D(\underline{a}_{\mathrm{Nis}})\underline{\lambda}_P (C \otimes_{K(\underline{\mathbf{M}}\mathbf{PST})} D) \simeq \underline{\lambda}_N K(\underline{a}_{\mathrm{Nis}})(C \otimes_{K(\underline{\mathbf{M}}\mathbf{PST})} D) \\ &\qquad \simeq \underline{\lambda}_N (K(\underline{a}_{\mathrm{Nis}})C \otimes_{K(\underline{\mathbf{M}}\mathbf{NST})} K(\underline{a}_{\mathrm{Nis}})D) \end{aligned}$$

where $\underline{\lambda}_N : K(\underline{\mathbf{M}}\mathbf{NST}) \to D(\underline{\mathbf{M}}\mathbf{NST})$ is the localisation functor. Since $K(\underline{a}_{\mathrm{Nis}})$ is a localisation, this yields by Lemma A.3.3 the desired natural transformation

$$\underline{\lambda}_N C' \otimes_{D(\underline{\mathbf{M}}\mathbf{NST})} \underline{\lambda}_N D' \Rightarrow \underline{\lambda}_N (C' \otimes_{K(\underline{\mathbf{M}}\mathbf{NST})} D') \tag{6.4}$$

for $(C', D') \in K(\underline{\mathbf{M}}\mathbf{NST}) \times K(\underline{\mathbf{M}}\mathbf{NST})$.

It remains to check properties (iii) and (iv) of Hypothesis 5.6.1: (iii) is obvious by construction, and (iv) is true because it is already true in $D(\underline{\mathbf{M}}\mathbf{PST})$ by the representability of $\mathbb{Z}_{\mathrm{tr}}(\overline{\square})$, and $\underline{a}_{\mathrm{Nis}}$ is exact. $\square$

As in Definition 5.6.2, we then get an object

$$RC_*^{\overline{\square}}(K) = \underline{\mathrm{Hom}}(\mathbb{Z}_{\mathrm{tr}}(\overline{\square}_\nu^\bullet), K) \in D(\underline{\mathbf{M}}\mathbf{NST})$$

attached to any $K \in D(\underline{\mathbf{M}}\mathbf{NST})$: this is the *derived Suslin complex of* $K$. As a consequence of Theorem 5.6.3, we have:

**Theorem 6.8.2.** *For any $K \in D(\underline{\mathbf{M}}\mathbf{NST})$, we have an isomorphism*

$$\underline{j}^{\overline{\square}}\underline{L}^{\overline{\square}}(K) \simeq RC_*^{\overline{\square}}(K)$$

*where $\underline{L}^{\overline{\square}}, \underline{j}^{\overline{\square}}$ are as in Theorem 6.5.2.*

*Remarks* 6.8.3. a) By Proposition 6.4.1 (3), we have a natural isomorphism

$$D(\underline{a}_{\mathrm{Nis}})(C \otimes_{D(\underline{\mathbf{M}}\mathbf{PST})} D) \simeq D(\underline{a}_{\mathrm{Nis}})C \otimes_{D(\underline{\mathbf{M}}\mathbf{NST})} D(\underline{a}_{\mathrm{Nis}})D$$

for any $C, D \in D(\underline{\mathbf{M}}\mathbf{PST})$. In the special case $C = F[0]$, $D = G[0]$ for $F, G \in \underline{\mathbf{M}}\mathbf{PST}$, taking $H_0$ this gives

$$\begin{aligned} \underline{a}_{\mathrm{Nis}}F \otimes_{\underline{\mathbf{M}}\mathbf{NST}} \underline{a}_{\mathrm{Nis}}G &\simeq \underline{a}_{\mathrm{Nis}}(F \otimes_{\underline{\mathbf{M}}\mathbf{PST}} G) \\ &\simeq H_0(\underline{a}_{\mathrm{Nis}}F[0] \otimes_{D(\underline{\mathbf{M}}\mathbf{NST})} \underline{a}_{\mathrm{Nis}}G[0]). \end{aligned}$$

Thus we recover the tensor structure of Proposition 3.10.1 on $\underline{\mathbf{M}}\mathbf{NST}$ by "truncating" the tensor structure of $D(\underline{\mathbf{M}}\mathbf{NST})$. Conversely, proceeding as in [3, Proof of Prop. 4.1.22], it can be shown that the functor $\otimes_{D(\underline{\mathbf{M}}\mathbf{NST})}$ of Proposition 6.4.1 (3) is actually the total left derived functor of $\otimes_{\underline{\mathbf{M}}\mathbf{NST}}$ and that (6.4) is the corresponding universal map.
b) Theorem 5.6.3 also yields a version of Voevodsky's results for $\mathbf{DM}^{\mathrm{eff}}$ and $D(\mathbf{NST})$ [49, 30], where he uses simplicial objects rather than cubical objects. Comparing the two, we get an *a posteriori* proof that for any $K \in D(\mathbf{NST})$ the two "Suslin" complexes $RC_*^{\mathbf{A}^1}(K)$ based on simplicial or cubical sets are quasi-isomorphic. Hopefully this can be proven by an explicit chain computation.

On the other hand, the theory of intervals does not yield a simplicial theory in the case of $\mathbf{MCor}$, see Remark 5.2.6. Nevertheless it is possible to develop such a theory by using a more direct geometric approach.

6.9. **The categories $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ and $\mathbf{MDM}^{\mathrm{eff}}$.** We start with:

**Proposition 6.9.1.**

(1) *An object $K$ of $D(\mathbf{MNST})$ is compact if and only if $D(\tau_{\mathrm{Nis}})(K)$ is compact in $D(\underline{\mathbf{M}}\mathbf{NST})$.*

(2) *The category $D(\mathbf{MNST})$ is compactly generated, and $D(\mathbf{MNST})^c$ is the thick hull (Definition A.11.4 d)) of the essential image of the composite functor*

$$K^b(\mathbf{MCor}) \to D(\mathbf{MPST}) \xrightarrow{D(a_{\mathrm{Nis}})} D(\mathbf{MNST}).$$

(3) *There is a unique tensor structure on $D(\mathbf{MNST})$ for which $D(\tau_{\mathrm{Nis}})$ is monoidal with respect to the monoidal structure of $D(\underline{\mathbf{M}}\mathbf{NST})$ from Proposition 6.4.1 (3). It preserves compact objects, and $D(a_{\mathrm{Nis}})$ is monoidal. The conditions of Hypothesis 5.6.1 are verified.*

*Proof.* The "if" part of (1) follows from the strong additivity and the full faithfulness of $D(\tau_{\mathrm{Nis}})$ (Theorem 3.7.15 b)). Next we consider (2). By Example A.11.3, $K^b(\mathbf{MCor})$ generates $D(\mathbf{MPST})$; the adjunction $(D(a_{\mathrm{Nis}}), Ri_{\mathrm{Nis}})$ then shows that its essential image $\mathcal{I}$ generates $D(\mathbf{MNST})$. Moreover $D(\tau_{\mathrm{Nis}})(\mathcal{I}) \subset D(\underline{\mathbf{M}}\mathbf{NST})^c$ by Theorem A.11.9, hence $\mathcal{I} \subset D(\mathbf{MNST})^c$ by "if" in (1). The last statement of (2) then follows from Lemma A.11.12, and implies "only if" in (1).

Let us now prove (3). For the first statement, by the full faithfulness of $D(\tau_{\mathrm{Nis}})$ it suffices to show that if $K, L \in D(\mathbf{MNST})$, then $D(\tau_{\mathrm{Nis}})(K) \otimes D(\tau_{\mathrm{Nis}})(L)$ is in the essential image of $D(\tau_{\mathrm{Nis}})$. Write $K \simeq D(a_{\mathrm{Nis}})\tilde{K}$, $L \simeq D(a_{\mathrm{Nis}})\tilde{L}$. Then

$$\begin{aligned} D(\tau_{\mathrm{Nis}})D(a_{\mathrm{Nis}})(\tilde{K} \otimes \tilde{L}) &\simeq D(\underline{a}_{\mathrm{Nis}})D(\tau_!)(\tilde{K} \otimes \tilde{L}) \\ &\simeq D(\underline{a}_{\mathrm{Nis}})D(\tau_!)(\tilde{K}) \otimes D(\underline{a}_{\mathrm{Nis}})D(\tau_!)(\tilde{L}) \simeq D(\tau_{\mathrm{Nis}})(K) \otimes D(\tau_{\mathrm{Nis}})(L) \end{aligned}$$

which also yields the monoidality of $D(a_{\mathrm{Nis}})$. Suppose now that $K, L \in D(\mathbf{MNST})^c$. By (1), we may choose $\tilde{K}, \tilde{L}$ in $K^b(\mathbf{MCor})$; since $\tilde{K} \otimes \tilde{L} \in K^b(\mathbf{MCor})$, we find that $D(\tau_{\mathrm{Nis}})(K \otimes L) \in D(\underline{\mathbf{M}}\mathbf{NST})^c$, hence $K \otimes L \in D(\mathbf{MNST})^c$ by "if" in (1).

It remains to check the conditions of Hypothesis 5.6.1. They all follow from the corresponding conditions in Proposition 6.8.1, using the exactness, monoidality, strong additivity and full faithfulness of $\tau_{\mathrm{Nis}}$ and $D(\tau_{\mathrm{Nis}})$. To avoid tediousness, we only justify (ii): for $C, D \in K(\mathbf{MNST})$, applying (6.4) to $C' = K(\tau_{\mathrm{Nis}})C$, $D' = K(\tau_{\mathrm{Nis}})D$ and playing with the monoidality of $K(\tau_{\mathrm{Nis}})$ and $D(\tau_{\mathrm{Nis}})$ and the commutation $D(\tau_{\mathrm{Nis}})\lambda_N = \underline{\lambda}_N K(\tau_{\mathrm{Nis}})$ yields a natural transformation

$$D(\tau_{\mathrm{Nis}})(\lambda_N C \otimes_{D(\mathbf{MNST})} \lambda_N D) \Rightarrow D(\tau_{\mathrm{Nis}})(\lambda_N(C \otimes_{K(\mathbf{MNST})} D))$$

whence a natural transformation

$$\lambda_N C \otimes_{D(\mathbf{MNST})} \lambda_N D \Rightarrow \lambda_N (C \otimes_{K(\mathbf{MNST})} D)$$

by the full faithfulness of $D(\tau_{\mathrm{Nis}})$. □

*Remark* 6.9.2. We cannot apply Theorem A.11.9 directly to the proof of Proposition 6.9.1, because we do not know whether $\operatorname{Ker} D(a_{\mathrm{Nis}})$ is generated by a set of compact objects of $D(\mathbf{MPST})$.

We now introduce analogous relations to those in §§6.2 and 6.3:

(CI1) for $M \in \mathbf{MSm}$,

$$[M \otimes \overline{\square}] \to [M];$$

(CI2) for $M \in \mathbf{MSm}$,

$$\mathbb{Z}_{\mathrm{tr}}(M \otimes \overline{\square}) \to \mathbb{Z}_{\mathrm{tr}}(M);$$

**Definition 6.9.3.** We write $\mathbf{MDM}^{\mathrm{eff}}$ for the localisation of $D(\mathbf{MNST})$ with respect to (CI2): this makes sense by Proposition 6.9.1 (3). We define $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ as $(\mathbf{MDM}^{\mathrm{eff}})^c$.

We thus have a naturally commutative diagram

$$\begin{array}{ccc} D(\mathbf{MNST}) & \xrightarrow{D(\tau_{\mathrm{Nis}})} & D(\underline{\mathbf{M}}\mathbf{NST}) \\ {\scriptstyle L^{\overline{\square}}}\downarrow & & \downarrow{\scriptstyle \underline{L}^{\overline{\square}}} \\ \mathbf{MDM}^{\mathrm{eff}} & \xrightarrow{\tau_{\mathrm{eff}}} & \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}} \end{array} \tag{6.5}$$

where the bottom row is obtained from the monoidality of $D(\tau_{\mathrm{Nis}})$ as in §5.7, and an induced functor

$$K^b(\mathbf{MCor})/\langle(\mathrm{CI1})\rangle \to \mathbf{MDM}^{\mathrm{eff}}\,. \tag{6.6}$$

**Theorem 6.9.4.**

(1) *The category* $\mathbf{MDM}^{\mathrm{eff}}$ *is compactly generated, and* $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ *is the thick hull of the image of* (6.6).

(2) *In* (6.5)*, the functor* $L^{\overline{\square}}$ *has a right adjoint* $j^{\overline{\square}}$*, which itself has a right adjoint* $R^{\overline{\square}}$*;* $L^{\overline{\square}}$ *induces a tensor structure on* $\mathbf{MDM}^{\mathrm{eff}}$ *and the essential image of* $j^{\overline{\square}}$ *consists of the* $\overline{\square}$*-local objects. The functor* $j^{\overline{\square}} L^{\overline{\square}}$ *is described by the formula*

$$j^{\overline{\square}} L^{\overline{\square}}(K) \simeq \underline{\mathrm{Hom}}_{D(\mathbf{MNST})}(\overline{\square}^{\bullet}_{\nu}, K)$$

*as in Theorem 6.8.2.*

(3) *The functor* $\tau_{\mathrm{eff}}$ *has a right adjoint* $\tau^{\mathrm{eff}}$*, and the base change morphism* $L^{\overline{\square}} \circ D(\tau^{\mathrm{Nis}}) \Rightarrow \tau^{\mathrm{eff}} \underline{L}^{\overline{\square}}$ (5.11) *is an isomorphism. (See Proposition 3.7.3 for* $\tau^{\mathrm{Nis}}$*, and Corollary 3.9.6 (3) for its exactness.)*

(4) *The functor $\tau_{\mathrm{eff}}$ is monoidal, fully faithful and carries compact objects to compact objects. It induces a full embedding*

$$\tau_{\mathrm{eff,gm}} : \mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}} \hookrightarrow \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,. \tag{6.7}$$

*Proof.* In view of Proposition 6.9.1, (1) and (2) are special cases of Proposition 5.4.2; the formula for $j^{\overline{\square}}L^{\overline{\square}}$ follows from Theorem 5.6.3. In (3), the existence of $\tau^{\mathrm{eff}}$ follows from (1) (Brown representability) since $\tau_{\mathrm{eff}}$ is strongly additive by the same fact for the other functors in (6.5), and the base change statement is a special case of Theorem 5.7.2. In (4), the monoidality of $\tau_{\mathrm{eff}}$ follows similarly from that of the 3 other functors in (6.5), and the preservation of compact objects follows from (1) and the description of the compact objects of $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ (Theorem 6.7.1).

Since $D(\tau_{\mathrm{Nis}})$ is fully faithful (Theorem 3.7.15 b)), the unit map

$$Id_{D(\mathbf{MNST})} \Rightarrow R\tau^{\mathrm{Nis}} D(\tau_{\mathrm{Nis}})$$

is a natural isomorphism. Applying $L^{\overline{\square}}$ and using the natural isomorphisms from (6.5) and (3), we get a natural isomorphism

$$L^{\overline{\square}} \overset{\sim}{\Rightarrow} \tau^{\mathrm{eff}}\tau_{\mathrm{eff}} L^{\overline{\square}}$$

whence a natural isomorphism $Id_{\mathbf{MDM}^{\mathrm{eff}}} \overset{\sim}{\Rightarrow} \tau^{\mathrm{eff}}\tau_{\mathrm{eff}}$, since $L^{\overline{\square}}$ is a localisation (Lemma A.3.3). This shows the full faithfulness of $\tau_{\mathrm{eff}}$. The last claim of (4) follows. ☐

**Definition 6.9.5.** Similar to Definition 6.2.1, we write $M$ for the composite functor $\mathbf{MCor} \to K^b(\mathbf{MCor}) \to \mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ given by Theorem 6.9.4 (1).

Note the trivial identity

$$\underline{M}(\tau\mathcal{X}) = \tau_{\mathrm{eff,gm}} M(\mathcal{X}), \quad \mathcal{X} \in \mathbf{MCor}\,.$$

*Remark* 6.9.6. There are nontrivial analogues of (MV1) and (MV2) in $\mathbf{MSm}$ and $\mathbf{MPST}$: the complex of $K^b(\mathbf{MCor})$ corresponding to the example of Corollary 4.3.11 vanishes under (6.6) (see also Theorem 7.5.2 (2)). However, we do not know if the kernel of (6.6) is generated by enough of these "Mayer-Vietoris relations". This is why the study of $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ and $\mathbf{MDM}^{\mathrm{eff}}$ is more delicate than that of $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ and $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$. This issue has now been successfully solved in [17], yielding a presentation of $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ and $\mathbf{MDM}^{\mathrm{eff}}$ in similar terms as $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ and $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$.

6.10. **Relationship with Voevodsky's categories.** Recall from [21, Definition 4.3.3] that $(\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}})_{\mathrm{Nis}}$ is defined to be the pseudo-abelian envelope of the localisation of $K^b(\mathbf{Cor})$ by the smallest thick subcategory containing all objects of the following type:

(HI1) for $X \in \mathbf{Sm}$,

$$[X \times \mathbf{A}^1] \to [X];$$

(MV1') for $X \in \mathbf{Sm}$ and an elementary Nisnevich cover $(U, V)$ of $X$,

$$[U \times_X V] \to [U] \oplus [V] \to [X].$$

*Remark* 6.10.1. Let $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ be Voevodsky's original category of effective geometric motives, which we recalled in the introduction. If $k$ is perfect, the obvious functor $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to (\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}})_{\mathrm{Nis}}$ is an equivalence of categories, see [21, Theorem 4.4.1]. This is a difficult theorem of Voevodsky that we do not want to use for the moment.

The functor $\underline{\omega} : \underline{\mathbf{M}}\mathbf{Cor} \to \mathbf{Cor}$ from (1.2) transforms (CI1) into (HI1) and (MV1) into (MV1'). Hence we get a functor $\underline{\omega}_{\mathrm{eff,gm}}$ in the following commutative diagram:

$$\begin{CD}
K^b(\mathbf{MCor}) @>{K^b(\tau)}>> K^b(\underline{\mathbf{M}}\mathbf{Cor}) @>{K^b(\underline{\omega})}>> K^b(\mathbf{Cor}) \\
@VVV @VVV @VVV \\
\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}} @>{\tau_{\mathrm{eff,gm}}}>> \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}} @>{\underline{\omega}_{\mathrm{eff,gm}}}>> (\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}})_{\mathrm{Nis}},
\end{CD}$$

where the vertical arrows are localisation functors and $\tau_{\mathrm{eff,gm}}$ is as in (6.7).

Let $\mathbf{NST}$ be the category of Nisnevich sheaves with transfers. By [21, Proposition 4.3.4] and [6, Proposition in §4.2.1], we may define $\mathbf{DM}^{\mathrm{eff}}$ as the localisation of $D(\mathbf{NST})$ by its localising subcategory generated by

(HI2) for $X \in \mathbf{Sm}$,

$$\mathbb{Z}_{\mathrm{tr}}(X \times \mathbf{A}^1) \to \mathbb{Z}_{\mathrm{tr}}(X).$$

Let $L^{\mathbf{A}^1} : D(\mathbf{NST}) \to \mathbf{DM}^{\mathrm{eff}}$ be the localisation functor. Let $j^{\overline{\square}}$ (resp. $j^{\mathbf{A}^1}$) be the right adjoint of $LC^{\overline{\square}}$ (resp. $L^{\mathbf{A}^1}$) (see Theorem 6.5.2). Recall from Proposition 3.8.1 that the functor $\underline{\omega}_{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NST} \to \mathbf{NST}$ has a right adjoint $\underline{\omega}^{\mathrm{Nis}} : \mathbf{NST} \to \underline{\mathbf{M}}\mathbf{NST}$.

**Proposition 6.10.2.** *There exist functors $\underline{\omega}_{\mathrm{eff}}$ and $\underline{\omega}^{\mathrm{eff}}$ fitting in commutative diagrams*

$$\begin{tikzcd} D(\underline{\mathbf{M}}\mathbf{NST}) \arrow[r, "D(\underline{\omega}_{\mathrm{Nis}})"] \arrow[d, "\underline{L}^{\overline{\square}}"'] & D(\mathbf{NST}) \arrow[d, "L^{\mathbf{A}^1}"] \\ \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}} \arrow[r, "\underline{\omega}_{\mathrm{eff}}"'] & \mathbf{DM}^{\mathrm{eff}}, \end{tikzcd} \qquad \begin{tikzcd} D(\underline{\mathbf{M}}\mathbf{NST}) & D(\mathbf{NST}) \arrow[l, "D(\underline{\omega}^{\mathrm{Nis}})"'] \\ \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}} \arrow[u, "\underline{j}^{\overline{\square}}"] & \mathbf{DM}^{\mathrm{eff}}. \arrow[l, "\underline{\omega}^{\mathrm{eff}}"] \arrow[u, "j^{\mathbf{A}^1}"'] \end{tikzcd}$$

*Moreover, $\underline{\omega}^{\mathrm{eff}}$ is right adjoint to $\underline{\omega}_{\mathrm{eff}}$, $\underline{\omega}^{\mathrm{eff}}$ is fully faithful, and $\underline{\omega}_{\mathrm{eff}}$ is a localisation and is monoidal.*

*Proof.* The functor $\underline{\omega}_{\mathrm{Nis}}$ (resp. $\underline{\omega}^{\mathrm{Nis}}$) transforms (CI2) into (HI2) (resp. $\mathbf{A}^1$-local objects into $\overline{\square}$-local objects), hence we get $\underline{\omega}_{\mathrm{eff}}$ (resp. $\underline{\omega}^{\mathrm{eff}}$). Note that $D(\underline{\omega}^{\mathrm{Nis}}) : D(\mathbf{NST}) \to D(\underline{\mathbf{M}}\mathbf{NST})$ is right adjoint to $D(\underline{\omega}_{\mathrm{Nis}})$ since both $\underline{\omega}_{\mathrm{Nis}}$ and $\underline{\omega}^{\mathrm{Nis}}$ are exact (see Proposition 3.8.1). This also implies that $D(\underline{\omega}^{\mathrm{Nis}})$ is fully faithful, because the image of a complex via $D(\underline{\omega}^{\mathrm{Nis}})$ (resp. $D(\underline{\omega}_{\mathrm{Nis}})$) can be computed by applying $\underline{\omega}^{\mathrm{Nis}}$ (resp. $\underline{\omega}_{\mathrm{Nis}}$) termwise, and $\underline{\omega}_{\mathrm{Nis}}\underline{\omega}^{\mathrm{Nis}} = \mathrm{Id}_{\mathbf{NST}}$. Hence $\underline{\omega}^{\mathrm{eff}}$ is fully faithful and right adjoint to $\underline{\omega}_{\mathrm{eff}}$, so that $\underline{\omega}_{\mathrm{eff}}$ is a localisation.

The monoidality of $\underline{\omega}_{\mathrm{eff}}$ will follow from that of the three other functors in the diagram. This is already known for the vertical ones (see Proposition 6.4.1 (3) for $\underline{L}^{\overline{\square}}$), so we are left to show the monoidality of $D(\underline{\omega}_{\mathrm{Nis}})$. By the same trick, the latter is reduced to the monoidality of $D(\underline{\omega}_!)$, which in turn follows from that of $\underline{\omega}_!$ (Proposition 2.3.1) and the definition of the tensor structures on $D(\underline{\mathbf{M}}\mathbf{PST})$ and $D(\mathbf{PST})$ (cf. §6.4). □

Composing with the functors of (6.5) and using the exactness of $\tau^{\mathrm{Nis}}$ (Corollary 3.9.6 (3)), we get commutative diagrams

$$\begin{tikzcd} D(\mathbf{MNST}) \arrow[r, "D(\omega_{\mathrm{Nis}})"] \arrow[d, "\underline{L}^{\overline{\square}}"'] & D(\mathbf{NST}) \arrow[d, "L^{\mathbf{A}^1}"] \\ \mathbf{MDM}^{\mathrm{eff}} \arrow[r, "\omega_{\mathrm{eff}}"'] & \mathbf{DM}^{\mathrm{eff}}, \end{tikzcd} \qquad \begin{tikzcd} D(\mathbf{MNST}) & D(\mathbf{NST}) \arrow[l, "D(\omega^{\mathrm{Nis}})"'] \\ \mathbf{MDM}^{\mathrm{eff}} \arrow[u, "\underline{j}^{\overline{\square}}"] & \mathbf{DM}^{\mathrm{eff}}. \arrow[l, "\omega^{\mathrm{eff}}"] \arrow[u, "j^{\mathbf{A}^1}"'] \end{tikzcd}$$

where $\omega^{\mathrm{eff}}$ is right adjoint to $\omega_{\mathrm{eff}}$ and $\omega_{\mathrm{eff}}$ is monoidal. Moreover,

**Proposition 6.10.3.** *The functor $\omega^{\mathrm{eff}}$ is fully faithful, hence $\omega_{\mathrm{eff}}$ is a localisation.*

*Proof.* Same as for Proposition 6.10.2, using the full faithfulness of $\omega^{\mathrm{Nis}}$ (Propositions 2.2.1 and 3.8.1). □

We finally have the following commutative diagram

$$
\begin{tikzcd}
\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}} \arrow[r, "(6.7)"] \arrow[d, "\tau_{\mathrm{eff,gm}}"'] & \mathbf{MDM}^{\mathrm{eff}} \arrow[d, "\tau_{\mathrm{eff}}"] \\
\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}} \arrow[r, "(6.2)"] \arrow[d, "\underline{\omega}_{\mathrm{eff,gm}}"'] & \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}} \arrow[d, "\underline{\omega}_{\mathrm{eff}}"] \\
(\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}})_{\mathrm{Nis}} \arrow[r] & \mathbf{DM}^{\mathrm{eff}} .
\end{tikzcd}
\tag{6.8}
$$

in which the lower horizontal functor is given as the same way as (6.2) (it is denoted by $c$ in [21, (4.5)]). All rows are fully faithful by Theorem 6.9.4 (4), Theorem 6.7.1 and [21, (4.5)].

## 7. Main results and computations

### 7.1. Motivic cohomology with modulus.

**Theorem 7.1.1.** *For any $\mathcal{X}, \mathcal{Y} \in \underline{\mathbf{M}}\mathbf{Cor}$ and $i \in \mathbb{Z}$, we have an isomorphism (see Definition 6.2.1 for the functor $M$).*

$$\operatorname{Hom}_{\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}}(\underline{M}(\mathcal{X}), \underline{M}(\mathcal{Y})[i]) \simeq \mathbb{H}^i_{\mathrm{Nis}}(\overline{\mathcal{X}}, RC^{\overline{\square}}_*(\mathcal{Y})_{\mathcal{X}}).$$

*The same formula holds in $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$, for $\mathcal{X}, \mathcal{Y} \in \mathbf{MCor}$ and $M(\mathcal{X}), M(\mathcal{Y})$.*

*Proof.* We compute as follows:

$$
\begin{aligned}
\operatorname{Hom}_{\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}}(\underline{M}(\mathcal{X}), \underline{M}(\mathcal{Y})[i]) &\overset{(1)}{\simeq} \operatorname{Hom}_{\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}}(LC^{\overline{\square}}\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}), LC^{\overline{\square}}\mathbb{Z}_{\mathrm{tr}}(\mathcal{Y})[i]) \\
&\overset{(2)}{\simeq} \operatorname{Hom}_{D(\underline{\mathbf{M}}\mathbf{NST})}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}), j^{\overline{\square}}LC^{\overline{\square}}\mathbb{Z}_{\mathrm{tr}}(\mathcal{Y})[i]) \\
&\overset{(3)}{\simeq} \operatorname{Hom}_{D(\underline{\mathbf{M}}\mathbf{NST})}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}), RC^{\overline{\square}}_*\mathbb{Z}_{\mathrm{tr}}(\mathcal{Y})[i]) \\
&\overset{(4)}{\simeq} \mathbb{H}^i_{\mathrm{Nis}}(\overline{\mathcal{X}}, RC^{\overline{\square}}_*(\mathcal{Y})_{\mathcal{X}}).
\end{aligned}
$$

Here (1) is seen by Theorem 6.7.1, (2) by adjunction, (3) by Theorem 6.8.2, and (4) by Proposition 3.6.2. The last statement now follows from the full faithfulness of $\tau_{\mathrm{eff,gm}}$ (Theorem 6.9.4 (4)). □

### 7.2. Chow motives.

In [49], Voevodsky constructs a $\otimes$-functor $\mathbf{Chow}^{\mathrm{eff}} \to \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$, where $\mathbf{Chow}^{\mathrm{eff}}$ is the category of effective Chow motives. This functor sends a Chow motive $h(X)$ to $M^V(X)$, where $M^V : \mathbf{Cor} \to \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ is the canonical functor, and is shown to be fully faithful when $k$ is perfect in [6] (see [21, Th. 4.4.1 (3)]).

**Theorem 7.2.1.** *Write* $\omega_{\mathrm{eff,gm}} = \underline{\omega}_{\mathrm{eff},gm} \circ \tau_{\mathrm{eff},gm}$ *(see (6.8)). There is a unique* $\otimes$*-functor* $\Phi^{\mathrm{eff}} : \mathbf{Chow}^{\mathrm{eff}} \to \mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ *whose composition with* $\omega_{\mathrm{eff,gm}}$ *is Voevodsky's functor; it sends* $h(X)$ *to* $M(X, \emptyset)$*.*

*Proof.* Voevodsky's functor is defined as follows: let $\mathcal{H}(\mathbf{Cor})$ be the homotopy category of $\mathbf{Cor}$; its Hom groups are $h(X,Y) = \mathrm{Coker}(\mathbf{Cor}(X \times \mathbf{A}^1, Y) \to \mathbf{Cor}(X,Y))$. Obviously, the natural functor $\mathbf{Cor} \to \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ factors through $\mathcal{H}(\mathbf{Cor})$. There is also a map

$$h(X,Y) \to CH_{\dim X}(X \times Y) \tag{7.1}$$

which sends a finite correspondence to the corresponding cycle class. This map is an isomorphism when $X$ and $Y$ are projective [11, Th. 7.1]. Hence the functor.

For $X, Y$ as above, the inclusions

$$\mathbf{MCor}((X,\emptyset),(Y,\emptyset)) \subseteq \mathbf{Cor}(X,Y)$$
$$\mathbf{MCor}((X,\emptyset)\otimes \overline{\square},(Y,\emptyset)) \subseteq \mathbf{Cor}(X \times \mathbf{A}^1, Y)$$

are equalities. Hence we get the refined functor by using (6.6) and Theorem 6.9.4 (1). □

**Definition 7.2.2.** Let $\mathbb{Z}(1) := \Phi^{\mathrm{eff}}(\mathbb{L})[-2]$, where $\mathbb{L} \in \mathbf{Chow}^{\mathrm{eff}}$ is the Lefschetz motive. We write $\mathbf{MDM}_{\mathrm{gm}}$ (resp. $\underline{\mathbf{M}}\mathbf{DM}_{\mathrm{gm}}$) for the category obtained from $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ (resp. from $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$) by $\otimes$-inverting $\mathbb{Z}(1)$ (resp. $\tau_{\mathrm{eff}}\mathbb{Z}(1)$).

The $\otimes$-functor $\Phi^{\mathrm{eff}}$ of Theorem 7.2.1 extends canonically to a $\otimes$-functor

$$\Phi : \mathbf{Chow} \to \mathbf{MDM}_{\mathrm{gm}}$$

where $\mathbf{Chow}$ is the category of (all) Chow motives. Since the latter is rigid, we get

**Corollary 7.2.3.** *For any smooth projective variety* $X$*, the motive* $M(X,\emptyset)$ *is strongly dualisable in* $\mathbf{MDM}_{\mathrm{gm}}$*.* □

7.3. **Empty modulus.** We now compute $M(X, \emptyset)$.

**Theorem 7.3.1.** *Let* $X$ *be a smooth proper* $k$*-variety. Then*

$$\underline{\omega}^{\mathrm{eff}} M^V(X) \simeq \underline{M}(X,\emptyset)$$

*where* $\underline{\omega}^{\mathrm{eff}} : \mathbf{DM}^{\mathrm{eff}} \to \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ *is the functor from Proposition 6.10.2. Similarly*

$$\omega^{\mathrm{eff}} M^V(X) \simeq M(X,\emptyset).$$

*Proof.* By Proposition 2.3.1 and 3.8.1, we have $\underline{\omega}^{\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(X) = \mathbb{Z}_{\mathrm{tr}}(X, \emptyset)$. The result now follows from Theorem 5.7.2 applied to $LT = D(\underline{\omega}_{\mathrm{Nis}}) : D(\underline{\mathbf{M}}\mathbf{NST}) \to D(\mathbf{NST})$ and $RS = D(\underline{\omega}^{\mathrm{Nis}}) : D(\mathbf{NST}) \to D(\underline{\mathbf{M}}\mathbf{NST})$ (recall that $\underline{\omega}_{\mathrm{Nis}}$ and $\underline{\omega}^{\mathrm{Nis}}$ are exact and that $D(\underline{\omega}_{\mathrm{Nis}})$ is monoidal, still by Proposition 3.8.1).

For the second statement, it suffices to show that $\tau^{\mathrm{eff}}\underline{M}(X, \emptyset) = M(X, \emptyset)$: this is obvious from the full faihtfulness of $\tau_{\mathrm{eff}}$ (Theorem 6.9.4 (4)), since $\tau_{\mathrm{eff}}M(X, \emptyset) = \underline{M}(X, \emptyset)$. □

**Corollary 7.3.2.** *Let $p$ be the exponential characteristic of $k$. Then the functor $\underline{\omega}^{\mathrm{eff}} : \mathbf{DM}^{\mathrm{eff}} \to \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ induces a functor*

$$\underline{\omega}^{\mathrm{eff}}_{\mathrm{gm}} : \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}[1/p] \to \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$$

*which is right adjoint to the functor $\underline{\omega}_{\mathrm{eff,gm}} : \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}[1/p] \to \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$ of (6.8). The same holds replacing $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$ by $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$ and $\underline{\omega}^{\mathrm{eff}}$ by $\omega^{\mathrm{eff}}$, yielding $\omega^{\mathrm{eff}}_{\mathrm{gm}} : \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}[1/p] \to \mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}[1/p]$.*

*Proof.* Follows from Theorem 7.3.1 and resolution of singularities à la de Jong-Gabber. □

In the next applications of Theorem 7.3.1, $k$ is supposed to be perfect. Recall from §7.2 that we have the Tate object $\mathbb{Z}(1) \in \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. For $i \geq 0$ and $M \in \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$, we put $\mathbb{Z}(i) = \mathbb{Z}(1)^{\otimes i}$ and $M(i) = M \otimes \mathbb{Z}(i)$.

**Corollary 7.3.3.** *Assume $k$ is perfect. Let $X$ be a smooth proper $k$-variety of dimension $d$, $\mathcal{Y} \in \underline{\mathbf{M}}\mathbf{Cor}$ a modulus pair, and $i, j$ integers with $i \geq 0$. Then we have a canonical isomorphism:*

$$\mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}}(\underline{M}(\mathcal{Y}), \underline{M}(X, \emptyset)(i)[-j]) \simeq H^{2d-j}(\mathcal{Y}^{\mathrm{o}} \times X, \mathbb{Z}(d+i))$$

*where the right hand side is Voevodsky's motivic cohomology. In particular, this group is isomorphic to $CH^{d+i}(\mathcal{Y}^{\mathrm{o}} \times X, 2i+j)$ and vanishes if $2i + j < 0$ by [50, Cor. 2].*
*The same formula holds in $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ if $\mathcal{Y} \in \mathbf{MCor}$ (with $M$ instead of $\underline{M}$).*

*Proof.* By Theorem 6.7.1, we may compute the Hom in the left hand side using $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}$ instead of $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. Since $\underline{M}(\mathbf{P}^i, \emptyset) = \bigoplus_{s=0}^{i} \mathbb{Z}(s)[2s]$, Theorem 7.3.1 shows that $\underline{\omega}^{\mathrm{eff}}$ sends $\mathbb{Z}(i) \in \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ to $\mathbb{Z}(i) \in \underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. Using Theorem 7.3.1 again, we then have an isomorphism

$$\mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}}(\underline{M}(\mathcal{Y}), \underline{M}(X, \emptyset)(i)[j]) \simeq \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}}(M^V(\mathcal{Y}^{\mathrm{o}}), M^V(X)(i)[j]).$$

The result now follows from Poincaré duality for $X$ [6, Prop. 6.7.1]. The last statement now follows from the full faithfulness of $\tau_{\mathrm{eff}}$ (Theorem 6.9.4 (4)) and the last part of Theorem 7.3.1. □

**Corollary 7.3.4.** *Assume $k$ perfect. The functor $\Phi^{\mathrm{eff}} : \mathbf{Chow}^{\mathrm{eff}} \to \mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ from Theorem 7.2.1 is fully faithful.* □

In contrast to Theorem 7.3.1, we have

**Theorem 7.3.5.** *Let $X$ be smooth and quasi-affine. Then $\mathbb{Z}_{\mathrm{tr}}(X, \emptyset)$ is $\overline{\square}$-invariant (see §6.1). More generally, we have an isomorphism*

$$\mathbb{Z}_{\mathrm{tr}}(X, \emptyset) \xrightarrow{\sim} \underline{\mathrm{Hom}}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{Y}), \mathbb{Z}_{\mathrm{tr}}(X, \emptyset))$$

*induced by the projection $p_Y : \mathcal{Y} \to \operatorname{Spec} k$, for any proper modulus pair $\mathcal{Y} \in \mathbf{MCor}$ such that $\overline{\mathcal{Y}}$ is geometrically irreducible (and $\mathcal{Y}^{\mathrm{o}} \neq \emptyset$).*

*Proof.* Take $\mathcal{Z} \in \underline{\mathbf{M}}\mathbf{Cor}$. It suffices to show that the map

$$p_Y^* : \underline{\mathbf{M}}\mathbf{Cor}(\mathcal{Z}, (X, \emptyset)) \to \underline{\mathbf{M}}\mathbf{Cor}(\mathcal{Z} \otimes \mathcal{Y}, (X, \emptyset)) \tag{7.2}$$

induced by $p_Y$ is an isomorphism. For any closed point $y \in \mathcal{Y}^{\mathrm{o}}$, we find $\underline{\mathbf{M}}\mathbf{Cor}(\mathcal{Z}, (X, \emptyset)) \to \underline{\mathbf{M}}\mathbf{Cor}(\mathcal{Z} \otimes (y, \emptyset), (X, \emptyset))$ is injective, hence (7.2) is injective as well.

To show the surjectivity, let us take $V \in \underline{\mathbf{M}}\mathbf{Cor}(\mathcal{Z} \otimes \mathcal{Y}, (X, \emptyset))$ which is irreducible. Let $\overline{V}$ be the closure of $V$ in $\overline{\mathcal{Z}} \times \overline{\mathcal{Y}} \times X$. We claim that the image $\overline{V'}$ of $\overline{V}$ in $\overline{\mathcal{Z}} \times X$ is closed and finite surjective over $\overline{\mathcal{Z}}$. To prove this claim, we consider a commutative diagram

$$\begin{tikzcd}
\overline{V} \arrow[r, hook, "i"] \arrow[d, "\pi'"'] \arrow[dr, "\pi"] & \overline{\mathcal{Z}} \times \overline{\mathcal{Y}} \times X \arrow[r, "a"] \arrow[d, "b"] & \overline{\mathcal{Z}} \times \overline{\mathcal{Y}} \arrow[d, "c"] \\
\overline{V'} \arrow[r, hook, "i'"'] & \overline{\mathcal{Z}} \times X \arrow[r, "d"'] & \overline{\mathcal{Z}}.
\end{tikzcd}$$

Since $V \in \underline{\mathbf{M}}\mathbf{Cor}(\mathcal{Z} \otimes \mathcal{Y}, (X, \emptyset))$, $ai$ is proper and surjective. Since the same is true of $c$, we find that $cai = d\pi$ is proper surjective. This implies that $\overline{V'}$ is closed and, combined with the surjectivity of $\pi'$, that $di'$ is proper [EGA2, Cor. 5.4.3]. But $di'$ is also quasi-affine (since so is $d$), hence finite. This proves the claim.

Now $V' := \overline{V'} \cap (\mathcal{Z}^{\mathrm{o}} \times X)$ is an element of $\underline{\mathbf{M}}\mathbf{Cor}(\mathcal{Z}, (X, \emptyset))$. We clearly have $V \subset V' \times \mathcal{Y}^{\mathrm{o}}$, and $V' \times \mathcal{Y}^{\mathrm{o}}$ is irreducible because $\overline{\mathcal{Y}}$ is geometrically irreducible. By comparing dimensions, we get $V = V' \times \mathcal{Y}^{\mathrm{o}} = p_Y^*(V')$. This proves the surjectivity of (7.2). □

7.4. **Motives of vector bundles and projective bundles.** Let $\mathcal{Y} \in \underline{\mathbf{M}}\mathbf{Cor}$ be a modulus pair, and let $E$ be a vector bundle on $\overline{\mathcal{Y}}$ of rank $n$, with associated projective bundle $\mathbf{P}(E)$. We define modulus pairs $\mathcal{E}$ and $\mathcal{P}$ with total spaces $E$ and $\mathbf{P}(E)$ by pulling back $\mathcal{Y}^\infty$: the resulting morphisms $\mathcal{E} \to \mathcal{Y}$, $\mathcal{P} \to \mathcal{Y}$ are minimal in the sense of Definition 1.10.3 a).

(There may be more general notions of vector and projective bundles, but we do not consider them here.)

*Remark* 7.4.1. By applying Corollary 7.3.3 with $X = \mathrm{Spec}(k)$ and $j = 2i$, we get $CH^i(\mathcal{Y}^o) \simeq \mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}}(\underline{M}(\mathcal{Y}), \mathbb{Z}(i)[2i])$. In particular, if $P(t_1, \dots, t_n) \in \mathbb{Z}[t_1, \dots, t_n]$ is a homogeneous polynomial of weight $i$ (the weight of $t_s$ being $s$), then the Chern classes of $E$ yield a morphism in $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$

$$P(c_1(E), \dots, c_n(E)) : \underline{M}(\mathcal{Y}) \to \mathbb{Z}(i)[2i].$$

**Theorem 7.4.2.** *Assume $k$ is perfect. Suppose $\overline{\mathcal{Y}}$ smooth. The projection $\bar{p} : \mathcal{P} \to \mathcal{Y}$ yields a canonical isomorphism in $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$*

$$\rho_{\mathcal{Y}} : \underline{M}(\mathcal{P}) \xrightarrow{\sim} \bigoplus_{i=0}^{n-1} \underline{M}(\mathcal{Y})(i)[2i].$$

*The same holds in $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ if $\mathcal{Y} \in \mathbf{MCor}$ (with $M$ instead of $\underline{M}$).*

*Remark* 7.4.3. If $\mathrm{char}\, k = 0$ or $\dim \mathcal{Y} \leq 3$, the assumption on $\overline{\mathcal{Y}}$ is innocent in view of Corollary 1.10.6.

*Proof.* We follow the method of Voevodsky in [49, proof of Prop. 3.5.1], with a simplification and a complication. We first define $\rho_{\mathcal{Y}}$; its naturality in $\mathcal{Y}$ reduces us by Mayer-Vietoris to the case where the vector bundle $E$ is trivial; in this case, we have an isomorphism of modulus pairs

$$\mathcal{P} \simeq \mathcal{Y} \otimes (\mathbf{P}^n, \emptyset)$$

hence a corresponding isomorphism of motives. One may then use either Theorem 7.3.1 or, more directly, the functor $\Phi^{\mathrm{eff}}$ of Theorem 7.2.1 and the computation of the Chow motive of $\mathbf{P}^n$ to conclude, using the definition of $\rho_{\mathcal{Y}}$.

The complication is that Voevodsky's construction of $\rho_{\mathcal{Y}}$ (in the case of $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$) uses diagonal maps, which cause a problem here (see Warning 1.4.4). We bypass this problem by using the morphism

$$\tilde{\Delta} : \mathcal{P} \to \mathcal{P} \otimes (\mathbf{P}(E), \emptyset)$$

induced by the diagonal inclusion $\mathbf{P}(E)^o \hookrightarrow \mathbf{P}(E)^o \times \mathbf{P}(E)$: here, the modulus condition is obviously verified. Using the morphisms $\underline{M}(\mathbf{P}(E), \emptyset) \to \mathbb{Z}(i)[2i]$ induced by the powers of $c_1(O_{\mathbf{P}(E)}(1))$ (see Remark 7.4.1), we get the desired morphisms

$$\rho^i_{\mathcal{Y}} : \underline{M}(\mathcal{P}) \xrightarrow{\underline{M}(\tilde{\Delta})} \underline{M}(\mathcal{P}) \otimes \underline{M}(\mathbf{P}(E), \emptyset) \to \underline{M}(\mathcal{Y}) \otimes \mathbb{Z}(i)[2i].$$

□

*Question* 7.4.4. Can one similarly produce an isomorphism $\underline{M}(\mathcal{E}) \xrightarrow{\sim} \underline{M}(\mathcal{Y}) \otimes \underline{M}(\mathbf{A}^n, \emptyset)$?

## 7.5. Mayer-Vietoris exact triangles.

**Definition 7.5.1.** For $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$, we put

$$M(\mathcal{X}) = L^{\overline{\square}}(\tau^{\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(\mathcal{X})[0]) = \tau^{\mathrm{eff}}\underline{M}(\mathcal{X}) \in \mathbf{MDM}^{\mathrm{eff}}$$

where $L^{\overline{\square}} : D(\mathbf{MNST}) \to \mathbf{MDM}^{\mathrm{eff}}$ is the localisation functor from (6.5). (The second identity uses Theorem 6.9.4 (3) and Corollary 3.9.6 (3).)

Note that there is no reason for $M(\mathcal{X})$ to belong to $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$, unless $\mathcal{X} \in \tau\,\mathbf{MCor}$.

**Theorem 7.5.2.**

(1) *For any elementary Nisnevich square* (4.1)*, we have an exact triangle in* $\mathbf{MDM}^{\mathrm{eff}}$

$$M(\mathcal{W}) \to M(\mathcal{U}) \oplus M(\mathcal{V}) \to M(\mathcal{X}) \xrightarrow{+1}$$

(2) *Under the hypotheses of Corollary 4.3.11, we have an exact triangle in* $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$*:*

$$M(X, D') \to M(X, D_1) \oplus M(X, D_2) \to M(X, D) \xrightarrow{+1} .$$

*Proof.* This follows respectively from Theorem 4.1.1 (or Conditions (MV1), (MV2)) and Corollary 4.3.11. □

*Remark* 7.5.3. If one wants to exploit Theorem 4.2.1 as well, one has to work in a suitable triangulated category of pro-motives. We leave this development for a further work.

The exact triangle of Theorem 7.5.2 (ii) is interesting because it only involves proper modulus pairs, and even its existence is not obvious *a priori*. Can one find sufficiently many such triangles to "present" $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$, in the same way as $\underline{\mathbf{M}}\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ is presented in Definiton 6.2.1 (using also the relations of type CI)? The answer is positive, and is the main result of [17].

## Appendix A. Categorical toolbox

This appendix gathers known and less-known results that we use constantly.

A.1. **Pro-objects ([SGA4-I, I.8], [2, App. 2]).** Recall that a *pro-object* of a category $\mathcal{C}$ is a functor $F : A \to \mathcal{C}$, where $A$ is a small cofiltered category (dual of [32, IX.1]). They are denoted by $(X_\alpha)_{\alpha\in A}$ or by "$\varprojlim_{\alpha\in A}$"$X_\alpha$ (Deligne's notation), with $X_\alpha = F(\alpha)$. Pro-objects of $\mathcal{C}$ form a category pro–$\mathcal{C}$, with morphisms given by the formula

$$\text{pro–}\mathcal{C}((X_\alpha)_{\alpha\in A}, (Y_\beta)_{\beta\in B}) = \varprojlim_{\beta\in B} \varinjlim_{\alpha\in A} \mathcal{C}(X_\alpha, Y_\beta).$$

There is a canonical full embedding $c : \mathcal{C} \hookrightarrow \text{pro–}\mathcal{C}$, sending an object to the corresponding constant pro-object ($A = \{*\}$).

For the next lemma, we recall a special case of comma categories from Mac Lane [32, II.6]. If $F : \mathcal{A} \to \mathcal{B}$ is a functor and $b \in \mathcal{B}$, we write $b \downarrow F$ for the category whose objects are pairs $(a, f) \in \mathcal{A} \times \mathcal{B}(b, F(a))$; a morphism $(a_1, f_1) \to (a_2, f_2)$ is a morphism $g \in \mathcal{A}(a_1, a_2)$ such that $f_2 = F(g)f_1$. The category $F \downarrow b$ is defined dually (objects: systems $F(a) \xrightarrow{f} b$, etc.) According to [32, IX.3], $F$ is *final* if, for any $b \in \mathcal{B}$, the category $F \downarrow b$ is nonempty and connected; here we shall use the dual property *cofinal* (same conditions for $b \downarrow F$). As usual, we abbreviate $Id_{\mathcal{A}} \downarrow a$ and $a \downarrow Id_{\mathcal{A}}$ by $\mathcal{A} \downarrow a$ and $a \downarrow \mathcal{A}$.

Let $F : A \to \mathcal{C}$ be a pro-object. For each $\alpha \in A$, we have a "projection" morphism $\pi_\alpha : F \to c(X_\alpha)$ in pro–$\mathcal{C}$. This yields an isomorphism in pro–$\mathcal{C}$

$$F \xrightarrow{\sim} \varprojlim_{\alpha\in A} c(X_\alpha)$$

(explaining Deligne's notation) and a functor

$$\theta : A \to F \downarrow c.$$

**Lemma A.1.1.** *The functor $\theta$ is cofinal.*

*Proof.* This is a tautology: let $F \xrightarrow{f} c(X)$ be an object of $F \downarrow c$. An object of $\theta \downarrow (F \xrightarrow{f} c(X))$ is a pair $(\alpha, \varphi)$, with $\alpha \in A$ and $\varphi : F(\alpha) \to X$ such that $f = c(\varphi)\pi_\alpha$. The definition of morphisms in pro–$\mathcal{C}$ shows that this category is nonempty. Since $A$ is cofiltering, it suffices by the dual of [23, Prop. 3.2.2 (iii)] to show that for any pair $(t_1, t_2)$ of morphisms $(\alpha_1, \varphi_1) \to (\alpha_2, \varphi_2)$, there exists $t : \alpha \to \alpha_1$ such that $t_1 t = t_2 t$ : this condition is verified thanks to the definition of pro–$\mathcal{C}(F, X)$. □

(Warning: the use of co in (co)final and (co)filtered is opposite in [32] and in [23]. We use the convention of [32].)

A.2. **Pro-adjoints** [SGA4-I, I.8.11.5]**.** Let $u : \mathcal{C} \to \mathcal{D}$ be a functor: it induces a functor pro–$u$ : pro–$\mathcal{C} \to$ pro–$\mathcal{D}$.

**Proposition A.2.1** (dual of [SGA4-I, I.8.11.4])**.** *Consider the following conditions:*

(i) *The functor* pro–$u$ *has a left adjoint.*
(ii) *There exists a functor $v : \mathcal{D} \to$ pro–$\mathcal{C}$ and an isomorphism*

$$\text{pro–}\mathcal{C}(v(d), c) \simeq \mathcal{D}(d, u(c))$$

*contravariant in $d \in \mathcal{D}$ and covariant in $c \in \mathcal{C}$.*
(iii) *$u$ is left exact.*

*Then* (i) $\iff$ (ii) $\Rightarrow$ (iii)*, and* (iii) $\Rightarrow$ (i) *if $\mathcal{C}$ is essentially small and closed under finite inverse limits.* □

(The condition on finite inverse limits appears in [2, p. 158], but is skipped in [SGA4-I, I.8.11.4].)

**Definition A.2.2.** In Condition (ii) of Proposition A.2.1, we say that $v$ is *pro-left adjoint* to $u$.

A.3. **Localisation ([13, Ch. I], see also [23, Ch. 7]).** Let $\mathcal{C}$ be a category, and let $\Sigma \subset Ar(\mathcal{C})$ be a class of morphisms: following Grothendieck and Maltsiniotis, we call $(\mathcal{C}, \Sigma)$ a *localiser*. Consider the functors $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is invertible for all $s \in \Sigma$. This "2-universal problem" has a solution $Q : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$. One may choose $\mathcal{C}[\Sigma^{-1}]$ to have the same objects as $\mathcal{C}$ and $Q$ to be the identity on objects; then $\mathcal{C}[\Sigma^{-1}]$ is unique (not just up to unique equivalence of categories). If $\mathcal{C}$ is essentially small, then $\mathcal{C}[\Sigma^{-1}]$ is small, but in general the sets $\mathcal{C}[\Sigma^{-1}](X, Y)$ may be "large"; one can sometimes show that it is not the case (Corollary A.5.4). A functor of the form $Q : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ will be called a *localisation*. We have a basic result on adjoint functors [13, Prop. I.1.3]:

**Lemma A.3.1.** *Let $G : \mathcal{C} \leftrightarrows \mathcal{D} : D$ be a pair of adjoint functors ($G$ is left adjoint to $D$). Then the following conditions are equivalent:*

(i) *$D$ is fully faithful.*
(ii) *The counit $GD \Rightarrow Id_{\mathcal{D}}$ is a natural isomorphism.*
(iii) *$G$ is a localisation.*

*The same holds if $G$ is right adjoint to $D$ (replacing the counit by the unit).*

**Definition A.3.2.** Let $(\mathcal{C}, \Sigma)$ be a localiser, and let $Q : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ be the corresponding localisation functor. We write

$$\operatorname{sat}(\Sigma) = \{s \in Ar(\mathcal{C}) \mid Q(s) \text{ is invertible}\}.$$

This is the *saturation* of $\Sigma$; we say that $\Sigma$ is *saturated* if $\operatorname{sat}(\Sigma) = \Sigma$.

**Lemma A.3.3** ([13, Ch. I, Lemma 1.2])**.** *Let $(\mathcal{C}, \Sigma)$ be a localiser, $\mathcal{D}$ a category, $F, G : \mathcal{C}[\Sigma^{-1}] \to \mathcal{D}$ two functors and $u : F \circ Q \Rightarrow G \circ Q$ a natural transformation, where $Q : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ is the localisation functor. Then $u$ induces a unique natural transformation $\bar{u} : F \Rightarrow G$.*

*Proof.* Define $\bar{u}_X = u_X : F(X) \to G(X)$ for $X \in Ob\mathcal{C}[\Sigma^{-1}] = Ob\mathcal{C}$. We must show that $\bar{u}$ commutes with the morphisms of $\mathcal{C}[\Sigma^{-1}]$. This is obvious, since $u$ commutes with the morphisms of $\mathcal{C}$ and the morphisms of $\mathcal{C}[\Sigma^{-1}]$ are expressed as fractions in the morphisms of $\mathcal{C}$. □

A.4. **Presheaves and pro-adjoints.** Let $\mathcal{C}$ be a category. We write $\hat{\mathcal{C}}$ for the category of presheaves of sets on $\mathcal{C}$ (functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$); it comes with the Yoneda embedding

$$y : \mathcal{C} \to \hat{\mathcal{C}}$$

which sends an object to the corresponding representable presheaf. If $u : \mathcal{C} \to \mathcal{D}$ is a functor, we have the standard sequence of 3 adjoint functors

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{y_{\mathcal{C}}} & \hat{\mathcal{C}} \\
{\scriptstyle u}\Big\downarrow & & {\scriptstyle u_!}\Big\downarrow {\scriptstyle u^*}\Big\uparrow {\scriptstyle u_*}\Big\downarrow \\
\mathcal{D} & \xrightarrow{y_{\mathcal{D}}} & \hat{\mathcal{D}}
\end{array}$$

where $u_!$ extends $u$ through the Yoneda embeddings [SGA4-I, Exp. I, Prop. 5.4]; $u_!$ and $u_*$ are computed by the usual formulas for left and right Kan extensions (loc. cit., (5.1.1)). If $u$ has a left adjoint $v$, the sequence $(u_!, u^*, u_*)$ extends to

$$(v_!, v^* = u_!, v_* = u^*, u_*)$$

(ibid., Rk. 5.5.2). Recall standard terminology for the functoriality of limits (=inverse limits) and colimits (= direct limits):

**Definition A.4.1.** A functor $u : \mathcal{C} \to \mathcal{D}$ is *continuous* (resp. *co-continuous*, resp. *bi-continuous*) if it commutes with all limits (resp. colimits, resp. limits and colimits) representable in $\mathcal{C}$. It is *left exact* (resp. *right exact*, resp. *exact*) if it commutes with finite limits (resp. finite colimits, resp. finite limits and colimits).

**Proposition A.4.2.** *a) The functor $u_!$ (resp. $u^*$, $u_*$) is co-continous (resp. bi-continuous, continuous). If $u$ has a left adjoint, then $u_!$ is also continuous. If $u$ has a pro-left adjoint $v$, so does $u_!$ which is therefore exact. Moreover, $u_!$ is then given by the formula*

$$(u_!F)(Y) = \varinjlim(F \circ v(Y)), \quad F \in \hat{\mathcal{C}}, Y \in \mathcal{D}.$$

*b) If $u$ is fully faithful, so is $u_!$.*
*c) If $u$ is a localisation or is full and essentially surjective, then $u_!$ is a*

*localisation.*
*d) In the case of c), for $C \in \mathcal{C}$ the following conditions are equivalent:*

(i) *The representable functor $y_{\mathcal{C}}(C) \in \hat{\mathcal{C}}$ induces a functor on $\mathcal{D}$ via $u$.*
(ii) *The unit map $y_{\mathcal{C}}(C) \to u^* u_! y_{\mathcal{C}}(C) \simeq u^* y_{\mathcal{D}}(u(C))$ is an isomorphism.*
(iii) *For any $C' \in \mathcal{C}$, the map $\mathcal{C}(C', C) \to \mathcal{D}(u(C'), u(C))$ induced by $u$ is bijective.*

*Proof.* a) follows from general properties of adjoint functors, except for the case of a pro-left adjoint. Let $u$ admit a pro-left adjoint $v$, and let $Y \in \mathcal{D}$: so there is an isomorphism of categories $Y \downarrow u \simeq v(Y) \downarrow c$. Hence, we get by Lemma A.1.1 a cofinal functor

$$A \to Y \downarrow u$$

where $A$ is the indexing set of $v(Y)$. Thus, for $F \in \hat{\mathcal{C}}$, $u_! F(Y)$ may be computed as

$$u_! F(Y) = \varinjlim_{\alpha \in A} F(v(Y)(\alpha)) = \text{pro–}\mathcal{C}(y_{\mathcal{C}}(v(Y)), c(F)).$$

The first equality is the formula in the proposition. The second one shows that the pro-left adjoint $v_!$ of $u_!$ is defined at $Y$ by $y_{\mathcal{C}}(v(Y))$; since any object of $\hat{\mathcal{D}}$ is a colimit of representable objects, this shows that $v_!$ is defined everywhere.

For b), see [SGA4-I, Exp. I, Prop. 5.6]. In c), it is equivalent to show that $u^*$ is fully faithful. Let $F, G \in \hat{\mathcal{D}}$, and let $\varphi : u^* F \to u^* G$ be a morphism of functors. In both cases, $u$ is essentially surjective: given $X \in \mathcal{D}$ and an isomorphism $\alpha : X \xrightarrow{\sim} u(Y)$, we get a morphism

$$\psi_X : F(X) \xrightarrow{\alpha^{*-1}} F(u(Y)) \xrightarrow{\varphi_Y} G(u(Y)) \xrightarrow{\alpha^*} G(X).$$

The fact that $\psi_X$ is independent of $(Y, \alpha)$ and is natural in $X$ is an easy consequence of each hypothesis (see Lemma A.3.3 in the first case).

In d), the equivalence (ii) $\iff$ (iii) is tautological and (iii) $\Rightarrow$ (i) is obvious. The implication (i) $\Rightarrow$ (iii) was proven in [13, I.4.1.2] assuming that $u$ is a localisation enjoying a calculus of left fractions; let us prove (i) $\Rightarrow$ (ii) in general. Under (i), we have $y_{\mathcal{C}}(C) \simeq u^* F$ for some $F \in \hat{D}$; the unit map becomes

$$\eta_{u^* F} : u^* F \to u^* u_! u^* F.$$

On the other hand, the counit map $\varepsilon_F : u_! u^* F \to F$ is invertible by the full faithfulness of $u^*$. By the adjunction identities, we have $u^*(\varepsilon_F) \circ \eta_{u^* F} = 1_{u^* F}$. Hence the conclusion. □

We shall usually write $u^!$ for the pro-left adjoint of $u_!$, when it exists.

A.5. **Calculus of fractions.**

**Definition A.5.1** (dual of [13, I.1.2])**.** A localiser $(\mathcal{C}, \Sigma)$ (or simply $\Sigma$) enjoys a *calculus of right fractions* if:

(i) The identities of $\mathcal{C}$ are in $\Sigma$.
(ii) $\Sigma$ is stable under composition.
(iii) (Ore condition.) For each diagram $X' \xrightarrow{s} X \xleftarrow{u} Y$ where $s \in \Sigma$, there exists a commutative square

$$\begin{array}{ccc} Y' & \xrightarrow{u'} & X' \\ {\scriptstyle t}\big\downarrow & & {\scriptstyle s}\big\downarrow \\ Y & \xrightarrow{u} & X \end{array} \qquad \text{where } t \in \Sigma.$$

(iv) (Cancellation.) If $f, g : X \rightrightarrows Y$ are morphisms in $\mathcal{C}$ and $s : Y \to Y'$ is a morphism of $\Sigma$ such that $sf = sg$, there exists a morphism $t : X' \to X$ in $\Sigma$ such that $ft = gt$.

**Proposition A.5.2.** *Suppose that $\Sigma$ enjoys a calculus of right fractions. For $c \in \mathcal{C}$, let $\Sigma \downarrow c$ denote the full subcategory of the comma category $\mathcal{C} \downarrow c$ given by the objects $c' \xrightarrow{s} c$ with $s \in \Sigma$. Then*
*a) $\Sigma \downarrow c$ is cofiltered.*
*b)* [13, I.1.2.3] *For any $d \in \mathcal{C}$, the obvious map*

$$\varinjlim_{c' \in \Sigma \downarrow c} \mathcal{C}(c', d) \to \mathcal{C}[\Sigma^{-1}](c, d) \tag{A.1}$$

*is an isomorphism.*
*c) Any morphism in $\mathcal{C}[\Sigma^{-1}]$ is of the form $Q(f)Q(s)^{-1}$ for $f \in Ar(\mathcal{C})$ and $s \in \Sigma$; if $f_1, f_2$ are two parallel arrows in $\mathcal{C}$, then $Q(f_1) = Q(f_2)$ if and only if there exists $s \in \Sigma$ such that $f_1 s = f_2 s$.*

*Proof.* a) The dual of Condition (a) in [32, p. 211] (supremum of two objects) follows from Axioms (iii) and (ii) of Definition A.5.1; the dual of Condition (b) (equalizing parallel arrows) follows from Axioms (iv) and (ii).

b) First let us specify the "obvious map" (A.1): it sends a pair $(c' \xrightarrow{s} c, c' \xrightarrow{f} d)$ with $s \in \Sigma$ and $f \in \mathcal{C}(c', d)$ to $Q(f)Q(s)^{-1}$. We now follow the strategy of [13, pp. 13/14]: using Axioms (ii) and (iii), we define for 3 objects $c, d, e \in \mathcal{C}$ a composition

$$\varinjlim_{c' \in \Sigma \downarrow c} \mathcal{C}(c', d) \times \varinjlim_{d' \in \Sigma \downarrow d} \mathcal{C}(d', e) \to \varinjlim_{c' \in \Sigma \downarrow c} \mathcal{C}(c', e)$$

which is shown to be well-defined and associative thanks to Axiom (iv). Hence we get a category $\Sigma^{-1}\mathcal{C}$ with the same objects as $\mathcal{C}$ and Hom sets as above, and (A.1) yields a functor $\Sigma^{-1}\mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$. But the obvious map $\mathcal{C}(c,d) \to \varinjlim_{c' \in \Sigma\downarrow c} \mathcal{C}(c',d)$ also yields a functor $\mathcal{C} \to \Sigma^{-1}\mathcal{C}$, which is the identity on objects and is easily seen to have the universal property of $\mathcal{C}[\Sigma^{-1}]$. Hence (A.1) is an isomorphism for all $(c,d)$.

c) The first statement has already been observed; the second one follows readily from (A.1). □

**Notation A.5.3.** We shall write $\Sigma^{-1}\mathcal{C}$ instead of $\mathcal{C}[\Sigma^{-1}]$ if $\Sigma$ enjoys a calculus of fractions.

**Corollary A.5.4.** *If $\Sigma$ admits a calculus of right fractions and if for any $c \in \mathcal{C}$, the category $\Sigma \downarrow c$ contains a small cofinal subcategory, then the* Hom *sets of $\Sigma^{-1}\mathcal{C}$ are small.* □

**Corollary A.5.5.** *Let $(\mathcal{C},\Sigma)$ be a localiser such that $\Sigma$ enjoys a calculus of right fractions. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Suppose that $F$ inverts the morphisms of $\Sigma$ and that, for any $c, d \in \mathcal{C}$, the obvious map*

$$\varinjlim_{c' \in \Sigma\downarrow c} \mathcal{C}(c',d) \to \mathcal{D}(F(c),F(d))$$

*is an isomorphism. Then the functor $\Sigma^{-1}F : \Sigma^{-1}\mathcal{C} \to \mathcal{D}$ induced by $F$ is fully faithful.* □

**Proposition A.5.6.** *a) Let $(\mathcal{C},\Sigma)$ be a localiser. Assume that $\Sigma$ enjoys a calculus of right fractions. Then the localisation functor $Q : \mathcal{C} \to \Sigma^{-1}\mathcal{C}$ is left exact; if limits indexed by a finite category $I$ exist in $\mathcal{C}$, they also exist in $\Sigma^{-1}\mathcal{C}$.*
*b) Let $\mathcal{C}$ be an essentially small category closed under finite limits, and let $G : \mathcal{C} \to \mathcal{D}$ be a left exact functor. Let $\Sigma = \{s \in Ar(\mathcal{C}) \mid G(s)$ is invertible$\}$. Then $\Sigma$ enjoys a calculus of right fractions; the induced functor $\Sigma^{-1}\mathcal{C} \to \mathcal{D}$ is conservative and left exact.*

*Proof.* After passing to the opposite categories, a) is [13, Prop. I.3.1 and Cor. I.3.2] and b) is [13, Prop. I.3.4]. □

We also have a useful lemma:

**Lemma A.5.7.** *Let $G : \mathcal{C} \to \mathcal{D}$ be an exact functor between abelian categories. Then $\mathcal{B} = \operatorname{Ker} G$ is a Serre subcategory of $\mathcal{C}$; if $G$ is a localisation, the induced functor $\mathcal{C}/\mathcal{B} \to \mathcal{D}$ is an equivalence of categories.*

*Proof.* The first statement is obvious. For the second one, let $f$ be a morphism in $\mathcal{C}$. The exactness of $G$ shows that $G(f)$ is an isomorphism if and only if $\operatorname{Ker} f, \operatorname{Coker} f \in \mathcal{B}$. □

A.6. **Pro-$\Sigma$-objects.**

**Definition A.6.1.** Let $(\mathcal{C}, \Sigma)$ be a localiser. We write $\mathrm{pro}_\Sigma\text{–}\mathcal{C}$ for the full subcategory of the category pro–$\mathcal{C}$ of pro-objects of $\mathcal{C}$ consisting of filtering inverse systems whose transition morphisms belong to $\Sigma$. An object of $\mathrm{pro}_\Sigma\text{–}\mathcal{C}$ is called a *pro-$\Sigma$-object*.

**Proposition A.6.2.** *Suppose that $\Sigma$ has a calculus of right fractions and, for any $c \in \mathcal{C}$, the category $\Sigma \downarrow c$ contains a small cofinal subcategory. Then $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ has a pro-left adjoint $Q^!$, which takes an object $X \in \Sigma^{-1}\mathcal{C}$ to "$\varprojlim_{M \in \Sigma\downarrow X}$" $M$. In particular, $Q^!(\Sigma^{-1}\mathcal{C}) \subset \mathrm{pro}_{\mathrm{sat}(\Sigma)}\text{–}\mathcal{C}$, where $\mathrm{sat}(\Sigma)$ is the saturation of $\Sigma$ (Definition A.3.2).*

*Proof.* In view of Corollary A.5.4 and Proposition A.5.6, this follows from Proposition A.4.2 a). □

*Remark* A.6.3. Consider the localisation functor $Q : \mathcal{C} \to \Sigma^{-1}\mathcal{C}$: it has a left Kan extension $\hat{Q} : \mathrm{pro}_\Sigma\text{–}\mathcal{C} \to \Sigma^{-1}\mathcal{C}$ [32, Ch. X] along the constant functor $\mathcal{C} \to \mathrm{pro}_\Sigma\text{–}\mathcal{C}$, given by the formula

$$\hat{Q}(\text{“}\varprojlim\text{”} C_\alpha) = \varprojlim Q(C_\alpha).$$

(The right hand side makes sense as an inverse limit of isomorphisms.) Then one checks easily that $Q^!$ is left adjoint to $\hat{Q}$.

**Theorem A.6.4.** *Let $(\mathcal{C}, \Sigma)$ be a localiser; assume that $\Sigma$ has a calculus of right fractions. Let $Q : \mathcal{C} \to \Sigma^{-1}\mathcal{C}$ denote the localisation functor, as well as the string of adjoint functors $(Q_!, Q^*, Q_*)$ between $\hat{\mathcal{C}}$ and $\widehat{\Sigma^{-1}\mathcal{C}}$ from §A.4. Then:*

1. *$Q_!$ has a pro-left adjoint, and is therefore exact.*
2. *For $\mathcal{F} \in \hat{\mathcal{C}}$ and $Y \in \Sigma^{-1}\mathcal{C}$, we have*

$$Q_!\mathcal{F}(Y) = \varinjlim_{X \in \Sigma\downarrow Y} F(Y).$$

*Proof.* This follows from Propositions A.4.2 a) and A.6.2. □

A.7. **Monoidal categories** [32, VII.1]**.** Recall that a monoidal category $(\mathcal{C}, \otimes)$ is *closed* if $\otimes$ has a right adjoint $\underline{\mathrm{Hom}}$. We use the following lemma several times:

**Lemma A.7.1.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two closed monoidal categories, and let $u : \mathcal{C} \to \mathcal{D}$ be a lax $\otimes$-functor: this means that we have a natural transformation*

$$uX \otimes uY \to u(X \otimes Y). \tag{A.2}$$

*Assume that $u$ has a right adjoint $v$. Then, for any $(X, Y) \in \mathcal{C} \times \mathcal{D}$, there is a canonical morphism*

$$\underline{\mathrm{Hom}}_\mathcal{C}(X, vY) \to v\,\underline{\mathrm{Hom}}_\mathcal{D}(uX, Y)$$

*bivariant in $(X, Y)$, which is an isomorphism if (A.2) is a natural isomorphism.*

*Proof.* Applying $u$ to the evaluation morphism $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, vY) \otimes X \to vY$ and using the counit of the adjunction, we get a composite morphism $u\,\underline{\mathrm{Hom}}_{\mathcal{C}}(X, vY) \otimes uX \to uvY \to Y$, hence a morphism

$$u\,\underline{\mathrm{Hom}}_{\mathcal{C}}(X, vY) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(uX, Y)$$

and finally a morphism

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, vY) \to v\,\underline{\mathrm{Hom}}_{\mathcal{D}}(uX, Y)$$

which is checked by Yoneda's lemma to be an isomorphism when (A.2) is. □

A.8. **Additive and monoidal categories.** Let $\mathcal{A}$ be an essentially small additive category. Instead of presheaves of sets on $\mathcal{A}$, one usually uses the category $\mathrm{Mod}\text{--}\mathcal{A}$ of *additive presheaves of abelian groups*; the results of §A.4 transfer to this context, mutatis mutandis. If $(\mathcal{A}, \Sigma)$ is a localiser with $\mathcal{A}$ additive and $\Sigma$ enjoys a calculus of right fractions, then $\Sigma^{-1}\mathcal{A}$ is additive and so is the functor $Q : \mathcal{A} \to \Sigma^{-1}\mathcal{A}$ [13, I.3.3]. For future reference, we give the additive analogue of Theorem A.6.4:

**Theorem A.8.1.** *Let $(\mathcal{A}, \Sigma)$ be a localiser; assume that $\mathcal{A}$ is additive and that $\Sigma$ has a calculus of right fractions. Let $Q : \mathcal{A} \to \Sigma^{-1}\mathcal{A}$ denote the localisation functor, as well as the string of adjoint functors $(Q_!, Q^*, Q_*)$ between $\mathrm{Mod}\text{--}\mathcal{A}$ and $\mathrm{Mod}\text{--}\Sigma^{-1}\mathcal{A}$. Then:*

1. *$Q_!$ has a pro-left adjoint, and is therefore exact.*
2. *For $\mathcal{F} \in \mathrm{Mod}\text{--}\mathcal{A}$ and $Y \in \Sigma^{-1}\mathcal{A}$, we have*

$$Q_!\mathcal{F}(Y) = \varinjlim_{X \in \Sigma \downarrow Y} F(Y).$$

A.9. **A pull-back lemma.** We shall use the following lemma several times.

**Lemma A.9.1.** *Let*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i"] \arrow[d, "f_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "f"] \\ \mathcal{C}' \arrow[r, "i'"] & \mathcal{D}' \end{tikzcd}$$

*be a commutative diagram of categories, where $i'$ and $f$ are fully faithful and $\mathcal{C}$ is the full subcategory*

$$\mathcal{C} = \{X \in \mathcal{D} \mid f(X) \in \mathrm{Im}\, i'\}.$$

*Assume that $i'$ has a left adjoint $a'$, $f$ has a right adjoint $g$ and $\mathcal{D}$ has a set of generators by strict epimorphisms $(X_i)_{i\in I}$* [SGA4-I, I.7.1]*, which all belong to $\mathcal{C}$. Then* *a) $i$ has a left adjoint $a$ and the $X_i$ are a set of generators of $\mathcal{C}$ by strict epimorphisms. We have an isomorphism of functors*

$$a' f \simeq f_{\mathcal{C}} a. \tag{A.3}$$

*b) $f_{\mathcal{C}}$ has a right adjoint if and only if $fg(\mathcal{C}') \subseteq \mathcal{C}'$; it is then given by the restriction of $g$ to $\mathcal{C}'$.*
*c) If $f$ and $a'$ are exact, $f_{\mathcal{C}}$ is left exact and $a$ is exact. If $\mathcal{D}$ is closed under finite colimits, then $f_{\mathcal{C}}$ is exact.*
*d) If $f$ has a (pro)-left adjoint $\ell$, then $f_{\mathcal{C}}$ has the (pro)-left adjoint $\ell_{\mathcal{C}} = a\ell i'$; under c), $\ell$ exact $\Rightarrow \ell_{\mathcal{C}}$ exact.*
*e) If $\mathcal{D}$ is closed under small colimits, so is $\mathcal{C}$; if filtered colimits are exact in $\mathcal{D}$, so are they in $\mathcal{C}$.*

*Proof.* a) Let $X \in \mathcal{D}$. Let us show that $a'f(X)$ is in the essential image of $f_{\mathcal{C}}$. Since $a'$ and $f$ are left adjoints, they are cocontinuous and it is enough to check this for $X$ a generator (cf. [SGA4-I, I.7.2 (i) b)]). For such a generator $X_i$, we have $a'f(X_i) = a'i'f_{\mathcal{C}}(X_i) \simeq f_{\mathcal{C}}(X_i)$ by the full faithfulness of $i'$.

Choose $a(X) \in \mathcal{C}$ such that $fia(X) \simeq a'f(X)$. Since $f$ and $i$ are fully faithful, $X \mapsto a(X)$ defines a functor $a : \mathcal{D} \to \mathcal{C}$, which is easily checked to be left adjoint to $i$. The claim on generators for $\mathcal{C}$ now follows from the fact that $a$ is co-continuous.

b) Necessity is obvious and sufficiency is easy.

c) We first check that $f_{\mathcal{C}}$ is left exact. Since $i'$ is fully faithful and commutes with any limit (as a right adjoint), it suffices to show that $i'f_{\mathcal{C}} = fi$ is left exact, which is true since $f$ and $i$ are left exact (the latter, as a right adjoint). By (A.3), the left exactness of $a$ now follows by the same reasoning, and its right exactness is tautological since it is a left adjoint.

Finally, we show that $f_{\mathcal{C}}$ is right exact if $\mathcal{D}$ is closed under finite colimits. Let $(c_\alpha)_{\alpha\in A}$ be a finite direct system in $\mathcal{C}$, with colimit $c$. We must show that $f_{\mathcal{C}}c$ is a colimit of $(f_{\mathcal{C}}c_\alpha)$. Let $d = \varinjlim ic_\alpha$: by the exactness of $a$ we have $ad = c$. Since $a'f$ is exact,

$$f_{\mathcal{C}}c = f_{\mathcal{C}}ad = a'fd = \varinjlim a'fic_\alpha = \varinjlim a'i'f_{\mathcal{C}}c_\alpha = \varinjlim f_{\mathcal{C}}c_\alpha$$

as wanted.

d) Note that $\ell_{\mathcal{C}}$ is automatically right exact as a (pro)-left adjoint; it is then left exact as a composition of left exact functors.

e) The first (resp. second) claim follows from the co-continuity (resp. exactness) of $a$. □

A.10. **Homological algebra.** Recall Grothendieck's theorem [14, Th. 2.4.1]:

**Theorem A.10.1.** *Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be a string of left exact functors between abelian categories. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and that $F$ carries injectives of $\mathcal{A}$ to $G$-acyclics. Then, for any $A \in \mathcal{A}$, there is a convergent spectral sequence*

$$E_2^{p,q} = R^p G R^q F(A) \Rightarrow R^{p+q}(GF)(A).$$

*Examples* A.10.2. If $F$ has an exact left adjoint, it carries injectives to injectives. If $G$ is exact, the hypothesis on $F$ is automatically verified.

We shall also use the following standard result:

**Proposition A.10.3.** *Let $a : \mathcal{B} \leftrightarrows \mathcal{A} : i$ be a pair of adjoint functors between abelian categories ($a$ is left adjoint to $i$). Suppose that $\mathcal{A}$ has enough injectives and that $a$ is exact. Then, for any $(A, B) \in \mathcal{A} \times \mathcal{B}$, there is a convergent spectral sequence*

$$\operatorname{Ext}^p_{\mathcal{B}}(B, R^q iA) \Rightarrow \operatorname{Ext}^{p+q}_{\mathcal{A}}(aB, A).$$

*If $B$ is projective, this spectral sequence collapses to isomorphisms*

$$\mathcal{B}(B, R^q iA) \simeq \operatorname{Ext}^q_{\mathcal{A}}(aB, A).$$

*Proof.* Fix $B$. By adjunction, the composition of functors

$$\mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{\mathcal{B}(B,-)} \mathbf{Ab}$$

is isomorphic to $\mathcal{A}(aB, -)$. We then get the spectral sequence from Theorem A.10.1 and Example A.10.2. The last fact is obvious. □

A.11. **Brown representability and compact generation.** Recall the following definitions and results of Neeman:

**Definition A.11.1.** A triangulated category $\mathcal{T}$ has the *Brown representability property* if

(1) It is cocomplete.
(2) Any homological functor $H : \mathcal{T}^{\mathrm{op}} \to \mathbf{Ab}$ which converts infinite direct sums into products is representable.

**Lemma A.11.2** ([23, Cor. 10.5.3])**.** *If $\mathcal{T}$ has the Brown representability property, it is complete; a triangulated functor $F : \mathcal{T} \to \mathcal{T}'$ has a right adjoint $G$ if and only if it is strongly additive, and $G$ is triangulated.* □

*Example* A.11.3. Suppose $\mathcal{T}$ is cocomplete and let $\mathcal{R} \subset \mathcal{T}$ be a localising subcategory: $\mathcal{R}$ is triangulated and closed under direct sums. Then the inclusion functor $\mathcal{R} \hookrightarrow \mathcal{T}$ and the localisation functor $\mathcal{T} \to \mathcal{T}/\mathcal{R}$ are strongly additive [9, Lemma 1.5].

**Definition A.11.4.** Let $\mathcal{T}$ be a triangulated category.
a) An object $X \in \mathcal{T}$ in is *compact* if the functor $Y \mapsto \mathcal{T}(X,Y)$ is strongly additive (Definition 3.2.3). We write $\mathcal{T}^c$ for the thick subcategory of $\mathcal{T}$ consisting of compact objects.
b) A subset $\mathcal{X}$ of $Ob(\mathcal{T})$ *generates* $\mathcal{T}$ if its right orthogonal is 0.
c) $\mathcal{T}$ is *compactly generated* if it is cocomplete and generated by a (small) set of compact objects.
d) Given a subset $\mathcal{X}$ of $Ob(\mathcal{T})$, the *thick hull of* $\mathcal{X}$ *in* $\mathcal{T}$ is the smallest triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{X}$ and is closed under direct summands.

*Remark* A.11.5. Suppose that $\mathcal{T}$ is cocomplete. Then a set $\mathcal{X} \subset Ob(\mathcal{T})$ of compact objects generates $\mathcal{T}$ in the sense of Definition A.11.4 b) if and only if the smallest localising subcategory of $\mathcal{T}$ containing $\mathcal{X}$ is equal to $\mathcal{T}$ [45, Lemma 2.2.1].

*Example* A.11.6. Let $\mathcal{A}$ be an essentially small additive category and $\mathcal{B} = \mathrm{Mod}\text{–}\mathcal{A}$. Then $\mathcal{T} = D(\mathcal{B})$ is compactly generated and $K^b(\mathcal{A}) \xrightarrow{\sim} \mathcal{T}^c$ [21, Prop. A.4.1].

We have the following very nice result of Beilinson-Vologodsky [6, Proposition in §1.4.2 (see also §1.2)]:

**Theorem A.11.7.** *Let $\mathcal{T}$ be a cocomplete triangulated category and let $\mathcal{S} \subseteq \mathcal{T}$ be a localising subcategory which is generated by a set of compact objects of $\mathcal{T}$. Then the localisation functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ has a right adjoint whose essential image is the right orthogonal $\mathcal{S}^\perp$ of $\mathcal{S}$. In particular, $\mathcal{S} = \mathcal{T} \iff \mathcal{S}^\perp = 0$.*

The two main results on compactly generated triangulated categories are:

**Theorem A.11.8** ([36, Th. 4.1])**.** *Any compactly generated triangulated category has the Brown representability property.*

**Theorem A.11.9** ([35, Th. 2.1])**.** *Let $\mathcal{T}$ be a compactly generated triangulated category. Let $\mathcal{S} \subset \mathcal{T}$ be a localising subcategory generated by a set of compact objects of $\mathcal{T}$. Then $\mathcal{T}/\mathcal{S}$ is compactly generated and compact objects of $\mathcal{T}$ remain compact in $\mathcal{T}/\mathcal{S}$; the induced functor $\mathcal{T}^c/\mathcal{S}^c \to (\mathcal{T}/\mathcal{S})^c$ is fully faithful and $(\mathcal{T}/\mathcal{S})^c$ is the thick hull of $\mathcal{T}^c/\mathcal{S}^c$ in $\mathcal{T}/\mathcal{S}$.*

**Corollary A.11.10** (cf. [21, Th. A.2.6])**.** *In the situation of Theorem A.11.9, the localisation functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ has a right adjoint, which also has a right adjoint.* $\square$

*Remark* A.11.11. Compact generation is not a necessary condition for the validity of Brown's representability theorem: a weaker sufficient condition is *good generation*, studied by Neeman [37, Ch. 8] and Krause [28]. For example, $D(\mathcal{A})$ is well-generated for any Grothendieck category $\mathcal{A}$, but not necessarily compactly generated ([38], see also Theorem A.13.1 a)). All "large" triangulated categories appearing in this paper are compactly generated; this can be proven by applying Theorems A.11.8 and A.11.9 to Example A.11.3, except for $D(\mathbf{MNST})$, for which the proof uses its good generation!

We shall also use the following lemma of Neeman, a special case of [21, Lemma 4.4.5]:

**Lemma A.11.12.** *Let $\mathcal{T}$ be a cocomplete triangulated category and let $\mathcal{X} \subset Ob(\mathcal{T})$ be a set of compact objects. If $\mathcal{X}$ generates $\mathcal{T}$ (see Definition A.11.4 and Remark A.11.5), then the thick hull of $\mathcal{X}$ is $\mathcal{T}^c$.*

A.12. **Grothendieck categories.** Recall that a *Grothendieck abelian category* (for short, a Grothendieck category) is an abelian category verifying Axiom AB5 of [14]: small colimits are representable and exact, and having a set of generators (equivalently, a generator). These generators are generators by strict epimorphisms as in Lemma A.9.1 a). We have the folllowing basic facts:

**Theorem A.12.1.** *a) Any Grothendieck category is complete and has enough injectives.*
*b) Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a functor, where $\mathcal{C}$ is a Grothendieck category. Then $F$ has a right adjoint if and only if it is cocontinuous.*
*c) Let $\mathcal{C}$ be a Grothendieck category, $\mathcal{B} \subset \mathcal{C}$ be a Serre subcategory, $\mathcal{D} = \mathcal{C}/\mathcal{B}$ and $G : \mathcal{C} \to \mathcal{D}$ the (exact) localisation functor. Then $G$ has a right adjoint $D$ if and only if $\mathcal{B}$ is stable under infinite direct sums. In this case, $\mathcal{B}$ and $\mathcal{D}$ are Grothendieck.*
*d) Let $G : \mathcal{C} \leftrightarrows \mathcal{D} : D$ be a pair of adjoint additive functors between additive categories, with $D$ fully faithful. If $\mathcal{C}$ is Grothendieck and $G$ is exact, $\mathcal{D}$ is Grothendieck.*

*Proof.* a) See [14, Th. 1.10.1], [SGA4-I, V.0.2.1] or [23, Th. 8.3.27 (i) and 9.6.2]. b) See [23, Prop. 8.3.27 (iii)]. c) See [12, Ch. III, Prop. 8 and 9]. d) Let $\mathcal{B}$ be the kernel of $G$. Then $\mathcal{B}$ is easily seen to be a Serre subcategory (e.g. [12, Ch. III, Prop. 5]), so the claim follows from c). □

A.13. **Unbounded derived categories.**

**Theorem A.13.1** ([23, Th. 14.3.1])**.** *Let $\mathcal{A}$ be a Grothendieck category.*

*a) The unbounded derived category $D(\mathcal{A})$ has the Brown representability property.*
*b) The localisation functor $\lambda_{\mathcal{A}} : K(\mathcal{A}) \to D(\mathcal{A})$ has a right adjoint $\rho_{\mathcal{A}}$, whose essential image is (by definition) the full subcategory of* homotopically injective *complexes.*[5]
*c) Let $F : K(\mathcal{A}) \to \mathcal{T}$ be a triangulated functor. Then $F$ has a (universal) right Kan extension $RF$ relative to $\lambda_{\mathcal{A}}$, given by $RF = F \circ \rho_{\mathcal{A}}$. In particular, any left exact functor $F : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is another abelian category, has a total right derived functor $RF : D(\mathcal{A}) \to D(\mathcal{B})$ given by $RF = \lambda_{\mathcal{B}} \circ K(F) \circ \rho_{\mathcal{A}}$.*
*d) The restriction of $RF$ to $D^+(\mathcal{A})$ is the total derived functor $R^+F$ (cf.* [46, §2, Rem. 1.6]*).*

Let us justify d), which is not in [23]: the point is that $\rho_{\mathcal{A}}$ carries $D^+(\mathcal{A})$ into $K^+(\mathcal{A})$ [23, Th. 13.3.7].

**Theorem A.13.2.** *Let $\mathcal{A}$ be an additive category.*
*a)* Mod –$\mathcal{A}$ *is a Grothendieck category with a set of projective generators. The category $D(\text{Mod}-\mathcal{A})$ is left complete (see Definition A.13.11).*
*b) If $\mathcal{A}$ is monoidal, its tensor structure canonically extends to* Mod –$\mathcal{A}$ *through the additive Yoneda functor, and provides* Mod –$\mathcal{A}$ *with the structure of a closed additive monoidal category.*
*c) The $\otimes$-structure of $\mathcal{A}$ extends uniquely to a $\otimes$-triangulated structure on the homotopy category $K^b(\mathcal{A})$.*
*d) The $\otimes$-structure of* Mod –$\mathcal{A}$ *has a total left derived functor, which is strongly additive and provides $D(\text{Mod}-\mathcal{A})$ with a closed $\otimes$-triangulated structure.*
*e) If $u : \mathcal{A} \to \mathcal{B}$ is a monoidal functor, $u_! : \text{Mod}-\mathcal{A} \to \text{Mod}-\mathcal{B}$ is monoidal, and so are the functors $K^b(u) : K^b(\mathcal{A}) \to K^b(\mathcal{B})$ and $Lu_! : D(\text{Mod}-\mathcal{A}) \to D(\text{Mod}-\mathcal{B})$.*

*Proof.* a) See e.g. [1, Prop. 1.3.6] for the first statement; the projective generators are given by $\mathcal{E} = \{y(A) \mid A \in \mathcal{A}\}$. For the left completeness, apply Lemma A.13.12 below with this $\mathcal{E}$. b) is [30, Def. 8.2] or [22, A.8]. c) is easy (define $\otimes$ termwise). d) This applies to any right exact covariant bifunctor $T : \text{Mod}-\mathcal{A} \times \text{Mod}-\mathcal{A} \to \mathcal{C}$, where $\mathcal{C}$ is abelian and cocomplete: by a) and [23, Th. 14.4.3], $K(\text{Mod}-\mathcal{A})$ has enough homotopically projective objects (K-projective in the sense of Spaltenstein [44]), which means that the localisation functor $\lambda : K(\text{Mod}-\mathcal{A}) \to D(\text{Mod}-\mathcal{A})$ has a left adjoint $\gamma$. Then the formula

$$LT(C, D) := \lambda T(\gamma C, \gamma D)$$

[5] This is the same notion as Spaltenstein's $K$-injective [44].

provides the desired total left derived functor. By Example A.11.3, $\lambda$ and $\gamma$ are strongly additive; thus if $T$ is strongly additive, so is $LT$. Similarly, a left exact contravariant/covariant bifunctor $S$ has a total right derived functor $RS$ given by the formula

$$RS(C,D) = \lambda S(\gamma C, \rho D)$$

where $\rho$ is right adjoint to $\lambda$ (also apply a) and Theorem A.13.1 b)). In the case $T = \otimes_{\mathrm{Mod}-\mathcal{A}}$, $S = \underline{\mathrm{Hom}}_{\mathrm{Mod}-\mathcal{A}}$, these formulas immediately imply that $LT$ is left adjoint to $RS$, which gives a second justification of the strong biadditiveness of $\otimes_{\mathrm{Mod}-\mathcal{A}}$.

e) is [22, A.12] for the first statement; the second one is easy and the third follows from the universal property of left derived functors as Kan extensions. □

**Definition A.13.3.** Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between Grothendieck categories. An object $C \in K(\mathcal{A})$ is *$F$-acyclic* if the morphism

$$\lambda_{\mathcal{B}} K(F) C \to RF \lambda_{\mathcal{A}} C = \lambda_{\mathcal{B}} K(F) \rho_{\mathcal{A}} \lambda_{\mathcal{A}} C$$

given by the unit map of the adjunction $(\lambda_{\mathcal{A}}, \rho_{\mathcal{A}})$ is an isomorphism.

*Example* A.13.4. If $F$ is exact, every object of $K(\mathcal{A})$ is $F$-acyclic.

Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be a chain of left exact functors between Grothendieck categories. The unit map of the adjunction $(\lambda_{\mathcal{B}}, \rho_{\mathcal{B}})$ yields a natural transformation

$$R(GF) \Rightarrow RG \circ RF. \tag{A.4}$$

The following lemma is made tautological by Definition A.13.3:

**Lemma A.13.5.** *(A.4) is a natural isomorphism if and only if $F$ carries homotopically injectives to $G$-acyclics. In particular, (A.4) is a natural isomorphism provided $G$ is exact (see Example A.13.4).* □

Here is a first application:

**Proposition A.13.6.** *Assume $\mathcal{C} = \mathcal{A}$, $F$ right adjoint to $G$, and $G$ exact. Then $RF$ is right adjoint to $RG = D(G)$. If further $F$ is fully faithful, then $D(G)$ is a localisation.*

*Proof.* The first statement is a special case of [23, Th. 14.4.5]. For the second one, by Lemma A.3.1 we have to show that the counit morphism $D(G)RF \Rightarrow Id_{\mathcal{A}}$ is an isomorphism, which follows from Lemma A.13.5. □

We come back to the general situation. Suppose that $F$ carries injectives of $\mathcal{A}$ to $G$-acyclics. Then Theorem A.10.1 implies that (A.4) is an isomorphism when restricted to $D^+(\mathcal{A})$ ([46, §2, Prop. 3.1], [23, Th. 13.3.7 and Prop. 13.3.13]). This is not true on $D(\mathcal{A})$ in general, as pointed out by Ayoub and Riou:

*Example* A.13.7. Let $\mathcal{B} = \mathrm{Mod}-\mathbb{Z}[\mathbb{Z}/2]$, $\mathcal{A} = \mathcal{B}^{\mathbb{N}}$ and $\mathcal{C} = \mathbf{Ab}$; let $F = \bigoplus_{\mathbb{N}}$ and $G = H^0(\mathbb{Z}/2, -)$. The above hypotheses are verified: since $\mathbb{Z}[\mathbb{Z}/2]$ is Noetherian, a direct sum of injectives is injective. Let $M = (\mathbb{Z}/2[n])_{n\in\mathbb{N}} \in D(\mathcal{A})$. We claim that the map

$$R(GF)(M) \to RGRF(M) \tag{A.5}$$

is not an isomorphism. Indeed, $GF = F'G'$ where $G' : \mathcal{A} \to \mathbf{Ab}^{\mathbb{N}}$ is $H^0(\mathbb{Z}/2, -)$ and $F' : \mathbf{Ab}^{\mathbb{N}} \to \mathbf{Ab}$ is $\bigoplus_{\mathbb{N}}$. Let $C = RG(\mathbb{Z}/2)$, so that $H^q(C) = \mathbb{Z}/2$ for $q \geq 0$ and $H^q(C) = 0$ for $q < 0$. Then, by Lemma A.13.5:

$$R(GF)(M) = R(F'G')(M) = RF'RG'(M) = \bigoplus_{n\in\mathbb{N}} C[n].$$

On the other hand,

$$RGRF(M) = RG(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/2[n]).$$

But, in $D(\mathcal{B})$, we have $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/2[n] \xrightarrow{\sim} \prod_{n\in\mathbb{N}} \mathbb{Z}/2[n]$, and $RG$ commutes with products as a right adjoint. Hence

$$RGRF(M) = \prod_{n\in\mathbb{N}} C[n].$$

For $q \in \mathbb{Z}$, we have $H^q(\bigoplus\limits_{n\in\mathbb{N}} C[n]) = \bigoplus\limits_{q+n\geq 0} \mathbb{Z}/2$ and $H^q(\prod\limits_{n\in\mathbb{N}} C[n]) = \prod\limits_{q+n\geq 0} \mathbb{Z}/2$.

However, we have the following lemma of Ayoub:

**Lemma A.13.8.** *Suppose that $F$ carries injectives to $G$-acyclics and that $RF, RG$ and $R(GF)$ are strongly additive (see Definition 3.2.3). Then* (A.4) *is an isomorphism.*

(In example A.13.7, $RG$ is not strongly additive.)

*Proof (Ayoub).* Let $M \in D(\mathcal{A})$. We have to show that (A.5) is an isomorphism. Viewing $M$ as an object of $K(\mathcal{A})$, we have an isomorphism

$$\mathrm{hocolim}_n\, \sigma_{\geq n} M \xrightarrow{\sim} M$$

where $\sigma_{\geq n}$ is the stupid truncation. This isomorphism still holds in $D(\mathcal{A})$, because $\lambda_{\mathcal{A}}$ is strongly additive (Example A.11.3). By the hypothesis, this reduces us to the case where $M \in D^+(\mathcal{A})$, and therefore to Grothendieck's theorem (cf. Theorem A.13.1 d)). □

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between Grothendieck categories. In view of Lemma A.13.8, we need a practical sufficient condition to ensure that $RF$ is strongly additive. The following one is adapted to the context of this paper:

**Proposition A.13.9.** *a) If $F$ is strongly additive and exact, $RF = D(F)$ is strongly additive.*
*b) Suppose that*

(i) *For any $p \geq 0$, $R^pF$ is strongly additive.*
(ii) *There exists a set $\mathcal{E}$ of compact projective generators of $\mathcal{B}$ such that, for any $E \in \mathcal{E}$, there is an integer $cd_F(E)$ such that*
$$\mathcal{B}(E, R^pF(A)) = 0 \text{ for } p > cd_F(E) \text{ and for all } A \in \mathcal{A}.$$

*Then $RF$ is strongly additive.*

*Proof.* a) The strong additivity of $F$ easily implies that of $K(F)$, which in turn implies that of $D(F)$ since $\lambda_{\mathcal{B}}$ is strongly additive as a left adjoint.

b) Let $(C_i)_{i \in I} \in D(\mathcal{A})^I$. We must show that the map
$$\bigoplus_{i \in I} RF(C_i) \to RF(\bigoplus_{i \in I} C_i)$$
is an isomorphism. Since the $E[n]$, $E \in \mathcal{E}$, $n \in \mathbb{Z}$, are a set of generators of $D(\mathcal{B})$, it suffices to check this after applying $D(\mathcal{B})(E[n], -)$ for all $(E, n)$. Since $E$ is projective, we have an isomorphism
$$D(\mathcal{B})(E[n], D) \simeq \mathcal{B}(E, H^n(D))$$
for any $D \in D(\mathcal{B})$; since $E$ is compact in $\mathcal{B}$, this formula shows that $E[n]$ is compact in $D(\mathcal{B})$. Therefore we must show that the homomorphisms
$$\bigoplus_{i \in I} \mathcal{B}(E, H^n(RFC_i)) \to \mathcal{B}(E, H^n(RF \bigoplus_{i \in I} C_i)).$$
are bijective. By (ii), the spectral sequence
$$\mathcal{B}(E, R^pFH^q(C)) \Rightarrow \mathcal{B}(E, H^{p+q}(RFC))$$
converges for any $C \in D(\mathcal{A})$. Thus it suffices to show that the homomorphisms
$$\bigoplus_{i \in I} \mathcal{B}(E, R^pFH^q(C_i)) \to \mathcal{B}(E, R^pFH^q(\bigoplus_{i \in I} C_i))$$

are bijective. By (i), this follows from the compactness of $E$. □

Finally, we need a practical sufficient condition to ensure that, in Condition (i) of Proposition A.13.9, the case $p = 0$ implies the cases $p > 0$. This is given by the classical

**Lemma A.13.10.** *Suppose that $F$ is strongly additive and that, in $\mathcal{A}$, infinite direct sums of injectives are $F$-acyclic. Then $R^pF$ is strongly additive for any $p > 0$.*

*Proof.* Décalage. □

Let $\mathcal{T}$ be a triangulated category in which countable products are representable. We then have the notion of homotopy limit, dual to that of homotopy colimit introduced in [9]. If $\mathcal{T}$ is endowed with a $t$-structure, for any object $T \in \mathcal{T}$ we have a map

$$T \to \operatorname{holim}_n \tau_{\geq -n} T \tag{A.6}$$

where the truncations are relative to the $t$-structure. This map is well-defined up to non-unique isomorphism. The following notion was studied by J. Lurie:

**Definition A.13.11.** The $t$-category $\mathcal{T}$ is *left complete* if (A.6) is an isomorphism for all $T \in \mathcal{T}$.

Suppose $\mathcal{T} = D(\mathcal{A})$, where $\mathcal{A}$ is a Grothendieck category. We endow $\mathcal{T}$ with its canonical $t$-structure. By Theorem A.13.1 a) and Lemma A.11.2, small products are representable in $\mathcal{T}$. An example where $\mathcal{T}$ is not left complete was given by Neeman in [39]. On the other hand, we have the following sufficient condition of Ayoub:

**Lemma A.13.12.** *Suppose that $D(\mathcal{A})$ is generated by $\coprod_{i\in\mathbb{Z}} \mathcal{E}[i]$, where $\mathcal{E} \subseteq \mathcal{A}$ is a class of objects of finite* Ext*-dimension (i.e. $A \in \mathcal{E} \Rightarrow \operatorname{Ext}^p_{\mathcal{A}}(A, -) = 0$ for $p \gg 0$). Then $D(\mathcal{A})$ is left complete.*

*Proof (Ayoub).* For $T \in D(\mathcal{A})$, (A.6) is an isomorphism if and only if the induced map

$$D(\mathcal{A})(A[0], T[i]) \to D(\mathcal{A})(A[0], \operatorname{holim}_n(\tau_{\geq -n}T)[i]) \tag{A.7}$$

is an isomorphism for any $A \in \mathcal{E}$ and any $i \in \mathbb{Z}$. The right hand term can be computed for any $A \in \mathcal{A}$ by the Milnor exact sequence

$$\begin{aligned} &0 \to \varprojlim_n{}^1 D(\mathcal{A})(A[0], \tau_{\geq -n}T)[i-1]) \\ &\to D(\mathcal{A})(A[0], \operatorname{holim}_n(\tau_{\geq -n}T)[i]) \to \varprojlim_n D(\mathcal{A})(A[0], \tau_{\geq -n}T)[i]) \to 0. \end{aligned} \tag{A.8}$$

If $A \in \mathcal{E}$, the hyperext spectral sequence

$$\operatorname{Ext}^p_{\mathcal{A}}(A, H^q(C)) \Rightarrow D(\mathcal{A})(A, C[p+q])$$

is strongly convergent for any $C \in D(\mathcal{A})$. Since $q \mapsto H^q(\tau_{\geq -n}T)$ is stationary with value $H^q(T)$, this implies that $n \mapsto D(\mathcal{A})(A[0], \tau_{\geq -n}T)[i])$ is stationary with value $D(\mathcal{A})(A[0], T[i])$. Hence the conclusion. $\square$

*Remark* A.13.13 (cf. [20, proof of Prop. C.1.1]). In Lemma A.13.12, $D(\mathcal{A})$ is generated by $\coprod_{i\in\mathbb{Z}} \mathcal{E}[i]$ provided $\mathcal{A}$ is generated by $\mathcal{E}$. Indeed, suppose this is the case: by the dual of [15, p. 42, Ch. I, Lemma 4.6.1], the hypothesis implies that any $K \in C^-(\mathcal{A})$ is homotopy equivalent to a complex $L$ such that $L^n$ is a direct sum of objects of $\mathcal{E}$ for all $n \in \mathbb{Z}$. Let now $C \in D(\mathcal{A})$ be such that $D(\mathcal{A})(A[i], C) = 0$ for all $A \in \mathcal{E}$ and all $i \in \mathbb{Z}$. For any direct sum $\bigoplus_{j\in J} A_j$ of objects of $\mathcal{E}$and any $i \in \mathbb{Z}$, we have

$$D(\mathcal{A})(\bigoplus_{j\in J} A_j[i], C) = \prod_{j\in J} D(\mathcal{A})(A_j[i], C) = 0.$$

It follows that, for any $D \in K^b(\mathcal{A})$, we have $D(\mathcal{A})(D, C) = 0$. We extend this to any $D \in K^-(\mathcal{A})$ by the argument in the proof of Lemma A.13.8, and then to any $D \in D(\mathcal{A})$ by the isomorphism

$$\operatorname{hocolim} \tau_{\leq n} D \xrightarrow{\sim} D.$$

Applying this to $D = C$, we get $C = 0$.

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## Acronyms

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