MOTIVES WITH MODULUS, II: MODULUS SHEAVES WITH TRANSFERS FOR PROPER MODULUS PAIRS

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ABSTRACT. We develop a theory of sheaves and cohomology on the category of proper modulus pairs. This complements [4], where a theory of sheaves and cohomology on the category of non-proper modulus pairs has been developed.

CONTENTS

INTRODUCTION

This is a sequel to [4], where a theory of sheaves and cohomology on the category $\underline{\mathbf{M}}\mathbf{Cor}$ of non-proper modulus pairs has been developed. This paper complements it by using work from [2] and [10] to develop a theory of sheaves and cohomology on the category **MCor** of proper

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modulus pairs. This completes the repairs to the mistake in [3]. The basic aim of both works is to lay a foundation for a theory of *motives with modulus*, to be completed in [5], generalizing Voevodsky's theory of motives in order to capture non $\mathbf{A}^1$-invariant phenomena.

In [4], Voevodsky's category **Cor** of finite correspondences on smooth separated schemes of finite type over a fixed base field k, was enlarged to the larger category of (non-proper) *modulus pairs*, $\underline{\mathbf{M}}\mathbf{Cor}$: Objects are pairs $M = (\overline{M}, M^\infty)$ consisting of a separated k-scheme of finite type $\overline{M}$ and an effective (possibly empty) Cartier divisor M^∞ on it such that the complement $M^\circ := \overline{M} \setminus M^\infty$ is in **Sm** (we call it the *smooth interior*). The group $\underline{\mathbf{M}}\mathbf{Cor}(M, N)$ of morphisms is defined as the subgroup of $\mathbf{Cor}(M^\circ, N^\circ)$ consisting of finite correspondences between smooth interiors whose closures in $\overline{M} \times_k \overline{N}$ are proper[1] over $\overline{M}$ and satisfy certain admissibility conditions with respect to M^∞ and N^∞. Let $\mathbf{MCor} \subset \underline{\mathbf{M}}\mathbf{Cor}$ be the full subcategory consisting of objects $(\overline{M}, M^\infty)$ with $\overline{M}$ proper over k.

We then define $\underline{\mathbf{M}}\mathbf{PST}$ (resp. **MPST**) as the category of additive presheaves of abelian groups on $\underline{\mathbf{M}}\mathbf{Cor}$ (resp. **MCor**). We have a pair of adjunctions

$$\mathbf{MPST} \underset{\xrightarrow{\tau_!}}{\xleftarrow{\tau^*}} \underline{\mathbf{M}}\mathbf{PST}, \tag{0.0.1}$$

where τ^* is induced by the inclusion $\tau : \mathbf{MCor} \to \underline{\mathbf{M}}\mathbf{Cor}$ and $\tau_!$ is its left Kan extension (see Lemma 1.2.3).

The main aim of [4] was to develop a *sheaf theory* on $\underline{\mathbf{M}}\mathbf{Cor}$ generalizing Voevodsky's theory of sheaves on **Cor**.

Definition 1. We define $\underline{\mathbf{M}}\mathbf{NST}$ to be the full subcategory of $\underline{\mathbf{M}}\mathbf{PST}$ of such objects F that F_M is a Nisnevich sheaf on $\overline{M}$ for every $M = (\overline{M}, M^\infty) \in \underline{\mathbf{M}}\mathbf{Cor}$, where F_M is the presheaf on $\overline{M}_{\mathrm{Nis}}$ which associates $F(U, M^\infty \times_{\overline{M}} U)$ to an étale map $U \to \overline{M}$.

Definition 2. Let $\underline{\Sigma}^{\mathrm{fin}}$ be the subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$ which have the same objects as $\underline{\mathbf{M}}\mathbf{Cor}$ and such that a morphism $f \in \underline{\mathbf{M}}\mathbf{Cor}(M, N)$ belongs to $\underline{\Sigma}^{\mathrm{fin}}$ if and only if $f^o \in \mathbf{Cor}(M^o, N^o)$ is the graph of an isomorphism $M^o \simeq N^o$ in **Sm** that extends to a proper morphism $\overline{f} : \overline{M} \to \overline{N}$ of k-schemes such that $M^\infty = \overline{f}^* N^\infty$.

Now the main result of [4] is the following.

Theorem 1 ([4, Th. 2])**.** *The following assertions hold.*

[1]Here we stress that we do not assume it is finite over $\overline{M}$.

(1) *The inclusion* $\underline{\mathbf{M}}\mathbf{NST} \to \underline{\mathbf{M}}\mathbf{PST}$ *has an exact left adjoint* $\underline{a}_{\mathrm{Nis}}$ *such that*

$$(\underline{a}_{\mathrm{Nis}}F)(M) = \varinjlim_{N \in \underline{\Sigma}^{\mathrm{fin}} \downarrow M} (F_N)_{\mathrm{Nis}}(N)$$

for every $F \in \underline{\mathbf{M}}\mathbf{PST}$ *and* $M \in \underline{\mathbf{M}}\mathbf{Cor}$, *where* $(F_M)_{\mathrm{Nis}}$ *is the Nisnevich sheafification of the preshseaf* F_M *on* $\overline{M}_{\mathrm{Nis}}$. *In particular* $\underline{\mathbf{M}}\mathbf{NST}$ *is a Grothendieck abelian category.*

(2) *For* $M \in \underline{\mathbf{M}}\mathbf{Cor}$, *let* $\mathbb{Z}_{\mathrm{tr}}(M) = \underline{\mathbf{M}}\mathbf{Cor}(-, M) \in \underline{\mathbf{M}}\mathbf{PST}$ *be the associated representable presheaf. Then we have* $\mathbb{Z}_{\mathrm{tr}}(M) \in \underline{\mathbf{M}}\mathbf{NST}$ *and there is a canonical isomorphism for any* $i \geq 0$ *and* $F \in \underline{\mathbf{M}}\mathbf{NST}$*:*

$$\mathrm{Ext}^i_{\underline{\mathbf{M}}\mathbf{NST}}(\mathbb{Z}_{\mathrm{tr}}(M), F) \simeq \varinjlim_{N \in \underline{\Sigma}^{\mathrm{fin}} \downarrow M} H^i_{\mathrm{Nis}}(\overline{N}, F_N).$$

The aim of the present paper is to introduce a sheaf theory on $\mathbf{MCor}$.

Definition 3. We define $\mathbf{MNST}$ to be the full subcategory of $\mathbf{MPST}$ of such objects F that $\tau_! F \in \underline{\mathbf{M}}\mathbf{NST}$.

Note that by definition, $\tau_! : \mathbf{MPST} \to \underline{\mathbf{M}}\mathbf{PST}$ induces a functor

$$\tau_{\mathrm{Nis}} : \mathbf{MNST} \to \underline{\mathbf{M}}\mathbf{NST}\,.$$

Now the main result of this paper is the following.

Theorem 2 (see Lemmas 4.2.2 and 4.2.5, Theorems 4.2.4, 5.1.1 and 5.1.3)**.**

(1) *We have* $\tau^*(\underline{\mathbf{M}}\mathbf{NST}) \subset \mathbf{MNST}$. *Letting*

$$\tau^{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NST} \to \mathbf{MNST}$$

be the induced functor, the pair of adjoint functors $(*)$ *induces a pair of adjoint functors*

$$\mathbf{MNST} \underset{\xrightarrow{\tau_{\mathrm{Nis}}}}{\overset{\tau^{\mathrm{Nis}}}{\longleftarrow}} \underline{\mathbf{M}}\mathbf{NST},$$

where τ_{Nis} *is exact and fully faithful. The functor* τ^{Nis} *is also exact.*

(2) *The inclusion* $\mathbf{MNST} \to \mathbf{MPST}$ *has an exact left adjoint* a_{Nis} *such that* $\underline{a}_{\mathrm{Nis}}\tau_! = \tau_{\mathrm{Nis}} a_{\mathrm{Nis}}$. *In particular,* $\mathbf{MNST}$ *is a Grothendieck abelian category.*

(3) *For* $M \in \mathbf{MCor}$, *let* $\mathbb{Z}_{\mathrm{tr}}(M) = \mathbf{MCor}(-, M) \in \mathbf{MPST}$ *be the associated representable presheaf. Then we have* $\mathbb{Z}_{\mathrm{tr}}(M) \in$

MNST *and there is a canonical isomorphism for any* $i \geq 0$ *and* $F \in \mathbf{MNST}$*:*

$$\mathrm{Ext}^i_{\mathbf{MNST}}(\mathbb{Z}_{\mathrm{tr}}(M), F) \simeq \varinjlim_{N \in \Sigma^{\mathrm{fin}} \downarrow M} H^i_{\mathrm{Nis}}(\overline{N}, (\tau_{\mathrm{Nis}} F)_N),$$

where $\Sigma^{\mathrm{fin}} := \underline{\Sigma}^{\mathrm{fin}} \cap \mathbf{MCor}$.

Finally we explain relations between cohomologies for **MNST** and **NST**. We denote by **PST** (resp. **NST**) Voevodsky's category of presheaves (resp. Nisnevich sheaves) with transfers. The functor $\omega : \mathbf{MCor} \to \mathbf{Cor}$, $\omega(\overline{M}, M^\infty) = \overline{M} \setminus |M^\infty|$ induces a pair of adjunctions

$$\mathbf{MPST} \underset{\omega_!}{\overset{\omega^*}{\underset{\longrightarrow}{\longleftarrow}}} \mathbf{PST},$$

in a similar way as (0.0.1). (See §6.2 for details.)

Theorem 3 (see Proposition 6.2.1 and Theorem 6.3.2)**.** *The following assertions hold.*

(1) *We have*

$$\omega_!(\mathbf{MNST}) \subset \mathbf{NST} \quad \text{ and } \quad \omega^*(\mathbf{NST}) \subset \mathbf{MNST}.$$

The functors $\omega_!$ *and* ω^* *induce a pair of adjoint functors*

$$\mathbf{MNST} \underset{\omega_{\mathrm{Nis}}}{\overset{\omega^{\mathrm{Nis}}}{\underset{\longrightarrow}{\longleftarrow}}} \mathbf{NST},$$

such that

$$\omega_{\mathrm{Nis}} a_{\mathrm{Nis}} = a^V_{\mathrm{Nis}} \omega_!, \quad \omega^{\mathrm{Nis}} a^V_{\mathrm{Nis}} = a_{\mathrm{Nis}} \omega^*,$$

where $a^V_{\mathrm{Nis}} : \mathbf{PST} \to \mathbf{NST}$ *is Voevodsky's sheafification functor. Moreover,* ω^{Nis} *and* ω_{Nis} *are exact and* ω^{Nis} *is fully faithful.*

(2) *For any* $F \in \mathbf{MNST}$ *and* $X \in \mathbf{Sm}$ *and* $i \geq 0$*, we have a canonical isomorphism*

$$H^i_{\mathrm{Nis}}(X, \omega_{\mathrm{Nis}} F) \simeq \varinjlim_{M \in \mathbf{MSm}(X)} H^i_{\mathrm{Nis}}(\overline{M}, F_M)$$

where $\mathbf{MSm}(X) = \{M \in \mathbf{MCor} \mid M^o = X\}$ *is viewed as a cofiltered ordered set* [4, Lemma 1.7.4].

A key ingredient of the proofs is Theorem 3.2.2, which is based on the works [10] and [2].

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Notation and conventions. In the whole paper we fix a base field k. Let **Sch** be the category of separated schemes of finite type over k, and let **Sm** be its full subcategory of smooth schemes. We write **Cor** for Voevodsky's category of finite correspondences [14].

An additive functor between additive categories is called *strongly additive* if it commutes with all representable direct sums. A Grothendieck topology is called *subcanonical* if every representable presheaf is a sheaf.

Let $\mathcal{C}$ and $\mathcal{D}$ be sites, and $u : \mathcal{C} \to \mathcal{D}$ a functor. We say that u is *continuous* (resp. *cocontinuous*) if the functor $u^* : \hat{\mathcal{D}} \to \hat{\mathcal{C}}$ (resp. $u_* : \hat{\mathcal{C}} \to \hat{\mathcal{D}}$) between categories of presheaves carries sheaves to sheaves. (cf. [SGA4, III] and [2, §A.1].)

1. Review of presheaf theory on modulus pairs

1.1. **Categories of modulus pairs.** A *modulus pair* M consists of $\overline{M} \in \mathbf{Sch}$ and an effective Cartier divisor $M^\infty \subset \overline{M}$ such that the open subset $M^o := \overline{M} - |M^\infty|$ is smooth over k. (The case $|M^\infty| = \emptyset$ is allowed.) We say that M is *proper* if $\overline{M}$ is. Note that $\overline{M}$ is automatically reduced, and M^o is dense in $\overline{M}$ [4, Rem. 1.1.2 (3)].

Let M_1, M_2 be modulus pairs. Let $Z \in \mathbf{Cor}(M_1^o, M_2^o)$ be an elementary (= integral) finite correspondence in the sense of Voevodsky [14]. We write $\overline{Z}^N$ for the normalization of the closure $\overline{Z}$ of Z in $\overline{M}_1 \times \overline{M}_2$ and $p_i : \overline{Z}^N \to \overline{M}_i$ for the canonical morphisms for $i = 1, 2$. We say Z is *admissible* (resp. *left-proper*) for (M_1, M_2) if $p_1^* M_1^\infty \geq p_2^* M_2^\infty$ (resp. $\overline{Z}$ is proper over $\overline{M}_1$).

By [4, Prop.1.2.3 and 1.2.6], modulus pairs and left proper admissible correspondences define an additive category that we denote by $\underline{\mathbf{M}}\mathbf{Cor}$. We write $\mathbf{MCor}$ for the full subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$ whose objects are proper modulus pairs.

We write $\underline{\mathbf{M}}\mathbf{Sm}$ for the category with same objects as $\underline{\mathbf{M}}\mathbf{Cor}$ a morphism of $\underline{\mathbf{M}}\mathbf{Sm}(M_1, M_2)$ being a (scheme-theoretic) k-morphism

$f^o : M_1^o \to M_2^o$ whose graph belongs to $\underline{\mathbf{M}}\mathbf{Cor}(M_1, M_2)$. We write $\mathbf{MSm}$ for the full subcategory of $\underline{\mathbf{M}}\mathbf{Sm}$ whose objects are proper modulus pairs.

We write $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$ for the subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$ with the same objects and the following condition on morphisms: $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}(M, N)$ belongs to $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(M, N)$ if and only if, for any component Z of α, the projection $\overline{Z} \to \overline{M}$ is *finite*, where $\overline{Z}$ is the closure of Z in $\overline{M} \times \overline{N}$. We write $\mathbf{MCor}^{\mathrm{fin}}$ for the full subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$ whose objects are proper modulus pairs.

We write $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ for the subcategory of $\underline{\mathbf{M}}\mathbf{Sm}$ with the same objects and such that a morphism $f : M \to N$ belongs to $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ if and only if $f^o : M^o \to N^o$ extends to a k-morphism $\overline{f} : \overline{M} \to \overline{N}$. We write $\mathbf{MSm}^{\mathrm{fin}}$ for the full subcategory of $\underline{\mathbf{M}}\mathbf{Sm}$ whose objects are proper modulus pairs. A morphism $f : M \to N$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ is *minimal* if we have $f^*N^\infty = M^\infty$.

Remarks 1.1.1.

(1) For $M \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, set $M^N := (\overline{M}^N, M^\infty|_{\overline{M}^N})$ where $p : \overline{M}^N \to \overline{M}$ is the normalization and $M^\infty|_{\overline{M}^N}$ is the pull-back of M^∞ to $\overline{M}^N$. Then $p : M^N \to M$ is an isomorphism in $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$ and $\underline{\mathbf{M}}\mathbf{Sm}$ (but not in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ in general).

(2) Let $f : M \to N$ be a morphism in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. The reducedness of $\overline{M}$, the separatedness of $\overline{N}$ and the denseness of M^o in $\overline{M}$ imply that this extension $\overline{f}$ is unique. This yields a forgetful functor $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}} \to \mathbf{Sch}$, which sends M to $\overline{M}$.

We have the following commutative diagram of inclusion functors

$$\begin{array}{ccccc} \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}} & \xrightarrow{\underline{b}_s} & \underline{\mathbf{M}}\mathbf{Sm} & \xleftarrow{\tau_s} & \mathbf{MSm} \\ \downarrow{\scriptstyle \underline{c}^{\mathrm{fin}}} & & \downarrow{\scriptstyle \underline{c}} & & \downarrow{\scriptstyle c} \\ \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}} & \xrightarrow{\underline{b}} & \underline{\mathbf{M}}\mathbf{Cor} & \xleftarrow{\tau} & \mathbf{MCor} \end{array} \tag{1.1.1}$$

1.2. **Presheaves.**

Definition 1.2.1. By a presheaf we mean here an additive contravariant functor to the category of abelian groups. (A functor is called additive if it commutes with finite coproducts.)

(1) The category of presheaves on $\mathbf{MSm}$ (resp. $\underline{\mathbf{M}}\mathbf{Sm}$, $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$) is denoted by $\mathbf{MPS}$ (resp. $\underline{\mathbf{M}}\mathbf{PS}$, $\underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}$).

(2) The category of presheaves on $\mathbf{MCor}$ (resp. $\underline{\mathbf{M}}\mathbf{Cor}$, $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}$) is denoted by $\mathbf{MPST}$ (resp. $\underline{\mathbf{M}}\mathbf{PST}$, $\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$.)

(3) We write

$$\mathbb{Z}_{\mathrm{tr}} : \underline{\mathbf{M}}\mathbf{Cor} \to \underline{\mathbf{M}}\mathbf{PST}, \quad \mathbf{MCor} \to \mathbf{MPST},$$
$$\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}} : \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}},$$

for the associated representable presheaf functors.

(4) For $M \in \underline{\mathbf{M}}\mathbf{Sm}$, we denote by $\mathbb{Z}^p(M)$ the presheaf with values in abelian groups defined by

$$\underline{\mathbf{M}}\mathbf{Sm} \ni N \mapsto \mathbb{Z}\,\underline{\mathbf{M}}\mathbf{Sm}(N, M),$$

where for any set S we denote by $\mathbb{Z}S$ the free abelian group on S.

Diagram (1.1.1) induces a commutative diagram of functors on presheaf categories:

$$\begin{array}{ccccc} \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}} & \xleftarrow{\underline{b}_s^*} & \underline{\mathbf{M}}\mathbf{PS} & \xrightarrow{\tau_s^*} & \mathbf{MPS} \\ \uparrow{\scriptstyle \underline{c}^{\mathrm{fin}*}} & & \uparrow{\scriptstyle \underline{c}^*} & & \uparrow{\scriptstyle c^*} \\ \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}} & \xleftarrow{\underline{b}^*} & \underline{\mathbf{M}}\mathbf{PST} & \xrightarrow{\tau^*} & \mathbf{MPST} \end{array} \tag{1.2.1}$$

Lemma-Definition 1.2.2 ([4, Def. 1.8.1 and Lemma 1.8.2]). *For $M = (\overline{M}, M^\infty) \in \underline{\mathbf{M}}\mathbf{Sm}$, we denote by $\mathbf{Comp}(M)$ the category whose objects are morphisms $M \xrightarrow{j_N} N$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ such that*

(i) $N \in \mathbf{MSm}$;
(ii) $\overline{j_N} : \overline{M} \to \overline{N}$ *is a dense open immersion;*
(iii) j_N *is minimal, i.e.,* $M^\infty = \overline{j_N}^* N^\infty$;
(iv) *we have* $N^\infty = M_N^\infty + C$ *for some effective Cartier divisors* M_N^∞, C *on* $\overline{N}$ *satisfying* $\overline{N} \setminus |C| = j(\overline{M})$. *(Note that, therefore,* $M^\infty = \overline{j_N}^* M_N^\infty$*.)*

For $N_1, N_2 \in \mathbf{Comp}(M)$ we define

$$\mathbf{Comp}(M)(N_1, N_2) = \{\gamma \in \mathbf{MSm}(N_1, N_2) \mid \gamma \circ j_{N_1} = j_{N_2}\}.$$

The category $\mathbf{Comp}(M)$ is nonempty, ordered and cofiltered.

Lemma 1.2.3 ([4, Prop. 2.4.1 and Lemma 2.4.2]).

(1) *The functor $\tau : \mathbf{MCor} \to \underline{\mathbf{M}}\mathbf{Cor}$ of (1.1.1) yields a string of 3 adjoint functors $(\tau_!, \tau^*, \tau_*)$:*

$$\mathbf{MPST} \;\underset{\xrightarrow{\tau_*}}{\overset{\xrightarrow{\tau_!}}{\xleftarrow{\tau^*}}}\; \underline{\mathbf{M}}\mathbf{PST},$$

where $\tau_!, \tau_*$ are fully faithful, τ^* is a localisation and the adjunction map $\mathrm{Id} \to \tau^*\tau_!$ is an isomorphism. The functors $\tau_!$

and τ^* commute with all colimits and $\tau_!$ has a pro-left adjoint represented by **Comp**, *hence is exact.*

(2) *The same statements as (1) hold for the functor $\tau_s : \mathbf{MSm} \to \underline{\mathbf{M}}\mathbf{Sm}$.*

(3) *For $G \in \mathbf{MPST}$ and $M \in \underline{\mathbf{M}}\mathbf{Sm}$, we have*

$$\tau_! G(M) \simeq \varinjlim_{N \in \mathbf{Comp}(M)} G(N).$$

(4) *For $G \in \mathbf{MPS}$ and $M \in \underline{\mathbf{M}}\mathbf{Sm}$, we have*

$$\tau_{s!} G(M) \simeq \varinjlim_{N \in \mathbf{Comp}(M)} G(N).$$

Lemma 1.2.4 ([4, Prop. 2.5.1])**.** *Let $\underline{b} : \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{Cor}$ be the inclusion functor from (1.1.1). Then $\underline{b}$ yields a string of 3 adjoint functors $(\underline{b}_!, \underline{b}^*, \underline{b}_*)$:*

$$\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}} \begin{array}{c} \xrightarrow{\underline{b}_!} \\ \xleftarrow{\underline{b}^*} \\ \xrightarrow{\underline{b}_*} \end{array} \underline{\mathbf{M}}\mathbf{PST},$$

where $\underline{b}_!, \underline{b}_*$ are localisations; $\underline{b}^*$ is exact and fully faithful; $\underline{b}_!$ has a pro-left adjoint, hence is exact. For $F \in \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ and $M \in \underline{\mathbf{M}}\mathbf{Cor}$, we have (see Definition 2)

$$\underline{b}_! F(M) = \varinjlim_{N \in \underline{\Sigma}^{\mathrm{fin}} \downarrow M} F(N). \tag{1.2.2}$$

The same statements hold for $\underline{b}_s$ from (1.1.1).

Lemma 1.2.5 ([4, Prop. 2.6.1])**.** *Let $\underline{c} : \underline{\mathbf{M}}\mathbf{Sm} \to \underline{\mathbf{M}}\mathbf{Cor}$ be the functor from (1.1.1). Then $\underline{c}$ yields a string of 3 adjoint functors $(\underline{c}_!, \underline{c}^*, \underline{c}_*)$:*

$$\underline{\mathbf{M}}\mathbf{PS} \begin{array}{c} \xrightarrow{\underline{c}_!} \\ \xleftarrow{\underline{c}^*} \\ \xrightarrow{\underline{c}_*} \end{array} \underline{\mathbf{M}}\mathbf{PST},$$

where $\underline{c}^*$ is exact and faithful (but not full). The same statements hold for $\underline{c}^{\mathrm{fin}}$ and c from (1.1.1). We have

$$c^* \tau^* = \tau_s^* \underline{c}^*, \quad \underline{c}^* \tau_! = \tau_{s!} c^*, \tag{1.2.3}$$

$$\underline{c}^{\mathrm{fin}*} \underline{b}^* = \underline{b}_s^* \underline{c}^*, \quad \underline{b}_! \underline{c}_!^{\mathrm{fin}} = \underline{c}_! \underline{b}_{s!}, \quad \underline{c}^* \underline{b}_! = \underline{b}_{s!} \underline{c}^{\mathrm{fin}*}. \tag{1.2.4}$$

For the readers' convenience, we recall the following lemma from [4, Lemma A.8.1]:

Lemma 1.2.6. *Let $\mathcal{C}, \mathcal{D}$ be abelian categories and let $\mathcal{C}' \subset \mathcal{C}, \mathcal{D}' \subset \mathcal{D}$ be full abelian subcategories. Let $c : \mathcal{C} \to \mathcal{D}$ and $c' : \mathcal{C}' \to \mathcal{D}'$ be additive functors satisfying $ci_C = i_D c'$, where $i_C : \mathcal{C}' \to \mathcal{C}$ and $i_D : \mathcal{D}' \to \mathcal{D}$ are the inclusion functors.*

(1) *If c is faithful, so is c'.*
(2) *Suppose that i_D is strongly additive or has a strongly additive left inverse (for example, a left adjoint). If c and i_C are strongly additive, so is c'.*
(3) *Suppose that i_C has a left adjoint a_C. If c has a left adjoint d, then $d' = a_C d i_D$ is a left adjoint of c'. If d and $a_{\mathcal{C}}$ are exact, so is d'. Moreover, $a_{\mathcal{C}} d = d' a_{\mathcal{D}}$ if $i_{\mathcal{D}}$ has a left adjoint $a_{\mathcal{D}}$.*
(4) *Suppose that i_C and i_D have left adjoints a_C and a_D, that a_D is exact, and that $a_D c = c' a_C$. If c is exact, then so is c'.*

2. Review of sheaf theory on non-proper modulus pairs

In this section we recall some basic definitions and properties on sheaves on categories of non-proper modulus pairs from [4, §4.1 and §4.2].

Let **Sq** be the product category of $[0] = \{0 \to 1\}$ with itself, depicted as

$$\begin{array}{ccc} 00 & \longrightarrow & 01 \\ \downarrow & & \downarrow \\ 10 & \longrightarrow & 11. \end{array}$$

For any category $\mathcal{C}$, denote by $\mathcal{C}^{\mathbf{Sq}}$ for the category of functors from **Sq** to $\mathcal{C}$. A functor $f : \mathcal{C} \to \mathcal{C}'$ induces a functor $f^{\mathbf{Sq}} : \mathcal{C}^{\mathbf{Sq}} \to \mathcal{C}'^{\mathbf{Sq}}$.

Let $Q \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$. We write $Q(ij) = (\overline{Q}(ij), Q^\infty(ij))$ for all $i, j \in \{0, 1\}$. We also write $Q^o(ij) := \overline{Q}(ij) - |Q^\infty(ij)|$.

2.1. The $\underline{\mathrm{MV}}^{\mathrm{fin}}$ cd-structure.

Definition 2.1.1.

(1) A Cartesian square

$$\begin{array}{ccc} W & \xrightarrow{v} & V \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{u} & X \end{array} \tag{2.1.1}$$

in **Sch** is called an *elementary Nisnevich square* if u is an open embedding, p is étale and $p^{-1}(X \setminus U)_{\mathrm{red}} \to (X \setminus U)_{\mathrm{red}}$ is an isomorphism. In this situation, we say $U \sqcup V \to X$ is an *elementary Nisnevich cover.* Recall that an additive presheaf is a Nisnevich sheaf if and only if it transforms any elementary Nisnevich square into a cartesian square [15, Cor. 2.17], [16, Thm. 2.2].

(2) A diagram (2.1.1) in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ is called an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-*square* if it becomes an elementary Nisnevich square (in **Sch**) after replacing X, U, V, W by $\overline{X}, \overline{U}, \overline{V}, \overline{W}$ (cf. Remark 1.1.1(2)) and all morphisms are minimal.

Proposition 2.1.2 ([4, Prop. 3.2.3 (2)]). *The cd-structure $P_{\underline{\mathrm{MV}}^{\mathrm{fin}}}$ on $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ consisting of $\underline{\mathrm{MV}}^{\mathrm{fin}}$-squares is strongly complete and strongly regular, hence complete and regular in the sense of* [15] *(see* [4, Def. A.7.1 and A.7.4]*).*

We define $\underline{\mathbf{M}}\mathbf{NS}^{\mathrm{fin}}$ to be the full subcategory of $\underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}$ consisting of sheaves with respect to the Grothendieck topology associated to $P_{\underline{\mathrm{MV}}^{\mathrm{fin}}}$. Let

$$\underline{a}^{\mathrm{fin}}_{s,\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{NS}^{\mathrm{fin}} \tag{2.1.2}$$

be the sheafification functor, that is, the left adjoint of the inclusion functor $\underline{i}^{\mathrm{fin}}_{s,\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NS}^{\mathrm{fin}} \hookrightarrow \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}$. It exists for general reasons and is exact [SGA4, II.3.4].

Lemma 2.1.3 ([4, Lemma 3.1.3]). *Let $M \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. Let M_{Nis} be the category of morphisms $f : N \to M$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ such that $\overline{f}$ is étale and $\overline{f}^* M^\infty = N^\infty$, endowed with the topology induced by $P_{\underline{\mathrm{MV}}^{\mathrm{fin}}}$, and let $(\overline{M})_{\mathrm{Nis}}$ be the (usual) small Nisnevich site on $\overline{M}$. Then we have an isomorphism of sites*

$$M_{\mathrm{Nis}} \to (\overline{M})_{\mathrm{Nis}}, \qquad N \mapsto \overline{N},$$

whose inverse is given by $(p : X \to \overline{M}) \mapsto (X, p^*(M^\infty))$. (This isomorphism of sites depends on the choice of M^∞.) □

Notation 2.1.4. Let $M = (\overline{M}, M^\infty) \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ and $F \in \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}$. We write F_M for the presheaf on $(\overline{M})_{\text{ét}}$ which associates $F(U, U \times_{\overline{M}} M^\infty)$ to an étale map $U \to \overline{M}$. By Lemma 2.1.3, $F \in \underline{\mathbf{M}}\mathbf{NS}^{\mathrm{fin}}$ if and only if F_M is a sheaf on $(\overline{M})_{\mathrm{Nis}}$ for every $M \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$.

Let $\underline{\mathbf{M}}\mathbf{NST}^{\mathrm{fin}}$ be the full subcategory of $\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ consisting of all objects $F \in \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ such that $\underline{c}^{\mathrm{fin}*} F \in \underline{\mathbf{M}}\mathbf{NS}^{\mathrm{fin}}$, where $\underline{c}^{\mathrm{fin}*} : \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{PS}^{\mathrm{fin}}$ is from (1.2.1).

We write $\underline{i}^{\mathrm{fin}}_{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NST}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ for the inclusion functor and $\underline{c}^{\mathrm{finNis}} : \underline{\mathbf{M}}\mathbf{NST} \to \underline{\mathbf{M}}\mathbf{NS}$ for the functor induced by $\underline{c}^{\mathrm{fin}*}$. By definition, we have

$$\underline{c}^{\mathrm{fin}*} \underline{i}^{\mathrm{fin}}_{\mathrm{Nis}} = \underline{i}^{\mathrm{fin}}_{s,\mathrm{Nis}} \underline{c}^{\mathrm{finNis}}. \tag{2.1.3}$$

Theorem 2.1.5 ([4, Th. 3.5.3])**.** *The functor $\underline{i}_{\mathrm{Nis}}^{\mathrm{fin}}$ has an exact left adjoint*

$$\underline{a}_{\mathrm{Nis}}^{\mathrm{fin}} : \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{NST}^{\mathrm{fin}}$$

satisfying

$$\underline{c}^{\mathrm{finNis}} \underline{a}_{\mathrm{Nis}}^{\mathrm{fin}} = \underline{a}_{s\mathrm{Nis}}^{\mathrm{fin}} \underline{c}^{\mathrm{fin}*}. \tag{2.1.4}$$

In particular $\underline{\mathbf{M}}\mathbf{NST}^{\mathrm{fin}}$ is Grothendieck. Moreover, $\underline{\mathbf{M}}\mathbf{NST}^{\mathrm{fin}}$ is closed under infinite direct sums in $\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ and the inclusion functor $\underline{i}_{\mathrm{Nis}}^{\mathrm{fin}} : \underline{\mathbf{M}}\mathbf{NST}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$ is strongly additive.

2.2. The $\underline{\mathrm{MV}}$ cd-structure.

Definition 2.2.1 ([4, Def. 4.1.1])**.** A diagram in $\underline{\mathbf{M}}\mathbf{Sm}$ is called an $\underline{\mathrm{MV}}$*-square* if it is isomorphic in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$ to $\underline{b}_s^{\mathbf{Sq}}(Q)$ for some $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square Q in Definition 2.1.1, where $\underline{b}_s^{\mathbf{Sq}}$ is the functor induced on squares by $\underline{b}_s : \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{Sm}$ from (1.1.1). Let $P_{\underline{\mathrm{MV}}}$ be a cd-structure on $\underline{\mathbf{M}}\mathbf{Sm}$ given by the collection of $\underline{\mathrm{MV}}$-squares.

Theorem 2.2.2 ([4, Th. 4.1.2])**.** *The cd-structure $P_{\underline{\mathrm{MV}}}$ is strongly complete and strongly regular, in particular complete and regular (see* [4, Def. A.7.1 and A.7.4]*).*

Remark 2.2.3. In view of Lemma 2.1.3, the topology defined by $P_{\underline{\mathrm{MV}}^{\mathrm{fin}}}$ is subcanonical. This is also true for $P_{\underline{\mathrm{MV}}}$ by [10, Th. 4.5.1].

We write $\underline{\mathbf{M}}\mathbf{NS}$ for the full subcategory of $\underline{\mathbf{M}}\mathbf{PS}$ consisting of sheaves with respect to the Grothendieck topology on $\underline{\mathbf{M}}\mathbf{Sm}$ associated to$P_{\underline{\mathrm{MV}}}$. We denote by $\underline{i}_{s,\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NS} \to \underline{\mathbf{M}}\mathbf{PS}$ the inclusion functor.

Lemma 2.2.4. *The functor $\underline{i}_{s,\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NS} \to \underline{\mathbf{M}}\mathbf{PS}$ has an exact left adjoint $\underline{a}_{s,\mathrm{Nis}}$. In particular $\underline{\mathbf{M}}\mathbf{NS}$ is Grothendieck. Moreover, the following conditions are equivalent for $F \in \underline{\mathbf{M}}\mathbf{PS}$.*

(i) *$F \in \underline{\mathbf{M}}\mathbf{NS}$.*

(ii) *It transforms any $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square*

$$Q: \begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & M \end{array} \tag{2.2.1}$$

into an exact sequence

$$0 \to F(M) \to F(U) \oplus F(V) \to F(W).$$

Proof. The first two assertions follow from the general properties of Grothendieck topologies [SGA4, Exp. II]. The equivalence (i) $\iff$ (ii) follows from [15, Cor. 2.17] in view of Theorem 2.2.2. ☐

Lemma-Definition 2.2.5 ([4, Lemma 4.5.1])**.** *For $F \in \underline{\mathbf{M}}\mathbf{PST}$, one has $\underline{c}^*F \in \underline{\mathbf{M}}\mathbf{NS}$ if and only if $\underline{b}^*F \in \underline{\mathbf{M}}\mathbf{NST}^{\mathrm{fin}}$, where*

$$\underline{b}^* : \underline{\mathbf{M}}\mathbf{PST} \to \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}, \quad \underline{c}^* : \underline{\mathbf{M}}\mathbf{PST} \to \underline{\mathbf{M}}\mathbf{PS}$$

are from (1.2.1).
We define $\underline{\mathbf{M}}\mathbf{NST}$ to be the full subcategory of $\underline{\mathbf{M}}\mathbf{PST}$ consisting of those F enjoying these equivalent conditions. We denote by $\underline{i}_{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NST} \to \underline{\mathbf{M}}\mathbf{PST}$ the inclusion functor, and by $\underline{b}^{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NST} \to \underline{\mathbf{M}}\mathbf{NST}^{\mathrm{fin}}$ the functor induced by $\underline{b}^*$.

Recall the functor $\underline{b}_! : \underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{PST}$ from Lemma 1.2.4.

Proposition 2.2.6 ([4, Prop. 4.5.4])**.** *We have $\underline{b}_!(\underline{\mathbf{M}}\mathbf{NST}^{\mathrm{fin}}) \subset \underline{\mathbf{M}}\mathbf{NST}$. Let $\underline{b}_{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NST}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{NST}$ be the restriction of $\underline{b}_!$ so that we have*

$$\underline{b}_! \underline{i}^{\mathrm{fin}}_{\mathrm{Nis}} = \underline{i}_{\mathrm{Nis}} \underline{b}_{\mathrm{Nis}}. \tag{2.2.2}$$

Then $\underline{b}_{\mathrm{Nis}}$ is an exact left adjoint of $\underline{b}^{\mathrm{Nis}}$, and is fully faithful.

Theorem 2.2.7 ([4, Lemma 4.5.3, Th. 4.5.5 and Prop. 4.5.6])**.** *The category $\underline{\mathbf{M}}\mathbf{NST}$ contains $\mathbb{Z}_{\mathrm{tr}}(M)$ for any $M \in \underline{\mathbf{M}}\mathbf{Cor}$. The inclusion functor $\underline{i}_{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NST} \to \underline{\mathbf{M}}\mathbf{PST}$ has an exact left adjoint*

$$\underline{a}_{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{PST} \to \underline{\mathbf{M}}\mathbf{NST}$$

given by $\underline{a}_{\mathrm{Nis}} = \underline{b}_{\mathrm{Nis}} \underline{a}^{\mathrm{fin}}_{\mathrm{Nis}} \underline{b}^*$. In particular, $\underline{\mathbf{M}}\mathbf{NST}$ is Grothendieck. Moreover, $\underline{\mathbf{M}}\mathbf{NST}$ is closed under infinite direct sums in $\underline{\mathbf{M}}\mathbf{PST}$, and $\underline{i}_{\mathrm{Nis}}$ is strongly additive. We have

$$\underline{b}_{\mathrm{Nis}} \underline{a}^{\mathrm{fin}}_{\mathrm{Nis}} = \underline{a}_{\mathrm{Nis}} \underline{b}_!, \qquad \underline{a}_{s,\mathrm{Nis}} \underline{c}^* = \underline{c}^{\mathrm{Nis}} \underline{a}_{\mathrm{Nis}}, \tag{2.2.3}$$

where $\underline{c}^{\mathrm{Nis}}$ is the functor determined by Lemma-Definition 2.2.5. This functor is exact, strongly additive and has a left adjoint $\underline{c}_{\mathrm{Nis}} = \underline{a}_{\mathrm{Nis}} \underline{c}_! \underline{i}_{s,\mathrm{Nis}}$.

Notation 2.2.8. Let $M \in \underline{\mathbf{M}}\mathbf{Cor}$ and $F \in \underline{\mathbf{M}}\mathbf{NST}$. Using Notation 2.1.4, we define $F_M := (\underline{b}^{\mathrm{Nis}} F)_M$, which is a sheaf on $(\overline{M})_{\mathrm{Nis}}$.

Proposition 2.2.9 ([4, Prop. 4.6.3])**.** *Let $F \in \underline{\mathbf{M}}\mathbf{NST}$, and let $M \in \underline{\mathbf{M}}\mathbf{Cor}$. Then there is a canonical isomorphism for any $i \geq 0$:*

$$\mathrm{Ext}^i_{\underline{\mathbf{M}}\mathbf{NST}}(\mathbb{Z}_{\mathrm{tr}}(M), F) \simeq \varinjlim_{N \in \underline{\Sigma}^{\mathrm{fin}} \downarrow M} H^i_{\mathrm{Nis}}(\overline{N}, F_N).$$

Moreover, we have

$$\varinjlim_{N \in \underline{\Sigma}^{\mathrm{fin}} \downarrow M} H^i_{\mathrm{Nis}}(\overline{N}, (R^q(\underline{b}^{\mathrm{Nis}}_s) \underline{c}^{\mathrm{Nis}} F)_N) = 0 \text{ for all } q > 0.$$

Corollary 2.2.10. *We have $\mathrm{Ext}^i_{\underline{\mathbf{M}}\mathbf{NST}}(\mathbb{Z}_{\mathrm{tr}}(M), F) = 0$ for $i > \dim \overline{M}$.*

Proof. For any $N \in \underline{\Sigma}^{\mathrm{fin}} \downarrow M$, we have $\dim \overline{N} = \dim \overline{M}$. Therefore the statement follows from Proposition 2.2.9 and the known bound for Nisnevich cohomological dimension. ☐

3. A cd-structure on $\mathbf{MSm}$

In this section we introduce a cd-structure on $\mathbf{MSm}$ and describe its main properties, following the works of Miyazaki [10] and Kahn-Miyazaki [2]. For this we need to start with the "off-diagonal" functor.

3.1. Off-diagonal.

Definition 3.1.1. Define $\mathbf{MEt}$ as the category such that

(1) objects are those morphisms $f : M \to N$ in $\mathbf{MSm}$ such that $f^{\mathrm{o}} : M^{\mathrm{o}} \to N^{\mathrm{o}}$ is étale, and
(2) a morphism from $f : M_1 \to N_1$ to $g : M_2 \to N_2$ is a pair of morphisms $(s : M_1 \to M_2, t : N_1 \to N_2)$ in $\mathbf{MSm}$ which commute with f, g and such that s^{o} and t^{o} are *open immersions*.

For modulus pairs M and N, we define *the disjoint union of* M *and* N by

$$M \sqcup N := (\overline{M} \sqcup \overline{N}, M^\infty \sqcup N^\infty).$$

Obviously, we have $(M \sqcup N)^{\mathrm{o}} = M^{\mathrm{o}} \sqcup N^{\mathrm{o}}$.

Theorem 3.1.2 ([10, Th. 3.1.3])**.** *There exists a functor*

$$\mathrm{OD} : \mathbf{MEt} \to \mathbf{MSm}$$

such that for any $f : M \to N$*, one has a functorial decomposition*

$$M \times_N M \cong M \sqcup \mathrm{OD}(f).$$

Moreover, we have $\mathrm{OD}(f)^{\mathrm{o}} = M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}} \setminus \Delta(M^{\mathrm{o}})$*, where* $\Delta : M^{\mathrm{o}} \to M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}}$ *is the diagonal morphism. In particular, if* f^{o} *is an open immersion, then* $\mathrm{OD}(f)^{\mathrm{o}} = \emptyset$*, hence* $\mathrm{OD}(f) = \emptyset$*. We call the functors* the off-diagonal functors.

3.2. The MV cd-structure.

Definition 3.2.1. Let T be an object of $\mathbf{MSm}^{\mathbf{Sq}}$ of the form

$$\begin{array}{ccc} T(00) & \xrightarrow{v_T} & T(01) \\ {\scriptstyle q_T}\downarrow & & \downarrow{\scriptstyle p_T} \\ T(10) & \xrightarrow{u_T} & T(11). \end{array} \tag{3.2.1}$$

Then T is called an MV*-square* if the following conditions hold:

(1) T is a pull-back square in $\mathbf{MSm}$.

(2) There exist an $\underline{\mathrm{MV}}$-square S (cf. Definition 2.2.1) such that $S(11) \in \mathbf{MSm}$, and a morphism $S \to T$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$ such that the induced morphism $S^{\mathrm{o}} \to T^{\mathrm{o}}$ is an isomorphism in $\mathbf{Sm}^{\mathbf{Sq}}$ and $S(11) \to T(11)$ is an isomorphism in $\mathbf{MSm}$. In particular, T^{o} is an elementary Nisnevich square.

(3) $\mathrm{OD}(q_T) \to \mathrm{OD}(p_T)$ is an isomorphism in $\mathbf{MSm}$.

We let P_{MV} be the cd-structure on $\mathbf{MSm}$ consisting of MV-squares.

The following are the main results of [10].

Theorem 3.2.2 ([10, Th. 4.3.1, 4.4.1 and 4.5.1])**.** *The cd-structure P_{MV} is complete and regular. The associated topology is subcanonical.*

Corollary 3.2.3. *Define $\mathbf{MNS} \subset \mathbf{MPS}$ as the full subcategory consisting of sheaves with respect to the Grothendieck topology on $\mathbf{MSm}$ associated with P_{MV}. Then there exists a pair of adjoint functors*

$$\mathbf{MPS} \underset{\overset{i_{s,\mathrm{Nis}}}{\longleftarrow}}{\overset{a_{s,\mathrm{Nis}}}{\longrightarrow}} \mathbf{MNS}, \tag{3.2.2}$$

where $i_{s,\mathrm{Nis}}$ is the natural inclusion and its left adjoint $a_{s,\mathrm{Nis}}$ is exact. Moreover, $\mathbf{MNS}$ is Grothendieck. For $F \in \mathbf{MPS}$, the following conditions are equivalent.

(1) *$F \in \mathbf{MNS}$.*

(2) *For any* MV*-square Q as (3.2.1), the associated sequence*

$$0 \to F(T(00)) \to F(T(10)) \times F(T(01)) \to F(T(11)) \tag{3.2.3}$$

is exact.

Proof. Same as for Lemma 2.2.4: the first two assertions are general facts on Grothendieck topologies, and the equivalence (i) $\iff$ (ii) follows from [15, Cor. 2.17] in view of Theorem 3.2.2 (1). □

The following are the main results of [2]. To state them, we need a definition.

Definition 3.2.4.

(1) For any square $S \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$, we define categories $\mathbf{Comp}(S)$ as the full subcategories of $S \downarrow \mathbf{MSm}^{\mathbf{Sq}}$ consisting of those objects $S \to T$ such that $S(ij) \to T(ij)$ belongs to $\mathbf{Comp}(S(ij))$ for any $(ij) \in \mathbf{Sq}$. Here, $\mathbf{Comp}(-)$ is from Definition 1.2.2.

(2) For an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square S in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, an object $S \to \tau^{\mathbf{Sq}}(Q)$ in $\mathbf{Comp}(S)$ is an MV-completion of S if Q is an MV-square. We write

$$\mathbf{Comp}(S)^{\mathrm{MV}} \subset \mathbf{Comp}(S)$$

for the full subcategory consisting of MV-completions of S.

Theorem 3.2.5. *The following assertions hold.*

(1) [2, Th. 1]. *The functor* $\tau_s : \mathbf{MSm} \to \underline{\mathbf{M}}\mathbf{Sm}$ *is continuous in the sense of* [SGA4, Exp. III] *for the topologies given by* P_{MV} *and* $P_{\underline{\mathrm{MV}}}$.

(2) [2, Th. 1.5.6]. *For any* $\underline{\mathrm{MV}}^{\mathrm{fin}}$*-square* S *such that* $\overline{S}(11)$ *is normal,* $\mathbf{Comp}(S)^{\mathrm{MV}}$ *is cofinal in* $\mathbf{Comp}(S)$.

Corollary 3.2.6. *Let* S *be an* $\underline{\mathrm{MV}}^{\mathrm{fin}}$*-square such that* $\overline{S}(11)$ *is normal. Then, for any* $i, j \in \{0, 1\}$*, the subcategory of* $\mathbf{Comp}(S(ij))$ *defined by*

$$\{T(ij) \mid T \in \mathbf{Comp}(S)^{\mathrm{MV}}\}$$

is cofinal in $\mathbf{Comp}(S(ij))$.

Proof. By Theorem 3.2.5 (1), it suffices to prove that the subcateogry $\{T(ij) \mid T \in \mathbf{Comp}(S)\}$ is cofinal in $\mathbf{Comp}(S(ij))$ for any $i, j \in \{0, 1\}$. To show this we need the following

Lemma 3.2.7. *For any morphism* $f : V \to U$ *in* $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ *and for any* $M \in \mathbf{Comp}(U)$*, there exists* $N \in \mathbf{Comp}(V)$ *such that* f *induces a morphism* $N \to M$ *in* $\mathbf{MSm}^{\mathrm{fin}}$.

Proof. Take any $N \in \mathbf{Comp}(V)$. Let Γ be the graph of the rational map $\overline{N} \dashrightarrow \overline{M}$ and let $\pi : \Gamma \to \overline{N}$. Then π induces an isomorphism $N' := (\Gamma, \pi^* N^\infty) \cong N$ and $N' \in \mathbf{Comp}(V)$. Therefore, by replacing N with N', we may assume that $\overline{f} : \overline{U} \to \overline{V}$ extends to a morphism of schemes $\overline{p} : \overline{N} \to \overline{M}$. Moreover, by taking blow up $\mathbf{Bl}_{\overline{N}-\overline{V}}(\overline{N}) \to \overline{N}$ and pulling back the divisor, we may assume that there exists an effective Cartier divisor D on $\overline{N}$ with $\overline{N} - \overline{V} = |D|$. Since the admissiblity of $V \to U$ implies $\overline{p}^* M^\infty|_{\overline{V}^N} = \overline{f}^* U^\infty|_{\overline{V}^N} \leq V^\infty|_{\overline{V}^N} = N^\infty|_{\overline{V}^N}$, [9, Lemma 3.14] shows that there exists a positive integer n such that $\overline{p}^* M^\infty|_{\overline{N}^N} \leq (N^\infty + nD)|_{\overline{N}^N}$. Then $N'' := (\overline{N}, N^\infty + nD) \in \mathbf{Comp}(V)$, it dominates N and $\overline{p}$ induces a morphism $N'' \to M$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, as desired. □

The corollary immediately follows from the lemma when $(i, j) = (1, 1)$. We prove the case $(i, j) = (1, 0)$. Take any $N \in \mathbf{Comp}(S(10))$ and any $T \in \mathbf{Comp}(S)$. Since $\mathbf{Comp}(S(10))$ is filtered, there exists $T'(10) \in \mathbf{Comp}(S(10))$ which dominates both N and $T(10)$. By the lemma there exists $T'(00) \in \mathbf{Comp}(S(00))$ such that $S(00) \to S(10)$ extends to $T'(00) \to T(10)$. Since $\mathbf{Comp}(S(00))$ is ordered we may assume that $T'(00)$ dominates $T(00)$. Then the resulting diagram

$$\begin{array}{ccc} T'(00) & \longrightarrow & T(01) \\ \downarrow & & \downarrow \\ T'(10) & \longrightarrow & T(11) \end{array}$$

is an object of $\mathbf{Comp}(S)$ dominating T, and $T'(10)$ dominates N by construction. This proves the case $(i,j) = (1,0)$. The proof for $(i,j) = (0,1)$ is completely the same.

Finally we prove the case $(i,j) = (0,0)$. Take any $N \in \mathbf{Comp}(S(00))$ and take any $T \in \mathbf{Comp}(S)$. Since $\mathbf{Comp}(S(00))$ is filtered, there exists $T'(00) \in \mathbf{Comp}(S(00))$ which dominates both N and $T(00)$. Then the square obtained by replacing $T(00)$ with $T'(00)$ dominates T. This finishes the proof of Corollary 3.2.6. □

Remark 3.2.8. The essential point of the above proof is the fact that the diagram category $\mathbf{Sq}$ does not have a loop, and therefore the use of the graph trick terminates in finitely many steps. We remark that we can generalize the proof to a much more abstract argument, cf. [1, Lemma C.6].

4. Sheaves on $\mathbf{MSm}$ and $\mathbf{MCor}$

4.1. Sheaves on MSm.

Lemma 4.1.1. *The category* $\mathbf{MNS}$ *is closed under infinite direct sums and the inclusion functor* $i_{s,\mathrm{Nis}}$ *is strongly additive.*

Proof. Indeed, the sheaf condition is tested on finite diagrams, hence the presheaf given by a direct sum of sheaves is a sheaf. □

Theorem 4.1.2. *The following assertions hold.*

(1) *We have* $\tau_s^*(\underline{\mathbf{M}}\mathbf{NS}) \subset \mathbf{MNS}$ *and* $\tau_{s!}(\mathbf{MNS}) \subset \underline{\mathbf{M}}\mathbf{NS}$*, where* τ_s^* *and* $\tau_{s!}$ *are from Lemma 1.2.3. Hence we obtain functors*

$$\tau_s^{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NS} \to \mathbf{MNS} \quad \text{and} \quad \tau_{s,\mathrm{Nis}} : \mathbf{MNS} \to \underline{\mathbf{M}}\mathbf{NS}$$

such that

$$i_{s,\mathrm{Nis}}\tau_s^{\mathrm{Nis}} = \tau_s^*\underline{i}_{s,\mathrm{Nis}}, \quad \tau_{s!}i_{s,\mathrm{Nis}} = \underline{i}_{s,\mathrm{Nis}}\tau_{s,\mathrm{Nis}}, \tag{4.1.1}$$

and that τ_s^{Nis} *is right adjoint to* $\tau_{s,\mathrm{Nis}}$*. (See (1.1.1) for* τ_s*.)*

(2) *For* $F \in \mathbf{MPS}$*, one has* $F \in \mathbf{MNS}$ *if and only if* $\tau_{s,!}F \in \underline{\mathbf{M}}\mathbf{NS}$*.*

(3) *We have*

$$\underline{a}_{s,\mathrm{Nis}}\tau_{s!} = \tau_{s,\mathrm{Nis}}a_{s,\mathrm{Nis}}, \tag{4.1.2}$$

(4) *The functor $\tau_{s,\mathrm{Nis}}$ is fully faithful and exact. Moreover, the functor τ_s^{Nis} preserves injectives.*

Proof. First we prove (1). The first assertion follows from the continuity of τ_s (Theorem 3.2.5 (1)). Similarly, the second assertion morally follows from the "continuity of $\tau_s^!$" (see [2, Rem. 1]): we give a proof based on Theorem 3.2.5 (2).

Take $F \in \mathbf{MNS}$. By Lemma 2.2.4, it suffices to show that the sequence

$$0 \to \tau_! F(M) \to \tau_! F(U) \times \tau_! F(V) \to \tau_! F(W)$$

is exact for any $Q \in P_{\underline{\mathrm{MV}}}$ as in (2.2.1). By Remark 1.1.1 (1), we may assume that $\overline{M}$ is normal. Since a filtered colimit of exact sequences of abelian groups is exact, the desired assertion follows from Lemma 1.2.3 (4), Corollary 3.2.3 and Corollary 3.2.6. Finally the adjointness follows from Lemma 1.2.3 (2). This completes the proof of (1).

(2) follows from (1) and $\mathrm{Id} \cong \tau_s^* \tau_{s,!}$ by Lemma 1.2.3 (2).

(3) follows from (4.1.1) by adjunction.

In (4), the exactness of $\tau_{s,\mathrm{Nis}}$ follows from Lemma 1.2.6 (4) applied to $c = \tau_{s,!}$, using (1), (3) and Lemma 2.2.4. It then implies the preservation of injectives by τ_s^{Nis}. The full faithfulness of $\tau_{s,\mathrm{Nis}}$ follows from that of $\tau_{s!}$ (Lemma 1.2.3 (2)) and (4.1.1). ☐

Remark 4.1.3. By [SGA4, II, Prop. 1.3], the continuity of τ_s implies *a priori* the existence of a left adjoint $\tau_{s,\mathrm{Nis}}$ to τ_s^{Nis} on sheaves of sets, having the following properties:

(1) (4.1.2);
(2) $\tau_{s,\mathrm{Nis}} = a_{s,\mathrm{Nis}} \tau_{s,!} i_{s,\mathrm{Nis}}$;
(3) $\tau_{s,\mathrm{Nis}}$ commutes with representable colimits;
(4) $\tau_{s,\mathrm{Nis}}$ sends a representable sheaf to the corresponding representable sheaf.

All this extends to sheaves of abelian groups by [SGA4, II, Prop. 6.3.1]. The one property which is missing is the right formula of (4.1.1). From (4.1.2), we deduce a base change morphism

$$\tau_{s!} i_{s,\mathrm{Nis}} \Rightarrow \underline{i}_{s,\mathrm{Nis}} \tau_{s,\mathrm{Nis}} \tag{4.1.3}$$

that we have shown above to be an isomorphism. By (4), this isomorphism is clear on the generators $\mathbb{Z}[M]$ ($M \in \mathbf{MSm}$) of the Grothendieck category $\mathbf{MNS}$, since both topologies on $\mathbf{MSm}$ and $\underline{\mathbf{M}}\mathbf{Sm}$ are subcanonical (Th. 2.2.2 and 3.2.2 (1)). But both sides of (4.1.3) are a composition of left and right exact functors, so this is not sufficient to get the general case. Thus the recourse to Theorem 3.2.5 (2) seems necessary to prove Theorem 4.1.2 (1).

4.2. **Sheaves on MCor.** The following is an analogue of Lemma-Definition 2.2.5:

Lemma-Definition 4.2.1. *For* $F \in \mathbf{MPST}$*, one has* $\tau_! F \in \underline{\mathbf{M}}\mathbf{NST}$ *if and only if* $c^* F \in \mathbf{MNS}$*. (See (1.1.1) for c.)*
We write $\mathbf{MNST}$ *for the full subcategory of* $\mathbf{MPST}$ *consisting of those* $F \in \mathbf{MPST}$ *that enjoy these equivalent conditions. Let* $i_{\mathrm{Nis}} : \mathbf{MNST} \to \mathbf{MPST}$ *be the inclusion functor and let* $\tau_{\mathrm{Nis}} : \mathbf{MNST} \to \underline{\mathbf{M}}\mathbf{NST}$ *and* $c^{\mathrm{Nis}} : \mathbf{MNST} \to \mathbf{MNS}$ *be such that*

$$\underline{i}_{\mathrm{Nis}} \tau_{\mathrm{Nis}} = \tau_! i_{\mathrm{Nis}}, \quad i_{s,\mathrm{Nis}} c^{\mathrm{Nis}} = c^* i_{\mathrm{Nis}}. \tag{4.2.1}$$

Proof. Let $F \in \mathbf{MPST}$. Then

$$\tau_! F \in \underline{\mathbf{M}}\mathbf{NST} \Rightarrow \tau_{s!} c^* F = \underline{c}^* \tau_! F \in c^* \, \underline{\mathbf{M}}\mathbf{NST} \subset \underline{\mathbf{M}}\mathbf{NS}$$

where we used (1.2.3) and Lemma-Definition 2.2.5; by Theorem 4.1.2 (2), this implies $c^* F \in \mathbf{MNS}$. This reasoning can be reversed. □

Lemma 4.2.2.

(1) *For any* $M \in \mathbf{MCor}$*, the presheaf* $\mathbb{Z}_{\mathrm{tr}}(M) \in \mathbf{MPST}$ *belongs to* $\mathbf{MNST}$*.*
(2) *We have a natural isomorphism*

$$\tau_{s,\mathrm{Nis}} c^{\mathrm{Nis}} \cong \underline{c}^{\mathrm{Nis}} \tau_{\mathrm{Nis}}. \tag{4.2.2}$$

Proof. (1) follows from Theorem 2.2.7, since $\tau_! \mathbb{Z}_{\mathrm{tr}}(M) = \mathbb{Z}_{\mathrm{tr}}(\tau M)$. (2) is a consequence of (4.2.1) and the corresponding isomorphism for presheaves (1.2.3). □

Lemma 4.2.3. *Let us consider the "2-Cartesian product category"* $\mathbf{MNS} \times_{\underline{\mathbf{M}}\mathbf{NS}} \underline{\mathbf{M}}\mathbf{NST}$*, that is, the category of triples* (F_s, F_t, φ) *consisting of* $F_s \in \mathbf{MNS}$*,* $F_t \in \underline{\mathbf{M}}\mathbf{NST}$ *and an isomorphism* $\varphi : \tau_{s,\mathrm{Nis}} F_s \xrightarrow{\sim} \underline{c}^{\mathrm{Nis}} F_t$ *in* $\underline{\mathbf{M}}\mathbf{NS}$*. The functor*

$$\mathbf{MNST} \to \mathbf{MNS} \times_{\underline{\mathbf{M}}\mathbf{NS}} \underline{\mathbf{M}}\mathbf{NST},$$

defined by $F \mapsto (c^{\mathrm{Nis}} F, \tau_{\mathrm{Nis}} F, \theta_F)$*, where* $\theta_F : \tau_{s,\mathrm{Nis}} c^{\mathrm{Nis}} F \xrightarrow{\sim} \underline{c}^{\mathrm{Nis}} \tau_{\mathrm{Nis}} F$ *is from (4.2.2), is an equivalence of categories.*

Proof. The same statement was proven for presheaves in [4, Lemma 2.7.1]; full faithfulness follows from this, and essential surjectivity follows from Lemma-Definition 4.2.1. □

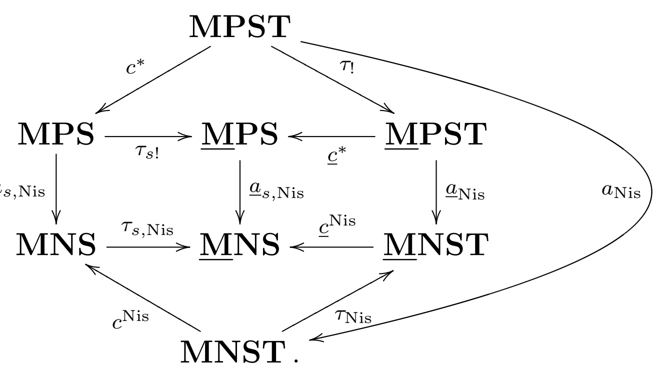
Theorem 4.2.4. *The following assertions hold.*

(1) *The functor* i_{Nis} *is strongly additive and has an exact left adjoint* a_{Nis}*. Consequently,* $\mathbf{MNST}$ *is Grothendieck. We have*

$$c^{\mathrm{Nis}} a_{\mathrm{Nis}} = a_{s,\mathrm{Nis}} c^*, \quad \tau_{\mathrm{Nis}} a_{\mathrm{Nis}} = \underline{a}_{\mathrm{Nis}} \tau_!. \tag{4.2.3}$$

(2) *The functor* c^{Nis} *has a left adjoint* $c_{\mathrm{Nis}} = a_{\mathrm{Nis}} c_! i_{\mathrm{Nis}}$*. Moreover,* c^{Nis} *is exact, strongly additive, and faithful.*

Proof. By Definition of $\mathbf{MNST}$, the strong addtivity of i_{Nis} follows from that of $\underline{i}_{\mathrm{Nis}}$ (Theorem 2.2.7 and Lemmma 1.2.6 (2)). We then use Lemma 4.2.3 and [4, Lemma 2.7.1] to construct a_{Nis} by patching $a_{s,\mathrm{Nis}}$ and $\underline{a}_{\mathrm{Nis}}$ over $\underline{a}_{s,\mathrm{Nis}}$, i.e., we want a_{Nis} to verify (4.2.3); a formal argument shows that such a patching is determined by the second isomorphism of (2.2.3) and by the one of (4.1.2).



The second isomorphism of (4.2.3) easily implies that a_{Nis} is left adjoint to i_{Nis}. Then (2) follows from Lemma 1.2.5, Lemma 4.1.1 and Lemma 1.2.6 (3).

Finally, the exactness of a_{Nis} is a consequence of the first isomorphism of (4.2.3) since c^{Nis} is faithfully exact as we have just shown. □

Lemma 4.2.5. *The following assertions hold.*

(1) *We have* $\tau^*(\underline{\mathbf{M}}\mathbf{NST}) \subset \mathbf{MNST}$*.*

(2) *Let* $\tau^{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NST} \to \mathbf{MNST}$ *be the functor characterized by*

$$\tau^* \underline{i}_{\mathrm{Nis}} = i_{\mathrm{Nis}} \tau^{\mathrm{Nis}}. \tag{4.2.4}$$

Then τ^{Nis} *is a right adjoint of* τ_{Nis}*, and* τ_{Nis} *is fully faithful, exact, and strongly additive. Moreover,* τ^{Nis} *preserves injectives and is strongly additive.*

Remark 4.2.6. We will see in Theorem 5.1.1 below that τ^{Nis} is also exact.

Proof. Let $F \in \underline{\mathbf{M}}\mathbf{NST}$ so that $\underline{c}^* F \in \underline{\mathbf{M}}\mathbf{NS}$. By Theorem 4.1.2 (1), we have $c^* \tau^* F = \tau_s^* \underline{c}^* F \in \mathbf{MNS}$. In view of Lemma-Definition 4.2.1, this proves that $\tau^* F \in \mathbf{MNST}$, whence (1).

In (2), the existence (and uniqueness) of τ^{Nis} follows from (1) (and the full faithfulness of $\underline{i}_{\mathrm{Nis}}$ and i_{Nis}). The adjointness is shown by using the full faithfulness of $\underline{i}_{\mathrm{Nis}}$ and i_{Nis}, the adjoint pair $(\tau_!, \tau^*)$, (4.2.1) and

(4.2.4). Similarly, the full faithfulness of τ_{Nis} follows from that of $\tau_!$ (Proposition 1.2.3) and (4.2.1). The strong additivity (resp. exactness) of τ_{Nis} follows from Proposition 1.2.3, Theorem 4.2.4 and Lemma 1.2.6 (2) (resp. (4)), applied with $c = \tau_!$; the latter implies that τ^{Nis} preserves injectives. Finally, its strong additivity is reduced to that of τ^* (Lemma 1.2.3 (1)), $\underline{i}_{\mathrm{Nis}}$ (Theorem 2.2.7) and i_{Nis} (Theorem 4.2.4) by the full faithfulness of i_{Nis}. □

5. Cohomology in MNST

5.1. Main result.

We begin with the following.

Theorem 5.1.1. *The functor $\tau^{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NST} \to \mathbf{MNST}$ from Lemma 4.2.5 is exact.*

The proof will be given later in this section (see Corollary 5.5.1). We now deduce its consequences.

Lemma 5.1.2. *For any $M \in \mathbf{MCor}$, $F \in \mathbf{MNST}$, $G \in \underline{\mathbf{M}}\mathbf{NST}$ and $q \geq 0$, we have natural isomorphisms*

$$\mathrm{Ext}^q_{\mathbf{MNST}}(\mathbb{Z}_{\mathrm{tr}}(M), \tau^{\mathrm{Nis}}G) \cong \mathrm{Ext}^q_{\underline{\mathbf{M}}\mathbf{NST}}(\mathbb{Z}_{\mathrm{tr}}(M), G),$$
$$\mathrm{Ext}^q_{\mathbf{MNST}}(\mathbb{Z}_{\mathrm{tr}}(M), F) \cong \mathrm{Ext}^q_{\underline{\mathbf{M}}\mathbf{NST}}(\mathbb{Z}_{\mathrm{tr}}(M), \tau_{\mathrm{Nis}}F).$$

Proof. By Theorem 5.1.1, τ^{Nis} is exact and it preserves injectives by Lemma 4.2.5. Hence we have

$$R^q(i_{\mathrm{Nis}})\tau^{\mathrm{Nis}}G = R^q(i_{\mathrm{Nis}}\tau^{\mathrm{Nis}})G = R^q(\tau^*\underline{i}_{\mathrm{Nis}})G = \tau^* R^q \underline{i}_{\mathrm{Nis}}G$$

for any $G \in \underline{\mathbf{M}}\mathbf{NST}$ by [4, Th. A.9.1]. Using the projectivity of $\mathbb{Z}_{\mathrm{tr}}(M)$ in $\mathbf{MPST}$ and that of $\tau_!\mathbb{Z}_{\mathrm{tr}}(M) = \mathbb{Z}_{\mathrm{tr}}(M)$ in $\underline{\mathbf{M}}\mathbf{PST}$, and using Prop. A.1.1 twice, we get isomorphisms

$$\begin{aligned}\mathrm{Ext}^q_{\mathbf{MNST}}(a_{\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(M), \tau^{\mathrm{Nis}}G) &\simeq \mathbf{MPST}(\mathbb{Z}_{\mathrm{tr}}(M), R^q(i_{\mathrm{Nis}})\tau^{\mathrm{Nis}}G)\\ &\simeq \mathbf{MPST}(\mathbb{Z}_{\mathrm{tr}}(M), \tau^* R^q\underline{i}_{\mathrm{Nis}}G) \simeq \underline{\mathbf{M}}\mathbf{PST}(\tau_!\mathbb{Z}_{\mathrm{tr}}(M), R^q\underline{i}_{\mathrm{Nis}}G)\\ &\simeq \underline{\mathbf{M}}\mathbf{PST}(\mathbb{Z}_{\mathrm{tr}}(M), R^q\underline{i}_{\mathrm{Nis}}G) \simeq \mathrm{Ext}^q_{\underline{\mathbf{M}}\mathbf{NST}}(\underline{a}_{\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(M), G).\end{aligned}$$

Moreover, $a_{\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(M) = \mathbb{Z}_{\mathrm{tr}}(M)$ and $\underline{a}_{\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(M) = \mathbb{Z}_{\mathrm{tr}}(M)$ by Theorem 2.2.7 and Lemma 4.2.2 (1), whence the first formula by evaluating both sides at M. The second one follows from the first by taking $G = \tau_{\mathrm{Nis}}F$, since $\tau^{\mathrm{Nis}}\tau_{\mathrm{Nis}} = \mathrm{Id}$. □

Let $M \in \mathbf{MCor}$ and $F \in \mathbf{MNST}$. Using Notation 2.2.8, we define $F_M := (\tau_{\mathrm{Nis}}F)_M$, which is a sheaf on $(\overline{M})_{\mathrm{Nis}}$.

Theorem 5.1.3. *For any $p \geq 0$, $M \in \mathbf{MCor}$ and $F \in \mathbf{MNST}$, we have a natural isomorphism*

$$\mathrm{Ext}^p_{\mathbf{MNST}}(\mathbb{Z}_{\mathrm{tr}}(M), F) \simeq \varinjlim_{N \in \Sigma^{\mathrm{fin}} \downarrow M} H^p_{\mathrm{Nis}}(\overline{N}, F_N). \tag{5.1.1}$$

Moreover, we have

$$\varinjlim_{N \in \underline{\Sigma}^{\mathrm{fin}} \downarrow M} H^p_{\mathrm{Nis}}(\overline{N}, (R^q(\underline{b}_s^{\mathrm{Nis}})\underline{c}^{\mathrm{Nis}}\tau_{\mathrm{Nis}} F)_N) = 0 \text{ for all } q > 0.$$

Proof. Combine Proposition 2.2.9, Lemma 4.2.5 and Lemma 5.1.2. □

Corollary 5.1.4. *We have $\mathrm{Ext}^q_{\mathbf{MNST}}(\mathbb{Z}_{\mathrm{tr}}(M), F) = 0$ for $q > \dim \overline{M}$.*

Proof. Same as for Corollary 2.2.10. □

5.2. **A generation lemma.** We now start proving Theorem 5.1.1. We need some preliminaries.

Lemma 5.2.1. *Let $F \in \underline{\mathbf{M}}\mathbf{PST}$ such that $\underline{a}_{\mathrm{Nis}} F = 0$. Then F may be written as a quotient of a direct sum of*

$$\mathbb{Z}_{\mathrm{tr}}(M/U) := \mathrm{Coker}\big(\mathbb{Z}_{\mathrm{tr}}(U) \to \mathbb{Z}_{\mathrm{tr}}(M)\big),$$

where $M \in \underline{\mathbf{M}}\mathbf{Sm}$, $U \to M$ is a covering for the Grothendieck topology on $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ associated to $P_{\underline{\mathrm{MV}}^{\mathrm{fin}}}$ from Proposition 2.1.2, and the cokernel is taken in $\underline{\mathbf{M}}\mathbf{PST}$. Moreover we have $\underline{a}_{\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(M/U) = 0$.

Proof. Let $G = c^*F \in \underline{\mathbf{M}}\mathbf{NS}$. Take $f \in G(L) = F(L)$ for $L \in \underline{\mathbf{M}}\mathbf{Sm}$. By (2.2.3) we have $\underline{a}_{s,\mathrm{Nis}} G = 0$. By [4, Lemma 4.3.2], we have $\varphi^* f = 0$ for a cover $\varphi : U \to M \to L$ in $\underline{\mathbf{M}}\mathbf{Sm}_{\mathrm{Nis}}$, where $U \to M$ is a strict Nisnevich cover and $M \to L$ is in $\underline{\Sigma}^{\mathrm{fin}}$. Hence the Yoneda map $\mathbb{Z}^p(L) \to G$ in $\underline{\mathbf{M}}\mathbf{PS}$ given by f (see Definition 1.2.1 (4) for $\mathbb{Z}^p(L)$) factors through

$$\mathbb{Z}^p(L/U) := \mathrm{Coker}(\mathbb{Z}^p(U) \to \mathbb{Z}^p(L))$$

(see Definition 1.2.1 (4) for $\mathbb{Z}^p(-)$). By the adjunction $(c_!, c^*)$ this induces a map $c_!\mathbb{Z}^p(L/U) \to F$, and

$$c_!\mathbb{Z}^p(L/U) \simeq \mathrm{Coker}(c_!\mathbb{Z}^p(U) \to c_!\mathbb{Z}^p(L)) \simeq \mathrm{Coker}(\mathbb{Z}_{\mathrm{tr}}(U) \to \mathbb{Z}_{\mathrm{tr}}(L)),$$

where the first isomorphism follows from the right exactness of $c_!$ as a left adjoint. This implies that the Yoneda map $y(f) : \mathbb{Z}_{\mathrm{tr}}(L) \to F$ in $\underline{\mathbf{M}}\mathbf{PST}$ factors through the cokernel of $\mathbb{Z}_{\mathrm{tr}}(U) \to \mathbb{Z}_{\mathrm{tr}}(M) \to \mathbb{Z}_{\mathrm{tr}}(L)$. Thus we get an induced map $\overline{y(f)} : \mathbb{Z}_{\mathrm{tr}}(M/U) \to F$. Since the map $\mathbb{Z}_{\mathrm{tr}}(M) \to \mathbb{Z}_{\mathrm{tr}}(L)$ is an isomorphism in $\underline{\mathbf{M}}\mathbf{PST}$, the image of $y(f)$ coincides with that of $\overline{y(f)}$. Collecting them over all pairs (L, f), this proves the lemma. Finally the last statement follows from [4, Th. 4.5.7]. □

5.3. The τ construction.

Definition 5.3.1. Let $M, N \in \underline{\mathbf{M}}\mathbf{Cor}$.

(1) We put
$$\mathbb{Z}_{\mathrm{tr}}(M)^\tau = \tau_! \tau^* \mathbb{Z}_{\mathrm{tr}}(M) \in \underline{\mathbf{M}}\mathbf{PST}\,.$$
Note $\mathbb{Z}_{\mathrm{tr}}(M)^\tau \in \underline{\mathbf{M}}\mathbf{NST}$ by Lemma 4.2.5.

(2) Let $\underline{\mathbf{M}}\mathbf{Cor}^\tau(N, M)$ be the subgroup of $\underline{\mathbf{M}}\mathbf{Cor}(N, M)$ generated by elementary correspondences Z in $\mathbf{Cor}(N^\circ, M^\circ)$ which satisfy the condition:

(♠) There exists a dense open immersion $j : \overline{N} \hookrightarrow \overline{L}$ with $\overline{L}$ proper such that the closure $\overline{Z}$ of Z in $\overline{L} \times \overline{M}$ is proper over $\overline{L}$.

Lemma 5.3.2. *For N, M as above, the condition (♠) is independent of the choice of $j : \overline{N} \hookrightarrow \overline{L}$, and we have*
$$\mathbb{Z}_{\mathrm{tr}}(M)^\tau(N) = \underline{\mathbf{M}}\mathbf{Cor}^\tau(N, M).$$

Proof. If $j' : \overline{N} \hookrightarrow \overline{L}'$ is another choice equipped with (proper) surjective $f : \overline{L} \to \overline{L}'$ such that $j' = fj$, writing $\overline{Z}' \subset \overline{L}' \times \overline{M}$ for the closure of Z, f induces a proper surjective map $\overline{Z}' \to \overline{Z}$. Then it is easy to see that $\overline{Z}'$ is proper over $\overline{L}'$ if and only if so is $\overline{Z}$ over $\overline{L}$. This proves the first assertion. To show the second assertion, we note that by Lemma 1.2.3 (3),
$$\mathbb{Z}_{\mathrm{tr}}(M)^\tau(N) = \varinjlim_{L \in \mathbf{Comp}(N)} \underline{\mathbf{M}}\mathbf{Cor}(L, M).$$
The second assertion follows from this using the first assertion (see the proof of [4, Lemma 1.8.3]). □

Lemma 5.3.3. *For $M, N \in \underline{\mathbf{M}}\mathbf{Cor}$, we put*
$$\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M)^\tau(N) = \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(N, M) \cap \underline{\mathbf{M}}\mathbf{Cor}^\tau(N, M) \subset \underline{\mathbf{M}}\mathbf{Cor}(N, M).$$
Then $\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M)^\tau$ defines an object of $\underline{\mathbf{M}}\mathbf{PST}^{\mathrm{fin}}$. Moreover we have
$$\underline{b}_! \mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M)^\tau = \mathbb{Z}_{\mathrm{tr}}(M)^\tau.$$

Proof. The first assertion follows from Lemma 5.3.2. For $N' \in \underline{\Sigma}^{\mathrm{fin}} \downarrow N$ (see Definition 2), Lemma 5.3.2 implies
$$\underline{\mathbf{M}}\mathbf{Cor}^\tau(N, M) = \mathbb{Z}_{\mathrm{tr}}(M)^\tau(N) = \mathbb{Z}_{\mathrm{tr}}(M)^\tau(N') = \underline{\mathbf{M}}\mathbf{Cor}^\tau(N', M),$$
where the second equality follows from the fact that $N' \simeq N$ in $\underline{\mathbf{M}}\mathbf{Cor}$. By definition this implies
$$\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M)^\tau(N') = \underline{\mathbf{M}}\mathbf{Cor}^\tau(N, M) \cap \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(N', M),$$

which proves the second assertion in view of the isomorphisms

$$\underline{b}_!\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M)^\tau(N) = \varinjlim_{N'\in\underline{\Sigma}^{\mathrm{fin}}\downarrow N} \mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M)^\tau(N'),$$

$$\underline{\mathbf{M}}\mathbf{Cor}(N,M) = \varinjlim_{N'\in\underline{\Sigma}^{\mathrm{fin}}\downarrow N} \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}(N',M),$$

which hold by (1.2.2) and [4, Prop. 1.9.2]. □

Remark 5.3.4. We can prove that $\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M)^\tau$ lies in $\underline{\mathbf{M}}\mathbf{NST}^{\mathrm{fin}}$ (so that we may remove $\underline{a}_{\mathrm{Nis}}^{\mathrm{fin}}$ in Theorem 5.4.1(1) below).

5.4. **Exactness of a certain Čech complex.**

Theorem 5.4.1. *Let $p: U \to M$ be a covering for the Grothendieck topology on $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ associated to $P_{\underline{\mathrm{MV}}^{\mathrm{fin}}}$ from Proposition 2.1.2. Denote by $U \times_M U$ the fiber product in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ (see* [4, Prop. 1.10.4 and Cor. 1.10.7]*).*

(1) *The Čech complex*

$$\cdots \to \underline{a}_{\mathrm{Nis}}^{\mathrm{fin}}\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U\times_M U)^\tau \to \underline{a}_{\mathrm{Nis}}^{\mathrm{fin}}\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U)^\tau \to \underline{a}_{\mathrm{Nis}}^{\mathrm{fin}}\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M)^\tau \to 0$$

is exact in $\underline{\mathbf{M}}\mathbf{NST}^{\mathrm{fin}}$.

(2) *The Čech complex*

$$\cdots \to \mathbb{Z}_{\mathrm{tr}}(U\times_M U)^\tau \to \mathbb{Z}_{\mathrm{tr}}(U)^\tau \to \mathbb{Z}_{\mathrm{tr}}(M)^\tau \to 0$$

is exact in $\underline{\mathbf{M}}\mathbf{NST}$.

Theorem 5.4.1 (2) follows from (1) by applying the exact functor $\underline{b}_{\mathrm{Nis}}$ from Proposition 2.2.9 and using isomorphisms

$$\underline{b}_{\mathrm{Nis}}\underline{a}_{\mathrm{Nis}}^{\mathrm{fin}}\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M)^\tau \simeq \underline{a}_{\mathrm{Nis}}\underline{b}_!\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M)^\tau \simeq \underline{a}_{\mathrm{Nis}}\mathbb{Z}_{\mathrm{tr}}(M)^\tau = \mathbb{Z}_{\mathrm{tr}}(M)^\tau,$$

where the first isomorphism follows from (2.2.3), the second from Lemma 5.3.3 and the last equality follows from the fact that $\mathbb{Z}_{\mathrm{tr}}(M)^\tau \in \underline{\mathbf{M}}\mathbf{NST}$ thanks to Lemma 4.2.5 (1).

We need some preliminaries for the proof of (1). It is inspired by that of [4, Th. 3.4.1], with some elaboration. Take $(X, D) \in \underline{\mathbf{M}}\mathbf{Cor}$ and a point $x \in X$. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be the filtered system of connected affine étale neighborhoods of $x \in X$. Let

$$\overline{S} = \varprojlim_{\lambda\in\Lambda} X_\lambda \tag{5.4.1}$$

be the henselization of X at x. Take $M \in \underline{\mathbf{M}}\mathbf{Cor}$ and let $\mathcal{D}$ be the category of diagrams

$$\overline{S} \xleftarrow{f} Z \xrightarrow{g} \overline{M} \tag{5.4.2}$$

of k-schemes with f quasi-finite such that $Z \to \overline{S} \times \overline{M}$ is a closed immersion and $V \not\subset \overline{S} \times M^\infty$ for any irreducible component V of Z. We denote (5.4.2) by (Z, f, g). A morphism from (Z, f, g) to (Z', f', g') is given by a morphism $\varphi : Z \to Z'$ which fits into a commutative diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{f} & & \searrow^{g} & \\ \overline{S} & & \downarrow \varphi & & \overline{M} \\ & \nwarrow_{f'} & & \nearrow_{g'} & \\ & & Z'. & & \end{array} \tag{5.4.3}$$

Note that φ is automatically a closed immersion, so $\mathcal{D}$ is a cofiltered ordered set as it is stable under unions. For $(Z, f, g) \in \mathcal{D}$ let $E(Z) = E(Z, f, g)$ be the set of irreducible components V of Z which belong to $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{fin}}((\overline{S}, D), M)$, i.e. such that $f|_V$ is finite and surjective over an irreducible component of $\overline{S}$ and satisfies the admissibility condition:

$$(fi_V v)^*(D \times_X \overline{S}) \geq (gi_V v)^*(M^\infty), \tag{5.4.4}$$

where $v : V^N \to V$ is the normalization and $i_V : V \hookrightarrow Z$ is the inclusion. Let $E^\tau(Z) \subset E(Z)$ be the subset of those V which belong to $\mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M)^\tau(\overline{S}, D)$, i.e. satisfying the following condition: there exists $\lambda \in \Lambda$ such that (Z, f, g) (resp. $V \hookrightarrow Z$) is the base change via $\overline{S} \to X_\lambda$ of

$$X_\lambda \xleftarrow{f_\lambda} Z_\lambda \xrightarrow{g_\lambda} \overline{M} \quad (\text{resp. } V_\lambda \hookrightarrow Z_\lambda), \tag{5.4.5}$$

where V_λ is an irreducible component of Z_λ satisfying the condition:

$(\clubsuit)_\lambda$ V_λ is finite over X_λ and satisfies the admissibility condition

$$(f_\lambda \circ i_{V_\lambda} \circ v_\lambda)^*(D \times_X X_\lambda) \geq (g_\lambda \circ i_{V_\lambda} \circ v_\lambda)^*(M^\infty), \tag{5.4.6}$$

similar to (5.4.4). Moreover, letting $\tilde{V_\lambda} = h_\lambda(V_\lambda)$ with $h_\lambda = (f_\lambda, g_\lambda) : Z_\lambda \to X_\lambda \times \overline{M}$ ($\tilde{V_\lambda}$ is finite over X_λ by the finiteness of $V_\lambda \to X_\lambda$), there exists a dense open immersion $X_\lambda \hookrightarrow \overline{X_\lambda}$ with $\overline{X_\lambda}$ proper such that the closure $\overline{\tilde{V_\lambda}}$ of $\tilde{V_\lambda}$ in $\overline{X_\lambda} \times \overline{M}$ is proper over $\overline{X_\lambda}$.

Let $L^\tau(Z)$ be the free abelian group on the set $E^\tau(Z)$.

Lemma 5.4.2. *Let V_λ be as in $(\clubsuit)_\lambda$ and $X_\mu \to X_\lambda$ $(\lambda, \mu \in \Lambda)$ be a map in the system of étale neighborhoods of $x \in X$. Let*

$$X_\mu \xleftarrow{f_\mu} Z_\mu \xrightarrow{g_\mu} \overline{M} \quad (\textit{resp. } V_\mu \hookrightarrow Z_\mu) \tag{5.4.7}$$

be the base change of (5.4.5) *(resp.* $V_\lambda \hookrightarrow Z_\lambda$*). If* $V_\lambda \subset Z_\lambda$ *satisfies* $(\clubsuit)_\lambda$*, then any component of* V_μ *satisfies* $(\clubsuit)_\mu$*.*

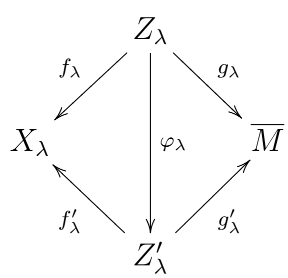
Proof. The finiteness over X_μ and the admissibility condition of $(\clubsuit)_\mu$ are clear. To check the last condition of $(\clubsuit)_\mu$, let $X_\mu \hookrightarrow \overline{X_\mu}$ be the normalization in X_μ of $\overline{X_\lambda}$ from $(\clubsuit)_\lambda$ and let $\tilde{V_\mu} = h_\mu(V_\mu)$ with $h_\mu = (f_\mu, g_\mu) : Z_\mu \to X_\mu \times \overline{M}$ ($\tilde{V_\mu}$ is finite over X_μ by the finiteness of $V_\mu \to X_\mu$). Then $\tilde{V_\mu} \subset \tilde{V_\lambda} \times_{X_\lambda} X_\mu$ so that the closure $\overline{\tilde{V_\mu}}$ of $\tilde{V_\mu}$ in $\overline{X_\mu} \times \overline{M}$ is contained in $\overline{\tilde{V_\lambda}} \times_{\overline{X_\lambda}} \overline{X_\mu}$, which is proper over $\overline{X_\mu}$ by the assumption. Hence $\overline{\tilde{V_\mu}}$ is also proper over $\overline{X_\mu}$, which implies the desired condition. □

Lemma 5.4.3. *For a commutative diagram* (5.4.3)*, there is a natural induced map*

$$\varphi_* : E^\tau(Z) \to E^\tau(Z')$$

which makes E^τ *a covariant functor on* $\mathcal{D}$*.*

Proof. Take $V \in E(Z)$ and let $V' = \varphi(V)$. By the finiteness of $V \to \overline{S}$, V' is finite over $\overline{S}$ and closed in Z'. The admissibility condition (5.4.4) for V implies that for V' by [4, Lemma 1.2.1]. Hence $V' \in E(Z')$. Suppose $V \in E^\tau(Z)$. To show $V' \in E^\tau(Z')$, take $\lambda \in \Lambda$ and V_λ as in $(\clubsuit)_\lambda$. Thanks to Lemma 5.4.2, we may assume that the diagram (5.4.3) is the base chage via $\overline{S} \to X_\lambda$ of



and $V' = V'_\lambda \times_{X_\lambda} \overline{S}$ with $V'_\lambda = \varphi_\lambda(V_\lambda)$. Since V_λ is finite and surjective over a component of X_λ, so is V'_λ, which implies that it is an irreducible component of Z'_λ. The admissibility condition (5.4.6) for V_λ implies that for V'_λ by [4, Lemma 1.2.1]. Letting $h'_\lambda = (f'_\lambda, g'_\lambda) : Z'_\lambda \to X_\lambda \times \overline{M}$, we have $h_\lambda = h'_\lambda \varphi_\lambda$ so that $h'_\lambda(V'_\lambda) = h_\lambda(V_\lambda)$. Hence V'_λ satisfies the last condition of $(\clubsuit)_\lambda$ since V_λ does. This implies $V' \in E^\tau(Z')$. □

Proof of Theorem 5.4.1 (1). It suffices to show the exactness of

$$\cdots \to \mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U \times_M U)^\tau(S) \to \mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(U)^\tau(S) \to \mathbb{Z}_{\mathrm{tr}}^{\mathrm{fin}}(M)^\tau(S) \to 0 \tag{5.4.8}$$

where $S = (\overline{S}, D \times_X \overline{S})$ with (X, D) and $\overline{S}$ as in (5.4.1). We first note that for a closed subscheme $Z \subset \overline{S}\times\overline{U}\times_{\overline{M}}\cdots\times_{\overline{M}}\overline{U}$ finite and surjective over an irreducible component of $\overline{S}$, the image of Z in $\overline{S}\times\overline{M}$ is finite over $\overline{S}$. From this fact we see that (5.4.8) is obtained as the inductive limit of

$$\cdots \to L^\tau(Z \times_{\overline{M}} (\overline{U} \times_{\overline{M}} \overline{U})) \to L^\tau(Z \times_{\overline{M}} \overline{U}) \to L^\tau(Z) \to 0 \tag{5.4.9}$$

where Z ranges over all closed subschemes of $\overline{S}\times\overline{M}$ that is finite surjective over an irreducible component of $\overline{S}$. It suffices to show the exactness of (5.4.9).

Since Z is finite over a henselian local scheme $\overline{S}$, Z is a disjoint union of henselian local schemes. Thus the Nisnevich cover $Z \times_{\overline{M}} \overline{U} \to Z$ admits a section $s_0 : Z \to Z \times_{\overline{M}} \overline{U}$. Define for $k \geq 0$

$$s_k := s_0 \times_{\overline{M}} \mathrm{Id}_{\overline{U}^k} : Z \times_{\overline{M}} \overline{U}^k \to Z \times_{\overline{M}} \overline{U} \times_{\overline{M}} \overline{U}^k = Z \times_{\overline{M}} \overline{U}^{k+1},$$

where $\overline{U}^k$ is the k-fold fiber product of $\overline{U}$ over $\overline{M}$. Then the maps

$$(s_k)_* : L^\tau(Z \times_{\overline{M}} \overline{U}^k) \to L^\tau(Z \times_{\overline{M}} \overline{U}^{k+1})$$

give us a homotopy from the identity to zero. □

5.5. **Proof of Theorem 5.1.1.**

Corollary 5.5.1.

(1) *Let $G \in \underline{\mathbf{M}}\mathbf{PST}$. If $\underline{a}_{\mathrm{Nis}}G = 0$, then $\underline{a}_{\mathrm{Nis}}\tau_!\tau^*G = 0$.*
(2) *The base change morphism $a_{\mathrm{Nis}}\tau^* \Rightarrow \tau^{\mathrm{Nis}}\underline{a}_{\mathrm{Nis}}$ is an isomorphism.*
(3) *The functor τ^{Nis} is exact.*

Proof. (1) Since $\underline{a}_{\mathrm{Nis}}$, $\tau_!$ and τ^* all commute with representable colimits as left adjoints, we are reduced by Lemma 5.2.1 to G of the form $\mathbb{Z}_{\mathrm{tr}}(M/U)$, which is equivalent to

$$\underline{a}_{\mathrm{Nis}}\big(\operatorname{Coker}(\mathbb{Z}_{\mathrm{tr}}(U)^\tau \to \mathbb{Z}_{\mathrm{tr}}(M)^\tau)\big) = 0,$$

where the cokernel is taken in $\underline{\mathbf{M}}\mathbf{PST}$. This follows from Theorem 5.4.1(2).

(2) Let $F \in \underline{\mathbf{M}}\mathbf{NST}$. The base change morphism $a_{\mathrm{Nis}}\tau^*F \to \tau^{\mathrm{Nis}}\underline{a}_{\mathrm{Nis}}F$ is defined as the composition

$$a_{\mathrm{Nis}}\tau^*F \xrightarrow{a_{\mathrm{Nis}}\tau^*(\eta_F)} a_{\mathrm{Nis}}\tau^*\underline{i}_{\mathrm{Nis}}\underline{a}_{\mathrm{Nis}}F \simeq a_{\mathrm{Nis}}i_{\mathrm{Nis}}\tau^{\mathrm{Nis}}\underline{a}_{\mathrm{Nis}}F \xrightarrow{\varepsilon_{\tau^{\mathrm{Nis}}\underline{a}_{\mathrm{Nis}}F}} \tau^{\mathrm{Nis}}\underline{a}_{\mathrm{Nis}}F$$

where η (resp. ε) is the unit (resp. counit) of the adjunction $(\underline{a}_{\mathrm{Nis}}, \underline{i}_{\mathrm{Nis}})$ (resp. $(a_{\mathrm{Nis}}, i_{\mathrm{Nis}})$). Since the second map is an isomorphism by the full faithfulness of i_{Nis}, it remains to show that the first one is an isomorphism. By the full faithfulness of τ_{Nis} (Lemma 4.2.5), it suffices to show it after applying this functor. But $\underline{a}_{\mathrm{Nis}}\tau_! \simeq \tau_{\mathrm{Nis}} a_{\mathrm{Nis}}$ by Theorem 4.2.4, so we are left to show that the map

$$\underline{a}_{\mathrm{Nis}}\tau_!\tau^* F \xrightarrow{\underline{a}_{\mathrm{Nis}}\tau_!\tau^*(\eta_F)} \underline{a}_{\mathrm{Nis}}\tau_!\tau^* \underline{i}_{\mathrm{Nis}}\underline{a}_{\mathrm{Nis}} F$$

is an isomorphism. This follows from (1), since $\underline{a}_{\mathrm{Nis}}\tau_!\tau^*$ is exact, and $\operatorname{Ker}\eta_F$ and $\operatorname{Coker}\eta_F$ are killed by $\underline{a}_{\mathrm{Nis}}$.

Finally, (3) This follows from (2), Lemma 1.2.6 (4) and the exactness of τ^*. ☐

Corollary 5.5.2. *The functor τ^{Nis} has a right adjoint.*

Proof. The category $\underline{\mathbf{M}}\mathbf{NST}$ is cocomplete and has a small set of generators, as a Grothendieck category (Theorem 2.2.7.) Moreover, τ^{Nis} respects all representable colimits as an exact, strongly additive functor (Lemma 4.2.5 and Corollary 5.5.1 (3)). Thus the dual hypotheses of the "special adjoint functor theorem" [7, V.8, Th. 2] are verified. ☐

This corollary is striking, since τ_s is not cocontinuous [2, Rem. 5.2.1].

6. Relation with NST

6.1. **MNS, $\underline{\mathbf{M}}$NS and NS.** We consider the functors

(6.1.1) $$\underline{\omega}_s : \underline{\mathbf{M}}\mathbf{Sm} \to \mathbf{Sm}, \qquad \omega_s : \mathbf{MSm} \to \mathbf{Sm},$$

defined by $\underline{\omega}_s(M) = M^o$ and $\omega_s(M) = M^o$, and the left adjoint to $\underline{\omega}_s$, defined by $\lambda_s(X) = (X, \emptyset)$. We have $\omega_s = \underline{\omega}_s \tau_s$, and:

Proposition 6.1.1 ([2, Th. 1])**.** *The functors $\omega_s : \mathbf{MSm}_{\mathrm{Nis}} \to \mathbf{Sm}_{\mathrm{Nis}}$, $\underline{\omega}_s : \underline{\mathbf{M}}\mathbf{Sm}_{\mathrm{Nis}} \to \mathbf{Sm}_{\mathrm{Nis}}$ and $\lambda_s : \mathbf{Sm} \to \underline{\mathbf{M}}\mathbf{Sm}$ are continuous and cocontinuous.*

Let **PS** (resp. **NS**) be the category of abelian presheaves (resp. Nisnevich sheaves) on **Sm**. The inclusion $i^V_{s,\mathrm{Nis}} : \mathbf{NS} \hookrightarrow \mathbf{PS}$ has a left adjoint $a^V_{s,\mathrm{Nis}}$. Let $\underline{\omega}^*_s : \mathbf{PS} \to \underline{\mathbf{M}}\mathbf{PS}$ and $\omega^*_s : \mathbf{PS} \to \mathbf{MPS}$ be the functors induced by $\underline{\omega}_s$ and ω_s. They have left adjoints $\underline{\omega}_{s,!}$ and $\omega_{s,!}$.

Proposition 6.1.2. *a) We have $\underline{\omega}_{s,!}(\underline{\mathbf{M}}\mathbf{NS}) \subset \mathbf{NS}$ and $\omega_{s,!}(\mathbf{MNS}) \subset \mathbf{NS}$.*
b) For $F \in \mathbf{PS}$, $\omega^*_s F \in \mathbf{MNS} \iff \underline{\omega}^*_s F \in \underline{\mathbf{M}}\mathbf{NS} \iff F \in \mathbf{NS}$.
c) Let $\omega^{\mathrm{Nis}}_s : \mathbf{NS} \to \mathbf{MNS}$ and $\omega_{s,\mathrm{Nis}} : \mathbf{MNS} \to \mathbf{NS}$ be the functors such that

(6.1.2) $$i_{s,\mathrm{Nis}}\omega^{\mathrm{Nis}}_s = \omega^*_s i^V_{s,\mathrm{Nis}}, \quad i^V_{s,\mathrm{Nis}}\omega_{s,\mathrm{Nis}} = \omega_{s,!} i_{s,\mathrm{Nis}},$$

which exist by b). Then $\omega_{s,\mathrm{Nis}}$ is left adjoint to ω^{Nis}_s; both functors are exact and ω^{Nis}_s is strongly additive. We have $\omega_{s,\mathrm{Nis}} = a^V_{s,\mathrm{Nis}}\omega_{s,!} i_{s,\mathrm{Nis}}$ and

(6.1.3) $$\omega_{s,\mathrm{Nis}} a_{s,\mathrm{Nis}} = a^V_{s,\mathrm{Nis}}\omega_{s,!}, \quad \omega^{\mathrm{Nis}}_s a^V_{s,\mathrm{Nis}} = a_{s,\mathrm{Nis}}\omega^*_s.$$

d) Let $\underline{\omega}_s^{\mathrm{Nis}} : \mathbf{NS} \to \underline{\mathbf{M}}\mathbf{NS}$ *and* $\underline{\omega}_{s,\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NS} \to \mathbf{NS}$ *be the functors defined in the same way as* ω_s^{Nis} *and* $\omega_{s,\mathrm{Nis}}$ *in c). Then the similar statements to c) hold for* $\underline{\omega}_s^{\mathrm{Nis}}$ *and* $\underline{\omega}_{s,\mathrm{Nis}}$.

Proof. a) Since λ_s is left adjoint to $\underline{\omega}_s$, we have $\underline{\omega}_{s,!} = \lambda_s^*$, hence the continuity of λ_s proves the first assertion. The second one follows from Theorem 4.1.2, as $\omega_{s,!} = \underline{\omega}_{s,!}\tau_{s,!}$.

b) If $\underline{\omega}_s^* F \in \underline{\mathbf{M}}\mathbf{NS}$, then $\omega_s^* F = \tau_s^* \underline{\omega}_s^* F \in \mathbf{MNS}$ by Theorem 4.1.2. If $\omega_s^* F \in \mathbf{MNS}$, then $\omega_{s,!}\omega_s^* F \xrightarrow{\sim} F \in \mathbf{NS}$ by a) since ω_s^* is fully faithful. If $F \in \mathbf{NS}$, then we have $\underline{\omega}_s^* F \in \underline{\mathbf{M}}\mathbf{NS}$ since $\underline{\omega}_s$ is continuous.

c) The second formula of (6.1.3) follows from the cocontinuity of ω_s (Proposition 6.1.1) and [SGA4, III, Prop. 2.3 (2)]. We prove the rest of the assertions by using Lemma 1.2.6 as follows. In the situation of Lemma 1.2.6, set $\mathcal{C} = \mathbf{PS}$, $\mathcal{C}' = \mathbf{NS}$, $\mathcal{D} = \mathbf{MPS}$, $\mathcal{D}' = \mathbf{MNS}$, $(i_{\mathcal{C}}, i_{\mathcal{D}}) = (i^V_{s,\mathrm{Nis}}, i_{s,\mathrm{Nis}})$ and $(c, c', d) = (\omega_s^*, \omega_s^{\mathrm{Nis}}, \omega_{s,!})$.

The assumption of Lemma 1.2.6 (2) is satisfied since $i_{s,\mathrm{Nis}}$ is strongly additive by Theorem 2.2.7, $i^V_{s,\mathrm{Nis}}$ is strongly additive by the quasi-compactness of the Nisnevich topology, and ω_s^* is strongly additive as a left adjoint. Hence ω_s^{Nis} is also strongly additive.

The assumption of Lemma 1.2.6 (3) is satisfied since $i^V_{s,\mathrm{Nis}}, i_{s,\mathrm{Nis}}$ have exact left adjoints $a^V_{s,\mathrm{Nis}}, a_{s,\mathrm{Nis}}$, and since $\omega_{s,!}$ is exact by [4, Prop. 2.2.1]. Hence the left adjoint $\omega_{s,\mathrm{Nis}}$ of ω_s^{Nis} is exact, and we have $\omega_{s,\mathrm{Nis}} = a^V_{s,\mathrm{Nis}}\omega_{s,!}i_{s,\mathrm{Nis}}$ and $a^V_{s,\mathrm{Nis}}\omega_{s,!} = \omega_{s,\mathrm{Nis}}a_{s,\mathrm{Nis}}$.

The assumption of Lemma 1.2.6 (4) is satisfied. Indeed, the formula $a_{\mathcal{D}}c = c' a_{\mathcal{C}}$ coincides with the second formula of (6.1.3) (which we have proven above), and ω_s^* is exact as a left and right adjoint. Hence ω_s^{Nis} is exact.

d) is shown by the same argument as c), by the cocontinuity of $\underline{\omega}_s$ (Proposition 6.1.1) and by Lemma 1.2.6 applied to $\mathcal{C} = \mathbf{PS}$, $\mathcal{C}' = \mathbf{NS}$, $\mathcal{D} = \underline{\mathbf{M}}\mathbf{PS}$, $\mathcal{D}' = \underline{\mathbf{M}}\mathbf{NS}$, $(i_{\mathcal{C}}, i_{\mathcal{D}}) = (i^V_{s,\mathrm{Nis}}, \underline{i}_{s,\mathrm{Nis}})$ and $(c, c', d) = (\underline{\omega}_s^*, \underline{\omega}_s^{\mathrm{Nis}}, \underline{\omega}_{s,!})$.

Indeed, $\underline{i}_{s,\mathrm{Nis}}$ is strongly additive (Lemma 4.1.1), $\underline{\omega}_s^*$ is exact as a left and right adjoint, $\underline{i}_{s,\mathrm{Nis}}$ has an exact left adjoint $\underline{a}_{s,\mathrm{Nis}}$, and $\underline{\omega}_{s,!}$ is exact. This finishes the proof. □

6.2. **MNST, $\underline{\mathbf{M}}$NST and NST.** Let **PST** be the abelian category of presheaves on **Cor**. The graph functor $c^V : \mathbf{Sm} \to \mathbf{Cor}$ induces an exact faithful functor $c^{V*} : \mathbf{PST} \to \mathbf{PS}$. Let **NST** be the full subcategory of **PST** consisting of $F \in \mathbf{PST}$ such that $c^{V*}F \in \mathbf{NS}$. The functor c^{V*} restricts to $c^{V,\mathrm{Nis}} : \mathbf{NST} \to \mathbf{NS}$. The inclusion $i^V_{\mathrm{Nis}} : \mathbf{NST} \hookrightarrow \mathbf{PST}$ has a left adjoint a^V_{Nis} by [14, Thm. 3.1.4]. By construction, it satisfies

$$c^{V,\mathrm{Nis}}a^V_{\mathrm{Nis}} = a^V_{s,\mathrm{Nis}}c^{V*}. \tag{6.2.1}$$

We consider the functors

$$\underline{\omega} : \underline{\mathbf{M}}\mathbf{Cor} \to \mathbf{Cor}, \qquad \omega : \mathbf{MCor} \to \mathbf{Cor}, \tag{6.2.2}$$

defined in the same way as (6.1.1). They induce $\underline{\omega}^* : \mathbf{PST} \to \underline{\mathbf{M}}\mathbf{PST}$ and $\omega^* : \mathbf{PST} \to \mathbf{MPST}$, which have left adjoints $\underline{\omega}_!$ and $\omega_!$. One has the obvious identifications

$$\underline{c}^* \underline{\omega}^* = \underline{\omega}_s^* c^{V*}, \quad c^* \omega^* = \omega_s^* c^{V*}. \tag{6.2.3}$$

One also sees from [4, (2.2.1)] (and its analogues for $\omega_s, \underline{\omega}_s$) that

$$c^{V*} \underline{\omega}_! = \underline{\omega}_{s,!} \underline{c}^*, \quad c^{V*} \omega_! = \omega_{s,!} c^*. \tag{6.2.4}$$

Proposition 6.2.1. *a) We have* $\underline{\omega}_!(\underline{\mathbf{M}}\mathbf{NST}) \subset \mathbf{NST}$ *and* $\omega_!(\mathbf{MNST}) \subset \mathbf{NST}$.
b) For $F \in \mathbf{PST}$, $\omega^* F \in \mathbf{MNST} \iff \underline{\omega}^* F \in \underline{\mathbf{M}}\mathbf{NST} \iff F \in \mathbf{NST}$.
c) Let $\omega^{\mathrm{Nis}} : \mathbf{NST} \to \mathbf{MNST}$ *and* $\omega_{\mathrm{Nis}} : \mathbf{MNST} \to \mathbf{NST}$ *be the functors such that*

$$i_{\mathrm{Nis}} \omega^{\mathrm{Nis}} = \omega^* i^V_{\mathrm{Nis}}, \quad i^V_{\mathrm{Nis}} \omega_{\mathrm{Nis}} = \omega_! i_{\mathrm{Nis}}, \tag{6.2.5}$$

where the second equality shows that $\omega_{\mathrm{Nis}} = a^V_{\mathrm{Nis}} \omega_! i_{\mathrm{Nis}}$ *by* $a^V_{\mathrm{Nis}} i^V_{\mathrm{Nis}} = \mathrm{Id}$. *Then* ω_{Nis} *is left adjoint to* ω^{Nis}, *and we have*

$$\omega_{\mathrm{Nis}} a_{\mathrm{Nis}} = a^V_{\mathrm{Nis}} \omega_!, \quad \omega^{\mathrm{Nis}} a^V_{\mathrm{Nis}} = a_{\mathrm{Nis}} \omega^*, \tag{6.2.6}$$

$$c^{\mathrm{Nis}} \omega^{\mathrm{Nis}} = \omega_s^{\mathrm{Nis}} c^{V,\mathrm{Nis}}. \tag{6.2.7}$$

Moreover, the functors ω_{Nis} *and* ω^{Nis} *are both exact,* ω^{Nis} *is fully faithful, strongly additive and preserves injectives.*
d) Let $\underline{\omega}^{\mathrm{Nis}} : \mathbf{NST} \to \underline{\mathbf{M}}\mathbf{NST}$ *and* $\underline{\omega}_{\mathrm{Nis}} : \underline{\mathbf{M}}\mathbf{NST} \to \mathbf{NST}$ *be the functors defined in the same way as* ω^{Nis} *and* ω_{Nis} *in c). Then the similar statements as c) holds for* $\underline{\omega}^{\mathrm{Nis}}$ *and* $\underline{\omega}_{\mathrm{Nis}}$.

Proof. a) First we prove $\underline{\omega}_!(\underline{\mathbf{M}}\mathbf{NST}) \subset \mathbf{NST}$. Since $\underline{c}^*(\underline{\mathbf{M}}\mathbf{NST}) \subset \underline{\mathbf{M}}\mathbf{NS}$ (see Lemma-Definition 2.2.5), we have

$$c^{V*} \underline{\omega}_!(\underline{\mathbf{M}}\mathbf{NST}) = \underline{\omega}_{s,!} \underline{c}^*(\underline{\mathbf{M}}\mathbf{NST}) \subset \underline{\omega}_{s,!}(\underline{\mathbf{M}}\mathbf{NS}) \subset \mathbf{NS},$$

where the first equality follows from the first equality of (6.2.4), and the last inclusion follows from Proposition 6.1.2 a). Then by definition of **NST** we have $\underline{\omega}_!(\underline{\mathbf{M}}\mathbf{NST}) \subset \mathbf{NST}$, as desired. The inclusion $\omega_!(\mathbf{MNST}) \subset \mathbf{NST}$ follows from this and Lemma-Definition 4.2.1.

b) follows from Proposition 6.1.2 b), (6.2.4) and the definitions of **NST**, $\underline{\mathbf{M}}\mathbf{NST}$ and **MNST**.

c) The full faithfulness of ω^{Nis} follows from that of ω^* [4, Prop. 2.2.1], and (6.2.7) follows from (6.2.3). The strategy of the rest of the proof is similar to that of c) in Proposition 6.1.2. In the situation of

Lemma 1.2.6, set $\mathcal{C} = \mathbf{PST}$, $\mathcal{C}' = \mathbf{NST}$, $\mathcal{D} = \mathbf{MPST}$, $\mathcal{D}' = \mathbf{MNST}$, $(i_{\mathcal{C}}, i_{\mathcal{D}}) = (i^V_{\mathrm{Nis}}, i_{\mathrm{Nis}})$ and $(c, c', d) = (\omega^*, \omega^{\mathrm{Nis}}, \omega_!)$.

The assumptions of Lemma 1.2.6 (2) and (3) are satisfied, since i_{Nis} is strongly additive by Theorem 4.2.4 (1), i^V_{Nis} is strongly additive by the quasi-compactness of Nisnevich cohomology, ω^* is strongly additive as a left and right adjoint, i^V_{Nis}, i_{Nis} have exact left adjoints $a^V_{\mathrm{Nis}}, a_{\mathrm{Nis}}$ by [8] and 4.2.4 (2), and $\omega_!$ is exact by [4, Prop. 2.2.1]. Therefore, the assertions follow, except for the second identity of (6.2.6) and the exactness of ω^{Nis}. (Note that the exactness of ω_{Nis} implies that ω^{Nis} preserves injectives.)

We can prove the second identity of (6.2.6) as follows. Its first identity yields a base change morphism

$$a_{\mathrm{Nis}}\omega^* \Rightarrow \omega^{\mathrm{Nis}} a^V_{\mathrm{Nis}}.$$

Let $F \in \mathbf{PST}$. We want to show that the morphism $a_{\mathrm{Nis}}\omega^* F \to \omega^{\mathrm{Nis}} a^V_{\mathrm{Nis}} F$ is an isomorphism. Since c^{Nis} is conservative as $\mathbf{MSm}$ and $\mathbf{MCor}$ have the same objects, it suffices to show that the induced morphism $c^{\mathrm{Nis}} a_{\mathrm{Nis}}\omega^* F \to c^{\mathrm{Nis}}\omega^{\mathrm{Nis}} a^V_{\mathrm{Nis}} F$ is an isomorphism. Since

$$c^{\mathrm{Nis}} a_{\mathrm{Nis}}\omega^* \overset{(4.2.3)}{=} a_{s,\mathrm{Nis}} c^* \omega^* \overset{(6.2.3)}{=} a_{s,\mathrm{Nis}}\omega_s^* c^{V*},$$
$$c^{\mathrm{Nis}}\omega^{\mathrm{Nis}} a^V_{\mathrm{Nis}} \overset{(6.2.7)}{=} \omega_s^{\mathrm{Nis}} c^{V,\mathrm{Nis}} a^V_{\mathrm{Nis}} \overset{(6.2.1)}{=} \omega_s^{\mathrm{Nis}} a^V_{s,\mathrm{Nis}} c^{V*},$$

the above morphism is rewritten as

$$a_{s,\mathrm{Nis}}\omega_s^* c^{V*} F \to \omega_s^{\mathrm{Nis}} a^V_{s,\mathrm{Nis}} c^{V*} F,$$

which is an isomorphism by Proposition 6.1.2 c).

Now, the formula we have proven now and the exactness of ω^* as a left and right adjoint show that the assumption of Lemma 1.2.6 (4) is satisfied. Hence ω^{Nis} is exact, as desired. This finishes the proof of c).

d). The proof is completely parallel to that of c). To see this, it suffices to observe the following: $\underline{i}_{\mathrm{Nis}}$ is strongly additive and has an exact left adjoint by [4, Lem. 4.5.3, Th. 4.5.5], $\underline{\omega}^*$ is strongly additive as a left and right adjoint, $\underline{\omega}_!$ is exact by [4, Prop. 2.3.1], $\underline{c}^{\mathrm{Nis}}$ is conservative since $\underline{\mathbf{M}}\mathbf{Sm}$ and $\underline{\mathbf{M}}\mathbf{Cor}$ have the same objects, the base change morphism $\underline{a}_{s,\mathrm{Nis}}\underline{\omega}_s^* \Rightarrow \underline{\omega}_s^{\mathrm{Nis}}\underline{a}^V_{s,\mathrm{Nis}}$ is an isomorphism by Proposition 6.1.2 d), and we have the following identifications:

$$\underline{c}^{\mathrm{Nis}}\underline{a}_{\mathrm{Nis}}\underline{\omega}^* \overset{(2.2.3)}{=} \underline{a}_{s,\mathrm{Nis}}\underline{c}^*\underline{\omega}^* \overset{(6.2.3)}{=} \underline{a}_{s,\mathrm{Nis}}\underline{\omega}_s^* c^{V*},$$
$$\underline{c}^{\mathrm{Nis}}\underline{\omega}^{\mathrm{Nis}} a^V_{\mathrm{Nis}} \overset{(6.2.7)}{=} \underline{\omega}_s^{\mathrm{Nis}} c^{V,\mathrm{Nis}} a^V_{\mathrm{Nis}} \overset{(6.2.1)}{=} \underline{\omega}_s^{\mathrm{Nis}} a^V_{s,\mathrm{Nis}} c^{V*}.$$

This finishes the proof. □

6.3. **Relation between cohomologies.** We now prove Theorem 3 (2) from the introduction.

Lemma 6.3.1. *Let* $I \in \mathbf{MNST}$ *be an injective object. Then* $\omega_{\mathrm{Nis}} I \in \mathbf{NST}$ *is flasque.*

Proof. Let $U \to X$ in $\mathbf{Sm}$ be an open dense immersion. We need to show the surjectivity of

$$\omega_{\mathrm{Nis}} I(X) = \varinjlim_{M \in \mathbf{MSm}(X)} I(M) \to \omega_{\mathrm{Nis}} I(U) = \varinjlim_{N \in \mathbf{MSm}(U)} I(U).$$

We fix $M \in \mathbf{MSm}(X)$ and will show that the composition $I(M) \to \omega_{\mathrm{Nis}} I(X) \to \omega_{\mathrm{Nis}} I(U)$ is already surjective. This follows from the functoriality of $\omega^!$, but for clarity we give an explicit argument. Take any $N \in \mathbf{MSm}(U)$. Let $\overline{N}'$ be the closure of the image of $U \to \overline{N} \times \overline{M}$, and let $\overline{L}$ be the blow-up of $\overline{N}'$ along $(\overline{N}' \setminus U)_{\mathrm{red}}$. Set $L := (\overline{L}, L^\infty) \in \mathbf{MCor}$ where L^∞ is the pull-back of N^∞ along the composition $\overline{L} \to \overline{N}' \to \overline{N}$. We have a commutative diagram in $\underline{\mathbf{M}}\mathbf{Cor}$

$$\begin{array}{ccccc} & & (U,\emptyset) & \longrightarrow & (X,\emptyset) \\ & \swarrow & \downarrow & & \downarrow \\ N & \longleftarrow & L^{(n)} & \longrightarrow & M \end{array}$$

for sufficiently large n (see [4, Def. 1.4.1]). Since $U \to X$ is an open dense immersion, $\mathbf{Cor}(V, U) \to \mathbf{Cor}(V, X)$ is injective for any $V \in \mathbf{Sm}$, which in turn implies the injectivity of $\mathbb{Z}_{\mathrm{tr}}(L^{(n)}) \to \mathbb{Z}_{\mathrm{tr}}(M)$ in $\mathbf{MNST}$. Since I is an injective object, we conclude that $I(M) \to I(L^{(n)})$ is surjective. This proves the lemma, as the canonical map $I(N) \to \omega_{\mathrm{Nis}} I(U)$ factors through $I(L^{(n)})$. □

Theorem 6.3.2.

(1) *For any* $M \in \mathbf{MCor}$, $G \in \mathbf{NST}$ *and* $p \geq 0$, *we have a canonical isomorphism*

$$\mathrm{Ext}^p_{\mathbf{MNST}}(\mathbb{Z}_{\mathrm{tr}}(M), \omega^{\mathrm{Nis}} G) \cong H^p_{\mathrm{Nis}}(M^{\mathrm{o}}, G).$$

(2) *For any* $X \in \mathbf{Sm}$, $F \in \mathbf{MNST}$ *and* $p \geq 0$, *we have a canonical isomorphism*

$$H^p_{\mathrm{Nis}}(X, \omega_{\mathrm{Nis}} F) \cong \varinjlim_{M \in \mathbf{MSm}(X)} H^p_{\mathrm{Nis}}(\overline{M}, F_M).$$

Proof. It follows from Proposition 6.2.1 and Theorem [4, Th. A.9.1] that

$$(R^p i_{\mathrm{Nis}}) \omega^{\mathrm{Nis}} = R^p(i_{\mathrm{Nis}} \omega^{\mathrm{Nis}}) = R^p(\omega^* i^V_{\mathrm{Nis}}) = \omega^* R^p i^V_{\mathrm{Nis}}.$$

By Prop. A.1.1 and the projectivity of $\mathbb{Z}_{\mathrm{tr}}(M)$ and $\mathbb{Z}^V_{\mathrm{tr}}(M^o)$, we get

$$\begin{aligned}&\mathrm{Ext}^p_{\mathbf{MNST}}(\mathbb{Z}_{\mathrm{tr}}(M), \omega^{\mathrm{Nis}}G) \cong \mathbf{MPST}(\mathbb{Z}_{\mathrm{tr}}(M), (R^p i_{\mathrm{Nis}})\omega^{\mathrm{Nis}}G)\\ &\cong \mathbf{MPST}(\mathbb{Z}_{\mathrm{tr}}(M), \omega^* R^p i^V_{\mathrm{Nis}} G) \cong \mathbf{PST}(\omega_! \mathbb{Z}_{\mathrm{tr}}(M), R^p i^V_{\mathrm{Nis}} G)\\ &\cong \mathbf{PST}(\mathbb{Z}^V_{\mathrm{tr}}(M^o), R^p i^V_{\mathrm{Nis}} G) \cong H^p_{\mathrm{Nis}}(M^o, G),\end{aligned}$$

whence (1).

Given $X \in \mathbf{Sm}$, we define functors $\Gamma^{\to}_X : \mathbf{MNST} \to \mathbf{Ab}$ and $\Gamma^V_X : \mathbf{NST} \to \mathbf{Ab}$ by

$$\Gamma^{\to}_X(F) = \varinjlim_{M \in \mathbf{MSm}(X)} F(M), \qquad \Gamma^V_X(G) = G(X).$$

We have $\Gamma^{\to}_X = \Gamma^V_X \omega_{\mathrm{Nis}}$. By [4, Th. A.9.1] and Lemma 6.3.1, we get $(R^p\Gamma^V_X)\omega_{\mathrm{Nis}} = R^p\Gamma^{\to}_X$ for any $p \geq 0$, since ω_{Nis} is exact. Taking an injective resolution $F \to I^\bullet$ in **MNST**, we proceed

$$\begin{aligned}R^p\Gamma^{\to}_X(F) &\cong H^p(\Gamma^{\to}_X I^\bullet) \cong H^p(\varinjlim_{M \in \mathbf{MSm}(X)} I^\bullet(M))\\ &\cong \varinjlim_{M \in \mathbf{MSm}(X)} H^p(I^\bullet(M)) \qquad \text{[4, Lemma 1.7.4]}\\ &\cong \varinjlim_{M \in \mathbf{MSm}(X)} \varinjlim_{N \in \Sigma^{\mathrm{fin}} \downarrow M} H^p_{\mathrm{Nis}}(\overline{N}, F_N) \qquad \text{(Theorem 5.1.3)}\\ &\cong \varinjlim_{M \in \mathbf{MSm}(X)} H^p_{\mathrm{Nis}}(\overline{M}, F_M).\end{aligned}$$

Here the last isomorphism holds since any $N \in \Sigma^{\mathrm{fin}} \downarrow M$ gives rise to an object $N \in \mathbf{MSm}(X)$. This concludes the proof of (2). □

7. Passage to derived categories

In this section, we extend the previous results to derived categories. The main result is an extension of Proposition 2.2.9 and Theorem 5.1.3 to unbounded complexes (Proposition 7.4.2 (2) and Theorem 7.5.1 (3)).

7.1. Compactness.

Proposition 7.1.1. *Let $M \in \underline{\mathbf{M}}\mathbf{Sm}$. Then the following objects are compact:*

1. $\mathbb{Z}_{\mathrm{tr}}(M)[0]$ *in* $D(\underline{\mathbf{M}}\mathbf{NST})$*, and in* $D(\mathbf{MNST})$ *if* M *is proper.*
2. $\mathbb{Z}(M)[0]$ *in* $D(\underline{\mathbf{M}}\mathbf{NS}^{\mathrm{fin}})$ *and* $D(\underline{\mathbf{M}}\mathbf{NS})$*.*

Proof. (1) follows from Proposition 2.2.9 (resp. from Theorem 5.1.3) and the known commutation of Nisnevich cohomology with filtered colimits of sheaves, via hypercohomology spectral sequences which are convergent and Corollary 2.2.10 (resp. 5.1.4). (2) is seen similarly, using [4, (3.6.1) and Prop. 4.4.2]. □

7.2. **Strong additivity.**

Theorem 7.2.1.

(1) *(cf.* [4, Prop. 4.3.3 (2)]*) The functor* $R\underline{b}_s^{\mathrm{Nis}} : D(\underline{\mathbf{M}}\mathbf{NS}) \to D(\underline{\mathbf{M}}\mathbf{NS}^{\mathrm{fin}})$ *is right adjoint to* $D(\underline{b}_{s,\mathrm{Nis}})$ *and is strongly additive. The counit map* $D(\underline{b}_{s,\mathrm{Nis}})R\underline{b}_s^{\mathrm{Nis}} \to \mathrm{Id}$ *is an isomorphism.*

(2) *The functor* $R(\underline{b}_s^{\mathrm{Nis}} \circ \underline{c}^{\mathrm{Nis}})$ *is strongly additive.*

Proof. (1) The adjunction statement follows from Proposition A.2.5. The second assertion implies the third by Lemma A.2.7. It remains to prove the strong additivity of $R\underline{b}_s^{\mathrm{Nis}}$. For this, we check that the hypothesis of Proposition A.2.8 c) are verified: we take the $\mathbb{Z}(M)$, $M \in \mathbf{MSm}^{\mathrm{fin}}$, as a set of generators. We have $\underline{b}_{s,\mathrm{Nis}}\mathbb{Z}(M) = \mathbb{Z}(M)$. Their compactness follows from Proposition 7.1.1 (2).

(2) Same argument as (1), using Proposition 7.1.1 (1) as well. □

7.3. **From** $D(\underline{\mathbf{M}}\mathbf{NS}^{\mathrm{fin}})$ **to** $D(\underline{\mathbf{M}}\mathbf{NS})$**.**

We first extend [4, Not. 4.4.1] from sheaves to complexes of sheaves:

Notation 7.3.1. Let $N \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ and let K be a complex on $\underline{\mathbf{M}}\mathbf{NS}^{\mathrm{fin}}$. We write K_N for the complex of sheaves on $(\overline{N})_{\mathrm{Nis}}$ deduced from K via the isomorphism of sites from [4, Lemma 3.1.3]. If K is a complex on $\underline{\mathbf{M}}\mathbf{NS}$, we write K_N for $(\underline{b}_s^{\mathrm{Nis}}K)_N$.

The following extends [4, Prop. 4.4.2] to unbounded complexes of sheaves.

Proposition 7.3.2. *Let* $M \in \underline{\mathbf{M}}\mathbf{Sm}$*. For* $K \in D(\underline{\mathbf{M}}\mathbf{NS})$*, we have natural isomorphisms*

$$\begin{aligned}\mathrm{Hom}_{D(\underline{\mathbf{M}}\mathbf{NS})}(\mathbb{Z}_{\mathrm{tr}}(M), K[i]) &\simeq \varinjlim_{N\in\underline{\Sigma}^{\mathrm{fin}}\downarrow M} \mathbb{H}^i_{\mathrm{Nis}}(\overline{N}, K_N)\\ &\simeq \varinjlim_{N\in\underline{\Sigma}^{\mathrm{fin}}\downarrow M} \mathbb{H}^i_{\mathrm{Nis}}(\overline{N}, (R\underline{b}_s^{\mathrm{Nis}}K)_N), \quad i \in \mathbb{Z}.\end{aligned}$$

Proof. Define as in loc. cit. functors $\Gamma^{\downarrow}_M : \underline{\mathbf{M}}\mathbf{NS}^{\mathrm{fin}} \to \mathbf{Ab}$ and $\underline{\Gamma}_M : \underline{\mathbf{M}}\mathbf{NS} \to \mathbf{Ab}$ by

$$\Gamma^{\downarrow}_M(G) = \varinjlim_{N\in\underline{\Sigma}^{\mathrm{fin}}\downarrow M} G(N), \quad \underline{\Gamma}_M(F) = F(M).$$

We have $\Gamma^{\downarrow}_M = \underline{\Gamma}_M\underline{b}_{s,\mathrm{Nis}}$ and we shall show that the natural transformation (A.2.1)

$$R\Gamma^{\downarrow}_M \Rightarrow R\underline{\Gamma}_M \circ D(\underline{b}_{s,\mathrm{Nis}}) \tag{7.3.1}$$

is invertible. For this, we apply Lemma A.2.7. Its first condition is given by [4, Lemma 4.4.3], which says that $\underline{b}_{s,\mathrm{Nis}}$ sends injectives to flabbys; by [4, Th. A.9.1 and Lemma A.9.3], this already yields isomorphisms

$$R^p\Gamma^{\downarrow}_M \xrightarrow{\sim} R^p\underline{\Gamma}_M \circ \underline{b}_{s,\mathrm{Nis}}, \quad p \geq 0. \tag{7.3.2}$$

We are now left to show the strong additivity of the three functors. For $D(\underline{b}_{s,\mathrm{Nis}})$, this follows from the strong additivity of $\underline{b}_{s,\mathrm{Nis}}$ as a left adjoint, and Proposition A.2.8 a).

For $R\Gamma^{\downarrow}_M$ and $R\underline{\Gamma}_M$, we check that the conditions of Proposition A.2.8 b) are verified. For $R\underline{\Gamma}_M$, Condition (ii) follows from the vanishing statement in (1) (use the compact projective generator $\mathbb{Z}$ of $\mathbf{Ab}$), and Condition (i) follows similarly from the known commutation of Nisnevich cohomology with filtered colimits of sheaves. The case of $R\Gamma^{\downarrow}_M$ is reduced to this one by (7.3.2).

Thus we get an isomorphism

$$\mathrm{Hom}_{D(\underline{\mathbf{M}}\mathbf{NS})}(\mathbb{Z}_{\mathrm{tr}}(M), \underline{b}_{s,\mathrm{Nis}}L[i]) \simeq \varinjlim_{N\in\underline{\Sigma}^{\mathrm{fin}}\downarrow M} \mathbb{H}^i_{\mathrm{Nis}}(\overline{N}, L_N), \quad i \in \mathbb{Z}$$

for any complex L on $\underline{\mathbf{M}}\mathbf{NS}^{\mathrm{fin}}$. Setting $L = \underline{b}_s^{\mathrm{Nis}}K$, we get the first isomorphism thanks to the isomorphism $\underline{b}_{s,\mathrm{Nis}}\underline{b}_s^{\mathrm{Nis}} \xrightarrow{\sim} \mathrm{Id}$ of [4, Prop. 4.3.3 (2)]. Composing (7.3.1) with $R\underline{b}_s^{\mathrm{Nis}}$ and using Theorem 7.2.1 (1), we get an isomorphism

$$R\Gamma^{\downarrow}_M \circ R\underline{b}_s^{\mathrm{Nis}} \xrightarrow{\sim} R\underline{\Gamma}_M \tag{7.3.3}$$

which yields the second isomorphism of Proposition 7.3.2. □

7.4. From $D(\underline{\mathbf{M}}\mathbf{NS})$ to $D(\underline{\mathbf{M}}\mathbf{NST})$.

Notation 7.4.1. Let $N \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ and let K be a complex on $\underline{\mathbf{M}}\mathbf{NST}^{\mathrm{fin}}$. We write K_N for $(\underline{b}_s^{\mathrm{Nis}}\underline{c}^{\mathrm{Nis}}K)_N$.

The following extends Proposition 2.2.9 to unbounded complexes of sheaves.

Proposition 7.4.2. *Let $M \in \underline{\mathbf{M}}\mathbf{Cor}$. For $K \in D(\underline{\mathbf{M}}\mathbf{NST})$, we have a natural isomorphism*

$$\begin{aligned}\mathrm{Hom}_{D(\underline{\mathbf{M}}\mathbf{NST})}(\mathbb{Z}_{\mathrm{tr}}(M), K[i]) &\simeq \varinjlim_{N\in\underline{\Sigma}^{\mathrm{fin}}\downarrow M} \mathbb{H}^i_{\mathrm{Nis}}(\overline{N}, K_N)\\ &\simeq \varinjlim_{N\in\underline{\Sigma}^{\mathrm{fin}}\downarrow M} \mathbb{H}^i_{\mathrm{Nis}}(\overline{N}, (R\underline{b}_s^{\mathrm{Nis}}D(\underline{c}^{\mathrm{Nis}})K)_N), \quad i \in \mathbb{Z}.\end{aligned}$$

Proof. Define as before functors $\underline{\Gamma}_M : \underline{\mathbf{M}}\mathbf{NS} \to \mathbf{Ab}$ and $\underline{\Gamma}_M^T : \underline{\mathbf{M}}\mathbf{NST} \to \mathbf{Ab}$ by

$$\underline{\Gamma}_M(F) = F(M), \quad \underline{\Gamma}_M^T(G) = G(M).$$

We have $\underline{\Gamma}_M^T = \underline{\Gamma}_M \circ \underline{c}^{\mathrm{Nis}}$ thanks to the adjunction $(\underline{c}_{\mathrm{Nis}}, \underline{c}^{\mathrm{Nis}})$ of Theorem 2.2.7. Moreover, $\underline{c}^{\mathrm{Nis}}$ is exact by this theorem. Let us show as before that the natural transformation (A.2.1)

$$R\underline{\Gamma}_M^T \Rightarrow R\underline{\Gamma}_M \circ D(\underline{c}^{\mathrm{Nis}}) \tag{7.4.1}$$

is invertible. We copy the argument of the previous subsection, using Lemma A.2.7.

Its first condition is given by [4, Lemma 4.6.1], which says that $\underline{c}^{\mathrm{Nis}}$ sends injectives to flabbys; by [4, Th. A.9.1 and Lemma A.9.3], this already yields isomorphisms

$$R^p\underline{\Gamma}_M^T \xrightarrow{\sim} R^p\underline{\Gamma}_M \circ \underline{c}^{\mathrm{Nis}}, \quad p \geq 0. \tag{7.4.2}$$

We are now left to show the strong additivity of the three functors. For $D(\underline{c}^{\mathrm{Nis}})$, this follows from the strong additivity of $\underline{c}^{\mathrm{Nis}}$ (Theorem 2.2.7) and Proposition A.2.8 a). For the two other functors, we need to check the conditions of Proposition A.2.8 b); this was done for $R\underline{\Gamma}_M$ in the previous section, and the case of $R\underline{\Gamma}_M^T$ is reduced to this one by (7.4.2).

Thus we get an isomorphism

$$\mathrm{Hom}_{D(\underline{\mathbf{M}}\mathbf{NST})}(\mathbb{Z}_{\mathrm{tr}}(M), L[i]) \xrightarrow{\sim} \mathrm{Hom}_{D(\underline{\mathbf{M}}\mathbf{NS})}(\mathbb{Z}_{\mathrm{tr}}(M), \underline{c}^{\mathrm{Nis}} L[i]) \quad i \in \mathbb{Z}$$

for any complex L on $\underline{\mathbf{M}}\mathbf{NST}$, and we get the first isomorphism by using Proposition 7.3.2. For the second one, composing (7.3.3) with (7.4.1), we obtain an isomorphism

$$R\underline{\Gamma}_M^T \simeq R\Gamma_M^{\downarrow} \circ R\underline{b}_s^{\mathrm{Nis}} \circ D(\underline{c}^{\mathrm{Nis}}) \tag{7.4.3}$$

which yields the second isomorphism of Proposition 7.4.2. □

7.5. From $D(\underline{\mathbf{M}}\mathbf{NST})$ to $D(\mathbf{MNST})$.

Theorem 7.5.1.

(1) *The functor $D(\tau_{\mathrm{Nis}}) : D(\mathbf{MNST}) \to D(\underline{\mathbf{M}}\mathbf{NST})$ is fully faithful.*

(2) *One has isomorphisms*

$$\begin{aligned}\mathrm{Hom}_{D(\mathbf{MNST})}(\mathbb{Z}_{\mathrm{tr}}(M), K[i]) &\simeq \varinjlim_{N \in \underline{\Sigma}^{\mathrm{fin}} \downarrow M} \mathbb{H}^i_{\mathrm{Nis}}(\overline{N}, K_N) \\ &\simeq \varinjlim_{N \in \underline{\Sigma}^{\mathrm{fin}} \downarrow M} \mathbb{H}^i_{\mathrm{Nis}}(\overline{N}, (R\underline{b}_s^{\mathrm{Nis}} D(\underline{c}^{\mathrm{Nis}} \tau_{\mathrm{Nis}}))K_N), \quad i \in \mathbb{Z}\end{aligned}$$

for any complex K on $\mathbf{MNST}$*, where $K_N := (\underline{b}_s^{\mathrm{Nis}} \underline{c}^{\mathrm{Nis}} \tau^{\mathrm{Nis}} K)_N$.*

Proof. Recall that τ_{Nis} and τ^{Nis} are both exact and strongly additive: see Lemma 4.2.5 (2) and Theorem 5.1.1 (τ_{Nis} is strongly additive as a left adjoint). Then we get from Proposition A.2.4 an adjunction $(D(\tau_{\mathrm{Nis}}), D(\tau^{\mathrm{Nis}}))$ and from Lemma A.2.7 an isomorphism

$$D(\tau^{\mathrm{Nis}})D(\tau_{\mathrm{Nis}}) \xleftarrow{\sim} D(\tau^{\mathrm{Nis}}\tau_{\mathrm{Nis}}) \xrightarrow{\sim} D(\mathrm{Id}) = \mathrm{Id}$$

which shows the full faithfulness of $D(\tau_{\mathrm{Nis}})$. This shows (1). (2) now follows from (1) and Proposition 7.4.2 (2). □

Remark 7.5.2. One can show that the essential image of $D(\tau_{\mathrm{Nis}})$ is

$$D_{\mathbf{MNST}}(\underline{\mathbf{M}}\mathbf{NST}) := \{K \in D(\underline{\mathbf{M}}\mathbf{NST}) \mid H^i(K) \in \tau_{\mathrm{Nis}}(\mathbf{MNST})\ \forall i \in \mathbb{Z}\}.$$

Since the proof involves delicate and lengthy arguments relying on the notion of left-completeness, we skip it (see [3].)

7.6. **$D(\mathbf{MNST})$, $D(\underline{\mathbf{M}}\mathbf{NST})$ and $D(\mathbf{NST})$.** We leave it to the reader to produce an unbounded version of Theorem 6.3.2.

Appendix A. Categorical toolbox, II

A.1. **A spectral sequence.** The following convenient proposition is used several times in the paper.

Proposition A.1.1. *Let $a : \mathcal{B} \leftrightarrows \mathcal{A} : i$ be a pair of adjoint functors between abelian categories (a is left adjoint to i). Suppose that $\mathcal{A}$ has enough injectives and that a is exact. Then, for any $(A, B) \in \mathcal{A} \times \mathcal{B}$, there is a convergent spectral sequence*

$$\mathrm{Ext}^p_{\mathcal{B}}(B, R^q iA) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{A}}(aB, A).$$

If B is projective, this spectral sequence collapses to isomorphisms

$$\mathcal{B}(B, R^q iA) \simeq \mathrm{Ext}^q_{\mathcal{A}}(aB, A). \tag{A.1.1}$$

Proof. Fix B. By adjunction, the composition of functors

$$\mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{\mathcal{B}(B,-)} \mathbf{Ab}$$

is isomorphic to $\mathcal{A}(aB, -)$. We then get the spectral sequence from [4, Th. A.9.1, Ex. A.9.2]. The last fact is obvious. □

A.2. **Unbounded derived categories.**

Theorem A.2.1 ([6, Th. 14.3.1])**.** *Let $\mathcal{A}$ be a Grothendieck category. a) Let $K(\mathcal{A})$ be the unbounded homotopy category of $\mathcal{A}$. The localisation functor $\lambda_{\mathcal{A}} : K(\mathcal{A}) \to D(\mathcal{A})$ has a right adjoint $\rho_{\mathcal{A}}$, whose essential image is (by definition) the full subcategory of* homotopically injective *complexes.*[2]

[2] This is the same notion as Spaltenstein's K-injective [12].

b) Let $F : K(\mathcal{A}) \to \mathcal{T}$ be a triangulated functor. Then F has a (universal) right Kan extension RF relative to $\lambda_\mathcal{A}$, given by $RF = F \circ \rho_\mathcal{A}$. In particular, any left exact functor $F : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is another abelian category, has a total right derived functor $RF : D(\mathcal{A}) \to D(\mathcal{B})$ given by $RF = \lambda_\mathcal{B} \circ K(F) \circ \rho_\mathcal{A}$.
c) The restriction of RF to $D^+(\mathcal{A})$ is the total derived functor R^+F (cf. [13, §2, Rem. 1.6]*).*

Let us justify d), which is not in [6]: the point is that $\rho_\mathcal{A}$ carries $D^+(\mathcal{A})$ into $K^+(\mathcal{A})$ [6, Th. 13.3.7].

Definition A.2.2. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between Grothendieck categories. An object $C \in K(\mathcal{A})$ is *F-acyclic* if the morphism

$$\lambda_\mathcal{B} K(F)C \to RF\lambda_\mathcal{A} C = \lambda_\mathcal{B} K(F)\rho_\mathcal{A}\lambda_\mathcal{A} C$$

given by the unit map of the adjunction $(\lambda_\mathcal{A}, \rho_\mathcal{A})$ is an isomorphism.

Example A.2.3. If F is exact, every object of $K(\mathcal{A})$ is F-acyclic.

Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be a chain of left exact functors between Grothendieck categories. The unit map of the adjunction $(\lambda_\mathcal{B}, \rho_\mathcal{B})$ yields a natural transformation

$$R(GF) \Rightarrow RG \circ RF. \tag{A.2.1}$$

The following lemma is made tautological by Definition A.2.2:

Lemma A.2.4. (A.2.1) *is a natural isomorphism if and only if F carries homotopically injectives to G-acyclics. In particular,* (A.2.1) *is a natural isomorphism provided G is exact (see Example A.2.3).* □

Here is a first application:

Proposition A.2.5. *Assume $\mathcal{C} = \mathcal{A}$, F right adjoint to G, and G exact. Then RF is right adjoint to $RG = D(G)$.*

Proof. This is a special case of [6, Th. 14.4.5]. □

We come back to the general situation. Suppose that F carries injectives of $\mathcal{A}$ to G-acyclics. Then [4, Th. A.9.1] implies that (A.2.1) is an isomorphism when restricted to $D^+(\mathcal{A})$ ([13, §2, Prop. 3.1], [6, Th. 13.3.7 and Prop. 13.3.13]). This is not true on $D(\mathcal{A})$ in general, as pointed out by Ayoub and Riou:

Example A.2.6. Let $\mathcal{B} = \mathrm{Mod}\text{–}\mathbb{Z}[\mathbb{Z}/2]$, $\mathcal{A} = \mathcal{B}^\mathbb{N}$ and $\mathcal{C} = \mathbf{Ab}$; let $F = \bigoplus_\mathbb{N}$ and $G = H^0(\mathbb{Z}/2, -)$. The above hypotheses are verified:

since $\mathbb{Z}[\mathbb{Z}/2]$ is Noetherian, a direct sum of injectives is injective. Let $M = (\mathbb{Z}/2[n])_{n\in\mathbb{N}} \in D(\mathcal{A})$. We claim that the map

$$R(GF)(M) \to RG \circ RF(M) \tag{A.2.2}$$

is not an isomorphism. Indeed, $GF = F'G'$ where $G' : \mathcal{A} \to \mathbf{Ab}^{\mathbb{N}}$ is $H^0(\mathbb{Z}/2, -)$ and $F' : \mathbf{Ab}^{\mathbb{N}} \to \mathbf{Ab}$ is $\bigoplus_{\mathbb{N}}$. Let $C = RG(\mathbb{Z}/2)$, so that $H^q(C) = \mathbb{Z}/2$ for $q \geq 0$ and $H^q(C) = 0$ for $q < 0$. Then, by Lemma A.2.4:

$$R(GF)(M) = R(F'G')(M) = RF' \circ RG'(M) = \bigoplus_{n\in\mathbb{N}} C[n].$$

On the other hand,

$$RG \circ RF(M) = RG(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/2[n]).$$

But, in $D(\mathcal{B})$, we have $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/2[n] \xrightarrow{\sim} \prod_{n\in\mathbb{N}} \mathbb{Z}/2[n]$, and RG commutes with products as a right adjoint. Hence

$$RG \circ RF(M) = \prod_{n\in\mathbb{N}} C[n].$$

For $q \in \mathbb{Z}$, we have $H^q(\bigoplus_{n\in\mathbb{N}} C[n]) = \bigoplus_{q+n\geq 0} \mathbb{Z}/2$ and $H^q(\prod_{n\in\mathbb{N}} C[n]) = \prod_{q+n\geq 0} \mathbb{Z}/2$.

However, we have the following lemma of Ayoub:

Lemma A.2.7. *Suppose that F carries injectives to G-acyclics and that RF, RG and $R(GF)$ are strongly additive. Then (A.2.1) is an isomorphism.*

(In example A.2.6, RG is not strongly additive.)

Proof (Ayoub). Let $M \in D(\mathcal{A})$. We have to show that (A.2.2) is an isomorphism. Viewing M as an object of $K(\mathcal{A})$, we have an isomorphism

$$\operatorname{hocolim}_n \sigma_{\geq n} M \xrightarrow{\sim} M$$

where $\sigma_{\geq n}$ is the stupid truncation. This isomorphism still holds in $D(\mathcal{A})$, because $\lambda_{\mathcal{A}}$ is strongly additive. By the hypothesis, this reduces us to the case where $M \in D^+(\mathcal{A})$, and therefore to Grothendieck's theorem (cf. Theorem A.2.1 c)). □

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between Grothendieck categories. In view of Lemma A.2.7, we need a practical sufficient condition to ensure that RF is strongly additive. The following ones are adapted to the context of this paper:

Proposition A.2.8. *a) If F is strongly additive and exact, $RF = D(F)$ is strongly additive.*
b) Suppose that

(i) *For any $p \geq 0$, R^pF is strongly additive.*
(ii) *There exists a set $\mathcal{E}$ of compact projective generators of $\mathcal{B}$ such that, for any $E \in \mathcal{E}$, there is an integer $cd_F(E)$ such that*

$$\mathcal{B}(E, R^pF(A)) = 0 \text{ for } p > cd_F(E) \text{ and for all } A \in \mathcal{A}.$$

Then RF is strongly additive.
c) Suppose that RF admits a left adjoint G which sends a set (E_α) of compact generators of $D(\mathcal{B})$ to compact objects of $D(\mathcal{A})$. Then RF is strongly additive.

Proof. a) The strong additivity of F easily implies that of $K(F)$, which in turn implies that of $D(F)$ since $\lambda_{\mathcal{B}}$ is strongly additive as a left adjoint.

b) Let $(C_i)_{i\in I} \in D(\mathcal{A})^I$. We must show that the map

$$\bigoplus_{i\in I} RF(C_i) \to RF(\bigoplus_{i\in I} C_i) \tag{A.2.3}$$

is an isomorphism. Since the $E[n]$, $E \in \mathcal{E}$, $n \in \mathbb{Z}$, are a set of generators of $D(\mathcal{B})$, it suffices to check this after applying $D(\mathcal{B})(E[n], -)$ for all (E, n). Since E is projective, we have an isomorphism

$$D(\mathcal{B})(E[n], D) \simeq \mathcal{B}(E, H^n(D))$$

for any $D \in D(\mathcal{B})$; since E is compact in $\mathcal{B}$, this formula shows that $E[n]$ is compact in $D(\mathcal{B})$. Therefore we must show that the homomorphisms

$$\bigoplus_{i\in I} \mathcal{B}(E, H^n(RFC_i)) \to \mathcal{B}(E, H^n(RF \bigoplus_{i\in I} C_i)).$$

are bijective. By (ii), the spectral sequence

$$\mathcal{B}(E, R^pFH^q(C)) \Rightarrow \mathcal{B}(E, H^{p+q}(RFC))$$

converges for any $C \in D(\mathcal{A})$. Thus it suffices to show that the homomorphisms

$$\bigoplus_{i\in I} \mathcal{B}(E, R^pFH^q(C_i)) \to \mathcal{B}(E, R^pFH^q(\bigoplus_{i\in I} C_i))$$

are bijective. By (i), this follows from the compactness of E.

c) Keep the notation of b). We may test (A.2.3) on the E_α's. By their compactness, we must show that the composition

$$\bigoplus_{i\in I} D(\mathcal{B})(E_\alpha, RF(C_i)[q]) \to D(\mathcal{B})(E_\alpha, \bigoplus_{i\in I} RF(C_i)[q]) \to D(\mathcal{B})(E_\alpha, RF(\bigoplus_{i\in I} C_i)[q])$$

is an isomorphism for all $q \in \mathbb{Z}$. By adjunction, it is transformed into

$$\bigoplus_{i\in I} D(\mathcal{A})(G(E_\alpha), C_i[q]) \to D(\mathcal{A})(G(E_\alpha), \bigoplus_{i\in I} C_i[q])$$

which is an isomorphism since the $G(E_\alpha)$ are compact. □

Finally, we need a practical sufficient condition to ensure that, in Condition (i) of Proposition A.2.8 b), the case $p = 0$ implies the cases $p > 0$. This is given by the classical

Lemma A.2.9. *Suppose that F is strongly additive and that, in $\mathcal{A}$, infinite direct sums of injectives are F-acyclic. Then R^pF is strongly additive for any $p > 0$.*

Proof. Décalage. □

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