TOWARDS A NON-ARCHIMEDEAN ANALYTIC ANALOG OF THE BASS–QUILLEN CONJECTURE

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Abstract. We suggest an analog of the Bass–Quillen conjecture for smooth affinoid algebras over a complete non-archimedean field. We prove this in the rank-1 case, i.e. for the Picard group. For complete discretely valued fields of residue characteristic 0 we prove a similar statement for the Grothendieck group of vector bundles K_0.

Introduction

For a ring A let us denote by $\mathrm{Vec}_r(A)$ the set of isomorphism classes of finitely generated projective modules of rank r. The Bass–Quillen conjecture predicts that for a regular noetherian ring A the inclusion into the polynomial ring $A[t_1,\dots,t_n]$ induces a bijection

$$\mathrm{Vec}_r(A) \xrightarrow{\sim} \mathrm{Vec}_r(A[t_1,\dots,t_n])$$

for all $n, r \geq 0$. Based on the work of Quillen and Suslin on Serre's problem the conjecture has been shown in case A is a smooth algebra over a field [13].

In this note we discuss a potential extension of this conjecture to affinoid algebras in the sense of Tate. Let K be a field which is complete with respect to a non-trivial non-archimedean absolute value and let A/K be a smooth affinoid algebra. In rigid geometry a building block is the ring of power series converging on the closed unit disc

$$A\langle t_1,\dots,t_n\rangle = \{f = \sum_{\underline{k}} c_{\underline{k}} t^{\underline{k}} \in A[\![t_1,\dots,t_n]\!] \mid c_{\underline{k}} \xrightarrow{|\underline{k}|\to\infty} 0\},$$

which serves as a replacement for the polynomial ring in algebra.

Using these convergent power series the following positive result in analogy with Serre's problem is obtained in [14].

Example 1 (Lütkebohmert)**.** *All finitely generated projective modules over $K\langle t_1,\dots,t_n\rangle$ are free.*

Unfortunately, over more general smooth affinoid algebras one has the following negative example [9, 4.2].

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Example 2 (Gerritzen)**.** *Assume the ring of integers K° of K is a discrete valuation ring with prime element π. For the smooth affinoid K-algebra $A = K\langle t_1, t_2\rangle/(t_1^2 - t_2^3 - \pi)$ the map*

$$\mathrm{Pic}(A) \to \mathrm{Pic}(A\langle t\rangle)$$

is not bijective.

This example shows that for our purpose the ring of convergent power series $A\langle t\rangle$ is not entirely appropriate. Let $\pi \in K \setminus \{0\}$ be an element with $|\pi| < 1$. As an improved non-archimedean analytic replacement for the polynomial ring over A we are going to use the pro-system of affinoid algebras "$\lim\limits_{t\mapsto\pi t}$" $A\langle t\rangle$. It represents an affinoid approximation of the non-quasi-compact rigid analytic space $(\mathbb{A}^1_A)^{\mathrm{an}}$ since

$$\lim_{t\mapsto\pi t} A\langle t\rangle = H^0((\mathbb{A}^1_A)^{\mathrm{an}}, \mathcal{O}).$$

Note that the latter non-affinoid K-algebra is harder to control, compare [10, Ch. 5] and [3].

As a non-archimedean analytic analog of the Bass–Quillen conjecture one might ask:

Question 3. Is the map

$$\mathrm{Vec}_r(A) \to \text{“}\lim_{t\mapsto\pi t}\text{''}\, \mathrm{Vec}_r(A\langle t\rangle)$$

a pro-isomorphism for A/K a smooth affinoid algebra?

We give a positive answer for $r = 1$.

Theorem 4. *For A/K a smooth affinoid algebra the map*

$$\mathrm{Pic}(A) \to \text{“}\lim_{t\mapsto\pi t}\text{''}\, \mathrm{Pic}(A\langle t\rangle)$$

is a pro-isomorphism.

The Picard group $\mathrm{Pic}(A)$ of an affinoid algebra A is isomorphic to the cohomology group $H^1(\mathrm{Sp}(A), \mathcal{O}^*)$.

In case the residue field of K has characteristic zero, one has the exponential isomorphism $\exp : \mathcal{O}(1) \xrightarrow{\sim} \mathcal{O}^*(1)$, where $\mathcal{O}(1) \subset \mathcal{O}$ is the subsheaf of rigid analytic functions f with $|f|_{\mathrm{sup}} < 1$ and $\mathcal{O}^*(1) \subset \mathcal{O}^*$ is the subsheaf of functions f with $|1 - f|_{\mathrm{sup}} < 1$. Based on this isomorphism [9, Satz 4] reduces Theorem 4 in case of characteristic zero to a vanishing result for the additive rigid cohomology group $H^1(\mathrm{Sp}(A), \mathcal{O}(1))$ which is established in [1]. As the articles [1] and [2] are written in German and are not easy to read, we give a simplified proof of their main results in Section 1 based on the cohomology theory of affinoid spaces [17].

However in case $\mathrm{ch}(K) > 0$ this approach using the exponential isomorphism does not apply. Instead, in Section 2 we explain how to pass from a vanishing result for the additive cohomology groups to a vanishing result for the multiplicative cohomology groups in the absence of an exponential isomorphism. Based on the latter vanishing the proof of Theorem 4 is given in Section 3.

In Section 4 we prove the following stable version of Question 3:

Theorem 5. *Let K be discretely valued, and assume that the residue field of K has characteristic zero. Let A/K be a smooth affinoid algebra. Then*

$$K_0(A) \to \text{“}\lim_{t \mapsto \pi t}\text{”}\, K_0(A\langle t\rangle)$$

is a pro-isomorphism.

The proof of Theorem 5 uses a regular model $\mathcal{X}/K^\circ$ of A, i.e. resolution of singularities, and “pro-cdh-descent” [15] for the K-theory spectrum of schemes, so it is rather non-elementary. Of course for such fields K Theorem 5 comprises Theorem 4, as there is a surjective determinant map $\det : K_0 \to \mathrm{Pic}$.

Notations. We denote the supremum seminorm [5, Sec. 3.1] of a rigid analytic function f on an affinoid space X by $|f|_{\sup}$. For a real number $r > 0$ we denote by $\mathcal{O}_X(r) \subseteq \mathcal{O}_X$ the subsheaf of functions of supremum seminorm $< r$. We often omit the subscript X if no confusion is possible. We write $\mathcal{O}^\circ \subseteq \mathcal{O}$ for the subsheaf of functions of supremum norm ≤ 1.

If $0 < r < 1$, functions of the from $1 + f$ with $|f|_{\sup} < r$ are invertible, and we denote by $\mathcal{O}^*(r) \subseteq \mathcal{O}^*$ the subsheaf of invertible functions of this form.

We use similar notations $K(r), K^\circ, K^*(r)$ for corresponding elements of the field K or complete valued extensions of K.

If a is an analytic point of an affinoid space [8, Sec. 2.1], we denote the completion of its residue field by F_a.

For the closed polydisk $\mathrm{Sp}(K\langle t_1, \dots, t_d\rangle)$ of radius 1 and dimension d over K we use the notation $\mathbb{B}^d_K$ or simply $\mathbb{B}^d$.

An affinoid algebra A/K is called smooth if $A \otimes_K K'$ is regular for all finite field extensions $K \subset K'$. As a general reference concerning the terminology of rigid spaces we refer to [5].

1. Vanishing of additive cohomology (after Bartenwerfer)

The aim of this section is to give new, more conceptual proofs of the main results of [1] and [2]. Our techniques are based on cohomology theory for affinoid spaces as developed by van der Put, see [17] and [8]. Let K be a field which is complete with respect to the non-archimedean absolute value $|\cdot| : K \to \mathbb{R}$. We assume that the absolute value $|\cdot|$ is not trivial. All affinoid spaces we consider in this section are assumed to be integral.

Let $\mathcal{M}, \mathcal{N}$ be sheaves of $\mathcal{O}^\circ$-modules on the affinoid space $X = \mathrm{Sp}(A)$. We say that $\mathcal{M}$ is weakly trivial if there exists $r \in (0,1)$ with $\mathcal{O}(r)\mathcal{M} = 0$. Note that this just means that there exists $f \in K^\circ \setminus \{0\}$ with $f\mathcal{M} = 0$. We say that an $\mathcal{O}^\circ$-morphism $u : \mathcal{M} \to \mathcal{N}$ is a weak isomorphism if $\mathrm{coker}(u)$ and $\ker(u)$ are weakly trivial. Note that the weak isomorphisms are exactly those morphisms which are invertible up to multiplication by elements of $K^\circ \setminus \{0\}$. We say that $\mathcal{M}$ is weakly locally free (wlf) if there is a finite affinoid covering $X = \cup_{i\in I} U_i$ and weak isomorphisms $(\mathcal{O}^\circ_{U_i})^{n_i} \simeq \mathcal{M}|_{U_i}$ for each $i \in I$.

Note that for $\mathcal{M}$ wlf the $\mathcal{O}_X$-module sheaf $\mathcal{M} \otimes_{\mathcal{O}^\circ_X} \mathcal{O}_X$ is coherent and locally free.

Lemma 6. *Let $\psi : \mathcal{M} \to \mathcal{N}$ be an $\mathcal{O}^\circ$-morphism of wlf sheaves on $X = \mathrm{Sp}(A)$ and assume that there exists $f \in A^\circ$ such that*

$$f \operatorname{coker}(\psi \otimes 1 : \mathcal{M} \otimes_{\mathcal{O}^\circ} \mathcal{O} \to \mathcal{N} \otimes_{\mathcal{O}^\circ} \mathcal{O}) = 0.$$

Then there exists $r \in (0,1)$ such that $f K(r) \operatorname{coker}(\psi) = 0$.

Proof. Without loss of generality $\mathcal{M} = (\mathcal{O}^\circ)^m$ and $\mathcal{N} = (\mathcal{O}^\circ)^n$. Let $\mathcal{C}$ be the cokernel of ψ. By Tate's acyclicity theorem [5, Cor. 4.3.11] we get an exact sequence

$$H^0(X, \mathcal{M} \otimes_{\mathcal{O}^\circ} \mathcal{O}) \to H^0(X, \mathcal{N} \otimes_{\mathcal{O}^\circ} \mathcal{O}) \to H^0(X, \mathcal{C} \otimes_{\mathcal{O}^\circ} \mathcal{O}),$$

where the right hand A-module is f-torsion by assumption. Let $e_1, \ldots, e_n \in \mathcal{N}(X)$ be the canonical basis elements. So we deduce that $fe_1, \ldots, fe_n$ have preimages $l_1, \ldots, l_n \in H^0(X, \mathcal{M} \otimes_{\mathcal{O}^\circ} \mathcal{O}) = A^m$. Choose $r \in (0,1)$ such that $K(r)l_1, \ldots, K(r)l_n \subset (A^\circ)^m$. □

Proposition 7. *Let $\mathcal{M}$ be an $\mathcal{O}^\circ$-module sheaf on $X = \mathrm{Sp}(A)$ such that $\mathcal{M} \otimes_{\mathcal{O}^\circ_X} \mathcal{O}_X$ is coherent and locally free as $\mathcal{O}_X$-module sheaf. Then the following are equivalent:*

(i) *$\mathcal{M}$ is wlf.*
(ii) *For each finite set of points $R \subset X$ there is an injective $\mathcal{O}^\circ$-linear morphism $\Psi : (\mathcal{O}^\circ)^n \to \mathcal{M}$ and $f \in \mathcal{O}^\circ(X)$ with $f(x) \neq 0$ for all $x \in R$ such that $f \operatorname{coker}(\Psi) = 0$.*
(iii) *For each point $x \in X$ there is an injective $\mathcal{O}^\circ(X)$-linear morphism $\Psi_x : (\mathcal{O}^\circ)^n \to \mathcal{M}$ and $f_x \in \mathcal{O}^\circ(X)$ with $f_x(x) \neq 0$ such that $f_x \operatorname{coker}(\Psi) = 0$.*

Proof. Clearly, (ii) implies (iii). We first prove (iii) implies (i). Choose for each point $x \in X$ a map Ψ_x and f_x as in (iii). There is a finite set of points $x_1, \ldots, x_k \in X$ such that we get a Zariski covering

$$X = \bigcup_{i \in \{1,\ldots,k\}} \{x \in X \mid f_{x_i}(x) \neq 0\}.$$

By [5, Lem. 5.1.8] there exists $\epsilon \in \sqrt{|K^\times|}$ such that the $U_i = \{x \in X \mid |f_{x_i}(x)| \geq \epsilon\}$ cover X. Then the morphisms $\Psi_{x_i}|_{U_i}$ are weak isomorphisms, so $\mathcal{M}$ is wlf.

We now prove that (i) implies (ii). As $\mathcal{M} \otimes_{\mathcal{O}^\circ_X} \mathcal{O}_X$ is locally free there exists a finitely generated projective A-module M with $M^\sim = \mathcal{M} \otimes_{\mathcal{O}^\circ_X} \mathcal{O}_X$, [5, Sec. 6.1]. By A_R we denote the semi-local ring which is the localization of A at the finitely many maximal ideals R. Choose a basis $b_1, \ldots, b_n$ of the free

A_R-module $M \otimes_A A_R$. Without loss of generality we can assume $b_1, \dots, b_n$ are induced by elements of $\mathcal{M}(X)$. We claim that the latter elements give rise to a morphism Ψ as in (ii). Indeed, by elementary algebra we find $f' \in A^\circ$ such that $f'(x) \neq 0$ for all $x \in R$ and such that

$$f' \operatorname{coker}(A^n \to M) = 0.$$

We conclude by Lemma 6. $\square$

Proposition 8. *Let $\phi : X \to Y$ be a finite étale morphism of affinoid spaces over K and let $\mathcal{M}$ be a wlf $\mathcal{O}_X^\circ$-module. Then $\phi_*\mathcal{M}$ is a wlf $\mathcal{O}_Y^\circ$-module.*

Proof. Let $X = \mathrm{Sp}(A)$ and $Y = \mathrm{Sp}(B)$. The $\mathcal{O}_Y$-module sheaf $\phi_*(\mathcal{M}) \otimes_{\mathcal{O}_Y^\circ} \mathcal{O}_Y = \phi_*(\mathcal{M} \otimes_{\mathcal{O}_X^\circ} \mathcal{O}_X)$ is coherent and locally free. For $y \in Y$ let R be the finite set $\phi^{-1}(y)$ and let $M \subset B$ be the maximal ideal corresponding to y. From Proposition 7 we deduce that there is an injective $\mathcal{O}_X^\circ$-linear morphism

$$\Psi : (\mathcal{O}_X^\circ)^n \to \mathcal{M}$$

whose cokernel is killed by some $f \in A^\circ$ which does not vanish on R. Then as the induced homomorphism $\phi^\sharp : B \to A$ is finite the prime ideals of B containing the ideal $I = (\phi^\sharp)^{-1}(Af)$ are exactly the preimages of the prime ideals in A which contain f, see [6, Sec. V.2.1]. So we can find $g \in I \cap B^\circ$ which is not contained in M. Then the cokernel of the injective morphism

$$\phi_*(\Psi) : \phi_*(\mathcal{O}_X^\circ)^n \to \phi_*(\mathcal{M}).$$

is g-torsion. By Proposition 7 we see that it suffices to show that $\phi_*(\mathcal{O}_X^\circ)$ is wlf.

Note that for $V \subset Y$ an affinoid subdomain $\mathcal{O}_X^\circ(\phi^{-1}(V))$ is the integral closure of $\mathcal{O}_Y^\circ(V)$ in $A \otimes_B \mathcal{O}_Y(V) = \mathcal{O}_X(\phi^{-1}(V))$ [5, Thm. 3.1.17]. As the field extension $Q(B) \to Q(A)$ is separable, it is not hard to bound this integral closure as follows. Let $b_1, \dots, b_d \in \mathcal{O}^\circ(X)$ induce a basis of the free B_M-module $A \otimes_B B_M$. This basis induces an injective $\mathcal{O}_Y^\circ$-linear morphism

$$\Psi : (\mathcal{O}_Y^\circ)^d \to \phi_*(\mathcal{O}_X^\circ).$$

Let δ be the discriminant of $b_1, \dots, b_d$. Then by [6, Lem. V.1.6.3] the cokernel of Ψ is δ-torsion.

As the point $y \in Y$ was arbitrary we conclude from Proposition 7 that $\phi_*(\mathcal{O}_X^\circ)$ is wlf. $\square$

For a sheaf $\mathcal{M}$ on X we write $\mathcal{M}^{\mathrm{oc}}$ for the associated overconvergent sheaf. The sheaf $\mathcal{M}^{\mathrm{oc}}$ is given on an affinoid open subdomain $U \subset X$ by

$$\mathcal{M}^{\mathrm{oc}}(U) = \mathrm{colim}_{U \subset U'} \mathcal{M}(U')$$

where U' runs through all wide neighborhoods of U in X. Note that there is a canonical morphism $\mathcal{M}^{\mathrm{oc}} \to \mathcal{M}$. For the definition and basic properties of overconvergent sheaves see [8, Sec. 2]. The following proposition is a simple consequence of Tate's acyclicity theorem [5, Cor. 4.3.11].

Proposition 9. *Let $X = \mathrm{Sp}(A)$ be an affinoid space.*

(i) *For any finite affinoid covering $\mathcal{U}$ of X the Čech cohomology groups $H^i(\mathcal{U}, \mathcal{O}^\circ)$ are weakly trivial (as K°-modules) for all $i > 0$.*
(ii) *The canonical map*

$$H^i(V, \mathcal{O}_X(r)^{\mathrm{oc}}|_V) \to H^i(V, \mathcal{O}_V(r))$$

is surjective for every affinoid subdomain $V \subset X$, every $r > 0$ and integer $i > 0$.

Proof. (i): Note that for each affinoid open subdomain U of X the Čech complex $(C(\mathcal{U}, \mathcal{O}), d)$ consists of complete normed K-vector spaces and the differential is continuous. To be concrete, we work with the supremum norm. The continuous morphism

$$d^{i-1} : C^{i-1}(\mathcal{U}, \mathcal{O}) \to Z^i(\mathcal{U}, \mathcal{O})$$

is surjective by [5, Cor. 4.3.11], so it is open according to [7, Thm. I.3.3.1]. In other words there exists $r \in (0, 1)$ such that $Z^i(\mathcal{U}, \mathcal{O}(r))$ is contained in $d^{i-1}(C^{i-1}(\mathcal{U}, \mathcal{O}^\circ))$. This means that $H^i(\mathcal{U}, \mathcal{O}^\circ)$ is $K(r)$-torsion.

(ii): In order to show part (ii) of the proposition it suffices to show that for each finite covering $\mathcal{U} = (U_l)_{l \in L}$ of V by rational subdomains of X the map

$$H^i(\mathcal{U}, \mathcal{O}_X(r)^{\mathrm{oc}}) \to H^i(\mathcal{U}, \mathcal{O}(r)) \tag{1}$$

is surjective. This is a consequence of

Claim 10.
(i) For $i > 0$ the image of $d^{i-1} : C^{i-1}(\mathcal{U}, \mathcal{O}(r)) \to Z^i(\mathcal{U}, \mathcal{O}(r))$ is open.
(ii) The image of $Z^i(\mathcal{U}, \mathcal{O}_X(r)^{\mathrm{oc}}) \to Z^i(\mathcal{U}, \mathcal{O}(r))$ is dense.

Part (i) of the claim is a consequence of Proposition 9(i). For part (ii) of the claim note that for each rational subdomain

$$U = \{|g_1| \le |g_0|, \dots, |g_r| \le |g_0|\}$$

of X the image of $\mathcal{O}_X^{\mathrm{oc}}(U) \to \mathcal{O}(U)$ is dense. To see this observe that for $\epsilon > 1$ and $\epsilon \in |K^*|^{\mathbb{Q}}$ the set U is a Weierstraß domain inside $\{|g_1| \le \epsilon|g_0|, \dots, |g_r| \le \epsilon|g_0|\}$.

For $\xi \in Z^i(\mathcal{U}, \mathcal{O}(r))$ we find $\xi' \in C^{i-1}(\mathcal{U}, \mathcal{O})$ with $d(\xi') = \xi$, using again [5, Cor. 4.3.11]. Find a sequence $\xi'_j \in C^{i-1}(\mathcal{U}, \mathcal{O}_X^{\mathrm{oc}})$ such that its image in $C^{i-1}(\mathcal{U}, \mathcal{O})$ converges to ξ'. Then $d(\xi'_j) \in Z^i(\mathcal{U}, \mathcal{O}^{\mathrm{oc}})$ is a sequence approximating ξ. By [8, Lem. 2.3.1] for large j we have $d(\xi'_j) \in Z^i(\mathcal{U}, \mathcal{O}_X(r)^{\mathrm{oc}})$. □

Theorem 11 (Bartenwerfer/van der Put)**.** *We have*

$$H^i(\mathbb{B}^d, \mathcal{O}(r)) = 0$$

for all $r > 0$ and integers $i > 0$.

Idea of proof (van der Put). Using Tate's acyclicity theorem the theorem is equivalent to the following two statements:

- for all $r > 0$ and integers $i > 0$ the cohomology group
$$H^i(\mathbb{B}^d, \mathcal{O}/\mathcal{O}(r)) = 0,$$
- $H^0(\mathbb{B}^d, \mathcal{O}) \to H^0(\mathbb{B}^d, \mathcal{O}/\mathcal{O}(r))$ is surjective.

Using the linear fibrations $\phi : \mathbb{B}^d \to \mathbb{B}^{d-1}$, base change [8, Thm. 2.7.4] and the fact that for any fibre $\phi^{-1}(a) \cong \mathbb{B}^1_{F_a}$ over an analytic point a of $\mathbb{B}^{d-1}$ we have
$$(\mathcal{O}_{\mathbb{B}^d}/\mathcal{O}_{\mathbb{B}^d}(r))|_{\phi^{-1}(a)} \cong \mathcal{O}_{\mathbb{B}^1_{F_a}}/\mathcal{O}_{\mathbb{B}^1_{F_a}}(r),$$
compare Lemma 23, we reduce the theorem to the case $d = 1$. In fact, by what is sayed and using the one-dimensional case of the theorem we get that
$$\phi_*(\mathcal{O}_{\mathbb{B}^d}/\mathcal{O}_{\mathbb{B}^d}(r)) = \bigoplus_{\mathbb{N}} \mathcal{O}_{\mathbb{B}^{d-1}}/\mathcal{O}_{\mathbb{B}^{d-1}}(r),$$
$$R^j\phi_*(\mathcal{O}_{\mathbb{B}^d}/\mathcal{O}_{\mathbb{B}^d}(r)) = 0 \quad (j > 0)$$
and we conclude by the Leray spectral sequence and by induction on d.

In the one-dimensional case the theorem follows from an explicit computation based on the Mittag–Leffler decomposition. For details see [17, Thm. 3.15]. □

Corollary 12. *The cohomology group*
$$H^i(\mathbb{B}^d, \mathcal{O}^\circ)$$
is $K(1)$-torsion for all integers $i > 0$.

Remark 13. In fact, in [4] Bartenwerfer shows that $H^i(\mathbb{B}^d, \mathcal{O}^\circ) = 0$ for every $i > 0$.

Lemma 14. *Let $X = \mathrm{Sp}(A)$ be an affinoid space such that the cohomology group $H^i(X, \mathcal{O}^\circ)$ is weakly trivial for some $i > 0$. Then for any wlf $\mathcal{O}^\circ$-module $\mathcal{M}$ the cohomology group $H^i(X, \mathcal{M})$ is weakly trivial.*

Proof. Below we are going to construct for every point $x \in X$ a function $f_x \in A^\circ$ with $f_x(x) \neq 0$ and with $f_x\, H^i(X, \mathcal{M}) = 0$. As the f_x generate the unit ideal in A, there exist finitely many points $x_1, \dots, x_r \in X$ and $c_1, \dots, c_r \in A^\circ$ with
$$c_1 f_{x_1} + \cdots + c_r f_{x_r} =: c \in K^\circ \setminus \{0\}.$$
Then $c\, H^i(X, \mathcal{M}) = 0$.

In order to construct such f_x for given $x \in X$ we use Proposition 7 in order to find an injective $\mathcal{O}^\circ_X$-linear morphism $\Psi : (\mathcal{O}^\circ)^n \to \mathcal{M}$ and $f' \in \mathcal{O}^\circ(X)$ with $f'(x) \neq 0$ and such that $f' \operatorname{coker}(\Psi) = 0$. From the long exact cohomology sequence corresponding to the short exact sequence
$$0 \to (\mathcal{O}^\circ)^n \xrightarrow{\Psi} \mathcal{M} \to \operatorname{coker}(\Psi) \to 0$$
it follows that we can take any nonzero $f_x \in K(r)f'$, where $r \in (0, 1)$ is chosen such that $K(r)\, H^i(X, \mathcal{O}^\circ) = 0$. □

Theorem 15. *For X/K a smooth affinoid space and for $\mathcal{M}$ a wlf $\mathcal{O}_X^\circ$-module the cohomology groups $H^i(X,\mathcal{M})$ are weakly trivial (as K°-modules) for all $i>0$.*

Proof. By Lemma 14 we can assume without loss of generality that $\mathcal{M}=\mathcal{O}^\circ$. We use induction on $i>0$. The base case $i=1$ is handled in the same way as the induction step, so let us assume $i>1$ and that we already know weak triviality of $H^j(U,\mathcal{O}^\circ)$ for all $0<j<i$ and smooth affinoid spaces U/K.

By [12, Satz 1.12] there exists a finite affinoid covering $\mathcal{U}=(U_l)_{l\in L}$ and finite étale morphisms $\phi_l: U_l\to\mathbb{B}^d$. From the Čech spectral sequence

$$E_2^{pq}=H^p(\mathcal{U},\underline{H}^q(\mathcal{O}^\circ))\Rightarrow H^{p+q}(X,\mathcal{O}^\circ)$$

we see that $H^i(X,\mathcal{O}^\circ)$ has a filtration whose associated graded piece gr^p is a subquotient of $H^p(\mathcal{U},\underline{H}^{i-p}(\mathcal{O}^\circ))$. By Proposition 9(i), gr^i is weakly trivial. By our induction assumption, $\underline{H}^{i-p}(\mathcal{O}^\circ)(U)$ is weakly trivial for $0<p<i$ and for U an intersection of opens in $\mathcal{U}$, hence gr^{i-p} is weakly trivial for these p. It thus suffices to show that gr^0 is weakly trivial or that $H^i(U_l,\mathcal{O}^\circ_{U_l})$ is weakly trivial for all $l\in L$.

So in order to show Theorem 15 we can assume without loss of generality that $\mathcal{M}=\mathcal{O}_X^\circ$ and that there exists a finite étale morphism $\phi: X\to\mathbb{B}^d$.

For all $j>0$ we get morphisms

$$R^j\phi_*(\mathcal{O}_X^\circ)\simeq R^j\phi_*(\mathcal{O}_X(1))\leftarrow R^j\phi_*(\mathcal{O}_X(1)^{\mathrm{oc}}). \tag{2}$$

with a weak isomorphism on the left and a surjective morphism on the right. The surjectivity follows from Proposition 9(ii). However, by base change [8, Thm. 2.7.4] the right hand side of (2) vanishes for $j>0$.

Combining this observation with the Leray spectral sequence we see that it suffices to show that $H^i(\mathbb{B}^d,\phi_*(\mathcal{O}_X^\circ))$ is weakly trivial for $i>0$. From Proposition 8 we deduce that $\phi_*(\mathcal{O}_X^\circ)$ is wlf as an $\mathcal{O}^\circ_{\mathbb{B}^d}$-module, so we conclude by using Theorem 11 and Lemma 14. □

The following corollary, which we will apply in the next sections, was first shown in [1] and [2].

Corollary 16 (Bartenwerfer)**.** *For X/K smooth affinoid there exists $s\in(0,1)$ such that the map*

$$H^i(X,\mathcal{O}(sr))\to H^i(X,\mathcal{O}(r)) \tag{3}$$

vanishes for all $r>0$ and integers $i>0$.

Proof. Choose $\pi\in K(1)\setminus\{0\}$ and write $s'=|\pi|$. By Theorem 15 we can assume without loss of generality that $\pi H^i(X,\mathcal{O}(1))=0$ for $i>0$. Now we claim $s=s'^2$ satisfies the requested property of the corollary. Indeed, for $r>0$ set $r'=\max\{|\pi|^n \mid n\in\mathbb{Z}, |\pi|^n\le r\}$. Then we get a commutative square

$$\begin{tikzcd} H^i(X, \mathcal{O}(s'r')) \arrow[r] \arrow[d, "\wr"] & H^i(X, \mathcal{O}(r')) \arrow[d, "\wr"] \\ H^i(X, \mathcal{O}(1)) \arrow[r, "=0"] & H^i(X, \mathcal{O}(1)) \end{tikzcd}$$

where the lower horizontal map is multiplication by π and the vertical maps are induced by the isomorphisms $\mathcal{O}(s'r') \cong \mathcal{O}(1)$ and $\mathcal{O}(r') \cong \mathcal{O}(1)$ given by multiplying with the appropriate powers of π. The morphism (3) is the composition of

$$H^i(X, \mathcal{O}(sr)) \to H^i(X, \mathcal{O}(s'r')) \xrightarrow{=0} H^i(X, \mathcal{O}(r')) \to H^i(X, \mathcal{O}(r)).$$

□

2. Vanishing of multiplicative cohomology

Given $r' < r$ we write $\mathcal{O}(r, r') := \mathcal{O}(r)/\mathcal{O}(r')$ and, if $r' < r \leq 1$, $\mathcal{O}^*(r, r') := \mathcal{O}^*(r)/\mathcal{O}^*(r')$.

Lemma 17. *For $r' < r \leq 1$ we have isomorphisms of sheaves of sets $\mathcal{O}(r) \xrightarrow{\sim} \mathcal{O}^*(r)$ and $\mathcal{O}(r, r') \xrightarrow{\sim} \mathcal{O}^*(r, r')$ given by $f \mapsto 1 + f$. If $r' \geq r^2$, the latter isomorphism is an isomorphism of abelian sheaves.*

Proof. Most of the claims are easy. To see that $f \mapsto 1 + f$ induces a map on the quotient sheaves $\mathcal{O}(r, r') \to \mathcal{O}^*(r, r')$ note that if f, g are functions of supremum seminorm < 1, then $|f - g|_{\sup} < r'$ if and only if $|(1 + f)(1 + g)^{-1} - 1|_{\sup} < r'$. Indeed, this follows from the computation $|f - g|_{\sup} = |(1+f)-(1+g)|_{\sup} = |((1+f)(1+g)^{-1}-1)(1+g)|_{\sup} = |(1+f)(1+g)^{-1}-1|_{\sup}$, where we used that $|1 + g|_{\sup} = |(1 + g)^{-1}|_{\sup} = 1$. □

Given an affinoid space X, we consider the following condition on the real number $0 < s \leq 1$:

$$\text{The map } H^i(X, \mathcal{O}(sr)) \to H^i(X, \mathcal{O}(r)) \text{ vanishes for all } r > 0 \text{ and integers } i > 0. \tag{4}$$

Proposition 18. *Let X/K be smooth affinoid. Assume that s satisfies (4). Then the map*

$$H^1(X, \mathcal{O}^*(sr)) \to H^1(X, \mathcal{O}^*(r))$$

vanishes for every $r \in (0, s)$.

Proof. We first prove:

Lemma 19. *Assume that s satisfies (4) for the affinoid space X. For any integer $i > 0$, $r \in (0, s)$, and $\xi \in H^i(X, \mathcal{O}^*(sr))$ there exists a decreasing zero sequence (r_n) in $(0, s)$ with $r_0 = r$ and a compatible system*

$$(\xi'_n) \in \lim_n H^i(X, \mathcal{O}^*(r_n))$$

such that $\xi'_0 \in H^i(X, \mathcal{O}^*(r))$ is equal to the image of ξ under $H^i(X, \mathcal{O}^*(sr)) \to H^i(X, \mathcal{O}^*(r))$.

Proof. Put $r_0 = r$ and inductively $r_{n+1} = r_n^2/s$. Explicitly, $r_n = (r/s)^{2^n} s$. Since $r < s$, the r_n form a decreasing zero sequence.

Put $\xi_0 = \xi$. We will inductively construct elements $\xi_n \in H^i(X, \mathcal{O}^*(sr_n))$ such that the images of ξ_n and ξ_{n+1} in $H^i(X, \mathcal{O}^*(r_n))$ coincide. Denote this common image by ξ'_n. Then $(\xi'_n)_{n\geq 0}$ is the desired compatible system.

Assume that we have already constructed ξ_n. From the commutative diagram with exact rows

$$\begin{tikzcd}
H^i(X,\mathcal{O}(sr_n)) \arrow[r] \arrow[d, equal] & H^i(X,\mathcal{O}(sr_n, s^2 r_{n+1})) \arrow[r] \arrow[d] & H^{i+1}(X,\mathcal{O}(s^2 r_{n+1})) \arrow[d, "=0 \text{ by } (4)"] \\
H^i(X,\mathcal{O}(sr_n)) \arrow[r] \arrow[d, "=0 \text{ by } (4)"] & H^i(X,\mathcal{O}(sr_n, sr_{n+1})) \arrow[r] \arrow[d] & H^{i+1}(X,\mathcal{O}(sr_{n+1})) \arrow[d, equal] \\
H^i(X,\mathcal{O}(r_n)) \arrow[r] & H^i(X,\mathcal{O}(r_n, sr_{n+1})) \arrow[r] & H^{i+1}(X,\mathcal{O}(sr_{n+1}))
\end{tikzcd}$$

we see that $H^i(X, \mathcal{O}(sr_n, s^2 r_{n+1})) \to H^i(X, \mathcal{O}(r_n, sr_{n+1}))$ vanishes for $i > 0$. Since $sr_{n+1} \geq r_n^2$ and $s^2 r_{n+1} = sr_n^2 \geq (sr_n)^2$, we may apply Lemma 17 to deduce that also $H^i(X, \mathcal{O}^*(sr_n, s^2 r_{n+1})) \to H^i(X, \mathcal{O}^*(r_n, sr_{n+1}))$ vanishes. From the commutative diagram with exact rows

$$\begin{tikzcd}
 & H^i(X,\mathcal{O}^*(sr_n)) \arrow[r] \arrow[d] & H^i(X,\mathcal{O}^*(sr_n, s^2 r_{n+1})) \arrow[d, "=0"] \\
H^i(X,\mathcal{O}^*(sr_{n+1})) \arrow[r] & H^i(X,\mathcal{O}^*(r_n)) \arrow[r] & H^i(X,\mathcal{O}^*(r_n, sr_{n+1}))
\end{tikzcd}$$

we deduce the existence of the desired element $\xi_{n+1} \in H^i(X, \mathcal{O}^*(sr_{n+1}))$ such that the images of ξ_n and ξ_{n+1} in $H^i(X, \mathcal{O}^*(r_n))$ coincide. $\square$

Lemma 20. *Let X/K be smooth affinoid, and let $(\xi_n) \in \lim_n H^1(X, \mathcal{O}^*(r_n))$ be a compatible system where the r_n form a decreasing zero sequence in $(0,1)$. Then there exists a finite affinoid covering $\mathcal{U}$ of X such that (ξ_n) lies in the image of $\lim_n H^1(\mathcal{U}, \mathcal{O}^*(r_n))$.*

Proof. Let $\mathcal{U}$ be a finite affinoid covering of X such that ξ_0 lies in the image of $H^1(\mathcal{U}, \mathcal{O}^*(r_0))$. We claim that then ξ_n lies in the image of $H^1(\mathcal{U}, \mathcal{O}^*(r_n))$ for all n. Recall that for any abelian sheaf $\mathcal{F}$ the map $H^1(\mathcal{U}, \mathcal{F}) \to H^1(X, \mathcal{F})$ is injective, and an element $\xi \in H^1(X, \mathcal{F})$ belongs to the image of this map if and only if $\xi|_U = 0$ in $H^1(U, \mathcal{F}|_U)$ for every $U \in \mathcal{U}$.

Fix $U \in \mathcal{U}$. We want to show that $\xi_n|_U = 0$ in $H^1(U, \mathcal{O}^*(r_n))$. By Corollary 16 there exists $m \geq n$ such that $H^1(U, \mathcal{O}(r_m)) \to H^1(U, \mathcal{O}(r_n))$ vanishes. Under the sequence of maps

$$H^1(U, \mathcal{O}^*(r_m)) \to H^1(U, \mathcal{O}^*(r_n)) \to H^1(U, \mathcal{O}^*(r_0))$$

we have $\xi_m|_U \mapsto \xi_n|_U \mapsto 0$. Hence the element $\xi_m|_U$ lifts to an element η_m in $H^0(U, \mathcal{O}^*(r_0, r_m))$. We claim that the image of η_m in $H^0(U, \mathcal{O}^*(r_0, r_n))$

has a preimage in $H^0(U, \mathcal{O}^*(r_0))$. In view of the commutative diagram with exact rows

$$\begin{tikzcd} H^0(U,\mathcal{O}^*(r_0)) \arrow[r] & H^0(U,\mathcal{O}^*(r_0,r_n)) \arrow[r] & H^1(U,\mathcal{O}^*(r_n)) \\ H^0(U,\mathcal{O}^*(r_0)) \arrow[r] \arrow[u, equal] & H^0(U,\mathcal{O}^*(r_0,r_m)) \arrow[r] \arrow[u] & H^1(U,\mathcal{O}^*(r_m)) \arrow[u] \end{tikzcd}$$

this will imply that $\xi_n|_U = 0$.

To prove the claim, note that Lemma 17 gives bijections $H^0(U, \mathcal{O}^*(r_0)) \cong H^0(U, \mathcal{O}(r_0))$ and $H^0(U, \mathcal{O}^*(r_0, r_n)) \cong H^0(U, \mathcal{O}(r_0, r_n))$ and similarly for r_n replaced by r_m. On the other hand, by the choice of m, the map $H^1(U, \mathcal{O}(r_m)) \to H^1(U, \mathcal{O}(r_n))$ vanishes. This implies the existence of the desired lift in view of the commutative diagram with exact rows

$$\begin{tikzcd} H^0(U,\mathcal{O}(r_0)) \arrow[r] & H^0(U,\mathcal{O}(r_0,r_n)) \arrow[r] & H^1(U,\mathcal{O}(r_n)) \\ H^0(U,\mathcal{O}(r_0)) \arrow[r] \arrow[u, equal] & H^0(U,\mathcal{O}(r_0,r_m)) \arrow[r] \arrow[u] & H^1(U,\mathcal{O}(r_m)). \arrow[u, "=0"'] \end{tikzcd}$$

□

We can now finish the proof of Proposition 18. Using the two preceding lemmas, it suffices to show that $\lim_n H^1(\mathcal{U}, \mathcal{O}^*(r_n))$ vanishes for every decreasing zero sequence (r_n). Consider an element $(\xi_n)_n$ in this inverse limit, and choose representing Čech 1-cocycles $\zeta_n \in Z^1(\mathcal{U}, \mathcal{O}^*(r_n))$. Then there exist 0-cochains $\eta_n \in C^0(\mathcal{U}, \mathcal{O}^*(r_n))$ such that $\zeta_n = \zeta_{n+1} \cdot \partial\eta_n$. Since (r_n) is a zero sequence, the product $\prod_{k=0}^{\infty} \eta_{n+k}$ converges in $C^0(\mathcal{U}, \mathcal{O}^*(r_n))$, and we get $\zeta_n = \partial(\prod_{k=0}^{\infty} \eta_{n+k})$, i.e., $\xi_n = 0$. □

Corollary 21. *For every $r \in (0, 1)$ we have $H^1(\mathbb{B}^d, \mathcal{O}^*(r)) = 0$.*

Proof. By Theorem 11, $s = 1$ satisfies condition (4) for $X = \mathbb{B}^d$. Hence by Proposition 18, the identity map on $H^1(\mathbb{B}^d, \mathcal{O}^*(r))$ vanishes. □

Corollary 22. *Let X/K be a smooth affinoid space. Then there exists $0 < r \le 1$ such that*

$$H^1(X, \mathcal{O}^*) \to H^1(X, \mathcal{O}^*/\mathcal{O}^*(r'))$$

is injective for every $r' \in (0, r)$.

Proof. By Corollary 16 there exists $0 < s \le 1$ satisfying (4). By Proposition 18 we can take $r = s^2$. □

3. Homotopy invariance of Pic

In this section we prove Theorem 4. Given $0 < r \le 1$, we set $\mathcal{O}^*(\infty, r) = \mathcal{O}^*/\mathcal{O}^*(r)$. Let $X = \mathrm{Sp}(A)$ be an affinoid space, and let $p : X \times \mathbb{B}^1 \to X$ be the projection, $\sigma : X \to X \times \mathbb{B}^1$ the zero section.

Lemma 23. *For any fibre $p^{-1}(a) \cong \mathbb{B}^1_{F_a}$ over an analytic point a of X we have*

$$\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty, r)|_{p^{-1}(a)} \cong \mathcal{O}^*_{\mathbb{B}^1_{F_a}}(\infty, r).$$

Proof. This follows easily from [8, Lemmas 2.7.1, 2.7.2]. □

Lemma 24. *We have $R^1p_*\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty, r) = 0$.*

Proof. The sheaf $\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty, r)$ and hence its higher direct images are overconvergent (see [17, 1.5.3], [8, Lem. 2.3.2]). Hence it suffices to prove that for any analytic point a of X the stalk $R^1p_*\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty, r)_a$ vanishes. By base change [8, Thm. 2.7.4] and Lemma 23, we have

$$R^1p_*\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty, r)_a \cong H^1(\mathbb{B}^1_{F_a}, \mathcal{O}^*_{\mathbb{B}^1_{F_a}}(\infty, r)).$$

In the exact sequence

$$H^1(\mathbb{B}^1_{F_a}, \mathcal{O}^*_{\mathbb{B}^1_{F_a}}) \to H^1(\mathbb{B}^1_{F_a}, \mathcal{O}^*_{\mathbb{B}^1_{F_a}}(\infty, r)) \to H^2(\mathbb{B}^1_{F_a}, \mathcal{O}^*_{\mathbb{B}^1_{F_a}}(r))$$

the group on the left vanishes because the Tate algebra is a UFD, the group on the right vanishes by dimension reasons. □

Fix $\pi \in K \setminus \{0\}$ with $|\pi| < 1$. Let t denote the coordinate on $\mathbb{B}^1$. Then $t \mapsto \pi t$ induces a map $p_*\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty, r) \to p_*\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty, r)$.

Lemma 25. *We have an isomorphism of pro-abelian sheaves*

$$\text{“}\lim_{t\mapsto\pi t}\text{”}\, p_*\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty, r) \cong \mathcal{O}^*_X(\infty, r)$$

Proof. Obviously, $\mathcal{O}^*_X(\infty, r) \xrightarrow{p^*} p_*\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty, r) \xrightarrow{\sigma^*} \mathcal{O}^*_X(\infty, r)$ is the identity. Choose n big enough such that $|\pi^n| \le r$. We claim that the map

$$p_*\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty, r) \to p_*\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty, r)$$

induced by $t \mapsto \pi^n t$ factors through $\mathcal{O}^*_X(\infty, r) \xrightarrow{p^*} p_*\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty, r)$. By overconvergence again it is enough to check this on the stalk at any analytic point a of X. By base change and Lemma 23 we have $p_*\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty, r)_a \cong H^0(\mathbb{B}^1_{F_a}, \mathcal{O}^*_{\mathbb{B}^1_{F_a}}(\infty, r))$. By Corollary 21 the natural map $H^0(\mathbb{B}^1_{F_a}, \mathcal{O}^*) \to H^0(\mathbb{B}^1_{F_a}, \mathcal{O}^*_{\mathbb{B}^1_{F_a}}(\infty, r))$ is surjective. Any element of $H^0(\mathbb{B}^1_{F_a}, \mathcal{O}^*)$ is of the form $u \cdot f(t)$ with $u \in F_a^*$, $f(0) = 1$, and $|f(t) - 1|_{\sup} < 1$ (see [5, Cor. 2.2.4]). But then $|f(\pi^n t) - 1|_{\sup} < |\pi^n| \le r$. This implies that the map

$$H^0(\mathbb{B}^1_{F_a}, \mathcal{O}^*_{\mathbb{B}^1_{F_a}}(\infty, r)) \to H^0(\mathbb{B}^1_{F_a}, \mathcal{O}^*_{\mathbb{B}^1_{F_a}}(\infty, r))$$

induced by $t \mapsto \pi^n t$ factors through $F_a^*/F_a^*(r) \hookrightarrow H^0(\mathbb{B}^1_{F_a}, \mathcal{O}^*_{\mathbb{B}^1_{F_a}}(\infty, r))$, concluding the proof. □

Proof of Theorem 4. Note that $\mathrm{Pic}(A) \cong H^1(X, \mathcal{O}^*)$. Since $X = \mathrm{Sp}(A)$ is assumed to be smooth, Corollary 22 implies that there exists $r \in (0, 1)$ such that the map $H^1(X\times\mathbb{B}^1,\mathcal{O}^*)\to H^1(X\times\mathbb{B}^1,\mathcal{O}^*(\infty,r))$ is injective. It thus suffices to show that

$$\sigma^*: \text{“}\lim_{t\mapsto\pi t}\text{”}\, H^1(X\times\mathbb{B}^1,\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty,r))\to H^1(X,\mathcal{O}^*_X(\infty,r))$$

is a pro-isomorphism.

Using the Leray spectral sequence, Lemma 24 yields an isomorphism

$$H^1(X\times\mathbb{B}^1,\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty,r))\cong H^1(X,p_*\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty,r)).$$

We combine this with the pro-isomorphism

$$\text{“}\lim_{t\mapsto\pi t}\text{”}\, H^1(X,p_*\mathcal{O}^*_{X\times\mathbb{B}^1}(\infty,r))\cong H^1(X,\mathcal{O}^*_X(\infty,r))$$

implied by Lemma 25 to finish the proof. □

4. K_0-invariance

In this section we assume that K is a complete discretely valued field whose residue field has characteristic zero. Let $\pi\in K^\circ$ be a prime element. Then for an affinoid algebra A/K the ring A° is noetherian excellent of finite Krull dimension and a quotient of a regular ring, for excellence see [11, Sec. I.9].

Let $\mathcal{X}\to\operatorname{Spec}A^\circ$ be a blow-up in an ideal whose cosupport is contained in $\operatorname{Spec}A^\circ/(\pi)$, i.e. an admissible blow-up in the sense of Raynaud [5, Ch. 8]. For an integer $n>0$ set $\mathcal{X}_n=\mathcal{X}\otimes_{K^\circ}K^\circ/(\pi^n)$.

Proposition 26. *There exists $n>0$ such that*

$$K_0(\mathcal{X})\to K_0(\mathcal{X}_n)$$

is injective.

Proof. Let $K(\mathcal{X},\mathcal{X}_n)$ be the homotopy fibre of the map $K(\mathcal{X})\to K(\mathcal{X}_n)$ between non-connective K-theory spectra [18, Sec. IV.10] and let $K_i(\mathcal{X},\mathcal{X}_n)$ be its homotopy groups. By “pro-cdh-descent” [15] the natural map

$$\text{“}\lim_n\text{”}\, K_0(A^\circ,A^\circ/(\pi^n))\to\text{“}\lim_n\text{”}\, K_0(\mathcal{X},\mathcal{X}_n)$$

is a pro-isomorphism. For each n we have an exact sequence

$$K_1(A^\circ)\to K_1(A^\circ/(\pi^n))\to K_0(A^\circ,A^\circ/(\pi^n))\to K_0(A^\circ)\xrightarrow{\sim}K_0(A^\circ/(\pi^n))$$

where the left map is surjective [18, Rmk. III.1.2.3] and the right map is an isomorphism [18, Lem. II.2.2], so $K_0(\mathcal{X},\mathcal{X}_n)$ vanishes as a pro-system in n. By the exact fibre sequence

$$K_0(\mathcal{X},\mathcal{X}_n)\to K_0(\mathcal{X})\to K_0(\mathcal{X}_n)$$

this finishes the proof of the proposition. □

Lemma 27. *If $\mathcal{X}$ is a regular scheme we obtain a natural exact sequence*

$$G_0(\mathcal{X}_1)\to K_0(\mathcal{X})\to K_0(A)\to 0,$$

where G_0 is the Grothendieck group of coherent sheaves.

Proof of Theorem 5. As A° contains $\mathbb{Q}$ and is excellent there exists an admissible blow-up $\mathcal{X} \to A^\circ$ such that $\mathcal{X}$ is a regular scheme [16, Thm. 1.1]. Let $A^\circ\langle t\rangle \subset A^\circ[\![t]\!]$ be those formal power series for which the coefficients converge to zero. Note that $A^\circ \to A^\circ\langle t\rangle$ is a regular ring homomorphism, so $\mathcal{X}' = \mathcal{X} \otimes_{A^\circ} A^\circ\langle t\rangle$ is a regular scheme with generic fibre $\mathrm{Spec}\,(A\langle t\rangle)$. Set $\mathcal{X}'_n = \mathcal{X}' \otimes_{K^\circ} K^\circ/(\pi^n)$.

Applying Lemma 27 to $\mathcal{X}$ and $\mathcal{X}'$ we get a commutative diagram with exact rows

$$\begin{tikzcd}
G_0(\mathcal{X}_1) \arrow[r] & K_0(\mathcal{X}) \arrow[r] & K_0(A) \arrow[r] & 0 \\
G_0(\mathcal{X}'_1) \arrow[r] \arrow[u, "\sigma^*", "\wr"'] & K_0(\mathcal{X}') \arrow[r] \arrow[u, "\sigma^*"] & K_0(A\langle t\rangle) \arrow[r] \arrow[u, "\sigma^*"] & 0
\end{tikzcd}$$

where σ is the zero-section induced by $t \mapsto 0$. The left vertical arrow is an isomorphism as $\mathcal{X}'_1 = \mathbb{A}^1_{\mathcal{X}_1}$, see [18, Thm. II.6.5]. In order to prove Theorem 5 we have to show that

$$\sigma^* : \text{“}\lim_{t\mapsto \pi t}\text{”}\, K_0(A\langle t\rangle) \to K_0(A)$$

is a pro-monomorphism. According to Proposition 26 we find $n > 0$ such that $K_0(\mathcal{X}') \to K_0(\mathcal{X}'_n)$ is injective. So by a diagram chase it suffices to show that

$$\sigma : \text{“}\lim_{t\mapsto \pi t}\text{”}\, K_0(\mathcal{X}'_n) \to K_0(\mathcal{X}_n)$$

is a pro-monomorphism, which is clear as the morphism $\mathcal{X}'_n \xrightarrow{t\mapsto \pi^n t} \mathcal{X}'_n$ factors through $\mathcal{X}_n$. $\square$

References

[1] Bartenwerfer, W. *Die erste “metrische” Kohomologiegruppe glatter affinoider Räume*, Nederl. Akad. Wetensch. Proc. Ser. A 40 (1978), no. 1, 1–14.

[2] Bartenwerfer, W. *Die höheren metrischen Kohomologiegruppen affinoider Räume*, Math. Ann. 241 (1979), no. 1, 11–34.

[3] Bartenwerfer, W. *Holomorphe Vektorraumbündel auf offenen Polyzylindern*, J. Reine Angew. Math. 326 (1981), 214–220.

[4] Bartenwerfer, W. *Die strengen metrischen Kohomologiegruppen des Einheitspolyzylinders verschwinden*, Nederl. Akad. Wetensch. Indag. Math. 44 (1982), no. 1, 101–106.

[5] Bosch, S. *Lectures on formal and rigid geometry*, Lecture Notes in Mathematics, 2105. Springer, 2014.

[6] Bourbaki, N. *Commutative algebra*, Elements of Mathematics, Springer-Verlag, Berlin, 1989.

[7] Bourbaki, N. *Topological vector spaces*, Elements of Mathematics, Springer-Verlag, Berlin, 1987.

[8] de Jong, J., van der Put, M. *Etale cohomology of rigid analytic spaces*, Doc. Math. 1 (1996), No. 01, 1–56

[9] Gerritzen, L. *Zerlegungen der Picard-Gruppe nichtarchimedischer holomorpher Räume*, Compositio Math. 35 (1977), no. 1, 23–38.

[10] Gruson, L. *Fibrés vectoriels sur un polydisque ultramétrique*, Ann. Sci. Ecole Norm. Sup. (4) 1 (1968), 45–89.

[11] *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents* Séminaire à l'Ecole Polytechnique 2006–2008, edited by Luc Illusie, Yves Laszlo and Fabrice Orgogozo, Astérisque No. 363–364 (2014) SMF, Paris.

[12] Kiehl, R. *Die de Rham Kohomologie algebraischer Mannigfaltigkeiten über einem bewerteten Körper*, Inst. Hautes Etudes Sci. Publ. Math. No. 33 (1967), 5–20.

[13] Lindel, H. *On the Bass–Quillen conjecture concerning projective modules over polynomial rings*, Invent. Math. 65 (1981/82), no. 2, 319–323.

[14] Lütkebohmert, W. *Vektorraumbündel über nichtarchimedischen holomorphen Räumen*, Math. Z. 152 (1977), no. 2, 127–143.

[15] Morrow, M. *Pro cdh-descent for cyclic homology and K-theory*, J. Inst. Math. Jussieu 15 (2016), no. 3, 539–567.

[16] Temkin, M. *Desingularization of quasi-excellent schemes in characteristic zero*, Adv. Math. 219 (2008), no. 2, 488–522.

[17] van der Put, M. *Cohomology on affinoid spaces*, Compositio Math. 45 (1982), no. 2, 165–198.

[18] Weibel, C. *The K-book*, Graduate Studies in Mathematics 145, AMS, Providence, RI, 2013.

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