RECIPROCITY SHEAVES AND LOGARITHMIC MOTIVES

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Contents

Abstract. We connect two developments aiming at extending Voevodsky's theory of motives over a field in such a way to encompass non-$\mathbf{A}^1$-invariant phenomina. One is theory of *reciprocity sheaves* introduced in [6] and [7] and developed in [15] and [2]. Another is theory of the triangulated category $\mathbf{logDM}^{\mathrm{eff}}$ of *logarithmic motives* launched in [1]. We prove that the Nisnevich cohomology of reciprocity sheaves is representable in $\mathbf{logDM}^{\mathrm{eff}}$.

Introduction

We fix once and for all a perfect base field k. The main purpose of this paper is to connect two developments aiming at extending Voevodsky's theory of motives over k in such a way to encompass non-$\mathbf{A}^1$-invariant phenomina. One is theory of *reciprocity sheaves* introduced in [6] and [7] and developed in [15] and [2]. Voevodsky's theory is based on the category $\mathbf{PST}$ of *presheaves with transers*, defined as the category of additive presheaves of abelian groups on the category $\mathbf{Cor}$ of finite correspondences: $\mathbf{Cor}$ has the same objects as the category $\mathbf{Sm}$ of separated smooth schemes of finite type over k and morphisms in $\mathbf{Cor}$ are finite correspondences. Let $\mathbf{NST} \subset \mathbf{PST}$ be the full subcategory of Nisnevich sheaves, i.e. those objects $F \in \mathbf{PST}$ whose restrictions F_X to the small étale site $X_{\text{ét}}$ over X are Nisnevich sheaves for all $X \in \mathbf{Sm}$. Voevodsky proved that $\mathbf{NST}$ is a Grothendieck abelian category and defined the triangulated category $\mathbf{DM}^{\mathrm{eff}}$ of effective motives as the localization of the derived category $D(\mathbf{NST})$ of complexes in $\mathbf{NST}$ with respect to an $\mathbf{A}^1$-*weak equivalence* (see [10, Def. 14.1]). It is equipped with a functor $M : \mathbf{Sm} \to \mathbf{DM}^{\mathrm{eff}}$ associating the motive $M(X)$ of $X \in \mathbf{Sm}$.

Let $\mathbf{HI}_{\mathrm{Nis}} \subset \mathbf{NST}$ be the full subcategory consisting of $\mathbf{A}^1$-invariant objects, namely such $F \in \mathbf{NST}$ that the projection $\pi_X : X \times \mathbf{A}^1 \to X$ induces an isomorphism $\pi_X^* : F(X) \simeq F(X \times \mathbf{A}^1)$ for any $X \in \mathbf{Sm}$. We say that $F \in \mathbf{HI}_{\mathrm{Nis}}$ is strictly $\mathbf{A}^1$-invariant if π_X induces isomorphisms

$$\pi_X^* : H^i_{\mathrm{Nis}}(X, F_X) \simeq H^i_{\mathrm{Nis}}(X \times \mathbf{A}^1, F_{X \times \mathbf{A}^1}) \text{ for all } i \geq 0.$$

The following theorem plays a fundamental role in Voevodsky's theory.

Theorem 0.1. *(Voevodsky [16]) Any $F \in \mathbf{HI}_{\mathrm{Nis}}$ is strictly $\mathbf{A}^1$-invariant and we have a natural isomorphism*

$$H^i_{\mathrm{Nis}}(X, F_X) \simeq \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}}(M(X), L^{\mathbf{A}^1} F[i]) \text{ for } X \in \mathbf{Sm}, \tag{0.1.1}$$

where $L^{\mathbf{A}^1} : D(\mathbf{NST}) \to \mathbf{DM}^{\mathrm{eff}}$ is the localization functor.

Note that there are interesting non-$\mathbf{A}^1$-invariant objects of $\mathbf{NST}$ such as the sheaves represented by smooth commutative algebraic groups which may have non-trivial unipotent parts (e.g. the additive group scheme $\mathbf{G}_a$), the sheaf of absolute Kähler differentials Ω^i and the de Rham-Witt sheaves $W_n\Omega^i$, and that (0.1.1) fails to hold for these sheaves since π_X induces an isomorphism $M(X \times \mathbf{A}^1) \simeq M(X)$ in $\mathbf{DM}^{\mathrm{eff}}$ but the maps induced on cohomology are not isomorphism in general for these sheaves. On the other hand, the category $\mathbf{RSC}_{\mathrm{Nis}}$ of reciprocity sheaves is a full subcategory of $\mathbf{NST}$ that contains $\mathbf{HI}_{\mathrm{Nis}}$ as well as the non-$\mathbf{A}^1$-invariant objects mentioned above (see [7, Th. 1 and Cor. 3.2.5]). Thus one may ask if it is possible to establish the formula (0.1.1) for $F \in \mathbf{RSC}_{\mathrm{Nis}}$ in some new category which enlarges $\mathbf{DM}^{\mathrm{eff}}$.

Recently Binda, Park and Østvær [1] launched theory of *logarithmic motives*, which has non-$\mathbf{A}^1$-invariant nature. Let $\mathrm{l}\mathbf{Sm}$ be the category of log smooth and separated fs log schemes of finite type over k and $\mathrm{l}\mathbf{Cor}$ be the category with the same objects as $\mathrm{l}\mathbf{Sm}$ and whose morphisms are log finite correspondences (see [1, Def. 2.1.1]). Let $\mathbf{PSh}^{ltr}$ be the category of additive presheaves of abelian groups on $\mathrm{l}\mathbf{Cor}$ and $\mathbf{Shv}^{ltr}_{dNis} \subset \mathbf{PSh}^{ltr}$ be the full subcategory consisting of those $\mathcal{F}$ whose restrictions to $\mathrm{l}\mathbf{Sm}$ are dividing Nisnevich sheaves (see [1, Def. 3.1.4]). It is shown in [1, §4 and Pr.4.6.6] that $\mathbf{Shv}^{ltr}_{dNis}$ is a Grothendieck abelian category, and the triangulated category $\mathbf{logDM}^{\mathrm{eff}}$ of *logarithmic motives* is defined as the localization of the derived category $D(\mathbf{Shv}^{ltr}_{dNis})$

of complexes in $\mathbf{Shv}^{ltr}_{dNis}$ with respect to a $\overline{\square}$-*weak equivalence*, where $\overline{\square}$ is $\mathbf{P}^1$ with the log-structure associated to the effective divisor $\infty \hookrightarrow \mathbf{P}^1$ (see [1, Def. 5.1.1][1]). It is equipped with a functor $M : \mathrm{l}\mathbf{Sm} \to \mathbf{logDM}^{\mathrm{eff}}$ associating the logarithmic motive $M(\mathfrak{X})$ of $\mathfrak{X} \in \mathrm{l}\mathbf{Sm}$.

Now we can state the main result of this paper.

Theorem 0.2. *(Theorems 6.1.1 and 6.3) There exists an exact and fully faithful functor*

$$\mathcal{L}og : \mathbf{RSC}_{\mathrm{Nis}} \to \mathbf{Shv}^{ltr}_{dNis} \;:\; F \to F^{\log} = \mathcal{L}og(F) \tag{0.2.1}$$

such that $F^{\log}$ for $F \in \mathbf{RSC}_{\mathrm{Nis}}$ is strictly $\overline{\square}$-invariant in the sense [1, Def. 5.2.7]. *For $X \in \mathbf{Sm}$ we have a natural isomorphism*

$$H^i_{\mathrm{Nis}}(X, F_X) \simeq \mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}}(M(X, \mathrm{triv}), L^{\overline{\square}} F^{\log}[i]), \tag{0.2.2}$$

where $L^{\overline{\square}} : D(\mathbf{Shv}^{ltr}_{dNis}) \to \mathbf{logDM}^{\mathrm{eff}}$ is the localization functor and (X, triv) is the log-scheme with the trivial log-structure.

We remark (see Remark 5.5) that for $F = \Omega^i$, $F^{\log}(\mathfrak{X})$ for $\mathfrak{X} \in \mathrm{l}\mathbf{Sm}$ whose underlying scheme is smooth, agrees with the sheaf of logarithmic differentials of $\mathfrak{X}$ at least assuming $\mathrm{ch}(k) = 0$ [2].

The theorem implies that the following formulas and formalism proved in [2] for the Nisnevich cohomology of smooth schemes with coefficient $F \in \mathbf{RSC}_{\mathrm{Nis}}$ [3] *have motivic origin*, i.e. arise from the corresponding formulas and formalism proved in [1] for $\mathbf{logDM}^{\mathrm{eff}}$. Take $F \in \mathbf{RSC}_{\mathrm{Nis}}$.

(*I*) (Smooth blowup formula) Let $i : Z \hookrightarrow X$ is a closed immersion in $\mathbf{Sm}$ and $\pi : X' \to X$ be the blowup of X along Z and $E = \pi^{-1}(Z)$ with the closed immersion $i_E : E \hookrightarrow X'$ and the projection $\pi_E : E \to Z$. Then there exists a long exact sequence

$$\cdots \to H^i_{\mathrm{Nis}}(X, F_X) \xrightarrow{\pi^* \oplus i^*} H^i_{\mathrm{Nis}}(X', F_{X'}) \oplus H^i_{\mathrm{Nis}}(Z, F_Z) \xrightarrow{i_E^* - \pi_E^*} H^i_{\mathrm{Nis}}(E, F_E) \to H^{i+1}_{\mathrm{Nis}}(X, F_X) \to \cdots$$

(*II*) (Projective bundle formula) Let $\mathcal{E}$ be a vector bundle of rank $n+1$ over $X \in \mathbf{Sm}$ and $\pi : P = \mathbf{P}(\mathcal{E}) \to X$ be the projection

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[1] In fact it is defined in loc.cite. as the localization of the homotopy category of complexes in $\mathbf{Shv}^{ltr}_{dNis}$ with respect to a $\overline{\square}$-local descent model structure.

[2] The assumption is necessary to use [12, Cor. 6.8] proved in case $\mathrm{ch}(k) = 0$. We expect that it is removed by using a forthcoming work of K. Rülling extending [12, Cor. 6.8] to the case $\mathrm{ch}(k) > 0$.

[3] In fact, in [2], they are proved not only for smooth schemes but also for *smooth modulus pairs*, which are pairs (X, D) of $X \in \mathbf{Sm}$ and (not necessary reduced) effective Cartier divisor D on X whose support is a normal crossing divisor.

of the corresponding projective bundle. Then there exists a natural isomorphism

$$\bigoplus_{0\le c\le n} H^{i-c}_{\mathrm{Nis}}(X,(\gamma^c F)_X) \simeq H^i_{\mathrm{Nis}}(P,F_P),$$

where $\gamma^c F = \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}^M_c, F)$ with $\mathcal{K}^M_c \in \mathbf{NST}$ the Milnor K-sheaf and $\underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}^M_c, -)$ the internal hom[4]. By [9, Cor. 0.3], we have $\gamma^c\Omega^{i+c} \simeq \Omega^i$ if $\mathrm{ch}(k)=0$. Thus the above formula gives a motivic interpretation of the projective bundle formula for sheaves of Kähler differentials.

(III) $((\mathbf{P}^n,\mathbf{P}^{n-1})$-invariance) For $X\in\mathbf{Sm}$ and an integer $n\ge 1$, let $\mathfrak{P}^n_X = (\mathbf{P}^n_X, \mathcal{M}_{\mathbf{P}^{n-1}_X}) \in \mathrm{l}\mathbf{Sm}$, where $\mathcal{M}_{\mathbf{P}^{n-1}_X}$ is the log-structure associated to the divisor $\mathbf{P}^{n-1}_X \subset \mathbf{P}^n_X$. Then we have a natural isomorphism

$$H^i_{\mathrm{Nis}}(X,F_X) \simeq H^i_{s\mathrm{Nis}}(\mathfrak{P}^n_X, F^{\log}),$$

where $H^i_{s\mathrm{Nis}}$ denotes the cohomology for the strict Nisnevich topology on $\mathrm{l}\mathbf{Sm}$ (see [1, §4]).

(IV) (Gysin sequence) Let $i: Y\hookrightarrow X$ be a closed immersion of pure codimension c in $\mathbf{Sm}$ and $\pi: X'\to X$ be the blowup of X along Y and $E=\pi^{-1}(Y)$. Put $\mathfrak{X}=(X',\mathcal{M}_E)\in \mathrm{l}\mathbf{Sm}$, where $\mathcal{M}_E$ is the log-structure associated to the divisor $E\subset X'$. Then there exists a long exact sequence

$$\cdots \to H^{i-c}_{\mathrm{Nis}}(Y,(\gamma^c F)_Y) \xrightarrow{i_*} H^i_{\mathrm{Nis}}(X,F_X) \to H^i_{s\mathrm{Nis}}(\mathfrak{X},F^{\log}) \to H^{i-c+1}_{\mathrm{Nis}}(Y,(\gamma^c F)_Y) \xrightarrow{i_*} \cdots .$$

The maps i_* are called Gysin maps.

(V) (Pushforward maps for projective morphisms) Let $f: Y\to X$ be a projective map where $X,Y\in\mathbf{Sm}$ are equi-dimensional. Let $c=\dim(Y)-\dim(X)$. If $c\ge 0$, then there exists a pushforward map

$$f_*: H^{i+c}_{\mathrm{Nis}}(Y,F_Y) \to H^i_{\mathrm{Nis}}(X,(\gamma^c F)_X.)$$

If $c\le 0$, then there exists a pushforward map

$$f_*: H^{i+c}_{\mathrm{Nis}}(Y,(\gamma^{-c}F)_Y) \to H^i_{\mathrm{Nis}}(X,F_X).$$

They satisfies the standard properties such as the coincidence with Gysin maps for closed immersions, the compatibility for compositions and the projection formula.

[4] γ preserves $\mathbf{RSC}_{\mathrm{Nis}}$

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1. Preliminaries

We fix once and for all a perfect base field k. In this section we recall the definitions and basic properties of modulus sheaves with transfers from [4] and [15] (see also [7] for a more detailed summary).

(1) Denote by **Sch** the category of separated schemes of finite type over k and by **Sm** the full subcategory of smooth schemes. For $X, Y \in \mathbf{Sm}$, an integral closed subscheme of $X \times Y$ that is finite and surjective over a connected component of X is called a *prime correspondence from X to Y*. The category **Cor** of finite correspondences has the same objects as **Sm**, and for $X, Y \in \mathbf{Sm}$, $\mathbf{Cor}(X, Y)$ is the free abelian group on the set of all prime correspondences from X to Y (see [16]). We consider **Sm** as a subcategory of **Cor** by regarding a morphism in **Sm** as its graph in **Cor**.

Let **PST** be the category of additive presheaves of abelian groups on **Cor** whose objects are called *presheaves with transfers*. Let $\mathbf{NST} \subseteq \mathbf{PST}$ be the category of Nisnevich sheaves with transfers and let

$$a^V_{\mathrm{Nis}} : \mathbf{PST} \to \mathbf{NST}$$

be Voevodsky's Nisnevich sheafification functor, which is an exact left adjoint to the inclusion $\mathbf{NST} \to \mathbf{PST}$. Let $\mathbf{HI} \subseteq \mathbf{PST}$ be the category of $\mathbf{A}^1$-invariant presheaves and put $\mathbf{HI}_{\mathrm{Nis}} = \mathbf{HI} \cap \mathbf{NST} \subseteq \mathbf{NST}$.

(2) Let $\mathbf{Sm}^{\mathrm{pro}}$ be the category of k-schemes X which are essentially smooth over k, i.e. X is a limit $\varprojlim_{i\in I} X_i$ over a filtered set I, where X_i is smooth over k and all transition maps are étale. Note $\operatorname{Spec} K \in \mathbf{Sm}^{\mathrm{pro}}$ for a function field K over k thanks to the assumption that k is perfect. We define $\mathbf{Cor}^{\mathrm{pro}}$ whose objects are the same as $\mathbf{Sm}^{\mathrm{pro}}$ and morphisms are defined as [12, Def. 2,2]. We extend $F \in \mathbf{PST}$ to a presheaf on $\mathbf{Cor}^{\mathrm{pro}}$ by $F(X) := \varinjlim_{i\in I} F(X_i)$ for X as above.

(3) We recall the definition of the category $\underline{\mathbf{M}}\mathbf{Cor}$ from [4, Definition 1.3.1]. A pair $\mathcal{X} = (X, D)$ of $X \in \mathbf{Sch}$ and an effective Cartier divisor D on X is called a *modulus pair* if $M -$

$|M_\infty| \in \mathbf{Sm}$. Let $\mathcal{X} = (X, D_X)$, $\mathcal{Y} = (Y, D_Y)$ be modulus pairs and $\Gamma \in \mathbf{Cor}(X - D_X, Y - D_Y)$ be a prime correspondence. Let $\overline{\Gamma} \subseteq X \times Y$ be the closure of Γ, and let $\overline{\Gamma}^N \to X \times Y$ be the normalization. We say Γ is *admissible* (resp. *left proper*) if $(D_X)_{\overline{\Gamma}^N} \geq (D_Y)_{\overline{\Gamma}^N}$ (resp. if $\overline{\Gamma}$ is proper over X). Let $\underline{\mathbf{M}}\mathbf{Cor}(\mathcal{X}, \mathcal{Y})$ be the subgroup of $\mathbf{Cor}(X - D_X, Y - D_Y)$ generated by all admissible left proper prime correspondences. The category $\underline{\mathbf{M}}\mathbf{Cor}$ has modulus pairs as objects and $\underline{\mathbf{M}}\mathbf{Cor}(\mathcal{X}, \mathcal{Y})$ as the group of morphisms from $\mathcal{X}$ to $\mathcal{Y}$.

(4) Let $\underline{\mathbf{M}}\mathbf{Cor}_{ls} \subset \underline{\mathbf{M}}\mathbf{Cor}$ be the full subcategory of $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$ with $X \in \mathbf{Sm}$ and $|D|$ a normal crossing divisor on X.

(5) Let $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}} \subset \underline{\mathbf{M}}\mathbf{Cor}$ be the full subcategory of the same objects such that $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(\mathcal{X}, \mathcal{Y})$ are generated by all admissible *finite* prime correspondences, where finite prime correspondences are defined by replacing the left properness in (3) by finiteness. We also define $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}_{ls} \subset \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}_{ls} \cap \underline{\mathbf{M}}\mathbf{Cor}_{ls}$.

(6) There is a canonical pair of adjoint functors $\lambda \dashv \underline{\omega}$:

$$\lambda : \mathbf{Cor} \to \underline{\mathbf{M}}\mathbf{Cor} \quad X \mapsto (X, \emptyset),$$

$$\underline{\omega} : \underline{\mathbf{M}}\mathbf{Cor} \to \mathbf{Cor} \quad (X, D) \mapsto X - |D|,$$

(7) There is a full subcategory $\mathbf{MCor} \subset \underline{\mathbf{M}}\mathbf{Cor}$ consisting of *proper modulus pairs*, where a modulus pair (X, D) is *proper* if X is proper. Let $\tau : \mathbf{MCor} \hookrightarrow \underline{\mathbf{M}}\mathbf{Cor}$ be the inclusion functor and $\omega = \underline{\omega}\tau$.

(8) We have two categories of *modulus presheaves with trasnfers*:

$$\mathbf{MPST} = Fun(\mathbf{MCor}, \mathbf{Ab}) \text{ and } \underline{\mathbf{M}}\mathbf{PST} = Fun(\underline{\mathbf{M}}\mathbf{Cor}, \mathbf{Ab}).$$

Let $\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}) = \underline{\mathbf{M}}\mathbf{Cor}(-, \mathcal{X}) \in \underline{\mathbf{M}}\mathbf{PST}$ be the representable presheaf for $\mathcal{X} \in \mathbf{MCor}$. In this paper we frequently write $\mathcal{X}$ for $\mathbb{Z}_{\mathrm{tr}}(\mathcal{X})$ for simplicity.

(9) By the same manner as (2), the category $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{pro}}$ is defined and $F \in \underline{\mathbf{M}}\mathbf{PST}$ is extended to a presheaf on $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{pro}}$ (see [12, §3.7]).

(10) The adjunction $\lambda \dashv \underline{\omega}$ induce a string of 4 adjoint functors $(\lambda_! = \underline{\omega}^!, \lambda^* = \underline{\omega}_!, \lambda_* = \underline{\omega}^*, \underline{\omega}_*)$:

$$\underline{\mathbf{M}}\mathbf{PST} \begin{array}{c} \xleftarrow{\underline{\omega}^!} \\ \xrightarrow{\underline{\omega}_!} \\ \xleftarrow{\underline{\omega}^*} \\ \xrightarrow{\underline{\omega}_*} \end{array} \mathbf{PST}$$

where $\underline{\omega}_!, \underline{\omega}_*$ are localisations and $\underline{\omega}^!$ and $\underline{\omega}^*$ are fully faithful.

(11) The functor τ yields a string of 3 adjoint functors $(\tau_!, \tau^*, \tau_*)$:

$$\mathbf{MPST} \underset{\overset{\longrightarrow}{\tau_*}}{\overset{\overset{\tau_!}{\longrightarrow}}{\underset{\longleftarrow}{\tau^*}}} \underline{\mathbf{M}}\mathbf{PST}$$

where $\tau_!, \tau_*$ are fully faithful and τ^* is a localisation; $\tau_!$ has a pro-left adjoint $\tau^!$, hence is exact. We will denote by $\underline{\mathbf{M}}\mathbf{PST}^\tau$ the essential image of $\tau_!$ in $\underline{\mathbf{M}}\mathbf{PST}$.

(12) The modulus pair $\overline{\square} := (\mathbf{P}^1, \infty)$ has an interval structure induced by the one of $\mathbf{A}^1$ (see [7, Lem. 2.1.3]). We say $F \in \mathbf{MPST}$ is $\overline{\square}$-invariant if $p^* : F(\mathcal{X}) \to F(\mathcal{X} \otimes \overline{\square})$ is an isomorphism for any $\mathcal{X} \in \mathbf{MCor}$, where $p : \mathcal{X} \otimes \overline{\square} \to \mathcal{X}$ is the projection. Let $\mathbf{CI}$ be the full subcategory of $\mathbf{MPST}$ consisting of all $\overline{\square}$-invariant objects and $\mathbf{CI}^\tau \subset \underline{\mathbf{M}}\mathbf{PST}$ be the essential image of $\mathbf{CI}$ under $\tau_!$.

(13) Recall from [7, Theorem 2.1.8] that $\mathbf{CI}$ is a Serre subcategory of $\mathbf{MPST}$, and that the inclusion functor $i^{\overline{\square}} : \mathbf{CI} \to \mathbf{MPST}$ has a left adjoint $h_0^{\overline{\square}}$ and a right adjoint $h^0_{\overline{\square}}$ given for $F \in \mathbf{MPST}$ and $\mathcal{X} \in \mathbf{MCor}$ by

$$h_0^{\overline{\square}}(F)(\mathcal{X}) = \mathrm{Coker}(i_0^* - i_1^* : F(\mathcal{X} \otimes \overline{\square}) \to F(\mathcal{X})),$$
$$h^0_{\overline{\square}}(F)(\mathcal{X}) = \mathrm{Hom}(h_0^{\overline{\square}}(\mathcal{X}), F).$$

For $\mathcal{X} \in \mathbf{MCor}$, we write $h_0^{\overline{\square}}(\mathcal{X}) = h_0^{\overline{\square}}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X})) \in \mathbf{CI}$, and by abuse of notation, we let $h_0^{\overline{\square}}(\mathcal{X})$ denote also for $\tau_! h_0^{\overline{\square}}(\mathcal{X}) \in \mathbf{CI}^\tau$.

(14) For $F \in \underline{\mathbf{M}}\mathbf{PST}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$, write $F_\mathcal{X}$ for the presheaf on the small étale site $X_{\text{ét}}$ over X given by $U \to F(\mathcal{X}_U)$ for $U \to X$ étale, where $\mathcal{X}_U = (U, D_{|U}) \in \underline{\mathbf{M}}\mathbf{Cor}$. We say F is a Nisnevich sheaf if so is $F_\mathcal{X}$ for all $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$ (see [4, Section 3]). We write $\underline{\mathbf{M}}\mathbf{NST} \subset \underline{\mathbf{M}}\mathbf{PST}$ for the full subcategory of Nisnevich sheaves and put

$$\mathbf{MNST}^\tau = \underline{\mathbf{M}}\mathbf{NST} \cap \mathbf{MPST}^\tau, \quad \mathbf{CI}^\tau_{\mathrm{Nis}} = \mathbf{CI}^\tau \cap \mathbf{MNST}^\tau.$$

By [4, Prop. 3.5.3] and [5, Theorem 2], the inclusion functor $i_{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NST} \to \underline{\mathbf{M}}\mathbf{PST}$ has an exact left adjoint $\underline{a}_{\mathrm{Nis}}$ such that $\underline{a}_{\mathrm{Nis}}(\mathbf{MPST}^\tau) \subset \mathbf{MNST}^\tau$. The functor $\underline{a}_{\mathrm{Nis}}$ has the following description: For $F \in \underline{\mathbf{M}}\mathbf{PST}$ and $\mathcal{Y} \in \underline{\mathbf{M}}\mathbf{Cor}$, let $F_{\mathcal{Y},\mathrm{Nis}}$ be the usual Nisnevich sheafification of $F_\mathcal{Y}$. Then, for $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$ we have

$$\underline{a}_{\mathrm{Nis}} F(X, D) = \varinjlim_{f : Y \to X} F_{(Y, f^*D),\mathrm{Nis}}(Y)$$

where the colimit is taken over all proper maps $f : Y \to X$ that induce isomorphisms $Y - |f^*D| \xrightarrow{\sim} X - |D|$.

(15) The functors $\underline{\omega}^*$ and $\underline{\omega}_!$ respect $\mathbf{\underline{M}NST}$ and $\mathbf{NST}$ and induce a pair of adjoint functors (which for simplicity we write $\underline{\omega}_!$ and $\underline{\omega}^*$). Moreover, we have

$$\underline{\omega}_! \underline{a}_{\mathrm{Nis}} = a^V_{\mathrm{Nis}} \underline{\omega}_!.$$

For $F \in \mathbf{PST}$, we have $F \in \mathbf{HI}$ (resp $F \in \mathbf{HI}_{\mathrm{Nis}}$) if and only if $\underline{\omega}^* F \in \mathbf{CI}^\tau$ (resp $\underline{\omega}^* F \in \mathbf{CI}^\tau_{\mathrm{Nis}}$).

(16) We say that $F \in \mathbf{\underline{M}PST}$ is *semi-pure* if the unit map

$$u : F \to \underline{\omega}^* \underline{\omega}_! F$$

is injective. For $F \in \mathbf{\underline{M}PST}$ (resp. $F \in \mathbf{\underline{M}NST}$), let $F^{sp} \in \mathbf{\underline{M}PST}$ (resp. $F^{sp} \in \mathbf{\underline{M}NST}$) be the image of $F \to \underline{\omega}^* \underline{\omega}_! F$ (called the semi-purification of F). For $F \in \mathbf{\underline{M}PST}$ we have

$$\underline{a}_{\mathrm{Nis}}(F^{sp}) \simeq (\underline{a}_{\mathrm{Nis}} F)^{sp}.$$

This follows from the fact that $\underline{a}_{\mathrm{Nis}}$ is exact and commutes with $\underline{\omega}^* \underline{\omega}_!$. For $F \in \mathbf{MPST}^\tau$ we have $F^{sp} \in \mathbf{MPST}^\tau$ since τ is exact and $\underline{\omega}^* \underline{\omega}_! \tau_! = \tau_! \omega^* \omega_!$.

(17) Let $\mathbf{CI}^{\tau,sp} \subset \mathbf{CI}^\tau$ be the full subcategory of semipure objects and consider the full subcategory

$$\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}} = \mathbf{CI}^{\tau,sp} \cap \mathbf{MNST}^\tau \subset \mathbf{CI}^\tau_{\mathrm{Nis}}.$$

By [15, Th. 0.1 and 0.4], we have $\underline{a}_{\mathrm{Nis}}(\mathbf{CI}^{\tau,sp}) \subset \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$.

(18) We write $\mathbf{RSC} \subseteq \mathbf{PST}$ for the essential image of $\mathbf{CI}$ under $\omega_!$ (which is the same as the essential image of $\mathbf{CI}^{\tau,sp}$ under $\underline{\omega}_!$ since $\omega_! = \underline{\omega}_! \tau_!$ and $\underline{\omega}_! F = \underline{\omega}_! F^{sp}$). Put $\mathbf{RSC}_{\mathrm{Nis}} = \mathbf{RSC} \cap \mathbf{NST}$. The objects of $\mathbf{RSC}$ (resp. $\mathbf{RSC}_{\mathrm{Nis}}$) are called reciprocity presheaves (resp. sheaves). We have $\mathbf{HI} \subseteq \mathbf{RSC}$ and it contains also smooth commutative group schemes (which may have non-trivial unipotent part), and the sheaf Ω^i of Kähler differentials, and the de Rham-Witt sheaves $W\Omega^i$ (see [6] and [7]).

(19) $\mathbf{NST}$ is a Grothendieck abelian category by [16, Lem. 3.1.6] and we can make $\mathbf{RSC}_{\mathrm{Nis}}$ its full sub-abelian category as follows: We define the kernel (resp. cokernel) of a map $\phi : F \to G$ in $\mathbf{RSC}_{\mathrm{Nis}}$ to be that of ϕ as a map in $\mathbf{NST}$. Here we need [15, Th. 0.1] to ensure that the cokernel of ϕ in $\mathbf{NST}$ stays in $\mathbf{RSC}_{\mathrm{Nis}}$. By definition, a sequence $0 \to F \to G \to H \to 0$ is exact in $\mathbf{RSC}_{\mathrm{Nis}}$ if and only if it is exact in $\mathbf{NST}$.

(20) By [7, Prop. 2.3.7] we have a pair of adjoint functors:

$$\mathbf{CI} \underset{\omega_!}{\overset{\omega^{\mathbf{CI}}}{\leftrightarrows}} \mathbf{RSC}, \tag{1.0.1}$$

where $\omega^{\mathbf{CI}} = h^0_{\square}\omega^*$ and it is fully faithful. It induces a pair of adjoint functors:

$$\mathbf{CI}^\tau \underset{\underline{\omega}_!}{\overset{\underline{\omega}^{\mathbf{CI}}}{\leftrightarrows}} \mathbf{RSC}, \tag{1.0.2}$$

where $\underline{\omega}^{\mathbf{CI}} = \tau_! h^0_{\square}\omega^*$ and it is fully faithful. Indeed, let $F = \tau_!\hat{F}$ for $\hat{F} \in \mathbf{CI}$ and $G \in \mathbf{RSC}$. In view of (13) and the exactness and full faithfulness of $\tau_!$, we have

$$\mathrm{Hom}_{\mathbf{CI}^\tau}(F, \tau_! h^0_{\square}\omega^* G) \simeq \mathrm{Hom}_{\mathbf{CI}}(\hat{F}, h^0_{\square}\omega^* G) \simeq$$
$$\mathrm{Hom}_{\mathbf{MPST}}(\hat{F}, \omega^* G) \simeq \mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(\tau_!\hat{F}, \underline{\omega}^* G) \simeq \mathrm{Hom}_{\mathbf{RSC}}(\underline{\omega}_! F, G).$$

(1.0.2) induce pair of adjoint functors:

$$\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}} \underset{\underline{\omega}_!}{\overset{\underline{\omega}^{\mathbf{CI}}}{\leftrightarrows}} \mathbf{RSC}_{\mathrm{Nis}}, \tag{1.0.3}$$

If $F \in \mathbf{CI}^\tau$, the adjunction induces a canonical map

$$F \to \underline{\omega}^{\mathbf{CI}}\underline{\omega}_! F$$

which is injective if $F \in \mathbf{CI}^{\tau,sp}$.

2. Purity with reduced modulus

For $F \in \underline{\mathbf{M}}\mathbf{PST}$, we put

$$F_{-1} = \mathrm{Ker}\left(\underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}((\mathbf{P}^1 - 0, \infty), F) \xrightarrow{i_1^*} F\right),$$

$$F^{(1)}_{-1} = \mathrm{Ker}\left(\underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}((\mathbf{P}^1, 0 + \infty), F) \xrightarrow{i_1^*} F\right),$$

Note that if $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, we have for $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$

(2.0.1)

$$F^{(1)}_{-1}(\mathcal{X}) = \mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\mathbf{P}^1, 0+\infty)^0, \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}), F)),$$

$$F_{-1}(\mathcal{X}) = \varinjlim_n \mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\mathbf{P}^1, n\cdot 0+\infty)^0, \underline{\mathrm{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}), F)),$$

where

$$h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\mathbf{P}^1, n\cdot 0+\infty)^0 = \mathrm{Coker}\left(\mathbb{Z} = \mathbb{Z}_{\mathrm{tr}}(\mathrm{Spec}\, k, \emptyset) \xrightarrow{i_1} h^{\overline{\square},\mathrm{sp}}_{0,\mathrm{Nis}}(\mathbf{P}^1, n\cdot 0+\infty)\right).$$

The existence of retractions in the following lemma was suggested by A. Merici.

Lemma 2.1. *For $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, the inclusion $F^{(1)}_{-1} \to F_{-1}$ admits a retraction $s_F : F_{-1} \to F^{(1)}_{-1}$ such that for any map $\phi : G \to F \to G$ in $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, the following diagram is commutative:*

$$\begin{array}{ccc} F_{-1} & \xrightarrow{s_F} & F^{(1)}_{-1} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ G_{-1} & \xrightarrow{s_F} & G^{(1)}_{-1} \end{array}$$

In particular $\tau^{(1)}F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ if $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$.

Proof. In view of (2.0.1), this follows from [2, Lem. 2.3]. □

Definition 2.2. For $e_1, \ldots, e_r \in \{0, 1\}$, put

$$\tau^{(e_1,\ldots,e_r)}F = \tau^{(e_r)} \cdots \tau^{(e_1)}F,$$

where

$$\tau^{(0)}F = F_{-1} \text{ and } \tau^{(1)}F = F_{-1}/F^{(1)}_{-1}.$$

By Lemma 2.1, $\tau^{(e_1,\ldots,e_r)}F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ if $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$.

Theorem 2.3. *Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. Let $\mathcal{X} = \operatorname{Spec} K\{t_1, \ldots, t_n\}$ and $D = \{t_1^{e_1} \cdots t_n^{e_n} = 0\} \subset \mathcal{X}$ with $e_1, \ldots, e_n \in \{0, 1\}$. For a subset $I \subset [1, n]$ let $i_{\mathcal{H}} : \mathcal{H} \hookrightarrow \mathcal{X}$ be the closed immersion defined by $\{t_i = 0\}_{i\in I}$ and $D_{\mathcal{H}} = \{\prod_{j\in[1,n]-I} t_j^{e_j} = 0\} \subset \mathcal{H}$. Then*

$$R^\nu i^!_{\mathcal{H}} F_{(\mathcal{X},D)} = 0 \ \text{ for } \nu \neq q := |I|, \tag{2.3.1}$$

and there is an isomorphism

$$(\tau^{(e_I)}F)_{(\mathcal{H},D_{\mathcal{H}})} \simeq R^q i^!_{\mathcal{H}} F_{(\mathcal{X},D)} \ \text{ with } e_I = (e_i)_{i\in I} \in \mathbb{Z}^q_{\geq 0}. \tag{2.3.2}$$

Proof. The proof is divided into two steps.

Step 1: We prove (2.3.1) and (2.3.2) in case $q = |I| = 1$.

For $\nu = 0$ (2.3.1) follows from the semipurity of F. Thus it suffices to show (2.3.1) only for $\nu > 1$. Let $J = \{j \in [1, n] \mid e_j \neq 0\}$ and $r = |J|$. If $\dim(\mathcal{X}) = 0$, the assertion is trivial. If $r = 0$, the assertion follows from [15, Cor. 8.6(3)]. Assume $r > 0$ and $\dim(\mathcal{X}) \geq 1$, and proceed by the double induction on r and $\dim(\mathcal{X})$. Without loss of generality, we may assume

(♠) $e_1 \neq 0$, and $\mathcal{H} = \{t_1 = 0\}$ if $\mathcal{H} \subset |D|$.

Let $\iota : \mathcal{Z} \hookrightarrow \mathcal{X}$ be the closed immersion defined by $\{t_1 = 0\}$ and $D_{\mathcal{Z}} = \{t_2^{e_2} \cdots t_r^{e_r} = 0\} \subset \mathcal{Z}$ and $D' = \{t_2^{e_2} \cdots t_r^{e_r} = 0\} \subset \mathcal{X}$. By [15, Lem. 7.1], we have an exact sequence sheaves on $\mathcal{X}_{\mathrm{Nis}}$:

$$0 \to F_{(\mathcal{X},D')} \to F_{(\mathcal{X},D)} \to \iota_*(F^{(e_1)}_{-1})_{(\mathcal{Z},D_{\mathcal{Z}})} \to 0,$$

which gives rise to a long exact sequence of sheaves on $\mathcal{H}_{\mathrm{Nis}}$:
(2.3.3)

$$\cdots \to R^\nu i^!_{\mathcal{H}} F_{(\mathcal{X},D')} \to R^\nu i^!_{\mathcal{H}} F_{(\mathcal{X},D)} \to R^\nu i^!_{\mathcal{H}} \iota_*(F^{(e_1)}_{-1})_{(\mathcal{Z},D_{\mathcal{Z}})} \to \cdots .$$

By the induction hypothesis, $R^\nu i^!_{\mathcal{H}} F_{(\mathcal{X},D')} = 0$ for $\nu > 1$. In case $\mathcal{H} \neq \mathcal{Z}$, we have a Cartesian diagram of closed immersions

$$\begin{array}{ccc} \mathcal{H} \cap \mathcal{Z} & \xrightarrow{\iota'} & \mathcal{H} \\ {\scriptstyle i_{\mathcal{H}\cap\mathcal{Z}}}\downarrow & & \downarrow{\scriptstyle i_{\mathcal{H}}} \\ \mathcal{Z} & \xrightarrow{\iota} & \mathcal{X} \end{array}$$

and we have an isomorphism

$$R^\nu i^!_{\mathcal{H}} \iota_*(F^{(e_1)}_{-1})_{(\mathcal{Z},D_{\mathcal{Z}})} \simeq \iota'_* R^\nu i^!_{\mathcal{H}\cap\mathcal{Z}}(F^{(e_1)}_{-1})_{(\mathcal{Z},D_{\mathcal{Z}})}.$$

By the induction hypothesis, $R^\nu i^!_{\mathcal{H}\cap\mathcal{Z}}(F^{(e_1)}_{-1})_{(\mathcal{Z},D_{\mathcal{Z}})} = 0$ for $\nu > 1$ noting $F^{(e_1)}_{-1} \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ by Lemma 2.1. So the desired vanishing follows from (2.3.3). Moreover, the assumptions (♠) and $\mathcal{H} \neq \mathcal{Z}$ imply that $\mathcal{H} \not\subset |D|$. Then (2.3.2) (with $q = 1$) follows from [15, Lem. 7.1(2)].

In case $\mathcal{Z} = \mathcal{H}$, we have

$$R^\nu i^!_{\mathcal{H}} \iota_*(F^{(e_1)}_{-1})_{(\mathcal{Z},D_{\mathcal{Z}})} = R^\nu \iota^! \iota_*(F^{(e_1)}_{-1})_{(\mathcal{Z},D_{\mathcal{Z}})},$$

which vanishes for $\nu > 0$. Hence (2.3.3) gives the desired vanishing together with an exact sequence:

$$0 \to (F^{(e_1)}_{-1})_{(\mathcal{H},D_{\mathcal{H}})} \xrightarrow{\delta} R^1 i^!_{\mathcal{H}} F_{(\mathcal{X},D')} \to R^1 i^!_{\mathcal{H}} F_{(\mathcal{X},D)} \to 0.$$

By [15, Lem. 7.1(2)] we have an isomorphism

$$(F_{-1})_{(\mathcal{H},D_{\mathcal{H}})} \simeq R^1 i^!_{\mathcal{H}} F_{(\mathcal{X},D')}$$

through which δ is identified with the map induced by the canonical map $F^{(e_1)}_{-1} \to F_{-1}$ by [15, Lem. 7.1(2)]. This proves the desired isomorphism (2.3.2) in case $\mathcal{Z} = \mathcal{H}$ and completes Step 1.

Step 2: We prove the theoerm by the induction on q assuming $q > 0$. Let $I = \{i_1, \ldots, i_q\} \subset [1, n]$ and $\mathcal{Y} \subset \mathcal{X}$ be the closed subscheme defined by $\{t_{i_1} = 0\}$. Let $i_{\mathcal{Y}} : \mathcal{Y} \hookrightarrow \mathcal{X}$ and $i_{\mathcal{H},\mathcal{Y}} : \mathcal{H} \to \mathcal{Y}$ be the induced closed immersions. By Step 1 we have $R^\nu i^!_{\mathcal{Y}} F_{(\mathcal{X},D)} = 0$ for $\nu \neq 1$ and we have an isomorphism

$$(\tau^{(e_{i_1})} F)_{(\mathcal{Y},D_{\mathcal{Y}})} \simeq R^1 i^!_{\mathcal{Y}} F_{(\mathcal{X},D)} \;\text{ with } D_{\mathcal{Y}} = \{t_1^{e_1} \cdots \overset{\vee}{t_{i_1}^{e_{i_1}}} \cdots t_n^{e_n} = 0\} \subset \mathcal{Y}.$$

Note $\tau^{(e_{i_1})} F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ by Lemma 2.1. Thus, by the induction hypothesis, we have $R^\nu i^!_{\mathcal{H},\mathcal{Y}} \tau^{(e_{i_1})} F_{(\mathcal{Y},D_{\mathcal{Y}})} = 0$ for $\nu \neq q - 1$. By the spectral sequence

$$E_2^{a,b} = R^b i^!_{\mathcal{H},\mathcal{Y}} R^a i^!_{\mathcal{Y}} F_{(\mathcal{X},D)} \Rightarrow R^{a+b} i^!_{\mathcal{H}} F_{(\mathcal{X},D)},$$

we get the desired vanishing (2.3.1) and an isomorphism

$$R^q i^!_{\mathcal{H}} F_{(\mathcal{X},D)} \simeq R^{q-1} i^!_{\mathcal{H},\mathcal{Y}} R^1 i^!_{\mathcal{Y}} F_{(\mathcal{X},D)} \simeq R^{q-1} i^!_{\mathcal{H},\mathcal{Y}} (\tau^{(e_{i_1})} F)_{(\mathcal{Y},D_{\mathcal{Y}})}$$
$$\simeq (\tau^{(e_{i_2},\dots,e_{i_q})}(\tau^{(e_{i_1})}F))_{(\mathcal{H},D_{\mathcal{H}})} \simeq (\tau^{(e_{i_1},e_{i_2},\dots,e_{i_q})}F)_{(\mathcal{H},D_{\mathcal{H}})},$$

where the third isomorphism holds by the induction hypothesis. This completes the proof of the theorem. □

We say $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$ reduced if so is D. The following corollaries 2.4 and 2.5 are immediate consequences of Theorem 2.3.

Corollary 2.4. *Take $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$ reduced. Let $x \in X^{(n)}$ with $K = k(x)$ and let $\mathcal{X} = X^h_{|x}$ be the henselization of X at x. Then*

$$H^i_x(X_{\mathrm{Nis}}, F_{(X,D)}) = 0 \quad \text{for } i \neq n.$$

Choosing an isomorphism

$$\varepsilon : \mathcal{X} \simeq \operatorname{Spec} K\{t_1, \dots, t_n\}$$

such that $D_{|\mathcal{X}} = \{t_1^{e_1} \cdots t_n^{e_n} = 0\} \subset \mathcal{X}$ with $e_1, \dots, e_n \in \{0, 1\}$, there exists an isomorphism depending on ε:

$$\theta_\varepsilon : \tau^{(e_1,e_2,\dots,e_n)} F(x) \simeq H^n_x(X_{\mathrm{Nis}}, F_{(X,D)}).$$

Corollary 2.5. *For $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$ reduced, the following sequence is exact:*

$$0 \to F(X, D) \to F(X - D, \emptyset) \to \bigoplus_{\xi \in D^{(0)}} \frac{F(X^h_{|\xi} - \xi, \emptyset)}{F(X^h_{|\xi}, \xi)}.$$

The idea of deducing the following corollary from the above is due to A. Merici.

Corollary 2.6. *Let $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$ be reduced.*

(1) Assume given an exact sequence in $\underline{\mathbf{M}}\mathbf{NST}$:

$$0 \to H \xrightarrow{\phi} G \xrightarrow{\psi} F \tag{2.6.1}$$

such that $F, G, H \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ and that $\underline{\omega}_!\psi$ is surjective in $\mathbf{NST}$. If X is henselian local,

$$0 \to H(\mathcal{X}) \to G(\mathcal{X}) \to F(\mathcal{X}) \to 0$$

is exact.

(2) Let $\gamma : F \to G$ be a map in $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ such that $\underline{\omega}_!\gamma$ is an isomorphism. Then $F(\mathcal{X}) \to G(\mathcal{X})$ is an isomorphism.

(3) For $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$, the unit map $u : F \to \underline{\omega}^{\mathbf{CI}}\underline{\omega}_! F$ induces an isomorphism $F(\mathcal{X}) \cong \underline{\omega}^{\mathbf{CI}}\underline{\omega}_! F(\mathcal{X})$.

Proof. To show (1), it suffices to show the surjectivity of $G(\mathcal{X}) \to F(\mathcal{X})$. Let $\eta \in X$ be the generic point and consider the following commutative diagram of the Cousin complexes:

$$\begin{array}{ccccccccc}
0 & \to & H(\mathcal{X}) & \to & H(\eta) & \to & \bigoplus_{x\in X^{(1)}} H^1_x(X, H_{\mathcal{X}}) & \to & \bigoplus_{y\in X^{(2)}} H^2_y(X, H_{\mathcal{X}}) \\
 & & \downarrow & & \downarrow{\scriptstyle \phi(\eta)} & & \downarrow{\scriptstyle H^1_x(\phi)} & & \downarrow{\scriptstyle H^2_y(\phi)} \\
0 & \to & G(\mathcal{X}) & \to & G(\eta) & \to & \bigoplus_{x\in X^{(1)}} H^1_x(X, G_{\mathcal{X}}) & \to & \bigoplus_{y\in X^{(2)}} H^2_y(X, G_{\mathcal{X}}) \\
 & & \downarrow & & \downarrow{\scriptstyle \psi(\eta)} & & \downarrow{\scriptstyle H^1_x(\psi)} & & \downarrow{\scriptstyle H^2_y(\psi)} \\
0 & \to & F(\mathcal{X}) & \to & F(\eta) & \to & \bigoplus_{x\in X^{(1)}} H^1_x(X, F_{\mathcal{X}}) & \to & \bigoplus_{y\in X^{(2)}} H^2_y(X, F_{\mathcal{X}})
\end{array}$$

By Corollary 2.4, the horizontal sequences are exact. By the assumption, $\psi(\eta)$ is surjective. By a diagram chase we are reduced to showing the following.

Claim 2.6.1. (i) For $x \in X^{(1)}$, the sequence

$$H^1_x(X, H_{\mathcal{X}}) \to H^1_x(X, G_{\mathcal{X}}) \to H^1_x(X, F_{\mathcal{X}})$$

is exact.

(ii) For $y \in X^{(2)}$, $H^2_y(\phi)$ is injective.

To show (i), by Corollary 2.4, it suffices to show the exactness of $\tau^{(e)}H \to \tau^{(e)}G \to \tau^{(e)}F$ for $e \in \{0,1\}$. The case $e = 0$ follows from the left exactness of the endofunctor $\underline{\mathrm{Hom}}_{\mathbf{MPST}}(\mathcal{X}, -)$ on $\underline{\mathbf{M}}\mathbf{NST}$ for any $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}$. We have a commutative diagram

$$\begin{array}{ccccc}
\tau^{(1)}H & \xrightarrow{\phi} & \tau^{(1)}G & \xrightarrow{\psi} & \tau^{(1)}F \\
{\scriptstyle p_H}\uparrow\downarrow{\scriptstyle s_H} & & {\scriptstyle p_G}\uparrow\downarrow{\scriptstyle s_G} & & {\scriptstyle p_F}\uparrow\downarrow{\scriptstyle s_F} \\
\tau^{(0)}H & \xrightarrow{\phi} & \tau^{(0)}G & \xrightarrow{\psi} & \tau^{(0)}F
\end{array}$$

where p_* are the projections and s_* is a right inverse of p_* coming from the retractions from Lemma 2.1. We have

$$\phi \circ p_H = p_G \circ \phi,\ \psi \circ p_G = p_F \circ \psi,\ \phi \circ s_H = s_G \circ \phi,\ \psi \circ s_G = s_F \circ \psi.$$

By an easy diagram chase, the case $e = 1$ now follows from the case $e = 0$. To show (ii), by Corollary 2.4, it suffices to show the injectivity of $\tau^{(\underline{e})}H \to \tau^{(\underline{e})}G$ for $\underline{e} \in \{(0,0),(0,1),(1,0),(1.1)\}$. The case $\underline{e} = (0,0)$

follows from the same left exactness as above, and the other cases from this case thanks to Lemma 2.1.

To show (2), we may assume $\mathcal{X}$ is henselian local. Then it follows from (1). (3) follows from (2) since $\underline{\omega}_! u$ is an isomorphism. This completes the proof of the corollary. □

3. Review on higher local symbols

3.1. Let X be an excellent separate scheme of pure dimension d. We denote by $X^{(c)}$ the set of points of codimension c. Let $x, y \in X$ be two points. We write

$$x > y :\Longleftrightarrow \overline{\{x\}} \supset \overline{\{y\}}, \text{ i.e. } , y \in \overline{\{x\}}.$$

A *specialization chain* (or just *chain*) in X is a sequence

$$x = (x_1, \ldots, x_n) \quad \text{with } x_1 > x_2 > \ldots > x_n \text{ and } x_i \in X^{(i)}.$$

We say a specialization chain x is *maximal* or *a Parsin chain* if $n = d$. We denote by $c(X)$ the set of specialization chains in X and

$$\mathrm{mc}(X) := \{x \in c(X) |\ x \text{ is maximal}\}.$$

If $d = 0$, the only element in $\mathrm{mc}(X)$ is the empty chain.

We say a specialization chain $x = (x_1, \ldots, x_n)$ is *a Q-chain with break at* r if $0 \leq r \leq n = d-1$ and $x_i \in X_{(i)}$ for $i \leq r$ and $x_i \in X_{(i+1)}$ for $i > r$. We denote

$$Q_r(X) := \{Q\text{-chains with break at } r \text{ in } X\}.$$

For $x \in Q_r(X)$, we denote by $B(x)$ the set of all $y \in X$ such that

$$x(y) := (x_1, \ldots, x_r, y, x_{r+1}, \ldots, x_{d-1}) \in \mathrm{mc}(X).$$

For a specialization chain x in X, let $\mathcal{O}^h_{X,x}$ be the henselization of X along x as defined in [3, Def. 1.6.2]. It is a finite product of local rings and we let $\mathfrak{m}_x \subset \mathcal{O}^h_{X,x}$ denote the product of its maximal ideals and $K^h_{X,x}$ denote its total fraction ring.

For $x \in Q_r(X)$ and $x(y)$ with $y \in B(x)$ as above, we have a natural inclusion of rings

$$\iota_y : K^h_{X,x} \to K^h_{X,x(y)}. \tag{3.1.1}$$

In what follows in this section, we fix $F \in \mathbf{RSC}_{\mathrm{Nis}}$ and write $\tilde{F} = \underline{\omega}^{\mathbf{CI}} F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ (cf. (1.0.3)). We also fix a function field K over the base field k. Let X be an integral K-scheme of dimension d. Recall from [13] that we have a collection of bilinear pairings

$$\{(-,-)_{X/K,x} : F(K(X)) \otimes K^M_d(K^h_{X,x}) \to F(K)\}_{x \in \mathrm{mc}(X)}, \tag{3.1.2}$$

where we recall $K^h_{X,x}$ is a finite product of fields and $K^M_n(K^h_{X,x})$ for an integer $n > 0$ denote the product of the fields. We also note that $\mathcal{O}^h_{X,x}$ is a finite product of henselian dvr's and let $K^M_n(\mathcal{O}^h_{X,x})$ denote the product of the dvr's.

For a local ring R and an ideal $I \subset R$, let $K^M_n(R, I) \subset K^M_n(R)$ denote the subgroup generated by symbols

$$\{1 + a, b_1, \dots, b_{n-1}\} \text{ with } a \in I,\ b_i \in R^\times.$$

The following properties hold for all $a \in F(K(X))$:

(HS1) Let $X \hookrightarrow X'$ be an open immersion where X' is an integral K-scheme of dimension d. Then we have $(a, \beta)_{X/K,x} = (a, \beta)_{X'/K,x}$ for all $\beta \in K^M_d(K^h_{X,x})$.

(HS2) Let $x = (x_1, \dots, x_d) \in \mathrm{mc}(X)$ and $X_1 \subset X$ be the closure of x_1 and $\delta = (x_2, \dots, x_d) \in \mathrm{mc}(X_1)$. Then for all $\beta \in K^M_d(K^h_{X,x})$

$$(a, \beta)_{X/K,x} = \begin{cases} \beta \cdot \mathrm{Tr}_{E/K}(a), & \text{if } d = 0; \\ (a_{|X_1}, \partial_{x_1}\beta)_{X_1/K,\delta}, & \text{if } d \geq 1 \text{ and } a \in F(\mathcal{O}_{X,x_1}); \end{cases}$$

where $a_{|X_1} \in F(K(X_1))$ is the restriction of a, and

$$\partial_{x_1} : K^M_d(K^h_{X,x}) \to K^M_{d-1}(K^h_{X_1,\delta})$$

is the tame symbol coming from the fact that $\mathcal{O}^h_{X,x}$ is a product of dvr's and $K^h_{X_1,\delta}$ is identified with the product of its residue fields.

(HS3) Let $D \subset X$ be an effective Cartier divisor such that $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{pro}}$ (cf. §1(9)). Let $I_D \subset \mathcal{O}_X$ be the ideal sheaf of D. Assume $a \in \tilde{F}(X, D)$. Then, for all $x \in \mathrm{mc}(X)$, we have

$$(a, \beta)_{X/K,x} = 0 \text{ for } \forall \beta \in K^M_d(\mathcal{O}^h_{X,x}, I_D\mathcal{O}^h_{X,x}).$$

(HS4) Let $x \in Q_r(X)$ with $0 \leq r \leq d - 1$. Then $(a, \iota_y(\beta))_{X/K,x(y)} = 0$ for almost all $y \in B(x)$. If $r < d - 1$ or X is projective over K,

$$\sum_{y \in B(x)} (a, \iota_y(\beta))_{X/K,x(y)} = 0, \quad \text{for all } \beta \in K^M_d(K^h_{X,x}),$$

where $\iota_y : K^h_{X,x} \to K^h_{X,x(y)}$ is the natural map.

(HS5) Let $f : Y \to X$ be a finite surjective K-morphism between two integral K-schemes. Let $x \in \mathrm{mc}(X)$ and $y \in \mathrm{mc}(Y)$ with $f(y) = x$. Then we have

$$(f^*a, \beta)_{Y/K,y} = (a, \mathrm{Nm}_{y/x}(\beta))_{X/K,x}\ (a \in F(K(X)),\ \beta \in K^M_d(K^h_{Y,y})),$$

$$(f_*b, \alpha)_{X/K,x} = \sum_{\substack{z \in \mathrm{mc}(X') \\ f(z)=x}} (b, i_{x/z}\alpha)_{Y/K,z}\ (b \in F(K(X')),\ \alpha \in K^M_d(K^h_{X,x})),$$

where $\mathrm{Nm}_{y/x} : K_d^M(K_{Y,y}^h) \to K_d^M(K_{X,x}^h)$ is the norm map on Milnor K-theory and $i_{x/z} : K_d^M(K_{X,x}^h) \to K_d^M(K_{Y,z}^h)$ is the inclusion map.

Now assume that $X \in \mathbf{Sm}$ of dimension d and $D \subset X$ be a reduced SNCD on X. For a function field K over k and a scheme Z over k, write $Z_K = Z \otimes_k K$ with $\phi_Z : Z_K \to Z$ the projection. If Z is integral, we denote by $K(Z)$ the function field of Z_K.

Proposition 3.2. *Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$ and $a \in F(X - |D|)$. Assume that there exists an open subset $U \subset X$ which contains all generic points of D such that the following condition holds: For any function field K over k and any $x = (x_1, \dots, x_d) \in \mathrm{mc}(U_K)$ with $x_1 \in D_K^{(0)}$, we have*

$$(\phi_X^*(a), \beta)_{U_K/K,x} = 0 \quad \textit{for } \forall \beta \in K_d^M(\mathcal{O}_{X_K,x}^h, \mathfrak{m}_x).$$

Then we have $a \in F(X, D)$.

Proof. By Corollary 2.5 there is an exact sequence

$$0 \to F(X, D) \to F(X - D) \to \bigoplus_{\eta \in D^{(0)}} \frac{F(\mathcal{O}_{X,\eta}^h - \eta)}{F(\mathcal{O}_{X,\eta}^h, \eta)}. \tag{3.2.1}$$

Moreover, fixing $\eta \in D^{(0)}$ and choosing a K-algebra isomorphism

$$\mathcal{O}_{X,\eta}^h \simeq K(\eta)\{t\} \text{ with } t \text{ variable},$$

the induced map $\mathrm{Spec}\, \mathcal{O}_{X,\eta}^h \to \mathbf{A}_\eta^1 = \mathrm{Spec}\, K(\eta)[t]$ induces an isomorphism:

$$\frac{\tilde{F}(\mathbf{P}_\eta^1 - 0_\eta, \infty_\eta)}{\tilde{F}(\mathbf{P}_\eta^1, 0_\eta + \infty_\eta)} \xrightarrow{\simeq} \frac{F(\mathcal{O}_{X,\eta}^h - \eta)}{F(\mathcal{O}_{X,\eta}^h, \eta)}. \tag{3.2.2}$$

In view of (3.2.1), fixing $\eta \in D^{(0)}$, it suffices to show that there is a Nisnevich neighborhood $f : (V, \xi) \to (U, \eta)$ such that $a \in F(V, \xi)$. In view of (3.2.2), we may assume that there exist an étale K-morphism $g : V \to \mathbf{A}_S^1 = \mathrm{Spec}\, R[t]$ with S a smooth affine model of η over K and $b \in \tilde{F}(\mathbf{P}_S^1 - 0_S, \infty_S)$ such that the closure of ξ in V coincides with $V \times_{\mathbf{A}_S^1} 0_S$ and g induces $V \times_{\mathbf{A}_S^1} 0_S \simeq 0_S$ and that $f^*(a) = g^*(b)$. Then it suffices to show $b \in F(\mathbf{A}_S^1)$. The last assertion follows from Claim 3.2.1(2) below and [13, Cor. 2.2]. □

Claim 3.2.1. Let the assumption be as in Proposition 3.2 and K be a function field over k.

(1) For any $x = (x_1, \dots, x_d) \in \mathrm{mc}(V_K)$ such that $x_1 = \xi_K$, we have

$$(f^*a, \alpha)_{V_K/K,x} = 0 \;\text{ for } \forall \alpha \in K_d^M(\mathcal{O}_{V_K,x}^h, \mathfrak{m}_x).$$

(2) For any $x = (x_1, \dots, x_d) \in \mathrm{mc}(\mathbf{A}^1_{S,K})$ with $x_1 = 0_K := 0_S \otimes_k K$,

$$(b, \alpha)_{\mathbf{A}^1_{S,K}/K,x} = 0 \quad \text{for } \forall \alpha \in K^M_d(\mathcal{O}^h_{\mathbf{A}^1_{S,K},x}, \mathfrak{m}_x).$$

Proof. Let $\overline{f} : \overline{V} \to U$ be the normalization of $f : V \to U$. Take $x \in \mathrm{mc}(V_K)$ as in (1) and let $\overline{x} = (\overline{x}_1, \dots, \overline{x}_d) = f(x) \in \mathrm{mc}(U_K)$. Note $\overline{x}_1 = \eta \in D^{(0)}$. By (HS5) we have

$$(f^*(a), \alpha)_{\overline{V}_K/K,x} = (a, \mathrm{Nm}_{x/\overline{x}}\, \alpha)_{U_K/K,\overline{x}} \quad \text{for } \alpha \in K^M_d(\mathcal{O}^h_{V_K,x}).$$

If $\alpha \in K^M_d(\mathcal{O}^h_{V_K,x}, \mathfrak{m}_x)$, then $\mathrm{Nm}_{x/\overline{x}}\, \alpha \in K^M_d(\mathcal{O}^h_{X_K,x}, \mathfrak{m}_x)$ so that we get $(f^*(a), \alpha)_{\overline{V}_K/K,x} = 0$ by the assumption of Proposition 3.2. This proves (1).

We prove (2). Let $\overline{g} : \overline{V} \to \mathbf{A}^1_S$ be the normalization of $g : V \to \mathbf{A}^1_S$. Take $x = (x_1, \dots, x_d) \in \mathrm{mc}(\mathbf{A}^1_{S,K})$ as (2) and $y = (y_1, \dots, y_d) \in \mathrm{mc}(\overline{V}_K)$ such that $y_1 = \xi_K$ and $x = g(y)$. By the construction, the closure of ξ in V coincides with $V \times_{\mathbf{A}^1_S} 0_S \simeq 0_S$ so that it is closed in $\overline{V}$. Hence the closure of ξ_K in $\overline{V}_K$ is contained in V_K and maps isomorphically to the closure of 0_K in $\mathbf{A}^1_{S,K}$. This implies that $y \in \mathrm{mc}(V_K)$ and $\mathcal{O}^h_{\mathbf{A}^1_{S,K},x} \to \mathcal{O}^h_{V_K,y}$ is a trivial extension of dvr's so that the local norm map

$$\mathrm{Nm}\, y/x : K^M_d(\mathcal{O}^h_{V_K,y}) \to K^M_d(\mathcal{O}^h_{\mathbf{A}^1_{S,K},x})$$

is an isomorphism. Thus, for a given $\alpha \in K^M_d(\mathcal{O}_{\mathbf{A}^1_{S,K},x}, \mathfrak{m}_x)$, we can find $\beta \in K^M_d(\mathcal{O}_{V_K,y}, \mathfrak{m}_y)$ such that $\alpha = \mathrm{Nm}_{y/x}(\beta)$. Then we get

$$(b, \alpha)_{\mathbf{A}^1_{S,K}/K,x} = (b, \mathrm{Nm}_{y/x}(\beta))_{\mathbf{A}^1_{S,K}/K,x} = (g^*b, \beta)_{V_K/K,y} = (f^*b, \beta)_{V_K/K,y},$$

where the second equality follows from (HS5). The last term vanishes by (1), which completes the proof of the claim. $\square$

4. Logarithmic cohomology of reciprocity sheaves

For $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$, we write $\mathcal{X}_{\mathrm{red}} = (X, |D|) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. We say $\mathcal{X} = (X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$ is reduced if $\mathcal{X} = \mathcal{X}_{\mathrm{red}}$.

Definition 4.1. Let $F \in \underline{\mathbf{M}}\mathbf{PST}$.

(1) We say that F is *log-semipure* if for any $\mathcal{X} \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$, the map $F(\mathcal{X}_{\mathrm{red}}) \to F(\mathcal{X})$ is injective. Note that if F is semipure, F is log-semipure.

(2) We say that F is *logarithmic* if it is log-semipure and satisfies the condition that for $\mathcal{X}, \mathcal{Y} \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$ with $\mathcal{X}$ reduced and $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(\mathcal{Y}, \mathcal{X})$, the image of $\alpha^* : F(\mathcal{X}) \to F(\mathcal{Y})$ is contained in $F(\mathcal{Y}_{\mathrm{red}}) \subset F(\mathcal{Y})$, where $\mathcal{Y} = (Y, E_{\mathrm{red}})$ for $\mathcal{Y} = (Y, E)$.

Let $\underline{\mathbf{M}}\mathbf{PST}_{\log}$ be the full subcategory of $\underline{\mathbf{M}}\mathbf{PST}$ consisting of logarithmic objects and put $\underline{\mathbf{M}}\mathbf{NST}_{\log} = \underline{\mathbf{M}}\mathbf{NST} \cap \underline{\mathbf{M}}\mathbf{PST}_{\log}$.

Theorem 4.2. *Any $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ is logarithmic, i.e. $\mathbf{CI}^{\tau,sp}_{\mathrm{Nis}} \subset \underline{\mathbf{M}}\mathbf{NST}_{\log}$.*

We need a preliminary for the proof of the theorem.

Lemma 4.3. *Let $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. Let $\mathbf{A}^n_K = \operatorname{Spec} K[x_1, \dots, x_n]$ be the affine space over a function field K over k and $X = \operatorname{Spec} K\{x_1, \dots, x_n\}$ be the henselization of $\mathbf{A}^n_K$ at the origin. Let $L_i = \{x_i = 0\} \subset \mathbf{A}^n$ and $\mathcal{L}_i = L_i \times_{\mathbf{A}^n_K} X$ for $i \in [1, n]$. For an integer $0 < r \le n$, the natural map*

$$K\{x_{r+1}, \dots, x_n\}[x_1, \dots, x_r] \to K\{x_1, \dots, x_n\}$$

induces a map in $\underline{\mathbf{M}}\mathbf{Cor}_{ls}$:

$$\rho_r : (\mathcal{X}, \mathcal{L}_1 + \cdots + \mathcal{L}_r) \to (\mathbf{A}^r_S, \{x_1 \cdots x_r = 0\}) \simeq (\mathbf{A}^1, 0)^{\otimes r} \otimes (S, \emptyset),$$

where $S = \operatorname{Spec} K\{x_{r+1}, \dots, x_n\}$. It induces

$$\rho_r^* : F(\mathbf{A}^r_S, \{x_1 \cdots x_r = 0\}) \to F(\mathcal{X}, \mathcal{L}_1 + \cdots + \mathcal{L}_r) \tag{4.3.1}$$

Then $F(\mathcal{X}, \mathcal{L}_1 + \cdots + \mathcal{L}_r)$ is generated by the image of ρ_r^* and

$$F(\mathcal{X}, \mathcal{L}_1 + \cdots + \overset{\vee}{\mathcal{L}_i} + \cdots \mathcal{L}_r) \;\; \text{for } i = 1, \dots, r.$$

Proof. For $\mathcal{Y} \in \underline{\mathbf{M}}\mathbf{Cor}$, let $F^{\mathcal{Y}} \in \underline{\mathbf{M}}\mathbf{PST}$ be given by $F^{\mathcal{Y}}(\mathcal{Z}) = F(\mathcal{Y} \otimes \mathcal{Z})$. One easily checks that $F^{\mathcal{Y}} \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$ for $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$. We prove the lemma by the induction on r. The case $r = 1$ holds since by [15, Lem. 7.1 and Lem 5.9], ρ_r induces an isomorphism

$$F^{(\mathbf{A}^1,0)}(S)/F^{(\mathbf{A}^1,\emptyset)}(S) \xrightarrow{\simeq} F(\mathcal{X}, \mathcal{L}_1)/F(\mathcal{X}).$$

By definition $\mathcal{L}_1 = \operatorname{Spec} K\{x_2, \dots, x_n\}$ and we have a map in $\underline{\mathbf{M}}\mathbf{Cor}$:

$$(\mathcal{X}, \mathcal{L}_1 + \cdots + \mathcal{L}_r) \to (\mathbf{A}^1, 0) \otimes (\mathcal{L}_1, \mathcal{L}_1 \cap (\mathcal{L}_2 + \cdots + \mathcal{L}_r))$$

induced by the natural map $K\{x_2, \dots, x_n\}[x_1] \to K\{x_1, \dots, x_n\}$. By [15, Lem. 7.1 and Lem 5.9], it induces an isomorphism

$$F^{(\mathbf{A}^1,0)}(\mathcal{L}_1, E)/F^{(\mathbf{A}^1,\emptyset)}(\mathcal{L}_1, E) \xrightarrow{\simeq} F(\mathcal{X}, \mathcal{L}_1 + \cdots + \mathcal{L}_r)/F(\mathcal{X}, \mathcal{L}_2 + \cdots + \mathcal{L}_r)$$

with $E = \mathcal{L}_1 \cap (\mathcal{L}_2 + \cdots + \mathcal{L}_r)$. By the induction hypothesis, $F^{(\mathbf{A}^1,0)}(\mathcal{L}_1, E)$ is generated by $F^{(\mathbf{A}^1,0)}(\mathcal{L}_1, E_j)$ with $E_j = \mathcal{L}_1 \cap (\mathcal{L}_2 \cdots + \overset{\vee}{\mathcal{L}_j} + \cdots \mathcal{L}_r)$ for $j = 2, \dots, r$ together with the image of the map

$$(F^{(\mathbf{A}^1,0)})^{(\mathbf{A}^1,0)^{\otimes r-1}}(S) = F^{(\mathbf{A}^1,0)^{\otimes r}}(S) \to F^{(\mathbf{A}^1,0)}(\mathcal{L}_1, E)$$

induced by

$$(\mathcal{L}_1, E) \to (\mathbf{A}^{r-1}_S, \{x_2 \cdots x_2 = 0\}) \simeq (\mathbf{A}^1, 0)^{\otimes r-1}$$

coming from the map $K\{x_{r+1},\dots,x_n\}[x_2,\dots,x_r] \to K\{x_2,\dots,x_d\}$. This proves the lemma. $\square$

Proof of Theorem 4.2: By Corollary 2.6, we may assume $F = \underline{\omega}^{\mathbf{CI}}G$ for $G \in \mathbf{RSC}_{\mathrm{Nis}}$. Take $\mathcal{X} = (X,D), \mathcal{Y} = (Y,E) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$ with $\mathcal{X}$ reduced and let $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(\mathcal{Y},\mathcal{X})$ be an elementary correspondence. We need to show that $\alpha^*(F(\mathcal{X})) \subset F(\mathcal{Y}_{\mathrm{red}})$. The question is Nisnevich local over X and Y. Hence we may assume $(X,D) = (\mathcal{X}, \mathcal{L}_1 + \cdots + \mathcal{L}_r) \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{pro}}$ from Lemma 4.3. If $r = 0$, we have $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}((Y,\emptyset),(X,\emptyset))$ by the assumption $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(\mathcal{Y},\mathcal{X})$ so that

$$\alpha^*(F(\mathcal{X})) = \alpha^*(F(X,\emptyset)) \subset F(Y,\emptyset) \subset F(\mathcal{Y}_{\mathrm{red}}).$$

Assume $r > 0$ and proceed by the induction on r. By Lemma 4.3, we may assume then

$$(X,D) = \mathcal{M} := (\mathbf{A}^1,0)^{\otimes r} \otimes (S,\emptyset) \ \text{ for } S \in \mathbf{Sm}\,.$$

On the other hand, by Corollary 2.5, we have an exact sequence

$$0 \to F(Y,E_{\mathrm{red}}) \to F(Y - E_{\mathrm{red}},\emptyset) \to \bigoplus_{\xi \in E_{(0)}} \frac{F(Y^h_{|\xi} - \xi,\emptyset)}{F(Y^h_{|\xi},\xi)}.$$

Hence we may replace Y with its Nisnevich neighborhood of a generic point ξ of E to assume that $Y = \operatorname{Spec} A$ is affine and E is irreducible and the zero-divisor of $\pi \in A$, and that α factors in $\underline{\mathbf{M}}\mathbf{Cor}$ as

$$\alpha : (Y,E) \xrightarrow{{}^t\Gamma_g} (Y',E') \xrightarrow{\alpha'} \mathcal{M},$$

where $g : Y' \to Y$ is a finite map such that $Y' \in \mathbf{Sm}$ and the reduced part of $E' := E \times_Y Y'$ is in $\mathbf{Sm}$ and irreducible and ${}^t\Gamma_g$ is the transpose of the graph of g, and α' is induced by a morphism $f : Y' \to (\mathbf{A}^1)^{\times r} \times S$ in $\mathbf{Sm}$. Then we get a commutative diagram

$$\begin{array}{ccc}
F(Y',E'_{\mathrm{red}}) & & \\
\big\downarrow{\scriptstyle\hookrightarrow} & & \\
F(Y',E_{\mathrm{red}} \times_Y Y') & \xrightarrow{g_*} & F(Y,E_{\mathrm{red}}) \\
\big\downarrow{\scriptstyle\hookrightarrow} & & \big\downarrow{\scriptstyle\hookrightarrow} \\
F(Y',E') & \xrightarrow{g_*} & F(Y,E)
\end{array}$$

Hence we may replace (Y,E) by (Y',E') to assume that α is induced by a morphism $f : Y \to \mathbf{A}^r \times S$. Then α factors in $\underline{\mathbf{M}}\mathbf{Cor}$ as

$$(Y,E) \xrightarrow{i} (\mathbf{A}^1,0)^{\otimes r} \otimes (Y,\emptyset) \to (\mathbf{A}^1,0)^{\otimes r} \otimes (S,\emptyset).$$

where the first map is induced by a section i of $\mathbf{A}^r \times Y \to Y$. Thus we are reduced to showing that $i^*(F((\mathbf{A}^1,0)^{\otimes r} \otimes (Y,\emptyset)) \subset F(Y, E_{\mathrm{red}})$. By Proposition 3.2 this follows from the following.

Claim 4.3.1. Assume $F = \underline{\omega}^{\mathbf{CI}}G$ for some $G \in \mathbf{RSC}_{\mathrm{Nis}}$. Take $a \in F((\mathbf{A}^1,0)^{\otimes r} \otimes (Y,\emptyset))$. Let K be a function field K over k. After replacing Y by an open subset containing ξ, we have

$$((a_{|Y})_K, \gamma)_{Y_K/K,\delta} = 0 \quad \text{for } \forall \gamma \in K_e^M(\mathcal{O}^h_{Y_K,\delta}, \mathfrak{m}_\delta)$$

for any $\delta = (\xi, \delta_1, \dots, \delta_{e-1}) \in \mathrm{mc}(Y_K)$, where $e = \dim(Y)$ and $\xi \in E$ is the generic point, and

$$(-,-)_{Y_K/K,\delta} : F(K(Y)) \otimes K_d^M(K^h_{Y_K,\delta}) \to F(K)$$

is from (3.1.2).

Proof. Write

$$\mathbf{A}^r \times Y = \mathrm{Spec}\, A[x_1, \dots, x_r], \ (\mathbf{A}^1, 0)^{\otimes r} \otimes (Y, \emptyset) = (\mathbf{A}^r_Y, \{x_1 \cdots x_r = 0\}).$$

After replacing Y by an open subset containing ξ, we can write

$$i(Y) = \bigcap_{1\le i\le r} \{x_i - u_i\pi^{m_i} = 0\} \quad \text{with } m_i \in \mathbb{Z}_{\ge 0},\ u_i \in A^\times.$$

Let $\delta = (\xi, \delta_1, \dots, \delta_{e-1})$ be as in the claim and put $\delta' = (\delta_1, \dots, \delta_{e-1}) \in \mathrm{mc}((E_{\mathrm{red}})_K)$. Let $X_K = \mathbf{A}^r \times Y_K$ and z_j for $1 \le j \le r$ be the generic point of

$$Z_j = \bigcap_{1\le i\le j} \{x_i - u_i\pi^{m_i} = 0\} \subset X_K,$$

and w_j be the generic point of

$$W_j = \{\pi = x_1 = \cdots = x_j = 0\} = F \cap Z_j \quad \text{with } F = \{\pi = 0\} \subset X_K.$$

The section i induces isomorphisms

$$Y_K \simeq Z_r \quad \text{and} \quad (E_{\mathrm{red}})_K \simeq W_r. \tag{4.3.2}$$

Take any $\gamma \in K_e^M(\mathcal{O}^h_{Y_K,\delta}, \mathfrak{m}_\delta)$ and put

$$\beta = \{\gamma, \frac{u_1\pi^{m_1} - x_1}{u_1\pi^{m_1}}, \dots, \frac{u_r\pi^{m_r} - x_r}{u_r\pi^{m_r}}\} \in K_d^M(K(X_\delta))\ (d = e + r),$$

where $K(X_\delta)$ is the function field of $X_\delta := \mathbf{A}^r \times \mathrm{Spec}(\mathcal{O}^h_{Y_K,\delta})$. Let $\sigma = (\eta_1, w_1, \dots, w_r, i(\delta')) \in \mathrm{mc}(X_K)$, where η_1 is the generic point of $D_1 =$

$\{x_1 = 0\} \subset X_K$ and $i(\delta') \in \mathrm{mc}(W_r)$ is the image of δ' under (4.3.2). For $a \in F((\mathbf{A}^1, 0)^{\otimes r} \otimes (Y, \emptyset))$ and its restriction $a_K \in F((\mathbf{A}^1, 0)^{\otimes r} \otimes (Y_K, \emptyset))$,

$$\begin{aligned} 0 = (a_K, \beta)_{X_K/K,\sigma} &= - \sum_{\substack{\tau \in X_K^{(1)} \\ \tau > w_1, \tau \neq \eta_1}} (a_K, \beta)_{X_K/K,(\tau, w_1, \dots, w_r, i(\delta'))} \\ &= -(a_K, \beta)_{X_K/K,(z_1, w_1, \dots, w_r, i(\delta'))} \\ &= \pm((a_K)_{|Z_1}, \beta_1)_{Z_1/K,(w_1, \dots, w_r, i(\delta'))}, \end{aligned}$$

where

$$\beta_1 = \{\gamma, \frac{u_2 \pi^{m_2} - x_2}{u_2 \pi^{m_2}}, \dots, \frac{u_r \pi^{m_r} - x_r}{u_r \pi^{m_r}}\} \in K^M_{d-1}(K(Z_{1,\delta}))$$

with $Z_{1,\delta} = Z_1 \times_{X_K} X_\delta$. The first equality follows from §3 (HS3) noting $\beta \in K^M_d(\mathcal{O}_{X_\delta,\eta}, \mathfrak{m}_\eta)$. The second follows from (HS4). The third equality holds since by (HS2), z_1 is the unique $\tau \in X_K^{(1)} - \{\eta_1\}$ such that $\tau > w_1$ and $(a_K, \beta)_{X_K/K,(\tau, w_1, \dots, w_r, i(\delta'))} \neq 0$ noting $\gamma_{|F} = 0$. Finally the last equality follows from (HS2). We further get

$$\begin{aligned} 0 = ((a_K)_{|Z_1}, \beta_1)_{Z_1/K,(w_1, w_2, \dots, w_r, i(\delta'))} &= - \sum_{\substack{\tau \in Z_1^{(1)} \\ \tau > w_2, \tau \neq w_1}} ((a_K)_{|Z_1}, \beta_1)_{Z_1/K,(\tau, w_2, \dots, w_r, i(\delta'))} \\ &= -((a_K)_{|Z_1}, \beta_1)_{Z_1/K,(z_2, w_2, \dots, w_r, i(\delta'))} \\ &= \pm((a_K)_{|Z_2}, \beta_2)_{Z_2/K,(w_2, \dots, w_r, i(\delta'))}. \end{aligned}$$

where

$$\beta_2 = \{\gamma, \frac{u_3 \pi^{m_3} - x_3}{u_3 \pi^{m_3}}, \dots, \frac{u_r \pi^{m_r} - x_r}{u_r \pi^{m_r}}\} \in K^M_{d-1}(K(Z_{2,\delta}))$$

with $Z_{2,\delta} = Z_2 \times_{X_K} X_\delta$. The above equalities hold by the same arguments as above except that for the third equality, there are a priori two $\tau \in Z_1^{(1)} - \{w_1\}$ with $\tau > w_2$ for which $((a_K)_{|Z_1}, \beta_1)_{Z_1/K,(\tau, w_2, \dots, w_r, i(\delta'))}$ may not vanish: One is z_2 and another is the generic point η_2 of $Z_1 \cap D_2$ with $D_2 = \{x_2 = 0\} \subset X_K$, but $((a_K)_{|Z_1}, \beta_1)_{Z_1/K,(\eta_2, w_2, \dots, w_r, i(\delta'))} = 0$. Indeed, $(a_K)_{|Z_1} \in F(\mathrm{Spec}(\mathcal{O}_{Z_1,\eta_2}), \eta_2)$ since Z_1 and D_2 intersect transversally in X_K. Hence it follows from (HS3) noting $\beta_1 \in K^M_d(\mathcal{O}_{Z_{1,\delta},\eta_2}, \mathfrak{m}_{\eta_2})$. Repeating the same arguments, we finally get

$$0 = ((a_K)_{|Z_r}, \gamma_{|Z_r})_{Z_r/K,(w_r, i(\delta'))} = ((a_K)_{|Y_K}, \gamma)_{Y_K/K,\delta},$$

where the second equality follows from (4.3.2). This completes the proof of the claim and Theorem 4.2. □

Definition 4.4. For $F \in \underline{\mathbf{M}}\mathbf{NST}_{\log}$ and an integer $i \geq 0$, consider the association

$$H^i_{\log}(-, F) : \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}_{ls} \to \mathbf{Ab} \; ; \; (X, D) \to H^i(X_{\mathrm{Nis}}, F_{(X, D_{\mathrm{red}})}).$$

By the definition this gives a presheaf on $\underline{\mathbf{M}}\mathbf{Cor}_{ls}^{\mathrm{fin}}$, which we call *the i-th logarithmic cohomology with coefficient F.*

5. Invariance of logarithmic cohomology under blowups

Let the notation be as in §4. Let $\Lambda_{ls}^{\mathrm{fin}}$ be the subcategory of $\underline{\mathbf{M}}\mathbf{Cor}_{ls}^{\mathrm{fin}}$ whose objects are the same as $\underline{\mathbf{M}}\mathbf{Cor}_{ls}^{\mathrm{fin}}$ and whose morphisms are those $\rho : (Y, E) \to (X, D)$ that are induced by blowups of X in smooth centers $Z \subset D$ which are normal crossing to D.

Theorem 5.1. *For $F \in \mathbf{CI}_{\mathrm{Nis}}^{\tau,sp}$ and $\rho : \mathcal{Y} \to \mathcal{X}$ in $\Lambda_{ls}^{\mathrm{fin}}$, we have*

$$\rho^* : H^i_{\log}(\mathcal{X}, F) \cong H^i_{\log}(\mathcal{Y}, F) \quad \text{for } \forall i \geq 0. \tag{5.1.1}$$

Proof. Writing $\mathcal{Y} = (Y, E)$ and $\mathcal{X} = (X, D)$, ρ is induced by a blowup $\rho : Y \to X$ in a smooth center $Z \subset D$ normal crossing to D. First we prove the theorem in case $i = 0$. We may assume that D is reduced and $E = \rho^* D$. Then ρ is invertible in $\underline{\mathbf{M}}\mathbf{Cor}$ so that $\rho^* : F(\mathcal{X}) \cong F(\mathcal{Y})$. Since this factors through $F(Y, E_{\mathrm{red}})$, we get (5.1.1) for $i = 0$.

To show (5.1.1) for $i > 0$, it now suffices to prove $R^i \rho_* F_{(Y,E_{\mathrm{red}})} = 0$. By [8, Lem. 9], Nisnevich locally around a point of Z, (X, D) is isomorphic to

$$(\mathbf{A}^c, L_1 + \cdots + L_r) \otimes \mathcal{W} \quad \text{with } \mathcal{W} = (W, W^\infty) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls},$$

where $\mathbf{A}^c = \operatorname{Spec} k[t_1, \ldots, t_c]$ with $c = \operatorname{codim}_z(Z, X)$ and $L_i = V(t_i)$ for $i = 1, \ldots, r$ with $1 \leq r \leq c$, and Z corresponds to $0 \times W$. Hence the theorem follows from the following proposition. □

Proposition 5.2. *Let $F \in \mathbf{CI}_{\mathrm{Nis}}^{\tau,sp}$ and $\mathcal{W} = (W, W^\infty) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$. Let $\mathbf{A}^n = \operatorname{Spec} k[t_1, \ldots, t_n]$ and put $L_i = V(t_i)$ for $1 \leq i \leq n$. Let $\rho : Y \to \mathbf{A}^n$ be the blow-up at the origin $0 \in \mathbf{A}^n$ and $\tilde{L}_i \subset Y$ be the strict transforms of L_i for $i = 1, \ldots, r$ and $E = \rho^{-1}(0) \subset Y$. Then*

$$R^i \rho_{W*} F_{(Y, \tilde{L}_1 + \cdots + \tilde{L}_r + E) \otimes \mathcal{W}} = 0 \quad \text{for } i \geq 1, \tag{5.2.1}$$

where $\rho_W := \rho \times \mathrm{id}_W : Y \times W \to \mathbf{A}^2 \times W$ is the base change of ρ.

Lemma 5.3. *Proposition 5.2 holds for $n = 2$.*

Proof. We can assume W is henselian local. The case $r = 1$ is proved in [2, Lem. 2.10] and we show the case $r = 2$.[5] Put $D = L_1 + L_2$. By the case $i = 0$ of Theorem 5.1, we get

$$F_{(\mathbf{A}^2, D) \otimes \mathcal{W}} \cong \rho_{W*} F_{(Y, \tilde{L}_1 + \tilde{L}_2 + E) \otimes \mathcal{W}}. \tag{5.3.1}$$

[5] The following argument is adopted from [2, Lem. 2.10], but the present case is easier.

Set
$$\mathcal{F} := F_{(Y,\tilde{L}_1+\tilde{L}_2+E)\otimes\mathcal{W}}.$$
Since $R^i\rho_{W*}\mathcal{F}$ for $i \geq 1$ is supported in $0 \times W$ we have
$$R^i\rho_{W*}\mathcal{F} = 0 \Longleftrightarrow H^0(\mathbf{A}^2_W, R^i\rho_{W*}\mathcal{F}) = 0,$$
where $\mathbf{A}^1_W = \mathbf{A}^1 \times W$, and
$$H^j(\mathbf{A}^2_W, R^i\rho_{W*}\mathcal{F}) = 0, \quad \text{for all } i, j \geq 1.$$
By (5.3.1) and [2, Lem. 2.8]
$$H^i(\mathbf{A}^2_W, \rho_{W*}\mathcal{F}) = H^i(\mathbf{A}^2_W, F_{(\mathbf{A}^2,D)\otimes\mathcal{W}}) = 0.$$
Thus the Leray spectral sequence yields
$$H^0(\mathbf{A}^2_W, R^i\rho_{W*}\mathcal{F}) = H^i(Y \times W, \mathcal{F}), \quad i \geq 0,$$
and we have to show, that this group vanishes for $i \geq 1$. We can write
$$\mathbf{A}^2 = \operatorname{Spec} k[x, y] \text{ and } L_1 = V(x),\ L_2 = V(y) \subset \mathbf{A}^2.$$
Then we have
$$Y = \operatorname{Proj} k[x, y][S, T]/(xT - yS) \subset \mathbf{A}^2 \times \mathbf{P}^1.$$
Denote by
$$\pi_0 : Y \hookrightarrow \mathbf{A}^2 \times \mathbf{P}^1 \to \mathbf{P}^1 = \operatorname{Proj} k[S, T]$$
the morphism induced by projection and let $\pi : Y \times W \to \mathbf{P}^1_W$ be its base change. Letting $E = \rho^{-1}(0) \subset Y$ be the exceptional, π_0 induces an isomorphism $E \simeq \mathbf{P}^1$, and we have
$$\tilde{L}_1 = \pi_0^{-1}(0), \quad \tilde{L}_2 = \pi_0^{-1}(\infty).$$
Set $s = S/T = x/y$ and write
$$\mathbf{P}^1 \setminus \{\infty\} = \mathbf{A}^1_s := \operatorname{Spec} k[s], \quad \mathbf{P}^1 \setminus \{0\} = \operatorname{Spec} k[\tfrac{1}{s}].$$
Set $U := \mathbf{A}^1_s \times W$ and $V := (\mathbf{P}^1 \setminus \{0\}) \times W$ and
$$\mathcal{U} := (\mathbf{A}^1_s, 0) \otimes \mathcal{W}, \quad \mathcal{V} := (\mathbf{P}^1 \setminus \{0\}, \infty) \otimes \mathcal{W}.$$
We have
$$\pi^{-1}(U) = \mathbf{A}^1_y \times U, \quad \pi^{-1}(V) = \mathbf{A}^1_x \times V,$$
and the restriction of π to these open subsets is given by projection. Furthermore, we have

(5.3.2) $$\mathcal{F}_{|\pi^{-1}(U)} = F_{(\mathbf{A}^1_y,0)\otimes\mathcal{U}}, \quad \mathcal{F}_{|\pi^{-1}(V)} = F_{(\mathbf{A}^1_x,0)\otimes\mathcal{V}}.$$

Thus [2, Lem. 2.8] yields
$$R^j\pi_*\mathcal{F} = 0 \text{ for } j \geq 1.$$
Thus it remains to show

(5.3.3) $$H^i(\mathbf{P}^1_W, \pi_*\mathcal{F}) = 0 \text{ for } i \geq 1.$$

where $\mathbf{P}^1_W = \mathbf{P}^1 \times W$. For this consider the map

$$a_0 : Y \to \mathbf{A}^1_x \times \mathbf{P}^1$$

which is the closed immersion $Y \hookrightarrow \mathbf{A}^2 \times \mathbf{P}^1$ followed by the projection $\mathbf{A}^2 \to \mathbf{A}^1_x$. Let $a : Y \times W \to \mathbf{A}^1_x \times \mathbf{P}^1 \times W$ be its base change. In view of (5.3.2), the map a induces a morphism in $\underline{\mathbf{M}}\mathbf{Cor}$:

$$\alpha : (Y, \tilde{L}_1 + \tilde{L}_2 + E) \otimes \mathcal{W} \to (\mathbf{A}^1_x, 0) \otimes (\mathbf{P}^1, \infty) \otimes \mathcal{W},$$

which is an isomorphism over $(\mathbf{A}^1_x, 0) \otimes (\mathbf{P}^1 - \{0\}, \infty) \otimes \mathcal{W}$. Setting

$$F_1 := \underline{\mathrm{Hom}}(\mathbb{Z}_{\mathrm{tr}}(\mathbf{A}^1_x, 0), F) \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}},$$

it induces a map of Nisnevich sheaves on $\mathbf{P}^1_W$:

$$\pi_*(\alpha^*) : F_{1,(\mathbf{P}^1,\infty)\otimes\mathcal{W}} \to \pi_*\mathcal{F},$$

which becomes an isomorphism over $(\mathbf{P}^1 - \{0\}) \times W$. Hence (5.3.3) follows from

$$H^i(\mathbf{P}^1_W, F_{1,(\mathbf{P}^1,\infty)\otimes\mathcal{W}}) = 0 \ \text{ for } i \geq 1,$$

which follows from [15, Th. 0.6]. □

Lemma 5.4. *Let $N > 2$ be an integer and assume that Proposition 5.2 holds for $n < N$. Let $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$ and $Z \subset X$ be a smooth integral closed subscheme with $2 \leq \mathrm{codim}(Z, Y) =: c < N$. Assume*

$$D = D_1 + \cdots + D_r + D' \ \text{ with } r \leq c,$$

where $D_1, \ldots, D_r$ are irreducible and reduced containing Z but not contained in D' and Z is transversal to $|D'|$. Let $\rho : Y \to X$ be the blow-up of X in Z and $\tilde{D}_i, \tilde{D}' \subset Y$ be the strict transforms of D_i and D' respectively and $E_Z = \rho^{-1}(Z)$. Then, for all $\mathcal{W} = (W, W^\infty) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$,

$$R^i \rho_{W*} F_{(Y, \tilde{D}_1 + \cdots + \tilde{D}_r + E_Z + \tilde{D}') \otimes \mathcal{W}} = 0 \ \text{ for } i \geq 1,$$

where $\rho_W : Y \times W \to X \times W$ denotes the base change of ρ.

Proof. [6] The question is Nisnevich local around the points in $Z \times W$. Let $z \in Z \times W$ be a point and set $A := \mathcal{O}^h_{X \times W, z}$. For $V \subset Y \times W$ we denote by $V_{(z)} := V \times_{X \times W} \operatorname{Spec} A$. By assumption we find a system of local parameters $t_1, \ldots, t_m$ of A, such that

$$(D_i \times W)_{(z)} = V(t_i) \ \text{ for } 1 \leq i \leq r, \ (Z \times W)_{(z)} = V(t_1, \ldots, t_c),$$

$$(D' \times W)_{(z)} = V(t_{c+1}^{e_{c+1}} \cdots t_{m_0}^{e_{m_0}}) \ \text{ with } c+1 \leq m_0 \leq m,$$

$$(X \times W^\infty)_{(z)} = V(t_{m_0+1}^{e_{m_0+1}} \cdots t_{m_1}^{e_{m_1}}) \ \text{ with } m_0 \leq m_1 \leq m.$$

Letting $K \hookrightarrow A$ be a coefficient field over k, we obtain an isomorphism

$$K\{t_1, \ldots, t_m\} \xrightarrow{\simeq} A.$$

[6] The proof is adopted from [2, Lem. 2.11].

Let $\rho_1 : \widetilde{\mathbf{A}^c} \to \mathbf{A}^c$ be the blow-up in 0. By the above

$$\rho_W : (Y, \tilde{D}_1 + \cdots + \tilde{D}_r + E_Z + \tilde{D}') \otimes \mathcal{W} \to (X, D) \otimes \mathcal{W}$$

is Nisnevich locally around z isomorphic over k to the morphism

$$(\widetilde{\mathbf{A}^c}, \tilde{L}_1 + \cdots + \tilde{L}_r + E) \otimes \mathcal{W}' \to (\mathbf{A}^c, L_1 + \cdots + L_r) \otimes \mathcal{W}',$$

$$(\mathcal{W}' = (\mathbf{A}_K^{m-c}, (\prod_{i=c+1}^{m_1} t_i^{e_i})))$$

which is induced by base change form ρ_1 (see Proposition 5.2 for $L_i, \tilde{L}_i, E$). Hence the statement follows from Theorem 5.2 for $n = c < N$. □

Proof of Proposition 5.2. The proof is by induction on $n \geq 2$. The case $n = 2$ follows from Lemma 5.3. Assume $n > 2$ and the theorem is proven for $\mathbf{A}^m$ with $m < n$. In case $r = 1$, Proposition 5.2 is proved in [2, Th. 2.9]. Assume $r \geq 2$. Let $Z := L_1 \cap L_2 \subset \mathbf{A}^n$ and $\tilde{Z} \subset Y$ be the strict transform of Z. Denote by $\rho' : Y' \to Y$ the blow-up of Y in $\tilde{Z}$ and $\tilde{L}'_i, E' \subset Y'$ be the strict transforms of $\tilde{L}, E$ respectively and $E'' = (\rho')^{-1}(\tilde{Z})$. Note that $\tilde{Z} = \tilde{L}_1 \cap \tilde{L}_2$ intersecting transversally with $\tilde{L}_3 + \cdots + \tilde{L}_r + E$ and $\mathrm{codim}(\tilde{Z}, Y) = 2$. Hence, by Lemma 5.4

$$R^i \rho'_{W*} F_{(Y', \tilde{L}'_1 + \cdots + \tilde{L}'_r + E' + E'') \otimes \mathcal{W}} = 0 \ \text{ for } i \geq 1.$$

Since Theorem 5.1 has been proved for $i = 0$, we have

$$\rho'_* F_{(Y', \tilde{L}'_1 + \cdots + \tilde{L}'_r + E' + E'') \otimes \mathcal{W}} = F_{(Y, \tilde{L}_1 + \cdots + \tilde{L}_r + E) \otimes \mathcal{W}}.$$

Hence we obtain

$$R^i \rho_{W*} F_{(Y, \tilde{L}_1 + \cdots + \tilde{L}_r + E) \otimes \mathcal{W}} = R^i (\rho\rho')_{W*} F_{(Y', \tilde{L}'_1 + \cdots + \tilde{L}'_r + \tilde{E} + E') \otimes \mathcal{W}}. \tag{5.4.1}$$

Denote by $\sigma : \hat{Y} \to \mathbf{A}^n$ the blow-up in Z and $\hat{L}_i \subset \hat{Y}$ be the strict transform of L_i and $\Xi = \sigma^{-1}(Z)$. By Lemma 5.4 we get

$$R^i \sigma_{W*} F_{(\hat{Y}, \hat{L}_1 + \cdots + \hat{L}_r + \Xi) \otimes \mathcal{W}} = 0 \ \text{ for } i \geq 1. \tag{5.4.2}$$

Denote by $\sigma' : \hat{Y}' \to \hat{Y}$ the blow-up in $\hat{Z} = \sigma^{-1}(0) \subset \Xi$ and $\hat{L}'_i$, $\Xi' \subset \hat{Y}'$ be the strict transforms of $\hat{L}_i$, Ξ respectively and $\Xi'' = \sigma'^{-1}(\hat{Z})$. Note that $\tilde{Z} \subset \hat{L}_3 \cap \cdots \cap \hat{L}_n \cap \Xi$ and $\mathrm{codim}(\tilde{Z}, \hat{Y}) = n - 1$ and $\tilde{Z}$ intersects transversally with $\hat{L}_1 + \hat{L}_2$. Thus by Lemma 5.4 and the case $i = 0$ of Theorem 5.1, we obtain

$$R\sigma'_{W*} F_{(\hat{Y}', \hat{L}'_1 + \cdots + \hat{L}'_r + \Xi' + \Xi'') \otimes \mathcal{W}} = F_{(\hat{Y}, \hat{L}_1 + \cdots + \hat{L}_r + \Xi) \otimes \mathcal{W}}. \tag{5.4.3}$$

Finally, by [2, Lem. 2.12], there is an isomorphism of $\mathbf{A}^n \times W$-schemes

$$(\hat{Y}', \hat{L}'_1, \ldots, \hat{L}_r, \Xi', \Xi'') \cong (Y', \tilde{L}'_1, \ldots, \tilde{L}'_r, E', E''). \tag{5.4.4}$$

Altogether we obtain for $i \geq 1$

$$\begin{aligned} R^i\rho_{W*}F_{(Y,\tilde{L}_1+\cdots+\tilde{L}_r+E)\otimes\mathcal{W}} &= R^i(\rho\rho')_{W*}F_{(Y',\tilde{L}'_1+\cdots+\tilde{L}'_r+E'+E'')\otimes\mathcal{W}}, && \text{by (5.4.1)},\\ &= R^i(\sigma\sigma')_{W*}F_{(\hat{Y}',\hat{L}'_1+\cdots+\hat{L}'_r+\Xi'+\Xi'')\otimes\mathcal{W}}, && \text{by (5.4.4)},\\ &= R^i\sigma_{W*}F_{(\hat{Y},\hat{L}_1+\cdots+\hat{L}_r+\Xi)\otimes\mathcal{W}}, && \text{by (5.4.3)},\\ &= 0, && \text{by (5.4.2)}. \end{aligned}$$

This completes the proof of the proposition. □

Remark 5.5. For simplicity, we write

$$H^i_{\log}(-,F) = H^i_{\log}(-,\underline{\omega}^{\mathbf{CI}}F) \ \text{ for } F \in \mathbf{RSC}_{\mathrm{Nis}}.$$

By [12, Cor. 6.8], if $\mathrm{ch}(k) = 0$ and $F = \Omega^i$, we have

$$H^i_{\log}(-,\Omega^i) = H^i(X, \Omega^i(\log |D|) \ \text{ for } (X,D) \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}\,.$$

Hence $H^i_{\log}(-,F)$ for $F \in \mathbf{RSC}_{\mathrm{Nis}}$ is a generalization of cohomology of sheaves of logarithmic differentials.

6. Relation with logarithmic sheaves with transfers

In this section we use the same notations as [1].

Let l$\mathbf{Sm}$ be the category of log smooth and separated fs log schemes of finite type over the base field k and $\mathbf{Sm}\mathrm{l}\mathbf{Sm} \subset \mathrm{l}\mathbf{Sm}$ be the full subcategory consisting of objects whose underlying schemes are smooth over k. Let l$\mathbf{Cor}$ be the category with the same objects as l$\mathbf{Sm}$ and whose morphisms are log correspondences defined in [1, Def. 2.1.1]. Let $\mathrm{l}\mathbf{Cor}_{\mathbf{Sm}\mathrm{l}\mathbf{Sm}} \subset \mathrm{l}\mathbf{Cor}$ be the full subcategory consisting of all objects in $\mathbf{Sm}\mathrm{l}\mathbf{Sm}$ (see [1, Def. 4.6.1]).

Let $\mathbf{PSh}^{ltr}$ be the category of additive presheaves of abelian groups on l$\mathbf{Cor}$ and $\mathbf{Shv}^{ltr}_{dNis} \subset \mathbf{PSh}^{ltr}$ be the full subcategory consisting of those $\mathcal{F}$ whose restrictions to l$\mathbf{Sm}$ are dividing Nisnevich sheaves (see [1, Def. 3.1.4]). It is shown in [1, §4 and Pr.4.6.6] that $\mathbf{Shv}^{ltr}_{dNis}$ is a Grothendieck abelian category and there is an equivalence of categories

$$\mathbf{Shv}^{ltr}_{dNis} \simeq \mathbf{Shv}^{ltr}_{dNis}(\mathbf{Sm}\mathrm{l}\mathbf{Sm}), \tag{6.0.1}$$

where the right hand side denotes the full subcategory of the category $\mathbf{PSh}^{ltr}(\mathbf{Sm}\mathrm{l}\mathbf{Sm})$ of additive presheaves of abelian groups on $\mathrm{l}\mathbf{Cor}_{\mathbf{Sm}\mathrm{l}\mathbf{Sm}}$ consisting of those $\mathcal{F}$ whose restrictions to $\mathbf{Sm}\mathrm{l}\mathbf{Sm}$ are dividing Nisnevich sheaves.

Now we construct a functor

$$\mathcal{L}og : \underline{\mathbf{M}}\mathbf{NST}_{\log} \to \mathbf{Shv}^{ltr}_{dNis}. \tag{6.0.2}$$

For $\mathfrak{X} = (X, \mathcal{M}) \in \mathbf{Sm}\mathrm{l}\mathbf{Sm}$, we put $\mathfrak{X}^{MP} = (X, \partial\mathfrak{X})$, where $\partial\mathfrak{X} \subset X$ is the closed subscheme consisting of the points where the log-structure $\mathcal{M}$ is not trivial. By [11, Theorem III.1.11.12], $\partial\mathfrak{X}$ with reduced structure is a normal crossing divisor on X so that we can view $\mathfrak{X}^{MP}$ as an objects of $\underline{\mathbf{M}}\mathbf{Cor}_{ls}$. For $F \in \underline{\mathbf{M}}\mathbf{PST}_{\log}$ and $\mathfrak{X} \in \mathbf{Sm}\mathrm{l}\mathbf{Sm}$, we put

$$F^{\log}(\mathfrak{X}) = F(\mathfrak{X}^{MP}). \tag{6.0.3}$$

Take $\mathfrak{Y} \in \mathbf{Sm}\mathrm{l}\mathbf{Sm}$ and $\alpha \in \mathrm{l}\mathbf{Cor}(\mathfrak{Y}, \mathfrak{X})$. By [1, Def. 2.1.1 and Rem. 2.1.1(iv)], we have

$$\alpha \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}((Y, n\cdot\partial\mathfrak{Y}), (X, \partial\mathfrak{X})) \text{ for some } n > 0,$$

where $n\cdot\partial\mathfrak{Y} \hookrightarrow Y$ is the n-th thickening of $\partial\mathfrak{Y} \hookrightarrow Y$. By the assumption $F \in \underline{\mathbf{M}}\mathbf{PST}_{\log}$, the induced map

$$F^{\log}(\mathfrak{X}) = F(\mathfrak{X}^{MP}) \xrightarrow{\alpha^*} F(Y, n\cdot\partial\mathfrak{Y})$$

factors through $F^{\log}(\mathfrak{Y}) = F(Y, \partial\mathfrak{Y}) \subset F(Y, n\cdot\partial\mathfrak{Y})$ and we get a map

$$\alpha^{*\log} : F^{\log}(\mathfrak{X}) \to F^{\log}(\mathfrak{Y}).$$

Moreover, for a map $\gamma : F \to G$ in $\underline{\mathbf{M}}\mathbf{PST}_{\log}$, the diagram

$$\begin{array}{ccc} F^{\log}(\mathfrak{X}) & \xrightarrow{\gamma} & G^{\log}(\mathfrak{X}) \\ \downarrow{\scriptstyle \alpha^{*\log}} & & \downarrow{\scriptstyle \alpha^{*\log}} \\ F^{\log}(\mathfrak{Y}) & \xrightarrow{\gamma} & G^{\log}(\mathfrak{Y}) \end{array}$$

is obviously commutative. Hence the assignment $\mathcal{X} \to F^{\log}(\mathcal{X})$ gives an object $F^{\log}$ of $\mathbf{PSh}^{ltr}(\mathbf{Sm}\mathrm{l}\mathbf{Sm})$ and we get a functor

$$\mathcal{L}og : \underline{\mathbf{M}}\mathbf{PST}_{\log} \to \mathbf{PSh}^{ltr}(\mathbf{Sm}\mathrm{l}\mathbf{Sm})\ ;\ F \to F^{\log}. \tag{6.0.4}$$

By the definitions of sheaves ([4, Def. 1] and [1, Def. 3.1.4]) and [4, Pr. 1.9.2], this induces a functor

$$\underline{\mathbf{M}}\mathbf{NST}_{\log} \to \mathbf{Shv}^{ltr}_{dNis}(\mathbf{Sm}\mathrm{l}\mathbf{Sm})$$

which induces the desired functor (6.0.2) using (6.0.1). By the construction, for $F \in \underline{\mathbf{M}}\mathbf{NST}_{\log}$ and $\mathfrak{X} \in \mathbf{Sm}\mathrm{l}\mathbf{Sm}$ with $\mathcal{X} = \mathfrak{X}^{MP} \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$, we have

$$H^i_{\mathrm{Nis}}(X, F_{\mathcal{X}}) = H^i_{s\mathrm{Nis}}(\mathfrak{X}, F^{\log})\ (F^{\log} = \mathcal{L}og(F)), \tag{6.0.5}$$

where the right hand side is the cohomology for the strict Nisnevich topology (see [1, Def. 4.3.1]).

Theorem 6.1. *For* $F \in \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}}$*,* $F^{\log} = \mathcal{L}og(F) \in \mathbf{Shv}^{ltr}_{dNis}$ *is strictly* $\overline{\square}$*-invariant in the sense* [1, Def. 5.2.7]*. For* $\mathfrak{X} \in \mathbf{SmlSm}$ *with* $\mathcal{X} = \mathfrak{X}^{MP} \in \underline{\mathbf{M}}\mathbf{Cor}_{ls}$*, we have a natural isomorphism*

$$H^i_{\mathrm{Nis}}(X, F_{\mathcal{X}}) \simeq \mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}}(M(\mathfrak{X}), F^{\log}[i]), \tag{6.1.1}$$

where $\mathbf{logDM}^{\mathrm{eff}}$ *is the triangulated category of logarithmic motives defined in* [1, Def. 5.1.1]*.*

Proof. We have isomorphisms

$$H^i_{\mathrm{Nis}}(X, F_{\mathcal{X}}) \overset{(6.0.5)}{\simeq} H^i_{s\mathrm{Nis}}(\mathfrak{X}, F^{\log}) \overset{(*1)}{\simeq} \varinjlim_{\mathfrak{Y}\in\mathfrak{X}^{Sm}_{div}} H^i_{s\mathrm{Nis}}(\mathfrak{X}, F^{\log}) \overset{(*2)}{\simeq} H^i_{d\mathrm{Nis}}(\mathfrak{X}, F^{\log}),$$

where $(*2)$ comes from [1, Th. 5.1.4] and $(*1)$ is a consequence of Theorem 5.1 in view of (6.0.5). Hence the strict $\overline{\square}$-invariance of $F^{\log}$ follows from [15, Th. 0.6]. Finally (6.1.1) follows from [1, Pr. 5.2.8]. □

Now we consider the composite functor

$$\mathcal{L}og' : \mathbf{RSC}_{\mathrm{Nis}} \xrightarrow{\omega^{\mathbf{CI}}} \mathbf{CI}^{\tau,sp}_{\mathrm{Nis}} \xrightarrow{\mathcal{L}og} \mathbf{Shv}^{ltr}_{dNis}.$$

Lemma 6.2. $\mathcal{L}og$ *and* $\mathcal{L}og'$ *have the same essential image.*

Proof. This follows directly from the construction and Corollary 2.6(3). □

In what follows, we let

$$\mathcal{L}og : \mathbf{RSC}_{\mathrm{Nis}} \to \mathbf{Shv}^{ltr}_{dNis} \;:\; F \to F^{\log} \tag{6.2.1}$$

denote $\mathcal{L}og'$ defined as above. By (6.0.3), we have

$$F^{\log}(X, \mathrm{triv}) = F(X) \;\text{ for } F \in \mathbf{RSC}_{\mathrm{Nis}},\; X \in \mathbf{Sm}, \tag{6.2.2}$$

where (X, triv) denotes the log-scheme with the trivial log structure.

Theorem 6.3. $\mathcal{L}og$ *is exact and fully faithful.*

Proof. First we prove the full faithfulness. The faithfulness follows from (6.2.2). Let $F, G \in \mathbf{RSC}_{\mathrm{Nis}}$ and $\gamma : F^{\log} \to G^{\log}$ be a map in $\mathbf{Shv}^{ltr}_{dNis}$. By (6.2.2) it induces maps $\gamma_X : F(X) \to G(X)$ for all $X \in \mathbf{Sm}$. They are compatible with the action of $\mathbf{Cor}$ since by [1, Rem 2.1.3(3)],

$$\mathbf{Cor}(Y, X) = \mathrm{l}\mathbf{Cor}(Y, \mathrm{triv}), (X, \mathrm{triv})) \;\text{ for } X, Y \in \mathbf{Sm}\,.$$

Thus γ_X for $X \in \mathbf{Sm}$ give a map $\gamma_{\mathbf{RSC}} : F \to G$ in $\mathbf{RSC}_{\mathrm{Nis}}$. To see $\mathcal{L}og(\gamma_{\mathbf{RSC}_{\mathrm{Nis}}}) = \gamma$, it suffices by (6.0.1) to show that $\mathcal{L}og(\gamma_{\mathbf{RSC}_{\mathrm{Nis}}})$ and γ induce the same map $F^{\log}(\mathfrak{X}) \to G^{\log}(\mathfrak{X})$ for $\mathfrak{X} \in \mathbf{SmlSm}$. If $\mathfrak{X}$ has the trivial log-structure, this follows immediately from the construction of $\gamma_{\mathbf{RSC}}$. The general case follows from this in view of the commutative diagram

$$\begin{array}{ccc} F^{\log}(\mathfrak{X}) & \xrightarrow{\gamma} & G^{\log}(\mathfrak{X}) \\ \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} \\ F^{\log}(X\backslash\partial\mathfrak{X}, \mathrm{triv}) & \xrightarrow{\gamma} & G^{\log}(X\backslash\partial\mathfrak{X}, \mathrm{triv}) \end{array}$$

where j^* are induced by the natural map $(X\backslash\partial\mathfrak{X}, \mathrm{triv}) \to \mathfrak{X}$ of log-schemes and injective by the construction and the semipurity of $\underline{\omega}^{\mathbf{CI}}F$. This completes the proof of the full faithfulness.

Next we show the exactness of $\mathcal{L}og$. It suffices to show the following.

Claim 6.3.1. Given an exact sequence $0 \to F \to G \to H \to 0$ in $\mathbf{RSC}_{\mathrm{Nis}}$, the induced sequence

$$0 \to F^{\log}(\mathfrak{X}) \to G^{\log}(\mathfrak{X}) \to H^{\log}(\mathfrak{X}) \to 0$$

is exact for every $\mathfrak{X} \in \mathbf{Sm}\mathrm{l}\mathbf{Sm}$ with X henselian local.

Indeed, by the definition of $\mathcal{L}og$, this is reduced to the exactness of

$$0 \to \underline{\omega}^{\mathbf{CI}}F(\mathfrak{X}^{MP}) \to \underline{\omega}^{\mathbf{CI}}G(\mathfrak{X}^{MP}) \to \underline{\omega}^{\mathbf{CI}}H(\mathfrak{X}^{MP}) \to 0,$$

which follows from Corollary 2.6(2). This completes the proof of Theorem 6.3. □

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