DERIVED LOG ALBANESE SHEAVES

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Abstract. We define higher pro-Albanese functors for every effective log motive over a field k of characteristic zero, and we compute them for every smooth log smooth scheme $X = (\underline{X}, \partial X)$. The result involves an inverse system of the coherent cohomology of the underlying scheme as well as a pro-group scheme $\mathrm{Alb}^{\log}(X)$ that extends Serre's semi-abelian Albanese variety of $\underline{X} - |\partial X|$. This generalizes the higher Albanese sheaves of Ayoub, Barbieri-Viale and Kahn.

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1. Introduction

Let k be a perfect field and let X be a smooth, projective and geometrically connected k-scheme. A very classical tool in the study of the geometry of X is given by the *Albanese variety* Alb_X of X, the universal Abelian variety receiving a map from X (up to the choice of a base point). When X is a smooth curve, the Albanese variety coincides with the Jacobian variety $\mathrm{Jac}(X)$ of X, and essentially every invariant of X can be recovered from it. In higher dimension, the Albanese variety is still an important tool for gathering information about the Chow group $\mathrm{CH}_0(X)$ of zero cycles of X. Extending the Albanese map by linearity, there is in fact a well-defined morphism (now independent on the choice of a base point)

$$a_X \colon \mathrm{CH}_0(X)^0 \to \mathrm{Alb}_X(k). \tag{1.0.1}$$

Much is known, at least conjecturally, about this map. If X is proper over an algebraically closed field, a famous theorem of Rojtman [Roj80] asserts that a_X is an isomorphism on torsion subgroups (at least modulo p-torsion in characteristic $p > 0$, which was later fixed by Milne [Mil82]). If k is finite, the kernel of a_X can be explicitly determined by geometric class field theory [KS83]. If k is the algebraic closure of a finite field, then a theorem of Kato and Saito (see again [KS83]) asserts that a_X is in fact an isomorphism, a statement that is conjectured to be true even when $k = \overline{\mathbb{Q}}$ as consequence of the Bloch-Beilinson conjectures. This is far from being true over the complex numbers, as shown by Mumford.

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When X is no longer proper, both sides of (1.0.1) need to be modified. It is already clear from the case of curves [Ser75] that one can consider a more general class of commutative algebraic groups as the targets of maps from X, including Abelian varieties, tori, and their extensions, i.e. semi-Abelian varieties. Serre [Ser60] (see also [FW84]) showed that the problem of finding a universal map to a semi-Abelian variety has always a solution. The corresponding universal object is now known as Serre's Albanese variety: it agrees with the usual Albanese variety if X is proper.

Using Serre's semi-Abelian Albanese variety it is possible to extend the Albanese morphism to every smooth quasi-projective variety[1], replacing the Chow group $\mathrm{CH}_0(X)$ with a different quotient of the free Abelian group of zero cycles of X, namely the Suslin homology group $H_0^S(X)$. As observed by Spieß and Szamuely [SS03], every semi-Abelian variety, seen as étale sheaf on the big site $\mathbf{Sm}(k)$, has a natural structure of étale sheaf with transfers, i.e. it enjoys an extra functoriality with respect to the category of finite correspondences $\mathbf{Cor}(k)$ introduced by Suslin and Voevodsky. Since every map from an affine space to a torus or to an Abelian variety is constant, such sheaves are moreover $\mathbf{A}^1$-homotopy invariant.

These two facts are essentially enough to show that the assignment $X \mapsto \mathbf{Alb}_X$ (here $\mathbf{Alb}_X$ is the non-connected algebraic group whose neutral component is exactly Serre's semi-Abelian Albanese[2]) can be promoted by left Kan extension to a motivic "realization" functor

$$L\,\mathrm{Alb}\colon \mathcal{DM}^{\mathrm{eff}}_{\text{ét}}(k,\mathbb{Q}) \longrightarrow \mathcal{D}(\mathbf{HI}_{\leq 1,\text{ét}}(k,\mathbb{Q})) \tag{1.0.2}$$

defined on the ∞-category of Voevodsky's effective motives $\mathcal{DM}^{\mathrm{eff}}_{\text{ét}}(k,\mathbb{Q})$, i.e. the full subcategory of the derived ∞-category of étale sheaves with transfers $\mathcal{D}(\mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\mathbb{Q}))$ whose objects are $\mathbb{A}^1$-local complexes, taking values in the derived ∞-category of the Abelian category $\mathbf{HI}_{\leq 1,\text{ét}}(k,\mathbb{Q})$ of 1-motivic sheaves with rational coefficients: this is the full subcategory of étale sheaves with transfers generated under colimits by lattices (i.e. étale sheaves L such that $L(\overline{k}) \cong \mathbb{Z}^r$) and semi-abelian varieties (see [ABV09, Prop. 1.3.8]). It is naturally a full subcategory of the abelian category of homotopy invariant sheaves with transfers.

This result, due to Ayoub and Barbieri-Viale [ABV09, Thm. 2.4.1] (extending Barbieri-Viale and Kahn [BVK16] to non necessarily geometric motives) has several consequences. First, it provides a construction of an Albanese map for arbitrary motives (in particular, for every separated k-scheme of finite type, not necessarily smooth or proper), giving for example vast generalizations of the theorem of Rojtman [BVK16, 13]. Second, the Albanese functor is now a *derived* functor: it has higher homotopy groups $L_i\,\mathrm{Alb}(M) = \pi_i(L\,\mathrm{Alb}(M))$ for every $M \in \mathcal{DM}^{\mathrm{eff}}_{\text{ét}}(k,\mathbb{Q})$, encoding information such as the Néron-Severi group of a variety (see [BVK16, Thm. 9.2.3]). Moreover, the functor $L\,\mathrm{Alb}$ in (1.0.2) can be identified with the left adjoint of the derived functor $i^{\mathbf{DM}^{\mathrm{eff}}}_{\leq 1}$ of the natural embedding $\mathbf{HI}_{\leq 1,\text{ét}}(k,\mathbb{Q}) \subset \mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\mathbb{Q})$. Using this fact, one can show that $i^{\mathbf{DM}^{\mathrm{eff}}}_{\leq 1}$ is fully faithful, and its essential image coincides with the stable ∞-category $\mathcal{DM}^{\mathrm{eff}}_{\leq 1}(k,\mathbb{Q})$ generated by the motives of curves. If we restrict ourselves to compact objects, $\mathcal{D}(\mathbf{HI}_{\leq 1,\text{ét}}(k,\mathbb{Q}))^{\omega}$ coincides with the (bounded) derived category of the Abelian category of Deligne 1-motives introduced in [Del74]. In fact, the properties of (1.0.2) are essential in the "motivic" proof of Deligne's conjectures on 1-motives, see [BVK16, Part 4] and [Vol13].

Our goal in this paper is to extend the picture sketched above in order to include a more general kind of algebraic groups in the definition of the Albanese variety. Thanks to Chevalley's structure theorem, any connected commutative algebraic group over a perfect field can be written as an extension of an Abelian variety by an affine smooth group scheme,

[1] at least after inverting the exponential characteristic of the ground field, in an appropriate sense.

[2] This is necessary in order to make the assignment independent from the choice of a base point

which splits as a product of a torus and a unipotent commutative group. As observed by Serre, however, the problem of finding a universal map from a smooth variety X to an arbitrary commutative algebraic group does *not* have a solution in general (namely, when X is not proper), whence the classical restriction to semi-Abelian varieties. A solution does, however, exist, if a bound on the dimension of the tangent spaces of the maps is imposed. Let us assume that k has characteristic zero (and keep this assumption throughout the rest of the Introduction, see Proposition 3.17). Faltings and Wüstholz [FW84] realized that when X admits a smooth compactification $\overline{X}$ with normal crossing boundary D, it is possible to use any finite dimensional subspace of the vector space $H^0(X,\Omega^1_X)$ to give such a bound. A natural choice is to use for $n\geq 1$ the subspaces $H^0(\overline{X},\Omega^1_{\overline{X}}(nD))$ of regular 1-forms on X having poles of order at most n along D. The resulting universal object $\mathrm{Alb}_{(\overline{X},nD)}$ depends on the pair $\mathfrak{X}^{(n)}:=(\overline{X},nD)$ in a functorial way. This gives a generalized Albanese morphism

$$a_{\mathfrak{X}}\otimes\mathbb{Q}\colon \mathbb{Q}_{tr}(X)\to \mathbf{Alb}_{\mathfrak{X}^{(n)}}\otimes_{\mathbb{Z}}\mathbb{Q},$$

which is a surjective morphism of étale sheaves with transfers with rational coefficients (here $\mathbb{Q}_{tr}(X)$ denotes the étale sheaf of $\mathbb{Q}$-vector spaces represented by X). The generalized Albanese $\mathbf{Alb}_{\mathfrak{X}^{(n)}}$ is an extension of Serre's semi-Abelian Albanese of X (independent on the choice of the compactification $\overline{X}$) by a unipotent group. If X is a curve, $\mathbf{Alb}_{\mathfrak{X}^{(n)}}=\mathrm{Jac}(\overline{X},nD)$ is exactly the generalised Jacobian variety of Rosenlicht and Serre [Ser75], and in higher dimension it is the generalised Albanese with modulus considered in [BS19], [BK18] (see also [Rus08], [Rus13]).

By varying n, we get a pro-object in the category of commutative algebraic groups up to isogeny "$\varprojlim_n$"$\mathbf{Alb}_{\mathfrak{X}^{(n)}}$, which satisfies an obvious universal property, see Prop. 3.18.

In fact, we can give a finer result. Let $\mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q})$ be the full (abelian) subcategory of the category of étale sheaves with transfers $\mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\mathbb{Q})$ generated under colimits by commutative connected k-group schemes of finite type and lattices. Note that we clearly have $\mathbf{HI}_{\text{ét},\leq 1}(k,\mathbb{Q})\subset \mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q})$. Write $\mathbf{Comp}(X)$ for the category of normal compactifications $\overline{X}$ of X such that the complement $\overline{X}-X$ is the support of an effective Cartier divisor.

Theorem 1.1. *(see Thm. 4.27) Assume that the characteristic of k is zero. The embedding $\mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q})\subseteq \mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\mathbb{Q})$ has a pro-left adjoint:*

$$\mathrm{Alb}\colon \mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k)\to \text{pro-}\mathbf{RSC}_{\text{ét},\leq 1}, \tag{1.1.1}$$

induced by colimit from

$$\mathbb{Q}_{\mathrm{tr}}(X)\mapsto \text{“}\varprojlim_n\text{”}\mathbf{Alb}_{\mathfrak{X}^{(n)}}$$

where $\mathfrak{X}=(\overline{X},D)$ for a choice of $X\hookrightarrow\overline{X}$ as above such that the support D of $\overline{X}-X$ is a normal crossing divisor.

It is natural to ask for a derived version of the above Theorem, in the spirit of the result of Ayoub, Barbieri-Viale and Kahn. However, since unipotent group schemes are $\mathbf{A}^1$-contractible, (1.1.1) cannot be extended to $\mathcal{DM}^{\mathrm{eff}}_{\text{ét}}(k,\mathbb{Q})$ in a non-trivial way, i.e. without simply collapsing to the subcategory $\mathbf{HI}_{\text{ét},\leq 1}(k,\mathbb{Q})$, recovering (1.0.2).

Our solution to this difficulty is to extend the construction to a framework in which $\mathbf{A}^1$-contractibility is no longer a problem. This is achieved by passing from the world of algebraic geometry to the world of *logarithmic algebraic geometry*, in the sense of Fontaine, Illusie, Kato and others. Over a field k (seen as log scheme with trivial log structure), we can, roughly speaking, replace schemes with log pairs $X:=(\underline{X},\partial X)$, where $\underline{X}$ is the *underlying* k-scheme and ∂X is a log structure supported on a Cartier divisor (the so-called compactifying log structure associated to the open embedding of schemes $\underline{X}-|\partial X|\hookrightarrow\underline{X}$).

For X a smooth log-smooth scheme (i.e. a log scheme such that the underlying scheme is smooth and the log structure is supported on a normal crossing divisor, see 2.1.1), we can modify a bit the previous construction and consider a system $(\overline{X}, |\partial X|_{\rm red} + nD)_{n\geq 1}$, where $\overline{X}$ is a smooth compactification of $\underline{X}$ (which always exists under our characteristic zero assumption) such that the reduced divisor $|\partial X|_{\rm red}$ associated to the log structure on X extends to a divisor on $\overline{X}$ and such that $D = \overline{X} - \underline{X}$ is an effective Cartier divisor with the property that $D + |\partial X|_{\rm red}$ has normal crossings. We have then a pro-algebraic group

$$\mathrm{Alb}^{\log}(X) := \text{“}\varprojlim_n\text{”}\mathbf{Alb}_{(\overline{X},|\partial X|_{\rm red}+nD)} \in \text{pro-}\mathbf{RSC}_{\acute{e}t,\leq 1} \tag{1.1.2}$$

which is an invariant of the log scheme X. Note that we recover the previous pro-Albanese in the case where $X = (\underline{X}, \mathrm{triv})$, i.e. the scheme $\underline{X}$ seen as log scheme with trivial log structure.

In order to exploit this formalism, we need another observation. Any commutative group scheme G (not just semi-Abelian varieties) gives rise to an étale sheaf with transfers (still denoted G) on $\mathbf{Sm}(k)$. As such, it belongs to the subcategory $\mathbf{RSC}_{\acute{e}t}(k,\mathbb{Z})$ of $\mathbf{Shv}^{\rm tr}_{\acute{e}t}(k,\mathbb{Z})$ of *reciprocity sheaves*. Its objects satisfy the property that each section $a \in F(X)$ for any $X \in \mathbf{Sm}(k)$ “has bounded ramification”, i.e. that the corresponding map $a\colon \mathbb{Z}_{tr}(X) \to F$ factors through a quotient $h_0(\mathfrak{X})$ associated to a pair $\mathfrak{X} = (\overline{X}, D)$ where $\overline{X}$ is a proper compactification of X and D is an effective Cartier divisor such that $X = \overline{X} - |D|$ (we refer to such a pair as a Cartier compactification of X). Thanks to [Sai21], every reciprocity sheaf F is *logarithmic*, i.e. it can be extended in a unique way to a functor $\mathcal{L}og(F)$ defined on the category $\mathbf{SmlSm}(k)$ of smooth log smooth log schemes over k (see also [BM21] for an alternative construction). In fact, we have that (with rational coefficients[3])

$$\mathcal{L}og(F) \in \mathbf{logCI}_{\rm d\acute{e}t} \subset \mathbf{Shv}^{\rm ltr}_{\rm d\acute{e}t}(k,\mathbb{Q})$$

where $\mathbf{Shv}^{\rm ltr}_{\rm d\acute{e}t}(k,\mathbb{Q})$ is the category of *dividing étale sheaves with log transfers* introduced in [BPØ20, Section 3] (see also [BPØ21]), i.e. sheaves for a certain Grothendieck topology on the category $\mathbf{lSm}(k)$ of log smooth log schemes over k, equipped with an extra transfer structure with respect to an extension of Voevodsky's category of finite correspondences. The topology is generated by étale covers of the underlying schemes together with admissible blow-ups with center contained in the locus where the log structure is non-trivial. The category $\mathbf{logCI}_{\rm d\acute{e}t}$ is the Grothendieck abelian category [BM21, Thm. 5.7] of strictly $\overline{\square} := (\mathbf{P}^1, \infty)$-invariant sheaves (here $(\mathbf{P}^1, \infty)$ denotes the log scheme $\mathbf{P}^1$ with compactifying log structure given by the open embedding $\mathbf{A}^1 \hookrightarrow \mathbf{P}^1$). Again by [BM21, Thm. 5.7], it is the heart of a t-structure, called the homotopy t-structure, on the ∞-category of effective log motives $\mathbf{log}\mathcal{DM}^{\rm eff}(k,\mathbb{Q})$ [BPØ20], i.e. the full subcategory of the derived ∞-category $\mathcal{D}(\mathbf{Shv}^{\rm ltr}_{\rm d\acute{e}t}(k,\mathbb{Q}))$ consisting of $\overline{\square}$-local complexes. By Saito's theorem, we have actually a fully faithful embedding (see (2.10.2))

$$\omega^{\mathbf{CI}}_{\leq 1}\colon\ \mathbf{RSC}_{\acute{e}t,\leq 1}(k,\mathbb{Q}) \hookrightarrow \mathbf{logCI}_{\rm d\acute{e}t} \subset \mathbf{Shv}^{\rm ltr}_{\rm d\acute{e}t}(k,\mathbb{Q}), \tag{1.1.3}$$

and passing to the derived categories, a functor

$$\omega^{\mathbf{log}\mathcal{DM}^{\rm eff}}_{\leq 1}\colon \mathcal{D}(\mathbf{RSC}_{\acute{e}t,\leq 1}(k,\mathbb{Q})) \to \mathcal{D}(\mathbf{Shv}^{\rm ltr}_{\rm d\acute{e}t}(k,\mathbb{Q})) \xrightarrow{L_{\overline{\square}}} \mathbf{log}\mathcal{DM}^{\rm eff}(k,\mathbb{Q}),$$

where $L_{\overline{\square}}$ is the localization functor.

If we put these facts together, we see that for each $(\underline{X}, \partial X) \in \mathbf{SmlSm}(k)$, each smooth compactification $(\overline{X}, |\partial X| + D) \in \mathbf{Comp}(X)$ and $n \geq 1$, we can construct a strictly $\overline{\square}$-invariant sheaf $\omega^{\mathbf{CI}}_{\leq 1}(\mathbf{Alb}_{(\overline{X},|\partial X|_{\rm red}+nD)})$ defined on the category of log smooth log schemes over k. With this input, the assignment $X \mapsto \mathrm{Alb}^{\log}(X)$ can be promoted to a functor defined on the motivic category in the following way.

[3] Note that the result holds integrally if we replace the étale topology with the Nisnevich topology

Theorem 1.2 (Theorems 5.1 and 5.15, Proposition 5.16)**.** *Assume that the characteristic of k is zero. The functor $\omega_{\le 1}^{\mathbf{log}\mathcal{DM}^{\mathrm{eff}}}$ has a pro-left adjoint, the log motivic Albanese functor:*

$$L\operatorname{Alb}^{\log}\colon \mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k,\mathbb{Q}) \to \text{Pro-}\mathcal{D}(\mathbf{RSC}_{\text{ét},\le 1}(k,\mathbb{Q})).$$

which fits in a commutative diagram:

$$\begin{tikzcd}
 & \mathcal{D}(\mathbf{HI}_{\le 1}(k,\mathbb{Q})) \arrow[r, "j"] & \text{Pro-}\mathcal{D}(\mathbf{RSC}_{\le 1}(k,\mathbb{Q})) \\
\mathcal{DM}^{\mathrm{eff}}_{\le 1}(k,\mathbb{Q}) \arrow[r, hook] \arrow[ur, bend left, "L\operatorname{Alb}_{\le 1}"] & \mathcal{DM}^{\mathrm{eff}}(k,\mathbb{Q}) \arrow[r, "\omega^*"] \arrow[u, "L\operatorname{Alb}"] & \mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k,\mathbb{Q}), \arrow[u, "L\operatorname{Alb}^{\log}"']
\end{tikzcd}$$

where $L\operatorname{Alb}_{\le 1}$ is the (restriction of the) functor $L\operatorname{Alb}$ of Ayoub, Barbieri-Viale and Kahn (1.0.2), ω^* is the natural comparison functor

$$\omega^*\colon \mathcal{DM}^{\mathrm{eff}}(k,\mathbb{Q}) \to \mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k,\mathbb{Q}),$$

which is fully faithful by [BPØ20, Thm.8.2.16]*. Moreover, the functor $\omega_{\le 1}^{\mathbf{log}\mathcal{DM}^{\mathrm{eff}}}$ is fully faithful and its essential image is the full stable ∞-subcategory of $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k,\mathbb{Q})$ generated by $\omega^*\mathcal{DM}^{\mathrm{eff}}_{\le 1}(k,\mathbb{Q})$ and $\mathbf{G}_a[n]$.*

The proof of the above theorem is fairly technical, and requires new ingredients compared to the argument given in [ABV09] (among which some very explicit computations). We would like to stress that the formalism of stable ∞-categories is essential to generalize the usual construction of derived functors (via resolutions) to pro-adjoint functors between derived categories, as we explain in Appendix A.

For $X \in \mathbf{SmlSm}(k)$, we determine the homotopy groups $\pi_i L\operatorname{Alb}^{\log}(X)$ completely:

Theorem 1.3 (Theorem 6.1)**.** *Let $X \in \mathbf{SmlSm}(k)$ geometrically connected and $(\overline{X}, D)$ a Cartier compactification of X. Then we have that*

$$\pi_i L\operatorname{Alb}^{\log}(X) \cong \begin{cases} \text{“}\varprojlim\text{”}\,(H^i(\overline{X},\mathcal{O}_{\overline{X}}(nD))^\vee)\otimes_k \mathbf{G}_a & \text{for } 2\le i\le \dim(X) \\ \text{“}\varprojlim\text{”}\left((H^1(\overline{X},\mathcal{O}_{\overline{X}}(nD))/H^1(\overline{X},\mathcal{O}_{\overline{X}}))^\vee \right)\otimes_k \mathbf{G}_a \\ \qquad\qquad \oplus \mathrm{NS}^*(\underline{X}-|\partial X|) & \text{for } i=1 \\ \operatorname{Alb}^{\log}(X) \quad \text{(see (1.1.2))} & \text{for } i=0 \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathrm{NS}^*(\underline{X}-|\partial X|)$ is the dual torus to the Néron-Severi group of $\underline{X}-|\partial X|$, and for V a k-vector space, $V^\vee$ denotes its linear dual.

As an application, we can identify the compact objects of $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}_{\le 1}(k,\mathbb{Q})$: our result generalizes [BVK16] on Deligne 1-motives to the case of étale Laumon 1-motives (or “Deligne 1-motives with additive part”, see 7):

Theorem 1.4 (Theorem 7.16)**.** *Assume that k has characteristic zero. The functor $\omega_{\le 1}^{\mathbf{log}\mathcal{DM}^{\mathrm{eff}}}$ preserves compact objects and it induces an equivalence*

$$\mathcal{D}^b(\mathcal{M}^a_{1,\text{ét}}\otimes\mathbb{Q}) \xrightarrow{\sim} \mathbf{log}\mathcal{DM}^{\mathrm{eff}}_{\le 1,\mathrm{gm}}(k,\mathbb{Q})$$

where the right hand side is the ∞-subcategory of compact objects of $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}_{\le 1}(k,\mathbb{Q})$

1.1. **Outline.** We now give a brief outline of the contents of the various sections of this paper.

In Section 2, we give a quick reminder of the theory of reciprocity sheaves and modulus sheaves with transfers as developed in [KMSY21a], [KMSY21b], [KSY21] and [Sai20]. We also give a quick recollection of the material in [BPØ20] and [BM21] on logarithmic motives and we prove some basic result with rational coefficients. In Section 3, we construct the Albanese map with modulus as a universal object in the category of reciprocity sheaves and compare it with the usual Albanese map and the Albanese group scheme of [BK18]. In Section 4 we introduce the categories of n-reciprocity sheaves by a suitable modification of the techniques of [ABV09]. We prove that the category of 0-reciprocity sheaves agrees with the category of 0-motivic sheaves of [ABV09]. We show the existence of a pro-left adjoint $\mathrm{Alb}^{\log}$ of the fully faithful embedding of 1-motivic sheaves into the category of dividing étale sheaves with log transfers, or "logarithmic sheaves" for short (again with rational coefficients).

Section 5 is the most technical one: we prove that the category of logarithmic sheaves admits enough BC-admissible objects (in the sense of the Appendix A) and deduce the existence of a pro-left derived functor $L\,\mathrm{Alb}^{\log}$. The techniques in particular are fairly different from the corresponding one in [ABV09], although the general structure of the proof is similar. Next, we prove that the functor $L\,\mathrm{Alb}^{\log}$ factors through $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}$ and that on $\mathcal{DM}^{\mathrm{eff}}_{\leq 1}$ it agrees with the motivic Albanese map of [ABV09] (note that this result is optimal, see Remark 6.10). After that, we compute $L\,\mathrm{Alb}^{\log}(\mathbf{G}_a)$ thanks to an explicit resolution (the Breen-Deligne resolution of the algebraic group $\mathbf{G}_a$), deducing the full faithfulness of the inclusion $\mathcal{D}(\mathbf{RSC}_{\leq 1,\text{ét}})(k,\mathbb{Q}) \to \mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k,\mathbb{Q})$.

In Section 6, we perform several computations, and we identify precisely $L\,\mathrm{Alb}(X)$ for $X \in \mathbf{SmlSm}(k)$, proving Theorem 1.3. We also pose some questions about the behaviour of the higher pro-Albanese sheaves in some special geometric situations. In Section 7 we consider the category of étale Laumon 1-motives, and prove that they are motivic in the sense that their bounded derived category agrees with the category of compact objects in the category of logarithmic 1-motives, as explained in Theorem 1.4.

Finally, in the Appendix A we introduce the notion of BC-admissible objects in a stable ∞-category and generalise the notion of a derived functor to pro-adjuntions between derived stable ∞-categories which are not in general induced by Quillen adjunctions.

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Warning 1.5. In the whole paper, we will commit the following abuse of notation: for G a smooth commutative group scheme, we still write G for the associated étale sheaf with transfers. For a ring Λ, we often write $G \in \mathbf{Shv}_{\text{ét}}(X,\Lambda)$ for the sheaf $G \otimes_{\mathbb{Z}} \Lambda$. Notice that if Λ is torsion free, the functor $\_ \otimes_{\mathbb{Z}} \Lambda$ is exact, hence if

$$\mathbf{1} \to H \to G \to Q \to \mathbf{1}$$

is an exact sequence of commutative algebraic groups, then

$$0 \to H \otimes_{\mathbb{Z}} \Lambda \to G \otimes_{\mathbb{Z}} \Lambda \to Q \otimes_{\mathbb{Z}} \Lambda \to 0$$

is an exact sequence of étale sheaves with transfers.

2. Reciprocity sheaves and logarithmic motives with rational coefficients

We work over a fixed ground field k, which is assumed to be perfect. Let Λ be a (commutative) ring of coefficients. In this section, we recall the main results on reciprocity sheaves and logarithmic motives and we state some general results on the categories with rational coefficients.

Let $\mathbf{Sm}(k)$ be the category of separated smooth schemes of finite type over k, and let $\mathbf{Cor}(k)$ be the additive category of finite correspondences. It has the same objects as $\mathbf{Sm}(k)$, and for $X, Y \in \mathbf{Sm}(k)$, the hom group $\mathbf{Cor}(X,Y)$ is the free abelian group on the set of integral closed subschemes of $X \times Y$ which are finite and surjective over a connected component of X (see [MVW06, Def. 1.1]). We denote by $\mathbf{PSh}^{\mathrm{tr}}(k,\Lambda)$ the category of additive presheaves of Λ-modules on $\mathbf{Cor}(k)$, whose objects are called *presheaves with transfers*. For $X \in \mathbf{Sm}(k)$, we let $\Lambda_{\mathrm{tr}}(X) = \mathbf{Cor}(-,X) \otimes_{\mathbb{Z}} \Lambda$ be the representable object. For τ the Nisnevich or the étale topology, we let $\mathbf{Shv}^{\mathrm{tr}}_{\tau}(k,\Lambda) \subseteq \mathbf{PSh}^{\mathrm{tr}}(k,\Lambda)$ be the category of τ-sheaves with transfers and we let

$$a^V_\tau \colon \mathbf{PSh}^{\mathrm{tr}}(k,\Lambda) \to \mathbf{Shv}^{\mathrm{tr}}_\tau(k,\Lambda)$$

be Voevodsky's τ-sheafification functor: it is induced by the classical sheafification functor defined on the category of presheaves of Λ-modules without transfers. Let $\mathbf{HI} \subseteq \mathbf{PSh}^{\mathrm{tr}}(k,\Lambda)$ be the category of $\mathbf{A}^1$-invariant presheaves, i.e. objects F such that the projection $X \times \mathbf{A}^1 \to X$ induces an isomorphism $F(X \times \mathbb{A}^1) \xrightarrow{\simeq} F(X)$ for every $X \in \mathbf{Sm}(k)$. Set $\mathbf{HI}_\tau = \mathbf{HI} \cap \mathbf{Shv}^{\mathrm{tr}}_\tau(k,\Lambda) \subseteq \mathbf{Shv}^{\mathrm{tr}}_\tau(k,\Lambda)$.

We recall the following result:

Proposition 2.1 ([MVW06], Cor. 14.22, Prop. 14.23)**.** *Let Λ be a $\mathbb{Q}$-algebra. Then for every $F \in \mathbf{PSh}^{\mathrm{tr}}(k,\Lambda)$ we have $a_{\mathrm{Nis}}F = a_{\mathrm{ét}}F$. Moreover, for all smooth X and $n > 0$ we have*

$$H^n_{\mathrm{Nis}}(X,F) = H^n_{\mathrm{ét}}(X,F).$$

2.1. **The ∞-category of logarithmic motives.** We recall the construction of the ∞-category of logarithmic motives of [BPØ20] and some properties. The standard reference for log schemes is [Ogu18]. We denote by $\mathbf{lSm}(k)$ the category of fine and saturated (fs for short) log smooth log schemes over $\mathrm{Spec}(k)$, considered as a log scheme with trivial log structure.

2.1.1. *Log geometry.* For $X \in \mathbf{lSm}(k)$, we write $\underline{X} \in \mathbf{Sch}(k)$ for the underlying k-scheme. We also write ∂X for the (closed) subset of $\underline{X}$ where the log structure of X is not trivial. Let $\mathbf{SmlSm}(k)$ be the full subcategory of $\mathbf{lSm}(k)$ having for objects $X \in \mathbf{lSm}(k)$ such that $\underline{X}$ is smooth over k. By e.g. [BPØ20, A.5.10], if $X \in \mathbf{SmlSm}(k)$, then ∂X is a strict normal crossing divisor on $\underline{X}$ and the log scheme X is isomorphic to $(\underline{X}, \partial X)$, i.e. to the compactifying log structure associated to the open embedding $(\underline{X} \setminus \partial X) \to \underline{X}$.

A morphism $f \colon X \to Y$ of fs log schemes is called *strict* if the log structure on X is the pullback log structure from Y. Geometrically, if both X and Y are objects in $\mathbf{SmlSm}(k)$, this amounts to require that there is an equality $\partial X = f^*(\partial Y)$ as reduced normal crossing divisors on X. For τ a Grothendieck topology on $\mathbf{Sch}(k)$, the *strict* topology $s\tau$ on $\mathbf{SmlSm}(k)$ is the Grothendieck topology generated by covers $\{e_i \colon X_i \to X\}$ such that $\underline{e_i} \colon \underline{X_i} \to \underline{X}$ is a τ-cover and each e_i is strict. Recall from [BPØ20, 3.1.4] that a cartesian square of fs log schemes

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

is a *dividing distinguished square* (or *elementary dividing square*) if $Y' = X' = \emptyset$ and f is a log modifications, in the sense of F. Kato [Kat21] (see [BPØ20, A.11.9] for more details on log modifications). The collection of dividing distinguished squares forms a cd structure on $\mathbf{SmlSm}(k)$, called the *dividing cd structure*. For τ a Grothendieck topoogy on $\mathbf{Sch}(k)$, the *dividing* topology $d\tau$ on $\mathbf{SmlSm}(k)$ is the topology on $\mathbf{SmlSm}(k)$ generated by the strict topology $s\tau$ and the dividing cd structure.

From now until the end of the section, we will consider $\tau \in \{\mathrm{Nis}, \mathrm{ét}\}$.

2.1.2. *Correspondences and transfers.* Following [BPØ20], we denote by $\mathbf{lCor}(k)$ the category of finite log correspondences over k. It is a variant of the Suslin–Voevodsky category of finite correspondences $\mathbf{Cor}(k)$. It has the same objects as $\mathbf{SmlSm}(k)$[4], and morphisms are given by the free abelian subgroup

$$\mathbf{lCor}(X,Y) \subseteq \mathbf{Cor}(X-\partial X, Y-\partial Y)$$

generated by elementary correspondences $V^o \subset (X-\partial X)\times(Y-\partial Y)$ such that the closure $V \subset \underline{X}\times\underline{Y}$ is finite and surjective over (a component of) $\underline{X}$ and such that there exists a morphism of log schemes $V^N \to Y$, where V^N is the fs log scheme whose underlying scheme is the normalization of V and whose log structure is given by the inverse image log structure along the composition $\underline{V^N} \to \underline{X}\times\underline{Y} \to \underline{X}$. See [BPØ20, 2.1] for more details, and for the proof that this definition gives indeed a category.

Additive presheaves (of Λ-modules) on the category $\mathbf{lCor}(k)$ will be called *presheaves (of Λ-modules) with log transfers*. Write $\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda)$ for the resulting category. As usual, for $X \in \mathbf{lCor}(k)$ we denote by $\Lambda_{\mathrm{ltr}}(X)$ the representable presheaf $\mathbf{lCor}(-,X)\otimes_{\mathbb{Z}}\Lambda$. As in [BM21], we let $\widetilde{\mathbf{SmlSm}}(k)$ be the category of fs log smooth k-schemes X which are essentially smooth over k, i.e. X is a limit $\varprojlim_{i\in I} X_i$ over a filtered set I, where $X_i \in \mathbf{SmlSm}(k)$ and all transition maps are strict étale (i.e. they are strict maps of log schemes such that the underlying maps $f_{ij}\colon \underline{X}_i \to \underline{X}_j$ are étale). For $X \in \mathbf{SmlSm}(k)$ and $x \in \underline{X}$, we put

$$X_x^h = (\underline{X}_x^h, \partial X_x^h) \in \widetilde{\mathbf{SmlSm}}(k) \tag{2.1.1}$$

where $(\underline{X})_x^h$ denotes the henselization of X at x and $(\partial X)_x^h$ denotes the pullback of ∂X along the henselization map. For $F \in \mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda)$ and $X \in \widetilde{\mathbf{SmlSm}}(k)$ such that $X = \varprojlim_{i\in I} X_i$ for $X_i \in \mathbf{SmlSm}(k)$ we put as usual $F(X) := \varinjlim_{i\in I} F(X_i)$.

We denote by $\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}(k,\Lambda) \subset \mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda)$ the subcategory of $d\tau$-sheaves. By [BPØ20, Prop. 4.5.4] and [BPØ20, Thm. 4.5.7], the inclusion $\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}(k,\Lambda) \subset \mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda)$ admits an exact left adjoint $a_{d\tau}$ (see [BPØ20, Prop. 4.2.10]), and the category $\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}(k,\Lambda)$ is a Grothendieck abelian category ([BPØ20, Prop. 4.2.12]). For $\tau \in \{\mathrm{Nis}, \mathrm{\acute{e}t}\}$, [BPØ20, Theorem 5.1.8] implies that for $F \in \mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda)$ and $X \in \mathbf{lSm}(k)$,

$$H^i_{d\tau}(X, a_{d\tau}F) = \varinjlim_{Y\in X^{\mathbf{Sm}}_{\mathrm{div}}} H^i_{s\tau}(Y, a_{s\tau}F). \tag{2.1.2}$$

where $X^{\mathbf{Sm}}_{\mathrm{div}}$ is the filtered category of log modifications $Y \to X$ such that $Y \in \mathbf{SmlSm}(k)$. The following statement can be shown by imitating the proof of [MVW06, Prop. 14.23] using (2.1.2).

Proposition 2.2. *Let Λ be a $\mathbb{Q}$-algebra and let F be an object of $\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda)$. Then there is a natural isomorphism*

$$H^n_{\mathrm{dNis}}(X, a_{\mathrm{dNis}}F) = H^n_{\mathrm{d\acute{e}t}}(X, a_{\mathrm{d\acute{e}t}}F),$$

for all $X \in \mathbf{SmlSm}(k)$ and $n \geq 0$.

Finally, recall from [BPØ20, (4.3.4)] that there is an adjunction

$$\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}(k,\Lambda) \underset{\omega^*_{\log}}{\overset{\omega^{\log}_\sharp}{\rightleftarrows}} \mathbf{Shv}^{\mathrm{tr}}_{\tau}(k,\Lambda) \tag{2.2.1}$$

where for $Y \in \mathbf{Cor}(k)$, $\omega^{\log}_\sharp F(Y) = F(Y,\mathrm{triv})$ and for $X \in \mathbf{lCor}(k)$, $\omega^*_{\log}F(X) = F(\underline{X} - |\partial X|)$. We will need later the following immediate result.

Proposition 2.3. *For all $A \in \mathbf{Shv}^{\mathrm{tr}}_\tau$ and $B \in \mathbf{Shv}^{\mathrm{ltr}}_{d\tau}$, we have that*

$$\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}(k,\Lambda)}(B, \omega^*_{\log}A) \cong \omega^*_{\log}\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}_{\tau}(k,\Lambda)}(\omega^{\log}_\sharp B, A).$$

[4]Notice that this notation conflicts with the notation of [BPØ20] where the objects were the same as $\mathbf{lSm}(k)$, although the categories of sheaves are the same in light of [BPØ20, Lemma 4.7.2]

2.1.3. *Log motives.* In light of Proposition 2.2, from now until the end of the section, we will consider one of the following situations:

- τ is the Nisnevich topology
- τ is the étale topology and Λ is a $\mathbb{Q}$-algebra.

Let $\mathcal{D}(\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}(k,\Lambda))$ be the derived stable ∞-category of the Grothendieck abelian category $\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}(k,\Lambda)$ as in [Lur17, Section 1.3.5]: it is equivalent to the underlying ∞-category of the model category $\mathbf{Cpx}(\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda))$ with the $d\tau$-local model structure used in [BPØ20] and [BM21].

The adjunction $(\omega^{\log}_{\sharp},\omega^{*}_{\log})$ of (2.2.1) induces the following adjunction of ∞-categories of sheaves (see [BPØ20, 4.3.4]):

$$L\omega^{\log}_{\sharp}\colon \mathcal{D}(\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}(k,\Lambda)) \rightleftarrows \mathcal{D}(\mathbf{Shv}^{\mathrm{tr}}_{\tau}(k,\Lambda)) : R\omega^{*}_{\log}. \tag{2.3.1}$$

Finally (see [BPØ20, Section 5.2]), let $\overline{\square} := (\mathbf{P}^1,\infty)$. Notice that $\omega(\overline{\square}) = \mathbf{A}^1$.

Definition 2.4. The stable ∞-category $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k,\Lambda)$ is the localization of the stable ∞-category $\mathcal{D}(\mathbf{Shv}^{\mathrm{ltr}}_{\tau}(k,\Lambda))$ with respect to the class of maps

$$(a_{d\tau}\Lambda(\overline{\square}\times X))[n] \to (a_{d\tau}\Lambda(X))[n]$$

for all $X\in \mathbf{lSm}(k)$ and $n\in\mathbb{Z}$. We let

$$L_{(d\tau,\overline{\square})}\colon \mathcal{D}(\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}(k,\Lambda)) \to \mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k,\Lambda)$$

be the localization functor. For $X\in\mathbf{SmlSm}(k)$, we will let $M(X) = L_{d\tau,\overline{\square}}(\Lambda_{\mathrm{ltr}}(X))$.

The interested reader can verify that this is equivalent to the underlying ∞-category of the model category $\mathbf{Cpx}(\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda))$ with the $(\overline{\square},d\tau)$-local model structure of [BPØ20, Def. 5.2.1] and [BM21, Def. 2.9]. The derived (triangulated) category of effective log motives $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$ is by definition the homotopy category of $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k,\Lambda)$.

We recall the following result, which follows naturally from [BM21, Thm. 5.7]:

Theorem 2.5. *The standard t-structure of $\mathcal{D}(\mathbf{Shv}^{\mathrm{ltr}}(k,\Lambda)$ induces an accessible t-structure on $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k,\Lambda)$ compatible with filtered colimits in the sense of* [Lur17, Def. 1.3.5.20]*, called the* homotopy t-structure.

We denote by $\mathbf{logCI}_{d\tau}$ its heart[5], which is then identified with the category of strictly $\overline{\square}$-invariant $d\tau$-sheaves and it is a Grothendieck abelian category. The inclusion

$$i^{\mathrm{ltr}}\colon \mathbf{logCI}_{d\tau} \hookrightarrow \mathbf{Shv}^{\mathrm{ltr}}_{d\tau}(k,\Lambda)$$

admits both a left adjoint $h^{\mathrm{ltr}}_0\colon F\mapsto \pi_0(L_{(d\tau,\overline{\square})}(F[0]))$ and a right adjoint h^0_{ltr} (see [BM21, Proposition 5.8]), in particular it is exact.

2.2. **Comparison with Voevodsky motives.** In this subsection, we assume that k admits resolution of singularities (see e.g. [BPØ20, Def. 7.6.3] for a precise definition). This assumption is always satisfied if $\mathrm{ch}(k)=0$.

Definition 2.6. Let $X\in\mathbf{Sm}(k)$. A *smooth Cartier compactification* or simply a *Cartier compactification* of X is a pair $(\overline{X},D)$ where $\overline{X}\in\mathbf{Sm}(k)$ is proper and $D\subseteq\overline{X}$ is an effective Cartier divisor with simple normal crossing such that $\overline{X}-|D|\cong X$.

Note that if k admits resolution of singularities, every $X\in\mathbf{Sm}(k)$ admits a (smooth) Cartier compactification. This definition is slightly different from the one used in [KMSY21a] and [KMSY21b], where the total space $\overline{X}$ is not required to be smooth over k, but simply normal. Under our assumption on k, this difference is irrelevant.

[5] In [BM21], it is denoted by $\mathbf{CI}^{\mathrm{ltr}}$

By [BPØ20, Prop. 8.2.12], the adjunction of (2.3.1) descends to an adjunction:

$$L^{\square}\omega_\sharp^{\log}\colon \mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k,\Lambda) \rightleftarrows \mathcal{DM}^{\mathrm{eff}}(k,\Lambda) : R^{\square}\omega_{\log}^{*}. \tag{2.6.1}$$

where the right hand side is the ∞-category of Voevodsky motives. By [BPØ20, Thm. 8.2.16 and Thm. 8.2.17], the functor $R^{\square}\omega^*_{\log}$ is fully faithful and for $X \in \mathbf{Sm}(k)$ and $(\overline{X}, D)$ a Cartier compactification we have a natural equivalence

$$R^{\square}\omega^*_{\log}M(X) \simeq M(\overline{X}, \partial X)$$

with ∂X supported on $|D|$. In particular, the essential image of $R^{\square}\omega^*_{\log}$ is the full subcategory spanned by $M(X)$ with $X \in \mathbf{SmlSm}(k)$ and $\underline{X}$ proper. Finally, by [BM21, Prop. 5.12] it is t-exact with respect to the homotopy t-structure of 2.5 on $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k,\Lambda)$ and the Morel-Voevodsky t-structure on $\mathcal{DM}^{\mathrm{eff}}(k,\Lambda)$. In particular, we have a fully faithful functor (still denoted by $\omega^*_{\log}$) $\mathbf{HI}_\tau \to \mathbf{logCI}_{d\tau}$ between the hearts that commutes with the inclusions. The following result will be crucial in the proof of Proposition 5.10:

Lemma 2.7. *The functor $\omega^*_{\log}\colon \mathbf{HI}_\tau \to \mathbf{logCI}_{d\tau}$ admits a right adjoint (in particular it commutes with all colimits) and is monoidal.*

Proof. Since $\omega^*_{\log}$ is fully faithful and by [BM21, Proposition 7.3] the composition $\mathbf{logCI}_{d\tau} \subseteq \mathbf{Shv}^{\mathrm{ltr}}_{d\tau} \xrightarrow{\omega_\sharp^{\log}} \mathbf{Shv}^{\mathrm{tr}}_\tau$ is also fully faithful, we have that for $F \in \mathbf{HI}_\tau$ and $G \in \mathbf{logCI}_{d\tau}$,

$$\mathrm{Hom}_{\mathbf{logCI}_{d\tau}}(\omega^*_{\log}F, G) = \mathrm{Hom}_{\mathbf{Shv}}(\omega_\sharp^{\log}\omega^*_{\log}F, \omega_\sharp^{\log}G) = \mathrm{Hom}_{\mathbf{HI}}(F, h^0_{\mathbf{A}^1}(\omega_\sharp^{\log}G)),$$

where $h^0_{\mathbf{A}^1} : \mathbf{Shv}^{\mathrm{tr}}_\tau \to \mathbf{HI}_\tau$ is right adjoint to the inclusion (see [RS21, 4.34]). This proves that $h^0_{\mathbf{A}^1}\omega_\sharp^{\log}$ is right adjoint to $\omega^*_{\log}$, which implies that $\omega^*_{\log}$ commutes with all colimits.

Since the functor $\omega^*_{\log}$ commutes with all colimits, it is enough to show that for $X, Y \in \mathbf{Sm}(k)$ we have

$$\omega^*_{\log}(h_0^{\mathbf{A}^1}(X \times Y)) = \omega^*_{\log}h_0^{\mathbf{A}^1}(X) \otimes_{\mathbf{logCI}} \omega^*_{\log}h_0^{\mathbf{A}^1}(Y),$$

where $h^0_{\mathbf{A}^1} : \mathbf{Shv}^{\mathrm{tr}}_\tau \to \mathbf{HI}_\tau$ is left adjoint to the inclusion (the 0-th Suslin homology sheaf). Thanks to [BPØ20, Proposition 8.2.4], for any choice of a smooth Cartier compactification $X \to \overline{X}$ and $Y \subseteq \overline{Y}$, by putting as log structure ∂X and ∂Y associated with the simple normal crossing divisor $\overline{X} - X$ and $\overline{Y} - Y$, we have equivalences in $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}$:

$$R^{\square}\omega^*_{\log}M^{\mathbf{A}^1}(X) = M(\overline{X}, \partial X) \quad \text{and} \quad R^{\square}\omega^*_{\log}M^{\mathbf{A}^1}(Y) = M(\overline{Y}, \partial Y).$$

By taking π_0, [BM21, Proposition 5.12] implies that

$$\omega^*h_0^{\mathbf{A}^1}(X) = h_0^{\mathrm{ltr}}(\overline{X}, \partial X) \quad \text{and} \quad \omega^*h_0^{\mathbf{A}^1}(Y) = h_0^{\mathrm{ltr}}(\overline{Y}, \partial Y).$$

Hence, we have that

$$\omega^*h_0^{\mathbf{A}^1}(X) \otimes_{\mathbf{logCI}} \omega^*h_0^{\mathbf{A}^1}(Y) = h_0^{\mathrm{ltr}}(\overline{X}, \partial X) \otimes_{\mathbf{logCI}} h_0^{\mathrm{ltr}}(\overline{Y}, \partial Y) = h_0^{\mathrm{ltr}}((\overline{X}, \partial X) \times (\overline{Y}, \partial Y)).$$

Finally, since the underlying scheme of $(\overline{X}, \partial X) \times (\overline{Y}, \partial Y)$ is $\overline{X} \times \overline{Y}$, which is proper, and the subscheme where the log structure is trivial is $X \times Y$, we have that the log scheme $(\overline{X}, \partial X) \times (\overline{Y}, \partial Y)$ is a Cartier compactification of $X \times Y$, hence again by [BPØ20, Proposition 8.2.4] we have

$$h_0^{\mathrm{ltr}}((\overline{X}, \partial X) \times (\overline{Y}, \partial Y)) \cong \omega^*h_0^{\mathbf{A}^1}(X \times Y),$$

which concludes the proof. $\square$

2.3. **The abelian category of reciprocity sheaves.** We recall the construction of the abelian category of reciprocity sheaves via modulus sheaves of [KMSY21a] and [KMSY21b] as done in [KSY21] and some properties.

A pair $\mathfrak{X} = (X, D)$ where X is proper scheme of finite type over k and D is an effective Cartier divisor on X is called a *proper modulus pair* if $X - |D| \in \mathbf{Sm}(k)$. Let $\mathfrak{X} = (X, D_X)$, $\mathcal{Y} = (Y, D_Y)$ be proper modulus pairs and $\Gamma \in \mathbf{Cor}(X - |D_X|, Y - |D_Y|)$ be a prime correspondence. Let $\overline{\Gamma} \subseteq X \times Y$ be the closure of Γ, and let $\overline{\Gamma}^N \to X \times Y$ be the normalization. We say that Γ is *admissible* if $(D_X)_{\overline{\Gamma}^N} \geq (D_Y)_{\overline{\Gamma}^N}$ as Weil divisors, where $(E)_{\overline{\Gamma}^N}$ denotes the pullback of the divisor E to the normalization $\overline{\Gamma}^N$. By [KMSY21a, Proposition 1.2.7], proper modulus pairs and admissible correspondences define an additive category, denoted $\mathbf{MCor}(k)$. For $\mathfrak{X} = (X, D)$ and $n \geq 0$, we let $\mathfrak{X}^{(n)} := (X, nD)$.

We denote by $\mathbf{MPST}(k, \Lambda)$ or simply $\mathbf{MPST}$ the category of additive presheaves of Λ-modules on $\mathbf{MCor}(k)$, whose objects are called *proper modulus presheaves with trasnfers*. For $\mathfrak{X} \in \mathbf{MCor}(k)$, we let $\Lambda_{tr}(\mathfrak{X}) = \mathbf{MCor}(-, \mathfrak{X}) \otimes_{\mathbb{Z}} \Lambda \in \mathbf{MPST}$ be the representable object.

Definition 2.8. For $X \in \mathbf{Sm}(k)$, we let $\mathbf{Comp}(X)$ be the cofiltered category given by modulus pairs $(\overline{X}, D)$ given by Cartier compactifications of X (see [KMSY21a, Lemma 1.8.2] and Definition 2.6).

There is a functor:

$$\omega : \mathbf{MCor}(k) \to \mathbf{Cor}(k) \quad (X, D) \mapsto X - |D|,$$

which induces adjoint functors (cf. [KMSY21a, Pr. 2.2.1]):

$$\omega_! : \mathbf{MPST}(k, \Lambda) \rightleftarrows \mathbf{PSh}^{\mathrm{tr}}(k, \Lambda) : \omega^*$$

where ω^* is fully faithful. For $\mathfrak{X} = (X, D) \in \mathbf{MCor}(k)$, we have

$$\omega^* F(\mathfrak{X}) = F(\omega(\mathfrak{X})) = F(X - |D|).$$

The functor $\omega_!$ is given by left Kan extension, so that for $X \in \mathbf{Sm}(k)$ and any choice of $\mathfrak{X} \in \mathbf{Comp}(X)$, we have

$$\omega_! F(X) = \varinjlim_{\mathfrak{Y} \in \mathbf{Comp}(X)} F(\mathfrak{Y}) \xrightarrow{\simeq} \varinjlim_{n} F(\mathfrak{X}^{(n)}). \tag{2.8.1}$$

where the displayed isomorphism follows from [Sai20, Lemma 1.27 (1)] (with $X^+_\infty = 0$), which implies that we have an isomorphism in pro-$\mathbf{MPST}$:

$$\text{“}\varprojlim_n\text{”} \mathbb{Z}_{\mathrm{tr}}(\mathfrak{X}^{(n)}) \cong \text{“}\varprojlim\text{”}_{\mathfrak{Y} \in \mathbf{Comp}(X)} \mathbb{Z}_{\mathrm{tr}}(\mathfrak{Y}).$$

As in the logarithmic case, let $\overline{\square} := (\mathbf{P}^1, \infty) \in \mathbf{MCor}(k)$ and for any $\mathfrak{X} = (X, D) \in \mathbf{MCor}(k)$ let (see [KMSY21a])

$$\mathfrak{X} \otimes \overline{\square} := (X \times \mathbf{P}^1, X \times \infty + D \times \mathbf{P}^1).$$

We say $F \in \mathbf{MPST}$ is $\overline{\square}$-invariant if for any $\mathfrak{X} \in \mathbf{MCor}(k)$, the projection $p : \mathfrak{X} \otimes \overline{\square} \to \mathfrak{X}$ induces an isomorphism

$$p^* : F(\mathfrak{X}) \to F(\mathfrak{X} \otimes \overline{\square}).$$

We let $\mathbf{CI}$ be the full subcategory of $\mathbf{MPST}$ consisting of all $\overline{\square}$-invariant objects. By [KSY21, Lemma 2.1.2], it is a Serre subcategory of $\mathbf{MPST}$ and that the inclusion functor $i^{\overline{\square}} : \mathbf{CI} \to \mathbf{MPST}$ has a left adjoint $h_0^{\overline{\square}}$ and a right adjoint $h^0_{\overline{\square}}$ given for $F \in \mathbf{MPST}$ and $\mathfrak{X} \in \mathbf{MCor}(k)$ by

$$h_0^{\overline{\square}}(F)(\mathfrak{X}) = \mathrm{Coker}(i_0^* - i_1^* : F(\mathfrak{X} \otimes \overline{\square}) \to F(\mathfrak{X})),$$

$$h^0_{\overline{\square}}(F)(\mathfrak{X}) = \mathrm{Hom}(h_0^{\overline{\square}}(\mathfrak{X}), F),$$

where for $a \in k$ the section $i_a : \mathfrak{X} \to \mathfrak{X} \otimes \overline{\square}$ is induced by the map $k[t] \to k[t]/(t-a) \cong k$. We write $\mathbf{RSC}(k,\Lambda) \subseteq \mathbf{PSh}^{\mathrm{tr}}(k,\Lambda)$ for the essential image of $\mathbf{CI}$ under $\omega_!$. It is an abelian subcategory of $\mathbf{PSh}^{\mathrm{tr}}(k,\Lambda)$.

Remark 2.9. By (2.8.1), for $F \in \mathbf{PSh}^{\mathrm{tr}}(k,\Lambda)$ the following conditions are equivalent:

(i) $F \in \mathbf{RSC}(k,\Lambda)$,
(ii) for every $X \in \mathbf{Sm}(k)$ and every section $a : \mathbb{Z}_{\mathrm{tr}}(X) \to F$, there exists $\mathfrak{Y} \in \mathbf{Comp}(X)$ such that a factors through $\mathbb{Z}_{\mathrm{tr}}(X) \to \omega_! h_0^{\overline{\square}}(\mathfrak{Y})$,
(iii) for every $X \in \mathbf{Sm}(k)$ and every section $a : \mathbb{Z}_{\mathrm{tr}}(X) \to F$, for any choice of $\mathfrak{X} \in \mathbf{Comp}(X)$ there exists n such that a factors through $\mathbb{Z}_{\mathrm{tr}}(X) \to \omega_! h_0^{\overline{\square}}(\mathfrak{X}^{(n)})$.

For τ the Nisnevich or the étale topology, we let $\mathbf{RSC}_\tau(k,\Lambda) := \mathbf{RSC}(k,\Lambda) \cap \mathbf{Shv}_\tau^{\mathrm{tr}}(k,\Lambda)$. The objects of $\mathbf{RSC}(k,\Lambda)$ (resp. $\mathbf{RSC}_\tau(k,\Lambda)$) are called reciprocity presheaves (resp. τ-reciprocity sheaves) of Λ-modules. By [Sai20, Thm. 0.1], the Nisnevich sheafification restricts to a functor

$$a^V_{\mathrm{Nis}} \colon \mathbf{RSC}(k,\Lambda) \to \mathbf{RSC}_{\mathrm{Nis}}(k,\Lambda),$$

which makes $\mathbf{RSC}_{\mathrm{Nis}}$ a Grothendieck abelian category (see [KSY21, Corollary 2.4.2]). Notice in particular that $\mathbf{RSC}_{\mathrm{Nis}}$ is closed under sub-objects and quotients in $\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k,\Lambda)$, and that the inclusion functor $i \colon \mathbf{RSC}_{\mathrm{Nis}} \to \mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k,\Lambda)$ is exact. As in [KSY21, Theorem 2.4.3 (1)], we denote by ρ the right adjoint to the inclusion i.

By [Sai20, Theorem 0.2], each $F \in \mathbf{RSC}_{\mathrm{Nis}}(k,\Lambda)$ satisfies *global injectivity*, i.e. for every $X \in \mathbf{Sm}$ connected with generic point η_X, the restriction map gives an injective map:

$$F(X) \hookrightarrow F(\eta_X).$$

By Proposition 2.1, if Λ is a $\mathbb{Q}$-algebra the étale sheafification coincides with the Nisnevich sheafification, hence it restricts to a functor

$$\begin{tikzcd} \mathbf{RSC}(k,\Lambda) \arrow[r, "a^V_{\mathrm{Nis}}"] \arrow[rr, bend right, "a^V_{\text{ét}}"'] & \mathbf{RSC}_{\mathrm{Nis}}(k,\Lambda) & \mathbf{RSC}_{\text{ét}}(k,\Lambda), \arrow[l, "\simeq"'] \end{tikzcd} \tag{2.9.1}$$

in particular, if Λ is a $\mathbb{Q}$-algebra, $\mathbf{RSC}_{\text{ét}}(k,\Lambda)$ is a Grothendieck abelian category and every $F \in \mathbf{RSC}_{\text{ét}}(k,\Lambda)$ satisfies global injectivity.

We have two important examples of reciprocity sheaves:

(1) Let $\mathcal{G}^*$ be the category of smooth commutative k-group schemes, locally of finite type over k (i.e. $G \in \mathcal{G}^*$ such that the connected component of the identity G^0 is a commutative algebraic gruop and $\pi_0(G)$ is finitely generated). It is classical that the corresponding étale sheaf has a unique structure of sheaf with transfers (see [SS03, Lemma 3.2]) and by [KSYR16, Theorem 4.4] it has reciprocity in the sense of [KSYR16, Definition 2.1.3]. This defines a functor $\mathcal{G}^* \to \mathbf{RSC}_{\text{ét}}$, generalizing the functor $\mathcal{G}^*_{\mathrm{sab}} \to \mathbf{HI}_{\text{ét}}$ considered in [BVK16].
(2) For $\mathfrak{C} = (C, C_\infty) \in \mathbf{MCor}(k)$ with $\dim(C) = 1$, let $\underline{\mathrm{Pic}}(C, C_\infty)$ denote the relative Picard group scheme. We would like to underline that C_∞ is not supposed to be reduced. By [RY16, Thm. 1.1] combined with [Sai20, Corollary 0.3], we have that

$$\omega_! h_0^{\overline{\square}}(\mathbb{Z}_{tr}(\mathfrak{C})) = \underline{\mathrm{Pic}}(C, C_\infty). \tag{2.9.2}$$

We end this subsection recalling the following result:

Proposition 2.10. *If Λ is a $\mathbb{Q}$-algebra, the "forgetting transfers" functor $\mathbf{RSC}_{\text{ét}}(k,\Lambda) \to \mathbf{Shv}_{\text{ét}}(k,\Lambda)$ is fully faithful and exact.*

Proof. The argument for homotopy invariant sheaves with transfers given in [BVK16, 3.9] works here as well, replacing the reference to Voevodsky's purity theorem for homotopy invariant sheaves with the global injectivity provided by [Sai20, Theorem 0.2]. $\square$

2.4. **Reciprocity sheaves and logarithmic motives.** In this subsection, we continue to assume that k satisfies resolution of singularities. Let τ and Λ be as in 2.1.3.

We denote by $\mathbf{logRec}_\tau \subseteq \mathbf{Shv}_\tau^{\mathrm{tr}}$ the essential image of the restriction of $\omega_\sharp^{\log}$ to $\mathbf{logCI}_{d\tau}$. As observed in [BM21, Definition 7.4], the functors $\omega_\sharp^{\log}$ and $\omega_{\log}^{\mathbf{CI}} := h^0_{\mathrm{ltr}}\omega^*_{\log}$ are an equivalence of categories between $\mathbf{logCI}_{d\tau}$ and $\mathbf{logRec}_\tau$. By [Sai21], there exists a fully faithful and exact functor

$$\mathcal{L}og : \mathbf{RSC}_{\mathrm{Nis}} \to \mathbf{logCI}_{\mathrm{dNis}}$$

such that $\omega_\sharp\mathcal{L}og \simeq id$. This implies (see [BM21, Theorem 7.6]) that the category $\mathbf{RSC}_{\mathrm{Nis}}$ is a full subcateogry of $\mathbf{logRec}_{\mathrm{Nis}}$ and the functor $\mathcal{L}og$ coincides with the composition:

$$\mathbf{RSC}_{\mathrm{Nis}} \subseteq \mathbf{logRec}_{\mathrm{Nis}} \xrightarrow{\omega_{\log}^{\mathbf{CI}}} \mathbf{logCI}_{\mathrm{dNis}} \hookrightarrow \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}\,. \tag{2.10.1}$$

If Λ is a $\mathbb{Q}$-algebra, by Proposition 2.2, we have similarly a fully faithful and exact functor:

$$\omega_{\log}^{\mathbf{CI}} : \mathbf{RSC}_{\mathrm{\acute{e}t}} \overset{(2.9.1)}{\cong} \mathbf{RSC}_{\mathrm{Nis}} \overset{(2.10.1)}{\to} \mathbf{logCI}_{\mathrm{dNis}} \cong \mathbf{logCI}_{\mathrm{d\acute{e}t}} \hookrightarrow \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{d\acute{e}t}}. \tag{2.10.2}$$

Remark 2.11. Since $\mathbf{logRec}_\tau \subseteq \mathbf{Shv}_\tau^{\mathrm{tr}}$ is fully faithful, the inclusion $\mathbf{RSC}_\tau \subseteq \mathbf{logRec}_\tau$ has a right adjoint given by the functor ρ of [KSY21, Theorem 2.4.3 (1)]. The composition $\mathbf{logCI}_{d\tau} \cong \mathbf{logRec}_\tau \xrightarrow{\rho} \mathbf{RSC}_\tau$ gives then a right adjoint to the inclusion $\mathbf{RSC}_\tau \subseteq \mathbf{logCI}_{d\tau}$. Since the last inclusions of (2.10.1) and (2.10.2) also have right adjoints h^0_{ltr} (which coincides under our assumption on Λ), the functors (2.10.1) and and (2.10.2) have right adjoints as well, in particular they commute with all colimits.

Remark 2.12. For every $F \in \mathbf{logCI}_{d\tau}$, the equivalence $\omega_\sharp^{\log}\colon \mathbf{logCI}_{d\tau} \simeq \mathbf{logRec}_\tau \colon \omega_{\log}^{\mathbf{CI}}$ induces an isomorphism of sheaves:

$$F \cong \omega_{\log}^{\mathbf{CI}}\omega_\sharp^{\log}F. \tag{2.12.1}$$

If $\omega_\sharp^{\log}F \in \mathbf{HI}_\tau$, we have that $\omega^*_{\log}\omega_\sharp^{\log}F \in \mathbf{logCI}_\tau$, hence

$$\omega_{\log}^{\mathbf{CI}}\omega_\sharp^{\log}F = h^0_{\mathrm{ltr}}\omega^*_{\log}\omega_\sharp^{\log}F \overset{(*)}{\simeq} \omega^*_{\log}\omega_\sharp^{\log}F,$$

where the equivalence $(*)$ strictly depends on the fact that $\omega^*_{\log}\omega_\sharp^{\log}F \in \mathbf{logCI}_{d\tau}$, which is not true unless $\omega_\sharp^{\log}F \in \mathbf{HI}_\tau$. We conclude that for every $F \in \mathbf{logCI}_{d\tau}$ such that $\omega_\sharp^{\log}F \in \mathbf{HI}_\tau$, we have an isomorphism of sheaves:

$$F \cong \omega^*_{\log}\omega_\sharp^{\log}F. \tag{2.12.2}$$

We finish this section with the following analogue of Proposition 2.3:

Proposition 2.13. *For all $A, B \in \mathbf{logCI}_{d\tau}$, then:*

$$\omega_\sharp^{\log}\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}}(A,B) \cong \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}_{\tau}}(\omega_\sharp^{\log}A, \omega_\sharp^{\log}B).$$

Proof. By [BM21, Theorem 5.10] and the fact that $\omega^*_{\log}$ is fully faithful, we have an exact sequence

$$0 \to B \to \omega^*_{\log}\omega_\sharp^{\log}B \to Q \to 0$$

with $\omega_\sharp^{\log}Q = 0$, which induces an exact sequence:

$$0 \to \omega_\sharp^{\log}\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}}(A,B) \to \omega_\sharp^{\log}\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}}(A,\omega^*_{\log}\omega_\sharp^{\log}B) \to \omega_\sharp^{\log}\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}}(A,Q).$$

By Proposition 2.3, we have that

$$\omega_\sharp\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}}(A,\omega^*_{\log}\omega_\sharp^{\log}B) \cong \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}_{\tau}}(\omega_\sharp^{\log}A, \omega_\sharp^{\log}B),$$

hence it is enough to show that $\omega_\sharp \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}}(A,Q)=0$. For $X\in\mathbf{Sm}(k)$ we have that

$$\begin{aligned}\omega_\sharp \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}}(A,Q)(X) &= \mathrm{Hom}_{\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}}(A, \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}}(\Lambda_{\mathrm{ltr}}(X,\mathrm{triv}),Q))\\ &= \mathrm{Hom}_{\mathbf{logRec}_\tau}(A, h^0_{\mathrm{ltr}}\,\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}}(\Lambda_{\mathrm{ltr}}(X,\mathrm{triv}),Q)),\end{aligned}$$

where $h^0_{\mathrm{ltr}}\colon \mathbf{Shv}^{\mathrm{ltr}}_{d\tau}\to\mathbf{logCI}_{d\tau}$ is the right adjoint of the inclusion. Since for all $Y\in\mathbf{Sm}(k)$ we have that

$$\omega_\sharp^{\log}\,\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}}(\Lambda_{\mathrm{ltr}}(X,\mathrm{triv}),Q)(Y) = Q(X\times Y,\mathrm{triv}) = \omega_\sharp^{\log}Q(X\times Y)=0,$$

by Lemma 2.14 below we conclude that $h^0_{\mathrm{ltr}}\,\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}}(\Lambda_{\mathrm{ltr}}(X,\mathrm{triv}),Q)=0$, which concludes the proof. □

Lemma 2.14. *For all $G\in\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}$ such that $\omega_\sharp^{\log}G=0$, we have $h^0_{\mathrm{ltr}}G=0$.*

Proof. By purity [BM21, Theoremn 5.10], it is enough to show that $h^0_{\mathrm{ltr}}Q(\eta_X,\mathrm{triv})=0$ for any generic point η_X of $X\in\mathbf{Sm}(k)$. We have that

$$h^0_{\mathrm{ltr}}Q(\eta_X,\mathrm{triv}) = \mathrm{Hom}(h_0^{\mathrm{ltr}}(\Lambda_{\mathrm{ltr}}(\eta_X,\mathrm{triv})),Q)$$

and as observed in the proof of [BM21, Lemma 7.2], it follows from [BPØ20, Proposition 8.2.2 and 8.2.4] that there is a surjective map

$$\Lambda_{\mathrm{tr}}(\eta_X,\mathrm{triv})\simeq\omega^*\Lambda_{\mathrm{tr}}(\eta_X)\twoheadrightarrow\omega^*h_0^{\mathbf{A}^1}(\eta_X)\simeq h_0^{\mathrm{ltr}}(\Lambda_{\mathrm{ltr}}(\eta_X,\mathrm{triv})).$$

In particular,

$$h^0_{\mathrm{ltr}}Q(\eta_X,\mathrm{triv})\hookrightarrow \mathrm{Hom}_{\mathbf{Shv}^{\mathrm{ltr}}_{d\tau}}(\Lambda_{\mathrm{ltr}}(\eta_X,\mathrm{triv}),Q)=\mathrm{Hom}_{\mathbf{Shv}^{\mathrm{tr}}_{\tau}}(\Lambda_{\mathrm{tr}}(\eta_X),\omega_\sharp^{\log}Q)=0$$

as required. □

3. Categories of rational maps and universal problems

3.1. **Commutative groups schemes and torsors under them.** We recall some well-known facts on commutative group schemes over a perfect field k and we fix some notation.

Let $\mathcal{G}^*$ be again the category of smooth commutative k-group schemes, locally of finite type over k (for short, a commutative k-group scheme). Write $\mathcal{G}$ for the subcategory of smooth commutative algebraic k-groups (i.e. objects of $\mathcal{G}^*$ which are of finite type). Given $G\in\mathcal{G}^*$, let G^0 be the connected component of the identity in G. Recall (see [BVK16, Definition 1.1.2] or [DG80, Proposition 5.1.4]) the following definition.

Definition 3.1. A group scheme $L\in\mathcal{G}^*$ is called *discrete* if $L^0=\mathrm{Spec}(k)$ and the abelian group $L(\overline{k})$ is finitely generated (equivalently, if L is étale over k). A discrete k-group scheme $L\in\mathcal{G}^*$ is called a *lattice* if $L(\overline{k})$ is torsion free.

As in [BVK16], we denote by ${}^t\mathcal{M}_0$ the subcategory of $\mathcal{G}^*$ consisting of discrete k-group schemes. By [BVK16, Lemma 1.1.3] it is a Serre subcategory of $\mathcal{G}^*$, hence it is an Abelian category. We denote by $\mathcal{M}_0$ the full subcategory of lattices. By [DG80, Proposition 5.1.8], for $G\in\mathcal{G}^*$, there is an exact sequence

$$0\to G^0\to G\to\pi_0(G)\to 0$$

where $\pi_0(G)$ is an étale k-group, which is universal for homomorphisms from G to discrete groups. The fibers of $G\to\pi_0(G)$ are the irreducible components of G.

Definition 3.2. Let $G\in\mathcal{G}$ be a smooth commutative algebraic k-group. By *k-torsor under G or for G* we mean a k-scheme P, locally of finite type over k, equipped with an action $P\times G\to P$ such that the induced morphism $(s,g)\mapsto(s,sg)\colon P\times G\to P\times P$ is an isomorphism. Write $\mathcal{P}_G$ for the category of *k-torsors under G*: morphisms between torsors are G-equivariant k-morphisms.

3.1.1. Given a k-torsor P under $G \in \mathcal{G}$, we can construct a commutative k-group scheme $P_G = \coprod_{n\in\mathbb{Z}} P^{\vee n} \in \mathcal{G}^*$ using the sum of torsors $\vee_G$ (see [Mil80, III.4.8.b]) following [Ram01, 1.2]. It fits in a short exact sequence

$$0 \to G \to P_G \xrightarrow{a_P} \underline{\mathbb{Z}} \to 0,$$

presenting P_G as extension of $\underline{\mathbb{Z}}$ by the group G. Moreover, we can identify the torsor P with the fiber of the section $1 \in \underline{\mathbb{Z}}$ along the map a_P, so that we have a natural inclusion $P \hookrightarrow P_G$.

3.1.2. Write $\mathcal{P}$ for the category whose objects are pairs (P, G), where $P \in \mathcal{P}_G$ for a $G \in \mathcal{G}$ a smooth commutative connected algebraic group. A morphism in $\mathcal{P}$ is the datum of a pair of morphisms $(f^1, f^0)\colon (P, G) \to (P', G')$, where $f^0\colon G \to G'$ is a k-morphism of algebraic groups and $f^1\colon P \to P'$ is f^0-equivariant. If X is a k-scheme, we write $X\backslash\mathcal{P}$ for the comma category over X: its objects are triples (u, P, G), where $(P, G) \in \mathcal{P}$ and $u\colon X \to P$ is a k-morphisms. Morphisms in $X\backslash\mathcal{P}$ are defined in the obvious way.

Definition 3.3. A *fibration to torsors* is the datum, for each $X \in \mathbf{Sm}$, of a full category $\mathcal{M}_X$ of $X\backslash\mathcal{P}$, contravariantly functorial in X. Similarly, a fibration to torsors *for proper modulus pairs* is the datum, for each $\mathfrak{X} = (\overline{X}, X_\infty) \in \mathbf{MCor}$, of a full subcategory $\mathcal{M}_{\mathfrak{X}}$ of $X\backslash\mathcal{P}$, where $X = \overline{X} \setminus X_\infty \in \mathbf{Sm}$, contravariantly functorial in $\mathfrak{X}$ for maps in $\mathbf{MSm}^{\mathrm{fin}}$ (see [KMSY21a, Definition 1.3.3.(2)]).

The initial object (if it exists) of $\mathcal{M}_X$ is called the *$\mathcal{M}$-Albanese torsor of X*. By definition, it is the datum of a smooth commutative connected algebraic group $\mathrm{Alb}^0_{\mathcal{M}}(X)$, a k-torsor $\mathrm{Alb}^1_{\mathcal{M}}(X)$ under $\mathrm{Alb}^0_{\mathcal{M}}(X)$ and a k-morphism $X \to \mathrm{Alb}^1_{\mathcal{M}}(X)$ which is universal for maps in $\mathcal{M}_X$. The algebraic group $\mathrm{Alb}^0_{\mathcal{M}}(X)$ is called the $\mathcal{M}$-Albanese variety of X. Similarly, if $\mathcal{M}_{-}$ is a fibration to torsors for proper modulus pairs, the initial object (if it exists) of $\mathcal{M}_{\mathfrak{X}}$ is called the *$\mathcal{M}$-Albanese torsor of* $\mathfrak{X}$. The corresponding algebraic group, $\mathrm{Alb}^0_{\mathcal{M}}(\mathfrak{X})$ will be called the $\mathcal{M}$-Albanese variety of $\mathfrak{X}$.

Example 3.4. For any X, let $\mathbf{SAb}_X$ be the full subcategory of $X\backslash\mathcal{P}$ consisting of maps to torsors P under semi-Abelian varieties. In this case, the existence of an initial object for $\mathbf{SAb}_X$ was proven by Serre [Ser60] in the case the base field k is algebraically closed. In [Wit08, Appendix A], a Galois descent argument is used to show that the Albanese variety, the Albanese torsor and the universal map

$$X \to \mathrm{Alb}^{(1)}_{\mathbf{SAb}}(X)$$

always exist, without any assumption on k.

If X is smooth and proper over k, the semi-abelian variety $\mathrm{Alb}^0_{\mathbf{SAb}}(X)$ is in fact an Abelian variety, and coincides with the classical Albanese variety of X, dual (as abelian variety) to the Picard variety $\mathrm{Pic}^{0,red}_X$.

3.2. **A universal construction.** We will discuss a number of situations in which the $\mathcal{M}$-Albanese torsor of a *proper* modulus pair $\mathfrak{X}$ exists. In fact, we will consider different fibrations to torsors $\mathcal{M}_{-}$ for proper modulus pairs, giving sufficient conditions for the initial object to exist. In the end, all the fibrations that we consider will turn out to be equivalent, giving then a *unique notion of Albanese torsor* for a proper modulus pair $\mathfrak{X}$.

Definition 3.5. Let $\mathfrak{X} = (\overline{X}, X_\infty)$ be a proper geometrically integral modulus pair, and write $U(\mathfrak{X}) = U(\overline{X}, X_\infty)$ for the k-vector space $H^0(\overline{X}, \Omega^1_{\overline{X},cl}(X_\infty))$, where $\Omega^1_{\overline{X},cl}$ denotes the subsheaf of closed forms. Let $\mathcal{M}^{\Omega}_{\mathfrak{X}}$ be the full subcategory of $X\backslash\mathcal{P}$ consisting of triples (u, P, G) with the following property: Let $k \subset \overline{k}$ be an algebraic closure of k and let $u_{\overline{k}}\colon X_{\overline{k}} \to P_{\overline{k}} \cong G_{\overline{k}}$ be the base change of u to $\overline{k}$. Then $(u, P, G) \in \mathcal{M}^{\Omega}_{\mathfrak{X}}$ if and only if $(u_{\overline{k}})^*\Omega(G_{\overline{k}}) \subseteq U(\overline{X}_{\overline{k}}, X_{\infty,\overline{k}})$, where $\Omega(G_{\overline{k}})$ denotes the space of invariant differential forms on $G_{\overline{k}}$. The assignment $\mathfrak{X} \mapsto \mathcal{M}^{\Omega}_{\mathfrak{X}}$ defines a fibration to torsors for proper modulus pairs in the sense of Definition 3.3

Remark 3.6. It follows immediately that $(u, P, G) \in X\backslash\mathcal{P}$ belongs to $\mathcal{M}^{\Omega}_{\mathfrak{X}}$ if and only if

$$u_L^*\Omega(G_L) \subseteq U(\overline{X}_L, X_{\infty,L})$$

for any algebraically closed field $L \subset k$.

Theorem 3.7. *For any $\mathfrak{X} \in$ **MCor**, the $\mathcal{M}^{\Omega}_{\mathfrak{X}}$-Albanese torsor of $\mathfrak{X}$ exists.*

Proof. Suppose that $k = \overline{k}$ is algebraically closed. Since any k-torsor over an algebraic group $G \in \mathcal{G}$ is trivial in this case, the category $\mathcal{M}^{\Omega}_{\mathfrak{X}}$ is equivalent to the category of morphisms $u\colon X \to G$ from X to algebraic groups satisfying the condition $u^*(\Omega(G)) \subseteq U(\mathfrak{X})$. Morphisms in $\mathcal{M}^{\Omega}_{\mathfrak{X}}$ are k-morphisms $f\colon G \to G'$ of *torsors* commuting with the structural morphisms $u\colon X \to G$ and $u'\colon X \to G'$ (where we view G and G' acting on themselves). Although the morphism f is not a homomorphism of algebraic groups in general, it can be written as $f = f_0 + \tau$, where f_0 is a group homomorphism and τ is a translation.

Following [Ser60], in order to check whether an initial object for $\mathcal{M}^{\Omega}_{\mathfrak{X}}$ exists, it is enough to restrict to the subcategory $\mathcal{M}^{\Omega,g}_{\mathfrak{X}}$ of morphisms which are *generating* (see [Ser60, Definition 1]). For them, one has the following simple

Lemma 3.8 (Lemma 6, p.198, [FW84])**.** *Let $u\colon X \to G \in \mathcal{M}^{\Omega}_{\mathfrak{X}}$ and suppose that u is generating in the sense of* [Ser60, Definition 1]*. Then the pullback map*

$$u^*\colon H^0(G, \Omega^1_G)^{\mathrm{inv}} = \Omega(G) \to H^0(X, \Omega^1_X)$$

is injective.

By e.g. [Ser75, Prop. III.16], the dimension of any $G \in \mathcal{G}$ agrees with the dimension of the k-vector space $\Omega(G)$ of invariant differential forms. The previous Lemma implies that for any $u\colon X \to G \in \mathcal{M}^{\Omega,g}_{\mathfrak{X}}$, one has $\dim G \le \dim U(\mathfrak{X})$. Since by [Ser60, Corollaire, p.05], a necessary and sufficient condition for the initial object to exist is a uniform bound on the dimensions of the groups appearing in $\mathcal{M}^{\Omega,g}_{\mathfrak{X}}$, we conclude.

Suppose now that k is any perfect field and let $\overline{k}$ be an algebraic closure of k. Write $\overline{X}_{\overline{k}}$ for the base change $\overline{X} \otimes_k \overline{k}$ and $\mathfrak{X}_{\overline{k}}$ for the pair $(\overline{X}_{\overline{k}}, (X_\infty)_{\overline{k}})$. According to the above argument, the category $\mathcal{M}^{\Omega}_{\mathfrak{X}_{\overline{k}}}$ admits a universal object,

$$\mathrm{alb}^{\Omega}_{\mathfrak{X}_{\overline{k}}}\colon X_{\overline{k}} \to \mathrm{Alb}^{\Omega}_{\mathfrak{X}_{\overline{k}}}.$$

The descent to the base field k can be done following the proof of Serre [Ser75, V.22] in the case of generalized Jacobians of curves to get a triple $(\mathrm{alb}^{\Omega}_{\mathfrak{X}}, \mathrm{Alb}^{\Omega,(1)}_{\mathfrak{X}}, \mathrm{Alb}^{\Omega,(0)}_{\mathfrak{X}})$ defined over k. □

3.3. **Cutting curves.** Assume now that $\mathfrak{X} = (\overline{X}, X_\infty) \in$ **MCor** is such that $\overline{X}$ is smooth over k. A finite morphism $\nu\colon \overline{C} \to \overline{X}$, with $\overline{C}$ a normal and geometrically integral curve, is *admissible* for $\mathfrak{X}$ if $\nu(\overline{C}) \not\subseteq X_\infty$. In this case, write C_∞ for the effective Cartier divisor ν^*X_∞ on $\overline{C}$. Write C for the open subset $\overline{C} \setminus C_\infty$. We will use the following Lemma, taken from [BS19].

Lemma 3.9 (Lemma 10.14, [BS19])**.** *Let γ be the restriction map*

$$\gamma\colon H^0(X, \Omega^1_X) \to \prod_{\nu\colon \overline{C} \to \overline{X}} H^0(C, \Omega^1_C)/H^0(\overline{C}, \Omega^1_{\overline{C}}(\nu^*X_\infty)),$$

where the product runs over the set of admissible curves $\nu\colon \overline{C} \to \overline{X}$. Then the kernel of γ agrees with $H^0(\overline{X}, \Omega^1_{\overline{X}}(X_\infty))$.

If $u\colon X \to P$ is a k-morphism from X to a torsor P for an algebraic group $G \in \mathcal{G}$, we get by composition a morphism

$$u_C\colon C := \overline{C} \times_{\overline{X}} X \to X \xrightarrow{u} P.$$

Write $\nu^*(u, P, G)$ for the corresponding object in $C\backslash\mathcal{P}$.

Lemma 3.10. *A triple $(u, P, G) \in X\backslash\mathcal{P}$ belongs to $\mathcal{M}^{\Omega}_{\mathfrak{X}}$ if and only if for any admissible curve $\nu\colon \overline{C} \to \overline{X}$, we have $\nu^*(u, P, G) \in \mathcal{M}^{\Omega}_{(\overline{C}, C_\infty)}$.*

Proof. The necessity of the condition is clear. According to Definition 3.5 and Remark 3.6, the statement of the Lemma can be checked over an algebraic closure $\overline{k}$ of k, so that we can assume $k = \overline{k}$. As above, the category $\mathcal{M}^{\Omega}_{\mathfrak{X}}$ is equivalent to the category of morphisms $\psi\colon X \to G$ from X to algebraic groups satisfying the condition $\psi^*(\Omega(G)) \subseteq U(\mathfrak{X})$. Let now $\psi\colon X \to G$ be a k-morphism, and let $\omega \in \Omega(G)$. We have to show that $\psi^*(\omega) \in H^0(\overline{X}, \Omega^1_{\overline{X}}(X_\infty))$ (note that $\psi^*(\omega)$ is automatically closed), i.e. that $\eta := \psi^*(\omega)$ has poles along $|X_\infty|$ of order bounded by the multiplicity of X_∞, assuming that this condition is satisfied after restriction admissible to curves. But this is precisely the content of Lemma 3.9. □

Proposition 3.11 (see [FW84])**.** *Let $\mathfrak{X}$ be a smooth proper modulus pair over a field k of characteristic 0. Let $\overline{k}$ be an algebraic closure of k. Then the $\mathcal{M}^{\Omega}_{\mathfrak{X}}$-Albanese torsor $(\mathrm{alb}^{\Omega}_{\mathfrak{X}}, \mathrm{Alb}^{\Omega,(1)}_{\mathfrak{X}}, \mathrm{Alb}^{\Omega,(0)}_{\mathfrak{X}})$ is characterized by the property that the pullback map*

$$u^*_{\overline{k}}\colon \Omega(\mathrm{Alb}^{\Omega,(0)}_{\mathfrak{X}})_{\overline{k}} \to H^0(X_{\overline{k}}, \Omega^1_{X_{\overline{k}}})$$

is injective, with image equal to the subspace $H^0(\overline{X}, \Omega^1_{\overline{X}}(X_\infty)) \otimes_k \overline{k}$.

3.4. **The universal regular quotient of the Chow group of zero cycles.** We start by recalling the definition of the Kerz-Saito Chow group of 0-cycles with modulus (see [KS16]). For an integral scheme $\overline{C}$ over k and for E a closed subscheme of $\overline{C}$, we set

$$\begin{aligned} G(\overline{C}, E) &= \bigcap_{x \in E} \mathrm{Ker}\big(\mathcal{O}^{\times}_{\overline{C},x} \to \mathcal{O}^{\times}_{E,x}\big) \\ &= \varinjlim_{E \subset U \subset \overline{C}} \Gamma(U, \ker(\mathcal{O}^{\times}_{\overline{C}} \to \mathcal{O}^{\times}_{E})), \end{aligned}$$

where U runs over the set of open subsets of $\overline{C}$ containing E (the intersection taking place in the function field $k(\overline{C})^{\times}$). We say that a rational function $f \in G(\overline{C}, E)$ satisfies the modulus condition with respect to E.

Let $\mathfrak{X} = (\overline{X}, X_\infty) \in \underline{\mathbf{M}}\mathbf{Sm}$ be a *proper* modulus pair and write X for the complement $\overline{X} \setminus |X_\infty|$. Let $Z_0(X)$ be the free abelian group on the set of closed points of X. Let $\overline{C}$ be an integral normal curve over k and let $\varphi_{\overline{C}}\colon \overline{C} \to \overline{X}$ be a finite morphism such that $\varphi_{\overline{C}}(\overline{C}) \not\subset X_\infty$ (so $\overline{C}$ is admissible in the sense of 3.3). The push forward of cycles along the restriction of $\varphi_{\overline{C}}$ to $C = \overline{C} \times_{\overline{X}} X$ gives a well defined group homomorphism

$$\tau_{\overline{C}}\colon G(\overline{C}, \varphi^*_{\overline{C}}(X_\infty)) \to Z_0(X),$$

sending a function f to the push forward of the divisor $\mathrm{div}_{\overline{C}}(f)$.

Definition 3.12 (Kerz-Saito)**.** We define the Chow group $\mathrm{CH}_0(\mathfrak{X}) = \mathrm{CH}_0(\overline{X}|X_\infty)$ of 0-cycles of $\overline{X}$ with modulus X_∞ as the cokernel of the homomorphism

$$\tau\colon \bigoplus_{\varphi_{\overline{C}}\colon \overline{C} \to \overline{X}} G(\overline{C}, \varphi^*_{\overline{C}}(X_\infty)) \to Z_0(X),$$

where the sum runs over the set of finite morphisms $\varphi_{\overline{C}}\colon \overline{C} \to X$ from admissible curves.

Definition 3.13. Let $X \in \mathbf{Sm}(k)$ be geometrically integral and $\mathfrak{X} \in \mathbf{Comp}(X)$ (see Definition 2.8). Let $\mathrm{CH}_0(\mathfrak{X})$ be the Chow group of 0-cycles with modulus of $\mathfrak{X}$. Let $\mathcal{M}^{\mathrm{CH}}_{\mathfrak{X}}$ be the full subcategory of $X\backslash\mathcal{P}$ consisting of triples (u, P, G) with the following property. For any algebraically closed field $L \supset k$, write $u_L\colon Z_0(X_L)^0 \to P_L(L) \cong G_L(L)$ for the induced morphism on zero-cycles of degree zero. Then $(u, P, G) \in \mathcal{M}^{\mathrm{CH}}_{\mathfrak{X}}$ if and only if u_L factors as

$$\begin{tikzcd} Z_0(X_L)^0 \arrow[d] \arrow[dr, "u_L"] & \\ \mathrm{CH}_0(\mathfrak{X}_L)^0 \arrow[r] & G_L(L), \end{tikzcd}$$

where $\mathrm{CH}_0(\mathfrak{X}_L)^0$ is the image of $Z_0(X_L)^0$.

Proposition 3.14. *Assume that $\mathfrak{X}$ is a smooth, proper and geometrically integral modulus pair over k. Then the $\mathcal{M}^{\mathrm{CH}}_{\mathfrak{X}}$-Albanese torsor $(\mathrm{alb}^{\mathrm{CH}}_{\mathfrak{X}}, \mathrm{Alb}^{\mathrm{CH},(1)}_{\mathfrak{X}}, \mathrm{Alb}^{\mathrm{CH},(0)}_{\mathfrak{X}})$ of $\mathfrak{X}$ exists. If $\dim \mathfrak{X} = 1$, it agrees with the Rosenlich-Serre generalized Jacobian $(\varphi_{X_\infty}, \mathrm{Jac}^{(1)}_{\overline{X},X_\infty}, \mathrm{Jac}^{(0)}_{\overline{X},X_\infty})$ of* [Ser75, V.4.20].

Proof. If $\dim \mathfrak{X} = 1$, this is precisely the content of [Ser75, V.Theorem 1], and the very definition of modulus for a rational map and local symbols. For the general case, as in the proof of Theorem 3.7, it is enough to show the existence in the case $k = \overline{k}$ is algebraically closed (the descent argument is identical). Similarly, we can restrict to the category $\mathcal{M}^{\mathrm{CH},g}_{\mathfrak{X}}$ of morphisms which are generating. According to [Ser60, Corollaire, p.05] it is then enough to show that there exists a uniform bound on the dimensions of the groups appearing in $\mathcal{M}^{\mathrm{CH},g}_{\mathfrak{X}}$. We do this by showing that $\mathcal{M}^{\mathrm{CH}}_{\mathfrak{X}}$ is a (full) subcategory of $\mathcal{M}^{\Omega}_{\mathfrak{X}}$. The required bound will be then provided by Lemma 3.8.

Suppose then that that $u\colon X \to G$ is a k-morphism from $X = \overline{X}\backslash|X_\infty|$ to a commutative connected algebraic group G such that the induce map on zero cycles factors through $\mathrm{CH}_0(\mathfrak{X})$. Let $\varphi\colon \overline{C} \to \overline{X}$ be a finite morphism from normal integral curve $\overline{C}$ such that $\varphi(\overline{C}) \not\subset X_\infty$. Put $C = \overline{C} \times_{\overline{X}} X$ and $C_\infty = \varphi^*(X_\infty)$. Let $u_C\colon C \to G$ be the composition $u \circ \varphi$. Then we have

$$u_C(\mathrm{div}_{\overline{C}}(f)) = u(\varphi_*(\mathrm{div}_{\overline{C}}(f))) = 0 \text{ in } G(k) \text{ for any } f \in G(\overline{C}, C_\infty).$$

In particular, the divisor C_∞ is a modulus in the sense of Rosenlicht-Serre for the rational map (still denoted u_C) $u_C\colon \overline{C} \dashrightarrow G$. Therefore we have then a factorization ([Ser75, V, Theorem 2])

$$\begin{tikzcd} C \arrow[d, "a"] \arrow[dr, "u_C"] & \\ \mathrm{Jac}_{\overline{C},C_\infty} \arrow[r, "\tilde{u}_C"'] & G \end{tikzcd},$$

where $a\colon C \to \mathrm{Jac}_{\overline{C},C_\infty}$ is the universal map from C to its generalized Jacobian (with respect to a chosen k-rational point). But now we have

$$u_C^*(\Omega(G)) = a^*(\tilde{u}_C^*(\Omega(G))) \subseteq a^*(\Omega(\mathrm{Jac}_{\overline{C},C_\infty})) = H^0(\overline{C}, \Omega^1_{\overline{C}} \otimes_{\overline{C}} \mathcal{O}_{\overline{C}}(C_\infty)),$$

where the last equality follows from [Ser75, V, Proposition 5]. This implies that $(u_C, G) \in \mathcal{M}^{\Omega}_{(\overline{C},C_\infty)}$ for any admissible curve and so, by Lemma 3.10, we deduce that $(u, G) \in \mathcal{M}^{\Omega}_{\mathfrak{X}}$. □

Remark 3.15. Suppose that $k = \overline{k}$ is algebraically closed and let $\mathfrak{X}$ be a smooth proper integral modulus pair. Then the morphism

$$\rho_{\mathfrak{X}}\colon \mathrm{CH}_0(\mathfrak{X})^0 \to \mathrm{Alb}_{\mathfrak{X}}(k)$$

to the Albanese variety $\mathrm{Alb}_{\mathfrak{X}} = \mathrm{Alb}^{\mathrm{CH},(0)}_{\mathfrak{X}}$ of $\mathfrak{X}$ induced by $\mathrm{alb}_{\mathfrak{X}}$ is surjective and regular. We can reformulate the universal property in $\mathcal{M}^{\mathrm{CH}}_{\mathfrak{X}}$ by saying that $\mathrm{Alb}_{\mathfrak{X}}$ is the universal regular quotient of the Chow group of zero cycles with modulus. As such it agrees, a posteriori, with the Albanese variety $\mathrm{Alb}(\overline{X}|X_\infty)$ of [BK18, Theorem 1.1]. If $k = \mathbb{C}$ and

$|X_\infty|$ is a strict normal crossing divisor, it agrees with the generalized Jacobian $J^d_{\overline{X}|X_\infty}$ (for $d = \dim \overline{X}$) studied in [BS19, 10.2]. Despite the fact that it is defined starting from a different modulus condition on algebraic cycles, it agrees also with the Albanese variety with modulus $\mathrm{Alb}(\overline{X}, X_\infty)$ defined by Russell [Rus13]. This is a consequence of Lemma 3.10 and the fact that both notions agree with the Rosenlicht-Serre generalized Jacobian in the one-dimensional case[6].

3.4.1. Let $\mathfrak{X}$ be as above. From the proof of Proposition 3.14, we deduce immediately the existence of a natural surjective map of torsors

$$\rho^{\Omega,\mathrm{CH}}_{\mathfrak{X}} \colon \mathrm{Alb}^{\Omega,(1)}_{\mathfrak{X}} \to \mathrm{Alb}^{\mathrm{CH},(1)}_{\mathfrak{X}}$$

equivariant with respect to a surjective homomorphism of algebraic k groups $\mathrm{Alb}^{\Omega,(0)}_{\mathfrak{X}} \to \mathrm{Alb}^{\mathrm{CH},(0)}_{\mathfrak{X}}$ such that $\mathrm{alb}^{\mathrm{CH}}_{\mathfrak{X}} = \rho^{\Omega,\mathrm{CH}}_{\mathfrak{X}} \circ \mathrm{alb}^{\Omega}_{\mathfrak{X}}$. We will see below that those maps are isomorphisms when the characteristic of the base field is zero.

To further relate the Chow groups of zero cycles with modulus of a pair $\mathfrak{X}$ with the $\mathcal{M}^{\Omega}_{\mathfrak{X}}$ Albanese construction of 3.2, we also recall the following

Proposition 3.16 ([Ser75, III, Proposition 10] or [KSYR16, Proposition 4.3.1])**.** *Let G be a commutative algebraic group over a field K of characteristic zero. Let $\overline{C}$ be a proper normal curve over K, C an open dense subscheme of $\overline{C}$ and $\psi\colon C \to G$ a K-morphism. Let D be an effective divisor on $\overline{C}$ supported on $\overline{C} \setminus C$ such that*

$$\psi^*(\Omega(G)) \subset H^0(\overline{C}, \Omega^1_{\overline{C}} \otimes_{\overline{C}} \mathcal{O}_{\overline{C}}(D)).$$

Then we have $\psi(\mathrm{div}_{\overline{C}}(f)) = 0$ in $G(K)$ for any $f \in G(\overline{C}, D)$.

Proposition 3.17. *Let k be a field of characteristic zero. Then the $\mathcal{M}^{\mathrm{CH}}_{\mathfrak{X}}$-Albanese torsor of $\mathfrak{X}$ agrees with the $\mathcal{M}^{\Omega}_{\mathfrak{X}}$-Albanese torsor $(\mathrm{alb}^{\Omega}_{\mathfrak{X}}, \mathrm{Alb}^{\Omega,(1)}_{\mathfrak{X}}, \mathrm{Alb}^{\Omega,(0)}_{\mathfrak{X}})$ of Theorem 3.7.*

Proof. It is enough to show that the two categories $\mathcal{M}^{\mathrm{CH}}_{\mathfrak{X}}$ and $\mathcal{M}^{\Omega}_{\mathfrak{X}}$ have the same objects (as they are both full subcategories of $X\backslash\mathcal{P}$). According to Definitions 3.13 and 3.5, it is enough to show the statement under the assumption that $k = \overline{k}$ is algebraically closed. We already know thanks to the proof of Proposition 3.14 that $\mathcal{M}^{\mathrm{CH}}_{\mathfrak{X}}$ is a full subcategory of $\mathcal{M}^{\Omega}_{\mathfrak{X}}$ (this does not require k to be of characteristic 0). To prove the other inclusion, let $u\colon X \to G \in \mathcal{M}^{\Omega}_{\mathfrak{X}}$. Again by Lemma 3.10, for any $\mathfrak{X}$-admissible morphism $\varphi\colon \overline{C} \to \overline{X}$ from a normal integral curve, the composition $u_C\colon \overline{C} \times_{\overline{X}} X \to X \to G$ satisfies $u_C^*(\Omega(G)) \subseteq H^0(\overline{C}, \Omega^1_{\overline{C}} \otimes_{\overline{C}} \mathcal{O}_{\overline{C}}(C_\infty))$, where C_∞ denotes as before the pullback $\varphi^*(X_\infty)$. By Proposition 3.16 above, we have $u_C(\mathrm{div}_{\overline{C}}(f)) = u(\varphi_* \mathrm{div}_{\overline{C}}(f)) = 0$ in $G(k)$ for any $f \in G(\overline{C}, C_\infty)$, so that the map induced by u on the group of zero cycles of degree zero $Z_0(X)^0$ factors through $\mathrm{CH}_0(\mathfrak{X})^0$. The same argument applies to any base-change to $L \supset k$ algebraically closed, so that $(u, G) \in \mathcal{M}^{\mathrm{CH}}_{\mathfrak{X}}$ as required. □

3.5. **The Albanese scheme with modulus.** We can now extend the construction of Ramachandran [Ram01] to the modulus setting. Let $X \in \mathbf{Sm}(k)$ be geometrically integral and $\mathfrak{X} \in \mathbf{Comp}(X)$. Thanks to Theorem 3.7, we have a map, defined over k,

$$\mathrm{alb}^{\Omega,(1)}_{\mathfrak{X}} \colon X \to \mathrm{Alb}^{\Omega,(1)}_{\mathfrak{X}} \tag{3.17.1}$$

universal for morphisms from X to torsors under commutative algebraic groups in $\mathcal{M}^{\Omega}_{\mathfrak{X}}$ (in the following, we shall say "to torsors under commutative algebraic groups with modulus $\mathfrak{X}$"). Let $\mathbf{Alb}^{\Omega}_{\mathfrak{X}} \in \mathcal{G}^*$ be the k-group scheme $\coprod_{n\in\mathbb{Z}} (\mathrm{Alb}^{\Omega,(1)}_{\mathfrak{X}})^{\otimes n}$ constructed in 3.1.1. The

[6] An independent (and explicit) proof of the fact that over $\mathbb{C}$ the generalized Jacobian $J^d_{\overline{X}|X_\infty}$ agrees with $\mathrm{Alb}(\overline{X}, X_\infty)$ defined by Russell has been given by T. Yamazaki [Yam17], using Hodge-theoretic methods.

universal map (3.17.1) composed with the natural inclusion of $\mathrm{Alb}^{\Omega,(1)}_{\mathfrak{X}}$ in $\mathbf{Alb}^{\Omega}_{\mathfrak{X}}$ gives then a canonical morphism

$$a_{\mathfrak{X}}\colon X \to \mathbf{Alb}^{\Omega}_{\mathfrak{X}}$$

which is now universal for morphisms to k-group schemes in the appropriate sense. By construction, the k-group scheme $\mathbf{Alb}^{\Omega}_{\mathfrak{X}}$ is an extension

$$0 \to \mathrm{Alb}^{\Omega,(0)}_{\mathfrak{X}} \to \mathbf{Alb}^{\Omega}_{\mathfrak{X}} \to \underline{\mathbb{Z}} \to 0.$$

If X has a k-rational point, the extension is split, i.e. we have an isomorphism $\mathbf{Alb}^{\Omega}_{\mathfrak{X}} \cong \mathrm{Alb}^{\Omega,(0)}_{\mathfrak{X}} \times \underline{\mathbb{Z}}$. This happens in particular when k is algebraically closed, and corresponds to the fact that we can trivialize the torsor $\mathrm{Alb}^{\Omega,(1)}_{\mathfrak{X}} \cong \mathrm{Alb}^{\Omega,(0)}_{\mathfrak{X}}$. Recall now the following Proposition, which follows from [KSYR16, Theorem 4.1.1] (while the transfer structure follows from [SS03, Proof of Lemma 3.2])

Proposition 3.18. *The k-group scheme $\mathbf{Alb}^{\Omega}_{\mathfrak{X}}$, regarded as étale sheaf on $\mathbf{Sm}(k)$, has a canonical structure of sheaf with transfers, and as such it has reciprocity in the sense of* [KSYR16, Definition 2.1.3].

Thanks to the Proposition, there is a unique map of preshaves with transfers

$$a_{\mathfrak{X}}\colon \mathbb{Z}_{tr}(X) \to \mathbf{Alb}^{\Omega}_{\mathfrak{X}}$$

extending the map $a_{\mathfrak{X}}\colon X \to \mathbf{Alb}^{\Omega}_{\mathfrak{X}}$ defined above.

3.5.1. When $\mathfrak{X}$ is moreover smooth over k, we can apply Proposition 3.14 to get a map, defined over k, $\mathrm{alb}^{\mathrm{CH},(1)}_{\mathfrak{X}}\colon X \to \mathrm{Alb}^{\mathrm{CH},(1)}_{\mathfrak{X}}$, universal for morphisms to torsors in $\mathcal{M}^{\mathrm{CH}}_{\mathfrak{X}}$. We can repeat the constructions of the previous point to get yet another k-group scheme $\mathbf{Alb}^{\mathrm{CH}}_{\mathfrak{X}} \in \mathcal{G}^*$ together with a canonical morphism

$$a^{\mathrm{CH}}_{\mathfrak{X}}\colon X \to \mathbf{Alb}^{\mathrm{CH}}_{\mathfrak{X}}.$$

The group $\mathbf{Alb}^{\mathrm{CH}}_{\mathfrak{X}}$ has by the same argument of Proposition 3.18 a canonical structure of étale sheaf with transfers, with reciprocity in the sense of [KSYR16, Definition 2.1.3]. This gives us a unique map of presheaves with transfers

$$a^{\mathrm{CH}}_{\mathfrak{X}}\colon \mathbb{Z}_{tr}(X) \to \mathbf{Alb}^{\mathrm{CH}}_{\mathfrak{X}}$$

extending $a^{\mathrm{CH}}_{\mathfrak{X}}$. When the base field has characteristic zero, the two constructions agree by Proposition 3.17, so that we canonically identify $\mathbf{Alb}^{\mathrm{CH}}_{\mathfrak{X}}$ with $\mathbf{Alb}^{\Omega}_{\mathfrak{X}}$. In general, we have a surjective map of étale sheaves $\mathbf{Alb}^{\Omega}_{\mathfrak{X}} \to \mathbf{Alb}^{\mathrm{CH}}_{\mathfrak{X}}$.

3.6. **The maximal semi-abelian quotient.** Let $X \in \mathbf{Sm}(k)$ be geometrically connected. Serre's Albanese map of Example 3.4 can be extended to a unique map of presheaves with transfers:

$$\mathbb{Z}_{tr}(X) \to \mathbf{Alb}_X \tag{3.18.1}$$

where $\mathbf{Alb}_X$ is the semi-abelian Albanese scheme of X, defined by [Ram01] or [SS03, Lemma 3.2] using the same recipe of Section 3.5. Since $\mathbf{Alb}_X$ is semi-abelian, it is a homotopy invariant étale sheaf with transfers. Thus, taking sections over any field $L \supset k$, we have a factorization of the map (3.18.1) through

$$h_0^{\mathbb{A}^1}(X_L) \to \mathbf{Alb}_X(L),$$

where $h_0^{\mathbb{A}^1}(X_L)$ denotes the zeroth Suslin homology group of $X_L = X \otimes_k L$. If X admits a k-point, the scheme $\mathbf{Alb}_X$ decomposes as $\underline{Z} \times \mathrm{Alb}^{(0)}_X$, where $\mathrm{Alb}^{(0)}_X$ denotes Serre's semi-abelian Albanese variety of X. In particular, we get for any $L \supset k$ algebraically closed an induced (surjective) map on the degree zero part

$$h_0^{\mathbb{A}^1}(X_L)^0 \to \mathrm{Alb}^{(0)}_{X_L}(L) = \mathrm{Alb}_{X_L}(L). \tag{3.18.2}$$

Let $\mathfrak{X} \in \mathbf{Comp}(X)$. By e.g. [BCKS17, Proposition 2.6], there is a natural surjection $\mathrm{CH}_0(\mathfrak{X}) \to h_0^{\mathbb{A}^1}(X)$, which can be composed with (3.18.2) to give a surjective homomorphism

$$\mathrm{CH}_0(\mathfrak{X}_L)^0 \to \mathrm{Alb}_{X_L}(L).$$

By Definition 3.13, we have then that the object $(\mathrm{alb}_X, \mathrm{Alb}_X^{(1)}, \mathrm{Alb}_X^{(0)})$ belongs to $\mathcal{M}_{\mathfrak{X}}^{\mathrm{CH}}$. By Proposition 3.14, the universal property of $\mathrm{Alb}_{\mathfrak{X}}^{\mathrm{CH},(0)}$ gives a unique surjection

$$\mathrm{Alb}_{\mathfrak{X}}^{\mathrm{CH},(0)} \to \mathrm{Alb}_X^{(0)}$$

(and similarly for $\mathrm{Alb}_{\mathfrak{X}}^{\mathrm{CH},(1)}$ and $\mathrm{Alb}_X^{(1)}$), which factors through the semi-abelian quotient $\mathrm{Alb}_{\mathbf{SAb},\mathfrak{X}}^{\mathrm{CH},(0)}$ of $\mathrm{Alb}_{\mathfrak{X}}^{\mathrm{CH},(0)}$. It is straightforward to show that if X has a k-rational point, the algebraic groups $\mathrm{Alb}_{\mathbf{SAb},\mathfrak{X}}^{\mathrm{CH},(0)}$ and $\mathrm{Alb}_X^{(0)}$ are isomorphic. We have therefore the following

Proposition 3.19. *Let $\mathfrak{X}$ be as above, and suppose that X has a k-rational point. Then the semi-abelian part of $\mathrm{Alb}_{\mathfrak{X}}^{\mathrm{CH},(0)}$ agrees with Serre's semi-abelian Albanese variety of X.*

3.7. **Universal problem for presheaves with transfers.** We continue with the notations of 3.5. As observed in Remark 2.9 (ii), for any $F \in \mathbf{RSC}$ and for every section $g : \mathbb{Z}_{tr}(X) \to F$, there exists $\mathfrak{X} \in \mathbf{Comp}(X)$ such that g factors through $\omega_! h_0^{\overline{\square}}(\mathfrak{X})$. In this case, we say that *g has modulus $\mathfrak{X}$*. We apply this to the case $\mathbf{Alb}_{\mathfrak{X}}^{\mathrm{CH}}$.

Proposition 3.20. *Let $X \in \mathbf{Sm}(k)$ be geometrically connected and $\mathfrak{X} \in \mathbf{Comp}(X)$. Then the canonical map $a_{\mathfrak{X}} \colon \mathbb{Z}_{tr}(X) \to \mathbf{Alb}_{\mathfrak{X}}^{\mathrm{CH}}$ factors through $\omega_! h_0^{\overline{\square}}(\mathfrak{X})$ and it is universal with respect to this property: for any smooth commutative k-group scheme G, seen as étale reciprocity sheaf, and for any section $g \colon \mathbb{Z}_{tr}(X) \to G$ with modulus $\mathfrak{X}$, there is a unique morphism $\tilde{g} \colon \mathbf{Alb}_{\mathfrak{X}}^{\mathrm{CH}} \to G$ in $\mathbf{PST}$ such that*

$$\begin{tikzcd} \mathbb{Z}_{tr}(X) \arrow[rr, "g"] \arrow[dr, "a_{\mathfrak{X}}"'] & & G \\ & \mathbf{Alb}_{\mathfrak{X}}^{\mathrm{CH}}. \arrow[ur, "\tilde{g}"'] & \end{tikzcd}$$

Proof. We first prove that $a_{\mathfrak{X}} \colon \mathbb{Z}_{tr}(X) \to \mathbf{Alb}_{\mathfrak{X}}$ factors through $\omega_! h_0^{\overline{\square}}(\mathfrak{X})$. We have to show that for any smooth k-scheme S, the map $\mathbb{Z}_{tr}(X)(S) \to \mathbf{Alb}_{\mathfrak{X}}(S)$ factors through $\omega_! h_0^{\overline{\square}}(\mathfrak{X})(S)$. Since $\mathbf{Alb}_{\mathfrak{X}}$, as any commutative k-group scheme, satisfies global injectivity, it is enough to check the factorization after passing to the function field $k(S)$ of S, and in fact even to its algebraic closure. Let then $K \supset k$ be an algebraically closed field, and look at the map $\mathbb{Z}_{\mathrm{tr}}(X)(K) \to \mathbf{Alb}_{\mathfrak{X}_K}(K)$. Let $\pi_0(X)$ be the spectrum of the integral closure of k in $\Gamma(X, \mathcal{O}_X)$. The assignment $X \mapsto \pi_0(X)$ is universal for morphisms from X into étale k-schemes. Since X is geometrically integral by assumption, we have $\mathbb{Z}_{\mathrm{tr}}(\pi_0(X)) = \underline{\mathbb{Z}}$ (as étale sheaves), and the map $\mathbb{Z}_{\mathrm{tr}}(X_K) \to \mathbf{Alb}_{\mathfrak{X}_K}$ induces then a map

$$\mathbb{Z}_{\mathrm{tr}}(X_K)^0 \to \mathrm{Alb}_{\mathfrak{X}_K}^{(0)},$$

where $\mathbb{Z}_{\mathrm{tr}}(X_K)^0$ denotes the kernel of $\mathbb{Z}_{\mathrm{tr}}(X_K) \to \mathbb{Z}_{\mathrm{tr}}(\pi_0(X_K))$. We can then identify $\mathbb{Z}_{tr}(X_K)^0(K)$ with the group of 0-cycles of degree zero $Z_0(X_K)^0$ of X_K. Since $\mathfrak{X}$ is a proper modulus pair, we have $\mathrm{CH}_0(\mathfrak{X}_K)^0 = h_0(\mathfrak{X}_K)^0(K)$ thanks to [KSY21, Remark 2.2.3], and thus the claim follows from Proposition 3.14. The same argument proves the universal property as well. □

From now on, we write simply $\mathbf{Alb}_{\mathfrak{X}}$ for the k-group scheme $\mathbf{Alb}_{\mathfrak{X}}^{\mathrm{CH}}$. We end this section with a the following result, which will be crucial for the construction of the category of 1-reciprocity sheaves.

Lemma 3.21. *Under the assumptions of Proposition 3.20, the map*

$$a_{\mathfrak{X}} \otimes_{\mathbb{Z}} \mathbb{Q} \colon \mathbb{Q}_{tr}(X) \to \mathbf{Alb}_{\mathfrak{X}} \otimes_{\mathbb{Z}} \mathbb{Q}$$

is a surjective morphism of étale sheaves with transfers with rational coefficients.

Proof. Let $\mathrm{Im}(a_{\mathfrak{X}}) \subset \mathbf{Alb}_{\mathfrak{X}}$ be the image of $a_{\mathfrak{X}}$ in $\mathbf{PST}$. Since $\mathbf{Alb}_{\mathfrak{X}}$ is a smooth k-group scheme, we have $\mathbf{Alb}_{\mathfrak{X}} \in \mathbf{RSC}_{\mathrm{Nis}}$ by [KSYR16, Theorem 4.1.1]. Let $C = \mathbf{Alb}_{\mathfrak{X}}/\mathrm{Im}(a_{\mathfrak{X}}) \in \mathbf{RSC}$. We have to show that $a^V_{\text{ét}}(C \otimes_{\mathbb{Z}} \mathbb{Q}) = 0$. By (2.9.1), we have that $a^V_{\text{ét}}(C \otimes_{\mathbb{Z}} \mathbb{Q}) \in \mathbf{RSC}_{\text{ét}}(k, \mathbb{Q})$, in particular it satisfies global injectivity by [Sai20, Theorem 0.2], i.e. for any $Y \in \mathbf{Sm}$ geometrically connected with function field $k(Y)$ with algebraic closure $\overline{k(Y)}$ there is an injective map

$$a^V_{\text{ét}}(C \otimes_{\mathbb{Z}} \mathbb{Q})(Y) \hookrightarrow a^V_{\text{ét}}(C \otimes_{\mathbb{Z}} \mathbb{Q})(k(Y)) \hookrightarrow (C \otimes_{\mathbb{Z}} \mathbb{Q})(\overline{k(Y)}) = C(\overline{k(Y)}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

To complete the proof, it is then enough to show that $C(\overline{k(Y)}) = 0$. This follows from the fact that for any $K \supset k$ algebraically closed, the map

$$\mathbb{Z}_{tr}(X \otimes_k K)(K) \to \mathbf{Alb}_{\mathfrak{X}}(K)$$

is surjective, since $\mathrm{CH}_0(\mathfrak{X}_K)^0 \to \mathrm{Alb}^{(0)}_{\mathfrak{X}}(K)$ is surjective. □

4. The Albanese functors

4.1. **n-reciprocity sheaves.** For any $n \geq 0$, let $\mathbf{Cor}(k)_{\leq n}$ be the category of finite correspondences on smooth k-schemes of dimension $\leq n$, and let $\mathbf{MCor}(k)_{\leq n}$ be the category of modulus correspondences on smooth proper modulus pairs $\mathfrak{X} = (\overline{X}, X_\infty)$ with $\dim(\overline{X}) \leq n$. We let $\mathbf{MPST}(k_{\leq n})$ be the category of additive presheaves on $\mathbf{MCor}(k)_{\leq n}$.

The natural inclusions of subcategory of objects of dimension $\leq n$ give rise to a standard string of adjoint functors between the category of presheaves

$$(\sigma_{n,!}, \sigma_n^*), \quad \sigma_{n,!} \colon \mathbf{MPST}(k_{\leq n}) \leftrightarrows \mathbf{MPST}\,(k) \colon \sigma_n^*. \tag{4.0.1}$$

Here, we follow the convention of [KMSY21a] for the left Kan extension of the restriction functor σ_n^*. Note that this is different from the one adopted in [ABV09].

Remark 4.1. Let

$$(\sigma^V_{n,!}, \sigma^{V,*}_n), \quad (\sigma^V_{n,!} \colon \mathbf{PST}(k_{\leq n}) \leftrightarrows \mathbf{PST}(k) \colon \sigma^{V,*}_n) \tag{4.1.1}$$

be the analogous adjoint functors from [ABV09]. Since the functor ω clearly restricts to a functor $\mathbf{MCor}(k)_{\leq n} \to \mathbf{Cor}(k)_{\leq n}$, for all $F \in \mathbf{PST}(k)$ we have that

$$\sigma_n^* \omega^* F \cong \omega^* \sigma_n^{V,*} F \text{ in } \mathbf{MPST}(k_{\leq n}).$$

By adjunction, we conclude that for all $F \in \mathbf{MPST}(k_{\leq n})$,

$$\sigma^V_{n,!} \omega_! F \cong \omega_! \sigma_{n,!} F \text{ in } \mathbf{PST}(k).$$

For $F \in \mathbf{MPST}(k)$ and $X \in \mathbf{Cor}(k)_{\leq n}$, for any modulus pair $\mathfrak{X} \in \mathbf{Comp}(X)$, we have $\mathfrak{X} \in \mathbf{MCor}(k)_{\leq n}$, hence $\sigma_n^* F(\mathfrak{X}) = F(\mathfrak{X})$

$$\omega_! \sigma_n^* F(X) = \varinjlim_{\mathfrak{X} \in \mathbf{Comp}(X)} \sigma_n^* F(\mathfrak{X}) = \omega_! F(X) = \sigma_n^{V,*}(\omega_! F)(X).$$

Finally, for every modulus pair $\mathfrak{X}$ and every $\alpha \colon Y \to \omega(\mathfrak{X}) \in \mathbf{Cor}(k)$, by [KMSY21b, Theorem 1.6.2] there exists a proper modulus pair $\mathfrak{Y}' \in \mathbf{Comp}(Y)$ and $\alpha' \colon \mathfrak{Y}' \to \mathfrak{X} \in \mathbf{MCor}(k)$ such that $\alpha = \omega(\alpha')$. In particular, since $\mathfrak{Y} \in \mathbf{MCor}(k)_{\leq n}$, the system $\{F(Y)\}$

for $Y \in \mathbf{Cor}(k)_{\leq n}$ running over the maps $Y \to \omega(X)$ is cofinal in the system $\{F(\omega(\mathfrak{Y}))\}$ for $\mathfrak{Y} \in \mathbf{MCor}(k)_{\leq n}$ running over the maps $\mathfrak{Y} \to \mathfrak{X}$. Hence we have that

$$\begin{aligned}\omega^*\sigma^V_{n,!}F(\mathfrak{X}) = \sigma^V_{n,!}F(\omega(\mathfrak{X})) &= \varinjlim_{\substack{(Y\to\omega(\mathfrak{X}))\\ Y\in\mathbf{Cor}(k)_{\leq n}}} F(Y)\\ &= \varinjlim_{\substack{(\mathfrak{Y}'\to\mathfrak{X})\\ Y\in\mathbf{Cor}(k)_{\leq n}}} F(\omega(\mathfrak{Y}'))\\ &= \varinjlim_{\substack{(\mathfrak{Y}\to\mathfrak{X})\\ \mathfrak{Y}\in\mathbf{MCor}(k)_{\leq n}}} F(\omega(\mathfrak{Y})) = \sigma_{n,!}\omega^*F(\mathfrak{X}).\end{aligned}$$

Remark 4.2. The functors $\sigma_{n,!}$ commute with colimits of presheaves since they are left adjoint. The functor σ_n^* also commutes with colimits of presheaves, since they are computed section-wise and $\sigma_n^*F(\mathfrak{X}) = F(\mathfrak{X})$ for all $\mathfrak{X} \in \mathbf{MCor}(k)_{\leq n}$. In particular for all diagrams $\{F_i\}$ in **MPST**, $\sigma_{n,!}\sigma_n^* \varinjlim F_i \cong \varinjlim \sigma_{n,!}\sigma_n^*F_i$

The following lemma is mutuated from [ABV09, Lemma 1.1.12]:

Lemma 4.3. *The unit map* $\mathrm{Id} \xrightarrow{\sim} \sigma_n^*\sigma_{n,!}$ *is invertible.*

Definition 4.4. We say that $F \in \mathbf{MPST}$ is *n-generated* (resp. *strongly n-generated*) if the counit map $\sigma_{n,!}\sigma_n^*F \to F$ is surjective (resp. an isomorphism).

Remark 4.5. If F is *n-generated* (resp. *strongly n-generated*), then $\omega_!F$ is n-generated (resp. strongly n-generated) in the sense of [ABV09]. Indeed, the functor $\omega_!$ is exact and $\omega_!\sigma_{n,!}\sigma_n^*F = \sigma^V_{n,!}\sigma_n^{*,V}\omega_!F$ by Remark 4.1.

For example, if $\mathfrak{X} = (\overline{X}, X_\infty)$ with $\dim(\overline{X}) \leq n$, then $\mathbb{Z}_{tr}(\mathfrak{X})$ is strongly n-generated. The proof of the following Lemma is a diagram chase.

Lemma 4.6. *Quotients and extensions of (strongly) n-generated sheaves are again (strongly) n-generated.*

Definition 4.7. Let $F \in \mathbf{CI}$. Following [ABV09, Definition 1.1.20], we say that F is an *n-modulus presheaf* if the natural map

$$h_{0,\acute{e}t}^{\mathbf{rec}}(\sigma_{n,!}\sigma_n^*F) \to h_{0,\acute{e}t}^{\mathbf{rec}}F = a^V_{\acute{e}t}\omega_{\mathbf{CI}}F$$

is an isomorphism of étale sheaves with transfers. Here, for any $G \in \mathbf{MPST}$, we denote by $h_{0,\acute{e}t}^{\mathbf{rec}}(G)$ the étale sheaf with transfers $a^V_{\acute{e}t}\omega_{\mathbf{CI}}h_0^{\overline{\square}}G$, where $a^V_{\acute{e}t}\colon \mathbf{PST} \to \mathbf{Shv}^{\mathrm{tr}}_{\acute{e}t}$ is Voevodksy's étale sheafification functor and $\omega_{\mathbf{CI}}$ is the composition $\omega_! \circ i^{\overline{\square}}$, where $i^{\overline{\square}}$ is the inclusion $\mathbf{CI} \to \mathbf{MPST}$, which has a right adjoint. Notice that the functor $h_{0,\acute{e}t}^{\mathbf{rec}}$ is a composition of left adjoints, hence it commutes with all colimits.

We write $\mathbf{CI}_{\leq n}$ for the full subcategory of n-modulus presheaves.

The following is identical to [ABV09, Remark 1.1.21]:

Lemma 4.8. *Let $F \in \mathbf{MPST}$ be strongly n-generated. Then $h_0^{\overline{\square}}(F)$ is an n-modulus sheaf.*

Remark 4.9. Notice that in this case, differently from [ABV09, Remark 1.1.21], if F is an n-modulus presheaf then it is not automatic that F is the $h_0^{\overline{\square}}$ of a strongly n-generated sheaf: we only know that the map

$$h_0^{\overline{\square}}\sigma_{n,!}\sigma_n^*F \to F$$

is an isomorphism after applying $a^V_{\acute{e}t}\omega_{\mathbf{CI}}$.

Definition 4.10. We define the category of *n-reciprocity sheaves* $\mathbf{RSC}_{\acute{e}t,\leq n}$ as the essential image of $\mathbf{CI}_{\leq n}$ via the functor $a^V_{\acute{e}t}\omega_{\mathbf{CI}}$.

The following result is immediate:

Lemma 4.11. *Let $F \in \mathbf{Shv}^{\mathrm{tr}}_{\mathrm{ét}}$. The following are equivalent*

(i) F is an n-reciprocity sheaf;
(ii) $F \cong h^{\mathbf{rec}}_{0,\mathrm{ét}}\sigma_{n,!}\sigma_n^*G$ for some $G \in \mathbf{MPST}$.

Moreover, if the above conditions hold, we can take $G \in \mathbf{CI}_{\leq n}$ in (ii).

Remark 4.12. If $F \in \mathbf{RSC}_{\mathrm{ét},\leq n}$, then F is an n-generated étale sheaf in the sense of [ABV09]. Indeed, for $F = h^{\mathbf{rec}}_{0,\mathrm{ét}}\sigma_{n,!}\sigma_n^*G$, then there is a surjective map in $\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{ét}}$

$$a^V_{\mathrm{ét}}\sigma^V_{n,!}\sigma^{V,*}_n(F) = a^V_{\mathrm{ét}}\omega_!\sigma_{n,!}\sigma_n^*G \twoheadrightarrow a^V_{\mathrm{ét}}\omega_{\mathbf{CI}}h^{\square}_{0,\mathrm{ét}}\sigma_{n,!}\sigma_n^*G \cong h^{\mathbf{rec}}_{0,\mathrm{ét}}\sigma_{n,!}\sigma_n^*G = F.$$

Recall that the category $F \in \mathbf{HI}_{\mathrm{ét},\leq n}$ of [ABV09, Definition 1.2.20], is the full subcategory of $\mathbf{HI}_{\mathrm{ét}}$ of objects F such that $a^V_{\mathrm{ét}}h^{\mathbf{A}^1}_0\sigma^V_{n,!}\sigma^{V,*}_nF \to a^V_{\mathrm{ét}}h^{\mathbf{A}^1}_0F = F$ is an isomorphism.

Proposition 4.13. *If $F \in \mathbf{HI}_{\mathrm{ét},\leq n}$, then $F \in \mathbf{RSC}_{\mathrm{ét},\leq n}$.*

Proof. Take $F \in \mathbf{HI}_{\mathrm{ét},\leq n}$. By Remark 4.1 we have

$$h^{\mathbf{rec}}_{0,\mathrm{ét}}\sigma_{n,!}\sigma_n^*\omega^*F = h^{\mathbf{rec}}_{0,\mathrm{ét}}\omega^*\sigma^V_{n,!}\sigma^{V,*}_nF. \tag{4.13.1}$$

Notice that $h^{\overline{\square}}_0\omega^* = \omega^*h^{\mathbf{A}^1}_0$, so by full faithfulness of ω^* we conclude that

$$h^{\mathbf{rec}}_{0,\mathrm{ét}}\omega^*\sigma^V_{n,!}\sigma^{V,*}_nF = a^V_{\mathrm{ét}}\omega_!h^{\overline{\square}}_0\omega^*\sigma^V_{n,!}\sigma^{V,*}_nF = a^V_{\mathrm{ét}}\omega_!\omega^*h^{\mathbf{A}^1}_0\sigma^V_{n,!}\sigma^{V,*}_nF = a^V_{\mathrm{ét}}h^{\mathbf{A}^1}_0\sigma^V_{n,!}\sigma^{V,*}_nF = F.$$

In view of (4.13.1) and Lemma 4.11, this implies $F \in \mathbf{RSC}_{\mathrm{ét},\leq n}$. □

The following lemma is analogue to [ABV09, Lemma 1.1.23]

Lemma 4.14. *For any $G \in \mathbf{CI}$, the natural map*

$$\sigma^{V,*}_nh^{\mathbf{rec}}_{0,\mathrm{ét}}(\sigma_{n,!}\sigma_n^*G) \to \sigma^{V,*}_na^V_{\mathrm{ét}}\omega_{\mathbf{CI}}G \tag{4.14.1}$$

induced by the counit map $\sigma_{n,!}\sigma_n^*G \to G$ is an isomorphism in $\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{ét}}(k_{\leq n})$

Corollary 4.15 (cfr. [ABV09, Corollary 1.1.26])**.** *Let $F \in \mathbf{RSC}$ such that $F = \omega_{\mathbf{CI}}G$ with $G \in \mathbf{CI}$ and consider the natural map*

$$h^{\mathbf{rec}}_{0,\mathrm{ét}}(\sigma_{n,!}\sigma_n^*G) = a^V_{\mathrm{ét}}\omega_{\mathbf{CI}}h^{\overline{\square}}_0\sigma_{n,!}\sigma_n^*G \to a^V_{\mathrm{ét}}\omega_{\mathbf{CI}}h^{\overline{\square}}_0G = a^V_{\mathrm{ét}}\omega_{\mathbf{CI}}G = a^V_{\mathrm{ét}}F$$

induced by the counit map $\sigma_{n,!}\sigma_n^*G \to G$. Let N be the kernel of the above map. If N is an n-generated étale sheaf in the sense of [ABV09], then it is zero.

Proof. Since the functor $\sigma^{V,*}_n$ is exact, we have by the definition of N an exact sequence of étale sheaves:

$$0 \to \sigma^{V,*}_n(N) \to \sigma^{V,*}_nh^{\mathbf{rec}}_{0,\mathrm{ét}}(\sigma_{n,!}\sigma_n^*G) \to \sigma^{V,*}_na^V_{\mathrm{ét}}F,$$

hence by Lemma 4.14 we conclude $\sigma^{V,*}_n(N) = 0$. Since N is n-generated, we have a surjective map of étale sheaves with transfers $\sigma^V_{n,!}\sigma^{V,*}_n(N) \to N$, showing that $N = 0$. □

4.2. **0-reciprocity sheaves.** We specialize the general results of the previous section to the case $n = 0$. By definition, the objects of the category $\mathbf{MCor}_{\leq 0}$ of smooth modulus pairs of dimension ≤ 0 are the finite étale extensions $\ell \supset k$ (with empty modulus divisor). The essential image of the restriction of the functor $\omega\colon \mathbf{MCor} \to \mathbf{Cor}$ to $\mathbf{MCor}_{\leq 0}$ induces an equivalence of categories

$$\omega_{|\leq 0}\colon \mathbf{MCor}_{\leq 0} \simeq \mathbf{Cor}_{\leq 0}$$

whose inverse is given by

$$\lambda_{|\leq 0}\colon \mathrm{Spec}(\ell) \mapsto (\mathrm{Spec}(\ell), \emptyset).$$

and induces an equivalence of categories

$$\omega_{|\leq 0,!}\colon \underline{\mathbf{M}}\mathbf{PST}(k_{\leq 0}) \simeq \mathbf{PST}(k_{\leq 0})\colon \lambda_{|\leq 0,!}. \tag{4.15.1}$$

Moreover, for any $\mathfrak{X} \in \mathbf{MCor}$ with $X = \omega(\mathfrak{X})$, we have

$$\mathbf{MCor}(\mathfrak{X}, (\mathrm{Spec}(\ell), \emptyset)) \cong \mathbf{Cor}(\omega(\mathfrak{X}), \mathrm{Spec}(\ell)) \cong \mathbf{Cor}(\pi_0(X), \mathrm{Spec}(\ell)) \cong \mathbb{Z}^{\pi_0(|X \otimes_k \ell|)},$$

where $\pi_0(X)$ is the spectrum of the integral closure of k in $\Gamma(X, \mathcal{O}_X)$ and $\pi_0(|Y|)$ for a scheme Y denotes the set of connected components of the underlying topological space $|Y|$. The first isomorphism follows from the fact that $\mathrm{Spec}(\ell) \in \mathbf{Cor}^{\mathrm{prop}}$ (see [KMSY21a, Lemma 1.5.1]), while the second and the third are classical (see [MVW06, Lecture 1] and [ABV09, 1.2.1]). From this we get an adjunction

$$\lambda_{|\leq 0} \circ \pi_0 \circ \omega \colon \mathbf{MCor} \leftrightarrows \mathbf{MCor}_{\leq 0} \colon \sigma_0. \tag{4.15.2}$$

We let Π_0 denote $\lambda_{|\leq 0} \circ \pi_0 \circ \omega$. Passing to the categories of presheaves, we have that the functor $\sigma_{0,!}$ of (4.0.1) has a left adjoint:

$$\Pi_{0,!} \colon \mathbf{MPST}(k) \leftrightarrows \mathbf{MPST}(k_{\leq 0}) \colon \Pi_0^* = \sigma_{0,!}. \tag{4.15.3}$$

In particular, $\sigma_{0,!}$ is given explicitly by

$$\sigma_{0,!}(F)(\mathfrak{X}) = F(\Pi_0(\mathfrak{X})) = F(\pi_0(\omega(\mathfrak{X})), \emptyset) \ \text{ for } F \in \mathbf{MPST}(k_{\leq 0}). \tag{4.15.4}$$

The following Corollary shows that the category of 0-reciprocity sheaves is simply equivalent to the category of 0-motivic sheaves in the sense of Ayoub–Barbieri-Viale.

Corollary 4.16. *Let $G \in \mathbf{MPST}(k)$, then $\sigma_{0,!}\sigma_0^* G \in \mathbf{CI}$. In particular, every 0-reciprocity sheaf is strongly 0-generated in the sense of [ABV09], and we have equivalences*

$$\mathbf{HI}_{\text{ét},\leq 0} \simeq \mathbf{RSC}_{\text{ét},\leq 0} \simeq \mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\leq 0}). \tag{4.16.1}$$

Proof. Let $\mathfrak{X} \in \mathbf{MCor}$ and $X = \omega(\mathfrak{X})$. By(4.15.4), we have that

$$\sigma_{0,!}\sigma_0^* G(\mathfrak{X} \otimes \mathbf{P}^1) = \sigma_0^* G(\pi_0(\omega(\mathfrak{X} \otimes \mathbf{P}^1)), \emptyset).$$

On the other hand, we have that $\omega(\mathfrak{X} \otimes \mathbf{P}^1) = X \times \mathbf{A}^1$ and as observed in the proof of [ABV09, Lemma 1.2.2], we have that $\pi_0(X \times \mathbf{A}^1) = \pi_0(X)$, hence

$$\sigma_{0,!}\sigma_0^* G(\mathfrak{X} \otimes \mathbf{P}^1) = \sigma_0^* G(\pi_0(\omega(\mathfrak{X} \otimes \mathbf{P}^1)), \emptyset) = \sigma_0^* G(\pi_0(\omega(\mathfrak{X})), \emptyset) = \sigma_{0,!}\sigma_0^* G(\mathfrak{X}).$$

Hence $\sigma_{0,!}\sigma_0^* G \in \mathbf{CI}$ proving the first assertion. Let $F \in \mathbf{RSC}_{\text{ét},\leq 0}$ and let $F = h^{\mathbf{rec}}_{0,\text{ét}} \sigma_{0,!}\sigma_0^* G'$ with $G' \in \mathbf{CI}(k)$ be as in Lemma 4.11(ii). Then $\sigma_{0,!}\sigma_0^* G' \in \mathbf{CI}$, so

$$F = a^V_{\text{ét}} \omega_{\mathbf{CI}} h_0^{\overline{\square}} \sigma_{0,!}\sigma_0^* G' \simeq a^V_{\text{ét}} \omega_! \sigma_{0,!}\sigma_0^* G' = a^V_{\text{ét}} \sigma^V_{0,!} \sigma_0^{V,*} \omega_! G'.$$

This implies that $F \in \mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\leq 0})$. On the other hand, by Proposition 4.13 we have

$$\mathbf{HI}_{\text{ét},\leq 0} \hookrightarrow \mathbf{RSC}_{\text{ét},\leq 0} \hookrightarrow \mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\leq 0}).$$

By [ABV09, Lemma 1.2.2], the composition above is an equivalence, hence we deduce the equivalences of (4.16.1). □

4.3. **1-reciprocity sheaves and Albanese functors.** In this subsection, we will assume that k has characteristic zero. In particular, k satisfies resolutions of singularities and for any smooth proper modulus pair, we can identify $\mathbf{Alb}^{\mathrm{CH}}_{\mathfrak{X}}$ with $\mathbf{Alb}^{\Omega}_{\mathfrak{X}}$, and we simply write $\mathbf{Alb}_{\mathfrak{X}}$ for the Albanese scheme of $\mathfrak{X}$.

Remark 4.17. If $\mathfrak{C} = (\overline{C}, C_\infty)$ denotes a 1-dimensional smooth and proper modulus pair, by Lemma 4.8, $h^{\mathbf{rec}}_{0,\text{ét}}(\mathfrak{C})$ is a 1-reciprocity sheaf. Moreover, as observed in (2.9.2), we have:

$$h^{\mathbf{rec}}_{0,\text{ét}}(\mathfrak{C}) = a^V_{\text{ét}} \omega_{\mathbf{CI}} h_0^{\overline{\square}}(\mathbb{Z}_{tr}(\mathfrak{C})) = a^V_{\text{ét}} h_0(\mathfrak{C}) = \underline{\mathrm{Pic}}(\overline{C}, C_\infty),$$

where $\underline{\mathrm{Pic}}(\overline{C}, C_\infty)$ denotes the relative Picard group scheme, whose connected component of the identity $\underline{\mathrm{Pic}}^0(\overline{C}, C_\infty)$ agrees with the Rosenlicht-Serre generalized Jacobian $\mathrm{Jac}(\overline{C}, C_\infty)$. By Proposition 3.20 and the proof of Proposition 3.14, we have $\underline{\mathrm{Pic}}(\overline{C}, C_\infty) = \mathbf{Alb}_{\mathfrak{C}}$. Hence we finally have that $h^{\mathbf{rec}}_{0,\text{ét}}(\mathfrak{C}) = \mathbf{Alb}_{\mathfrak{C}}$ is represented by a commutative group scheme.

Remark 4.18. More generally, let $\mathfrak{X}$ be a smooth and proper modulus pair. Then the sheaf $\mathbf{Alb}_{\mathfrak{X}}$ is 1-generated. In fact, $\mathbf{Alb}_{\mathfrak{X}}$ can be written as extension of a semi-abelian k-group scheme (i.e. a k-group scheme G such that G^0 is a semi-abelian variety) and a unipotent algebraic group U. In characteristic 0, the group U is a product of $\mathbf{G}_a$, and $\mathbf{G}_a$ is a direct summand of $h^{\mathbf{rec}}_{0,\text{ét}}(\mathbb{P}^1, 2\infty)$, so that it is 1-generated. The semi-abelian k-group scheme G is a quotient of the generalized Jacobian of a suitable curve contained in G, which is 1-generated by the previous remark (see [ABV09, 1.3]). Hence G itself is 1-generated by [ABV09, Lemma 1.1.15]. Applying again [ABV09, Lemma 1.1.15] to $\mathbf{Alb}_{\mathfrak{X}}$ we get the statement.

We deduce from the previous remarks the following analogue to [ABV09, Lemma 1.3.4].

Lemma 4.19. *Let $F \in \mathbf{RSC}_{\text{ét},\leq 1}$. Then any subsheaf of F is a 1-generated étale sheaf in the sense of* [ABV09].

Proof. We essentially follow the steps in the proof of [ABV09, Lemma 1.3.4], starting from the case of $F = h^{\mathbf{rec}}_{0,\text{ét}}(\mathfrak{C})$, for $\mathfrak{C}$ a smooth and proper modulus pair of dimension 1. This is a 1-reciprocity sheaf by Remark 4.17. Let $E \subset F$ be a subsheaf. Since colimits of 1-generated étale sheaves are 1-generated étale sheaves, (see 4.2), we can assume that E is the image of a map $a\colon \mathbb{Z}_{tr}(X) \to h^{\mathbf{rec}}_{0,\text{ét}}(\mathfrak{C})$, for $X \in \mathbf{Sm}$. By Remark 4.17, we have then a map

$$a\colon \mathbb{Z}_{tr}(X) \to \mathbf{Alb}_{\mathfrak{C}} \cong h^{\mathbf{rec}}_{0,\text{ét}}(\mathfrak{C}).$$

Since $\mathbf{Alb}_{\mathfrak{C}} \in \mathbf{RSC}$, by Remark 2.9 there exists a smooth proper modulus pair $\mathfrak{X}$ with $X = \mathfrak{X}^\circ$ such that a factors through $h_0(\mathfrak{X})$, and by Proposition 3.20, it uniquely factors through $a_{\mathfrak{X}}\colon h_0(\mathfrak{X}) \to \mathbf{Alb}_{\mathfrak{X}}$:

$$\begin{tikzcd} h_0(\mathfrak{X}) \arrow[d, "a_{\mathfrak{X}}"'] \arrow[dr, "a"] & \\ \mathbf{Alb}_{\mathfrak{X}} \arrow[r, "a'"] & \mathbf{Alb}_{\mathfrak{C}}. \end{tikzcd}$$

By Lemma 3.21, the motivic Albanese map $a_{\mathfrak{X}}$ is a surjective morphism of étale sheaves, hence the image of a agrees with the image of a'. Thus $E = \mathrm{Im}(a')$ is a 1-generated étale sheaf by [ABV09, Lemma 1.1.15] and Remark 4.18.

Now the general case. Let $E \subset F \cong h^{\mathbf{rec}}_{0,\text{ét}}(\sigma_{1,!}\sigma_1^* G)$ with $G \in \mathbf{MPST}$ (cf. Lemma 4.11(ii)). The sheaf $\sigma_1^* G \in \mathbf{MPST}(k_{\leq 1})$ can be written as colimit of representable sheaves in $\mathbf{MPST}(k_{\leq 1})$, i.e.

$$\sigma_1^* G = \varinjlim_{\mathfrak{C} \to \sigma_1^* G} \mathbb{Z}_{tr}(\mathfrak{C})_{\leq 1}, \quad \mathfrak{C} = (\overline{C}, C_\infty), \quad \dim(\overline{C}) = 1.$$

Since $\sigma_{1,!}$ and $h^{\mathbf{rec}}_{0,\text{ét}}$ commute with colimits (being left adjoints), we have then

$$F \cong h^{\mathbf{rec}}_{0,\text{ét}}(\sigma_{1,!}(\sigma_1^* G)) = \varinjlim h^{\mathbf{rec}}_{0,\text{ét}}(\sigma_{1,!}\mathbb{Z}_{tr}(\mathfrak{C})_{\leq 1}) = \varinjlim h^{\mathbf{rec}}_{0,\text{ét}}(\mathfrak{C}). \tag{4.19.1}$$

For any $\mathfrak{C} \in (\mathfrak{C} \to \sigma_1^* G)_{\leq 1}$, we have in particular a map $h^{\mathbf{rec}}_{0,\text{ét}}(\mathfrak{C}) \to F$, and thus a map from the fiber product

$$\varinjlim_{\mathfrak{C} \to \sigma_{1,*}F} (h^{\mathbf{rec}}_{0,\text{ét}}(\mathfrak{C}) \times_F E) \to E$$

which is surjective (the proof of surjectivity is formal and identical to the corresponding statement in the proof of [ABV09, Lemma 1.3.4]). Now it is enough to notice that each $h^{\mathbf{rec}}_{0,\text{ét}}(\mathfrak{C}) \times_F E \subset h^{\mathbf{rec}}_{0,\text{ét}}(\mathfrak{C})$ is a 1-generated étale sheaf by the previous step and the fact that 1-generated étale sheaves are stable by colimits. To conclude we apply again [ABV09, Lemma 1.1.15]. $\square$

Proposition 4.20. *Let $F \in \mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k, \mathbb{Q})$ be an étale sheaf of $\mathbb{Q}$-vector spaces which is 1-generated in the sense of* [ABV09]. *If it is a reciprocity sheaf, then it is a 1-reciprocity sheaf. In particular, any subsheaf of a 1-reciprocity sheaf of $\mathbb{Q}$-vector spaces is again* *a* 1*-reciprocity sheaf and the category* $\mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q})$ *is closed under taking subobjects, colimits and extensions in* $\mathbf{RSC}_{\text{ét}}(k,\mathbb{Q})$.

Proof. By Lemma 4.19, any subsheaf of a 1-reciprocity sheaf is again 1-generated, and by [KSY21, Corollary 2.4.2] any subsheaf of a reciprocity sheaf is a reciprocity sheaf. Then the second part of the Proposition follows from the first, since $\mathbf{RSC}_{\text{ét}}(k,\mathbb{Q})$ is an abelian category stable by colimits in $\mathbf{Shv}^{\text{tr}}_{\text{ét}}(k,\mathbb{Q})$ (here we are using the fact that we consider $\mathbb{Q}$-coefficients in order to exploit Proposition 2.1, since [KSY21, Corollary 2.4.2] is a statement about the Nisnevich sheafification) and 1-generated étale sheaves are stable by colimits and extensions by [ABV09, Lemma 1.1.15].

We now prove the first assertion. Let $F \in \mathbf{Shv}^{\text{tr}}_{\text{ét}}(k,\mathbb{Q})$ be a 1-generated étale sheaf of $\mathbb{Q}$-vector spaces and suppose that $F \in \mathbf{RSC}$, i.e. that there exists $G \in \mathbf{CI}$ such that $F = \omega_! G$. By Remark 4.1, we have that

$$\sigma^V_{1,!}\sigma_1^{V,*}F = \omega_!\sigma_{1,!}\sigma_1^*G$$

and the counit $\sigma^V_{1,!}\sigma_1^{V,*}F \to F$ is the image via $\omega_!$ of the counit $\sigma_{1,!}\sigma_1^*G \to G$. Since $G \in \mathbf{CI}$, the map $\sigma_{1,!}\sigma_1^*G \to G$ factors through $h_0^{\overline{\square}}\sigma_{1,!}\sigma_1^*G$, which induces a factorization

$$\begin{tikzcd}
a^V_{\text{ét}}\omega_!\sigma_{1,!}\sigma_1^*G \arrow[r, equal] \arrow[drr, bend right] & a^V_{\text{ét}}\sigma^V_{1,!}\sigma_1^{V,*}F \arrow[rr, "(*)"] & & F \\
& & h^{\mathbf{rec}}_{0,\text{ét}}(\sigma_{1,!}\sigma_1^*G). \arrow[ur] &
\end{tikzcd}$$

Since F is a 1-generated étale sheaf, the map $(*)$ is surjective, hence the induced map $h^{\mathbf{rec}}_{0,\text{ét}}(\sigma_{1,!}\sigma_1^*G) \to F$ is surjective. Let $N = \ker(h^{\mathbf{rec}}_{0,\text{ét}}(\sigma_{1,!}\sigma_1^*G) \to F)$. By Lemma 4.11, $h^{\mathbf{rec}}_{0,\text{ét}}(\sigma_{1,!}\sigma_1^*G) \in \mathbf{RSC}_{\text{ét},\leq 1}$ so that N is a 1-generated étale sheaf by Lemma 4.19. Hence $N = 0$ by Corollary 4.15 so that $h^{\mathbf{rec}}_{0,\text{ét}}(\sigma_{1,!}\sigma_1^*G) \cong F$, which concludes the proof by Lemma 4.11. □

Recall that $\mathbf{RSC}_{\text{ét}}(k,\mathbb{Q})$ is a Grothendieck abelian category by [KSY21, Corollary 2.4.2]. The following corollary is immediate from the previous proposition.

Corollary 4.21. *The inclusion* $\mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q}) \subset \mathbf{RSC}(k,\mathbb{Q})$ *(and consequently* $\mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q}) \subseteq \mathbf{Shv}^{\text{tr}}_{\text{ét}}(k,\mathbb{Q})$*) is exact, and the category* $\mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q})$ *is a Grothendieck abelian category (in particular, it has enough injectives).*

The proof of the following Lemma is identical to the proof of [ABV09, Lemma 1.3.6], using Lemma 4.19.

Lemma 4.22. *Let* $G \in \mathcal{G}^*$ *be a smooth commutative* k*-group scheme and let* F *be an étale subsheaf of* G *with transfers such that its sheaf of connected components* $\pi_0(F)$ *is zero. Then* F *is represented by a closed subgroup of* G.

Definition 4.23. A 1-reciprocity sheaf $F \in \mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q})$ is called *finitely generated* if there exists a commutative k-group scheme $G \in \mathcal{G}^*$ with $\pi_0(G)$ finitely generated together with a surjection $q\colon G \to F$. If the kernel of q is itself finitely generated, we say that F is *finitely presented.*

We write $\mathbf{RSC}^\star_{\text{ét},\leq 1} \subset \mathbf{RSC}_{\text{ét},\leq 1}$ for the full subcategory of finitely presented 1-reciprocity sheaves. An almost word-by-word translation of [ABV09, Proposition 1.3.8], using Lemma 4.19 and Lemma 4.22 gives the following canonical presentation of every finitely presented 1-reciprocity sheaf. This result will be repeatedly used in the rest of the paper.

Proposition 4.24. *Any* 1*-reciprocity sheaf is filtered colimit of finitely presented* 1*-reciprocity sheaves. If* F *is a finitely presented* 1*-reciprocity sheaf, then there is a unique and functorial exact sequence*

$$0 \to L \to G \to F \to 0$$

where $G \in \mathcal{G}^*$ is a smooth commutative k-group scheme and $L \in \mathbf{RSC}_{\text{ét},\leq 0}(k) \cong \mathbf{Shv}^{\text{tr}}_{\text{ét}}(k_{\leq 0})$ is a torsion free finitely generated 0-motivic sheaf.

Remark 4.25. For a smooth commutative k-group scheme $G \in \mathcal{G}^*$, write G^{sab} for its semi-abelian quotient: it is a smooth commutative k-group scheme such that the connected component of the identity $(G^{sab})^0$ is a semi-abelian variety. We have by Chevalley's theorem an extension

$$0 \to U \to G \to G^{sab} \to 0 \tag{4.25.1}$$

where U is a unipotent group. Since k is of characteristic zero, $U \cong \mathbf{G}_a^r$ where r is the *unipotent rank* of G. More generally, for any F finitely presented 1-motivic sheaf, by Proposition 4.24 we have a functorial commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[dr, "\varphi"] & G \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\ & & G^{sab} \arrow[r] & F^{\mathbf{A}^1} \arrow[r] & 0 \end{tikzcd} \tag{4.25.2}$$

where by Remark 4.25, $F^{\mathbf{A}^1}$ is isomorphic to $h^{\mathbf{A}^1}_{0,\text{ét}}(F)$, and since $\ker(G \to G^{sab})$ is unipotent, by [MVW06, Example 2.23] we also conclude that $C^{\mathbf{A}^1}_*(F) \simeq h^{\mathbf{A}^1}_{0,\text{ét}}(F)[0]$, where $C^{\mathbf{A}^1}_*(F)$ is the Suslin complex. Moreover, by the right exactness of $h^{\mathbf{A}^1}_{0,\text{ét}}$, the kernel of the map $G^{sab} \to F^{\mathbf{A}^1}$ is a quotient of the lattice L, hence it is itself a lattice.

Remark 4.26. Every $G \in \mathcal{G}^*$ is a compact object in $\mathbf{RSC}_{\text{ét}}$, since the string of forgetful functors $\mathbf{RSC}_{\text{ét}} \subseteq \mathbf{Shv}^{\text{tr}}_{\text{ét}} \to \mathbf{PST} \to \mathbf{PSh}(\mathbf{Sm}(k), \mathbf{Set})$ preserves filtered colimits and group schemes are compact in $\mathbf{PSh}(\mathbf{Sm}(k), \mathbf{Set})$. In particular, since compact objects are stable by finite colimits, every object of $\mathbf{RSC}^{\star}_{\text{ét},\leq 1}$ is compact in $\mathbf{RSC}_{\text{ét}}$.

We can now prove the following generalization to [ABV09, Proposition 1.3.11].

Theorem 4.27. *Let* $\text{ch}(k) = 0$. *The embedding* $\mathbf{RSC}_{\text{ét},\leq 1}(k, \mathbb{Q}) \subseteq \mathbf{Shv}^{\text{tr}}_{\text{ét}}(k, \mathbb{Q})$ *has a pro-left adjoint:*

$$\text{Alb}\colon \mathbf{Shv}^{\text{tr}}_{\text{ét}}(k) \to \text{pro-}\mathbf{RSC}_{\text{ét},\leq 1}\,.$$

induced by colimit from

$$\mathbb{Q}_{\text{tr}}(X) \mapsto \text{“}\varprojlim_n\text{”}\mathbf{Alb}_{\mathfrak{X}^{(n)}}$$

for any choice of $\mathfrak{X} \in \mathbf{Comp}(X)$ *smooth.*

Proof. For $X \in \mathbf{Sm}(k)$, recall from Definition 2.8 the cofiltered category of Cartier compactifications $\mathbf{Comp}(X)$. It is enough to show that for any X, for any choice of $\mathfrak{X} \in \mathbf{Comp}(X)$ with total space smooth (since $\text{ch}(k) = 0$, such choice exists), and for any $E \in \mathbf{RSC}_{\text{ét},\leq 1}(k, \mathbb{Q})$, we have

$$\varinjlim_n \text{Hom}_{\mathbf{Shv}^{\text{tr}}_{\text{ét}}(k)}(\mathbf{Alb}_{\mathfrak{X}^{(n)}}, E) = \text{Hom}_{\mathbf{Shv}^{\text{tr}}_{\text{ét}}(k)}(\mathbb{Q}_{\text{tr}}(X), E).$$

By Galois descent it is also enough to prove the claim for k algebraically closed. By Lemma 3.21, for any $\mathfrak{X}^{(n)}$ the Albanese map $a_{\mathfrak{X}}\colon \mathbb{Q}_{tr}(X) \to \mathbf{Alb}_{\mathfrak{X}^{(n)}}$ is a surjective morphism of $\mathbf{Shv}^{\text{tr}}_{\text{ét}}(k, \mathbb{Q})$ (here we are considering $\mathbf{Alb}_{\mathfrak{X}^{(n)}}$ after tensoring with $\mathbb{Q}$), hence since filtered colimits are exact, we only need to show that

$$\varinjlim_n \text{Hom}_{\mathbf{Shv}^{\text{tr}}_{\text{ét}}(k)}(\mathbf{Alb}_{\mathfrak{X}^{(n)}}, E) \to \text{Hom}_{\mathbf{Shv}^{\text{tr}}_{\text{ét}}(k)}(\mathbb{Q}_{\text{tr}}(X), E)$$

is surjective, i.e. that for any choice of $\mathfrak{X} \in \mathbf{Comp}(X)$, every map $\mathbb{Q}_{\text{tr}}(X) \to E$ factors through $\mathbf{Alb}_{\mathfrak{X}^{(n)}}$ for some n. Since $\mathbb{Q}_{\text{tr}}(X)$ is compact in $\mathbf{Shv}^{\text{tr}}_{\text{ét}}(k)$, by Proposition 4.24 we can suppose that E is finitely presented. Let $E = \text{coker}(L \hookrightarrow G)$ be as in Proposition 4.24. Then we get a long exact sequence

$$\ldots \to \mathrm{Hom}(\mathbb{Q}_{tr}(X), G) \to \mathrm{Hom}(\mathbb{Q}_{tr}(X), E) \to H^1_{\text{ét}}(X, L) \to \ldots.$$

Since $L \cong \mathbb{Q}^r$, being k separably closed, $H^1_{\text{ét}}(X, L) = H^1_{\mathrm{Nis}}(X, \mathbb{Q}^r) = 0$. Thus the map

$$\mathrm{Hom}(\mathbb{Q}_{tr}(X), G) \to \mathrm{Hom}(\mathbb{Q}_{tr}(X), E)$$

is surjective, i.e. every map $s\colon \mathbb{Q}_{tr}(X) \to E$ factors through G. Since $G \in \mathbf{RSC}_{\text{ét}}(k, \mathbb{Q})$, by Remark 2.9 (iii) for every s as above and for any choice of $\mathfrak{X} \in \mathbf{Comp}(X)$, there exists n such that s factors through $h^{\mathbf{rec}}_{0,\text{ét}}(\mathfrak{X}^{(n)})$, and by Proposition 3.20, s has to factor through $\mathbf{Alb}_{\mathfrak{X}^{(n)}}$, concluding the proof. □

4.4. **The log Albanese functor.** We now generalize this to the logarithmic setting. Let

$$\omega^{\mathbf{CI}}_{\log}\colon \mathbf{RSC}_{\text{ét}}(k, \mathbb{Q}) \to \mathbf{Shv}_{\text{dét}}(k, \mathbb{Q})$$

be the composition in (2.10.2). As observed in Remark 2.11, $\omega^{\mathbf{CI}}_{\log}$ is fully faithful, exact and commutes with all colimits.

Theorem 4.28. *Let* $\mathrm{ch}(k) = 0$. *The fully faithful exact functor*

$$\mathbf{RSC}_{\text{ét},\leq 1}(k, \mathbb{Q}) \subseteq \mathbf{RSC}_{\text{ét}}(k, \mathbb{Q}) \xrightarrow{\omega^{\mathbf{CI}}_{\log}} \mathbf{Shv}^{\mathrm{ltr}}_{\text{dét}}(k, \mathbb{Q})$$

has a pro-left adjoint, called the log Albanese functor

$$\mathrm{Alb}^{\log}\colon \mathbf{Shv}^{\mathrm{ltr}}_{\text{dét}}(k) \to \textit{pro-}\mathbf{RSC}_{\text{ét},\leq 1}.$$

induced by colimit from

$$\mathbb{Q}_{\mathrm{tr}}(X) \mapsto \text{“}\varprojlim_n\text{”}\mathbf{Alb}_{\mathfrak{X}^{(n)}}$$

for any choice of $\mathfrak{X} \in \mathbf{Comp}(X)$ *smooth.*

Proof. We proceed as in the non-log case. Since $\omega^{\mathbf{CI}}_{\log}$ commutes with filtered colimits, we can reduce to prove the adjunction for maps against E finitely presented, quotient of a smooth commutative group k-scheme G by a lattice L. As before, it is enough to prove the claim with k algebraically closed, by Galois descent. Since we are considering sheaves of $\mathbb{Q}$-vector spaces, we can assume that $L \cong \mathbb{Q}^r$. For $X \in \mathbf{SmlSm}(k)$ we have

$$H^1_{\text{dét}}(X, \omega^{\mathbf{CI}}_{\log} L) \cong H^1_{\text{ét}}(X - |\partial X|, \mathbb{Q}^r) = 0.$$

So following the steps of the previous proof, it is enough to show that for any G commutative group scheme and $X \in \mathbf{SmlSm}(k)$ any map

$$s : \mathbb{Q}_{\mathrm{ltr}}(X) \to \omega^{\mathbf{CI}}_{\log} G$$

factors through $\omega^{\mathbf{CI}}_{\log} \mathrm{Alb}_{\mathfrak{X}^{(n)}}$ for some n, and conclude by full faithfulness of $\omega^{\mathbf{CI}}_{\log}$. On the other hand, since $G \in \mathbf{RSC}_{\mathrm{Nis}}$, by [BM21, Lemma 6.6 and Theorem 7.6] we have that

$$\mathrm{Hom}_{\mathbf{Shv}^{\mathrm{ltr}}_{\text{dét}}}(\mathbb{Q}_{\mathrm{ltr}}(X), \omega^{\mathbf{CI}}_{\log} G) = \varinjlim_n \mathrm{Hom}_{\mathbf{RSC}_{\text{ét}}}(h^{\mathbf{rec}}_{0,\text{ét}}(\mathfrak{X}^{(n)}), G).$$

Since $\omega^{\mathbf{CI}}_{\log}$ is fully faithful, s factors through $\omega^{\mathbf{CI}}_{\log} h^{\mathbf{rec}}_{0,\text{ét}}(\mathfrak{X}^{(n)})$ for some n, hence again by Proposition 3.20 and full faithfulness of $\omega^{\mathbf{CI}}_{\log}$, it factors through $\omega^{\mathbf{CI}}_{\log} \mathbf{Alb}_{\mathfrak{X}^{(n)}}$, proving the claim. □

We finish this section by proving some results on extensions in $\mathbf{RSC}_{\text{ét},\leq 1}(k, \mathbb{Q})$:

Proposition 4.29. *The category* $\mathbf{RSC}_{\text{ét},\leq 1}(k, \mathbb{Q})$ *is closed under extensions in* $\mathbf{Shv}_{\text{ét}}(k, \mathbb{Q})$.

Proof. Let $F_1, F_2 \in \mathbf{RSC}_{\text{ét},\leq 1}$. Since $\mathbf{RSC}_{\text{ét},\leq 1}(k, \mathbb{Q})$ is a full subcategory of $\mathbf{Shv}_{\text{ét}}(k, \mathbb{Q})$ by Proposition 2.10, we have that an extension in $\mathbf{RSC}_{\text{ét},\leq 1}(k, \mathbb{Q})$ splits in $\mathbf{Shv}_{\text{ét}}(k, \mathbb{Q})$ if and only if it splits in $\mathbf{RSC}_{\text{ét},\leq 1}(k, \mathbb{Q})$, so

$$\mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q})}(F_1, F_2) \hookrightarrow \mathrm{Ext}^1_{\mathbf{Shv}_{\text{ét}}(k,\mathbb{Q})}(F_1, F_2).$$

By Proposition 4.24, we can write $F_1 = \varinjlim F_{i_1}$ with $F_{i_1} \in \mathbf{RSC}^{\star}_{\text{ét},\leq 1}$. In particular, there is a surjective map $\oplus F_{i_1} \twoheadrightarrow F_1$. Let K be its kernel: we have the following commutative diagram with exact rows:

(4.29.1)

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{RSC}_{\text{ét},\leq 1}}(K, F_2) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\leq 1}}(F_1, F_2) \arrow[r] \arrow[d, "(*)"] & \prod \mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\leq 1}}(F_{i_1}, F_2) \arrow[r] \arrow[d, "(**)"] & \mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\leq 1}}(K, F_2) \arrow[d, hook] \\
\mathrm{Hom}_{\mathbf{Shv}_{\text{ét},\leq 1}}(K, F_2) \arrow[r] & \mathrm{Ext}^1_{\mathbf{Shv}_{\text{ét},\leq 1}}(F_1, F_2) \arrow[r] & \prod \mathrm{Ext}^1_{\mathbf{Shv}_{\text{ét},\leq 1}}(F_{i_1}, F_{i_2}) \arrow[r] & \mathrm{Ext}^1_{\mathbf{Shv}_{\text{ét},\leq 1}}(K, F_2)
\end{tikzcd}$$

so by the five-lemma $(*)$ is surjective if $(**)$ is surjective, hence we can suppose $F_1 \in \mathbf{RSC}^{\star}_{\text{ét},\leq 1}$. Since F_1 is compact in both $\mathbf{RSC}_{\text{ét},\leq 1}$ and $\mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}$ and filtered colimits are exact, again by 4.24 it is enough to suppose that $F_2 \in \mathbf{RSC}^{\star}_{\text{ét},\leq 1}$. Let $F_i = \mathrm{coker}(L_i \hookrightarrow G_i)$ as in Proposition 4.24, we have a commutative square with exact rows:

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\leq 1}}(G_1, G_2) \arrow[r] \arrow[d, hook, "(1)"] & \mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\leq 1}}(G_1, F_2) \arrow[r] \arrow[d, hook, "(3)"] & \mathrm{Ext}^2_{\mathbf{RSC}_{\text{ét},\leq 1}}(G_1, L_2) \arrow[d] \\
\mathrm{Ext}^1_{\mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}}(G_1, G_2) \arrow[r, "(2)"] & \mathrm{Ext}^1_{\mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}}(G_1, F_2) \arrow[r] & \mathrm{Ext}^2_{\mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\mathbb{Q})}(G_1, L_2).
\end{tikzcd}$$

By [Mil17, Exercise 5-10] we have that $\mathcal{G}^*$ is closed by extensions in fppf sheaves. This implies that (1) is an isomorphism. Moreover, $\mathrm{Ext}^2_{\mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\mathbb{Q})}(G_1, L_2) = 0$ by [Mil70, Corollary 1.], which implies that (2) is surjective, so (3) is surjective, hence an isomorphism. We have now the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{RSC}_{\text{ét},\leq 1}}(L_1, F_2) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\leq 1}}(F_1, F_2) \arrow[r] \arrow[d, hook] & \mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\leq 1}}(G_1, F_2) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\leq 1}}(L_1, F_2) \arrow[d, hook] \\
\mathrm{Hom}_{\mathbf{Shv}_{\text{ét},\leq 1}}(L_1, F_2) \arrow[r] & \mathrm{Ext}^1_{\mathbf{Shv}_{\text{ét},\leq 1}}(F_1, F_2) \arrow[r] & \mathrm{Ext}^1_{\mathbf{Shv}_{\text{ét},\leq 1}}(G_1, F_2) \arrow[r] & \mathrm{Ext}^1_{\mathbf{Shv}_{\text{ét},\leq 1}}(L_1, F_2).
\end{tikzcd}$$

and we conclude using the 5-lemma. □

Lemma 4.30. *For all $F_1, F_2 \in \mathbf{RSC}_{\text{ét},\leq 1}$ we have*

$$\mathrm{Ext}^i_{\mathbf{Shv}_{\text{ét}}(k,\mathbb{Q})}(F_1, F_2) = 0 \quad \textit{for } i \geq 2.$$

Proof. By the same argument of 4.29, we can suppose $F_1, F_2 \in \mathbf{RSC}^{\star}_{\text{ét},\leq 1}$ (namely, the exact sequence of (4.29.1) allows to consider $F_1 \in \mathbf{RSC}^{\star}_{\text{ét},\leq 1}$ and since now F_1 is a compact object in $\mathbf{Shv}_{\text{ét},\leq 1}(k, \mathbb{Q})$ and filtered colimits are exact, we can assume that $F_2 \in \mathbf{RSC}^{\star}_{\text{ét},\leq 1}$). Let $F_i = \mathrm{coker}(L_i \hookrightarrow G_i)$ as in Proposition 4.24. Since L_1 is a lattice, $\mathrm{Ext}^i(L_1, F_2) = 0$ for $i \geq 1$, so we can suppose $F_1 = G_1$. Since by [Mil70, Cor. 1] we have

$$\mathrm{Ext}^i_{\mathbf{Shv}_{\text{ét}}(k,\mathbb{Q})}(G_1, L_2) = \mathrm{Ext}^i_{\mathbf{Shv}_{\text{ét}}(k,\mathbb{Q})}(G_1, G_2) = 0 \quad \text{for } i \geq 2,$$

we can conclude. □

Proposition 4.31. *For $F_1, F_2 \in \mathbf{RSC}_{\text{ét},\leq 1}(k, \mathbb{Q})$, we have*

$$\mathrm{Ext}^i_{\mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q})}(F_1, F_2) \simeq \mathrm{Ext}^i_{\mathbf{Shv}_{\text{ét}}(k,\mathbb{Q})}(F_1, F_2) \ \ \textit{for all } i \geq 0.$$

In particular the category $\mathbf{RSC}_{\text{ét},\leq 1}(k, \mathbb{Q})$ has cohomological dimension 1, i.e. for $i \geq 2$ we have

$$\mathrm{Ext}^i_{\mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q})}(F_1, F_2) = 0.$$

Proof. The second part follows from the first in light of Lemma 4.30. Let us proceed by induction: the case $i = 0$ is Proposition 2.10 and $i = 1$ is Proposition 4.29. Let $i \geq 2$: in light of Lemma 4.30, we need to show that $\mathrm{Ext}^i_{\mathbf{RSC}_{\acute{e}t,\leq 1}(k,\mathbb{Q})}(F_1, F_2) = 0$. Let $0 \to F_2 \to I \to B \to 0$ with I injective in $\mathbf{RSC}_{\acute{e}t,\leq 1}(k, \mathbb{Q})$. We conclude since:

$$0 \overset{(1)}{=} \mathrm{Ext}^i_{\mathbf{Shv}_{\acute{e}t}(k,\mathbb{Q})}(F_1, B) \overset{(2)}{\cong} \mathrm{Ext}^i_{\mathbf{RSC}_{\acute{e}t,\leq 1}(k,\mathbb{Q})}(F_1, B) \overset{(3)}{\cong} \mathrm{Ext}^{i+1}_{\mathbf{RSC}_{\acute{e}t,\leq 1}(k,\mathbb{Q})}(F_1, F_2)$$

where (1) is Lemma 4.30, (2) is the induction hypothesis and (3) comes from the fact that I is injective. □

5. The derived Albanese functor

Assume that k has characteristic zero. The goal of this section is to show the existence of a left derived Albanese functor in the sense of Definition A.10. To ease the notation, we will write $\mathbf{Shv}^{\mathrm{tr}}$ (resp. $\mathbf{Shv}^{\mathrm{ltr}}$, resp. $\mathbf{logCI}$) for $\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k, \mathbb{Q})$ (resp. $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k, \mathbb{Q})$, resp. $\mathbf{logCI}_{\mathrm{dNis}}$) and we will identify it with $\mathbf{Shv}^{\mathrm{tr}}_{\acute{e}t}(k, \mathbb{Q})$ (resp. $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{d\acute{e}t}}(k, \mathbb{Q})$, resp. $\mathbf{logCI}_{\mathrm{d\acute{e}t}}$) by Proposition 2.1 (resp. Proposition 2.2). We also write $\mathbf{RSC}_{\acute{e}t,\leq 1} \subset \mathbf{RSC}_{\acute{e}t}$ for $\mathbf{RSC}_{\acute{e}t,\leq 1}(k, \mathbb{Q}) \subset \mathbf{RSC}_{\acute{e}t}(k, \mathbb{Q})$. In this section we let $\omega^{\mathbf{CI}}_{\log}$ denote the functor from Theorem 4.28:

$$\mathbf{RSC}_{\acute{e}t,\leq 1} \hookrightarrow \mathbf{Shv}^{\mathrm{ltr}}. \tag{5.0.1}$$

By Remark 2.11 and Proposition 4.20 this functor is fully faithful, exact and commutes with all colimits.

We announce the main theorem (cfr. [ABV09, Thm. 2.4.1]), whose proof will occupy the rest of the Section.

Theorem 5.1. *The functor* $\mathrm{Alb}^{\log}$ *of Theorem 4.28 has a pro-left derived functor* $L\,\mathrm{Alb}^{\log}$ *which factors through the stable* ∞*-category of effective log motives, giving rise to the log motivic Albanese functor (still denoted* $L\,\mathrm{Alb}^{\log}$*):*

$$L\,\mathrm{Alb}^{\log}\colon \mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k, \mathbb{Q}) \to \text{Pro-}\mathcal{D}(\mathbf{RSC}_{\acute{e}t,\leq 1}).$$

which is a pro-left adjoint of the functor

$$\omega^{\mathbf{logDM}^{\mathrm{eff}}}_{\leq 1}\colon \mathcal{D}(\mathbf{RSC}_{\acute{e}t,\leq 1}) \xrightarrow{\mathcal{D}(\omega^{\mathbf{CI}}_{\log})} \mathcal{D}(\mathbf{Shv}^{\mathrm{ltr}}) \xrightarrow{L_{\bar{\square}}} \mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k, \mathbb{Q}).$$

5.1. **Some preliminary results.** We collect now some technical lemmas that will be used in the proof of the main theorem.

Recall the pair of adjoint functors from (2.2.1)

$$\omega^*_{\log}\colon \mathbf{Shv}^{\mathrm{tr}}_{\acute{e}t}(k, \mathbb{Q}) \rightleftarrows \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{d\acute{e}t}}(k, \mathbb{Q})\colon \omega^{\log}_{\sharp}$$

where $\omega^*_{\log}F(X) = F(\underline{X} - |\partial X|)$. Recall that $\omega^{\log}_{\sharp}\omega^*_{\log} = \mathrm{id}$ and that by [BPØ20, Proposition 8.2.8], for $F \in \mathbf{HI}_{\acute{e}t}$ we have that $\omega^*_{\log}(F) \cong \omega^{\mathbf{CI}}_{\log}F$.

Lemma 5.2. *For any finitely presented* 1*-reciprocity sheaf* $F \in \mathbf{RSC}_{\acute{e}t,\leq 1}$*, we have an isomorphism:*

$$\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F) \cong \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F^{\mathbf{A}^1}).$$

Moreover, we have that

$$\omega^{\mathbf{CI}}_{\log}\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F) \cong \omega^*_{\log}\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F_i^{\mathbf{A}^1}).$$

Proof. First notice that $\mathbf{G}_m \cong \omega^{\log}_{\sharp}\omega^*_{\log}\mathbf{G}_m$, and that $\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F^{\mathbf{A}^1}) \in \mathbf{HI}$, hence

$$\omega^{\mathbf{CI}}_{\log}\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F^{\mathbf{A}^1}) \cong \omega^*_{\log}\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F^{\mathbf{A}^1})$$

so the second isomorphism follows from the first. To prove the first isomorphism, consider the presentation $0 \to L \to G \to F \to 0$ of Proposition 4.24. Since $\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, \mathbf{G}_a^r) = \underline{\mathrm{Ext}}^1_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, \mathbf{G}_a) = 0$, we conclude from the sequence (4.25.1) that

$$\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, G) = \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, G^{sab}).$$

By [BVK16, Lemma 3.1.4] and Proposition 2.10, we have

$$\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, G) \cong \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F),$$

which allows to conclude. □

Let $\mathfrak{X} = (\overline{X}, D)$ be a smooth proper and geometrically integral modulus pair, and let $\mathbf{Alb}^{sab}_{\mathfrak{X}}$ be the maximal semi-abelian quotient of $\mathbf{Alb}_{\mathfrak{X}}$. Suppose that $X = \overline{X} \setminus D$ has a k-rational point. Then, by Proposition 3.19, $\mathbf{Alb}^{sab}_{\mathfrak{X}}$ agrees with Serre's semi-abelian Albanese variety of X, Alb_X. Let $U(\mathfrak{X})$ be as in Definition 3.5: as observed in [BS19, Lemma 10.7] we have an extension

$$0 \to U(\mathfrak{X}) \otimes_k \mathbf{G}_a \to \mathbf{Alb}_{\mathfrak{X}} \to \mathbf{Alb}^{sab}_{\mathfrak{X}} = \mathrm{Alb}_X \to 0 \tag{5.2.1}$$

Since $\mathrm{Alb}_X \in \mathbf{HI}_{\leq 1,\text{ét}}$, which is a Serre subcategory of $\mathbf{RSC}_{\leq 1,\text{ét}}$, the natural surjection of (5.2.1) (with rational coefficients) gives then rise to a map

$$\mathrm{Ext}^1_{\mathbf{HI}_{\text{ét},\leq 1}}(\mathrm{Alb}_X, \mathbf{G}_m) \to \mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\leq 1}}(\mathrm{Alb}_{\mathfrak{X}}, \mathbf{G}_m). \tag{5.2.2}$$

Lemma 5.3. *The map (5.2.2) is an isomorphism.*

Proof. We need to show that

$$\mathrm{Hom}_{\mathbf{RSC}_{\text{ét},\leq 1}}(U, \mathbf{G}_m) = 0, \quad \text{and} \quad \mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\leq 1}}(U, \mathbf{G}_m) = 0.$$

which by Proposition 4.31 follow form analogous vanishing in $\mathbf{Shv}_{\text{ét}}$, which are well-known (see e.g. [Ser75, VII, Proposition 7]). □

Notice that for every $G \in \mathbf{logCI}$, we have that $\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}}(\omega^*_{\log}\mathbf{G}_m, G) \in \mathbf{logCI}$ since

$$\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}}(\omega^*_{\log}\mathbf{G}_m, G) = \pi_0(R\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}}(\omega^*_{\log}\mathbf{G}_m, G))$$

and $R\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}}(\omega^*_{\log}\mathbf{G}_m, F)$ is clearly $\overline{\square}$-local.

Lemma 5.4. *For any finitely presented 1-reciprocity sheaf $F \in \mathbf{RSC}_{\text{ét},\leq 1}$, we have an isomorphism:*

$$\omega^*_{\log}\mathbf{G}_m \otimes_{\mathbf{logCI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\omega^*\mathbf{G}_m, F) \xrightarrow{\simeq} \omega^*_{\log}\big(\mathbf{G}_m \otimes_{\mathbf{HI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F^{\mathbf{A}^1})\big).$$

where $\otimes_{\mathbf{logCI}}$ is the tensor product of the category **logCI**.

Proof. By Proposition 2.13, we have that

$$\omega^{\log}_{\sharp}\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}_{\text{ét}}(k,\mathbb{Q})}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}F) \cong \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\mathbb{Q})}(\mathbf{G}_m, F).$$

By Lemma 5.2 the right hand side is isomorphic to $\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\mathbb{Q})}(\mathbf{G}_m, F^{\mathbf{A}^1})$, and moreover it is an object of $\mathbf{HI}_{\text{ét}}(k, \mathbb{Q})$, hence by (2.12.2) we have an isomorphism

$$\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}F) \cong \omega^*_{\log}\omega^{\log}_{\sharp}\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}F) \cong \omega^*_{\log}\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F^{\mathbf{A}^1}),$$

so we conclude since by Lemma 2.7 we have that

$$\omega^*_{\log}\mathbf{G}_m \otimes_{\mathbf{logCI}} \omega^*_{\log}\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F^{\mathbf{A}^1}) \cong \omega^*_{\log}\big(\mathbf{G}_m \otimes_{\mathbf{HI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F^{\mathbf{A}^1})\big).$$

□

Remark 5.5. Thanks to Proposition 4.24, we can write every $F \in \mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q})$ as filtered colimit of finitely presented 1-reciprocity sheaves, $F = \varinjlim_i F_i$. Since the functor $\omega^{\mathbf{CI}}_{\log}$ from (5.0.1) commutes with all colimits, we have $\omega^{\mathbf{CI}}_{\log}(F) = \varinjlim_i \omega^{\mathbf{CI}}_{\log} F_i$. By the isomorphism

$$\omega^*_{\log}\mathbf{G}_m \cong \operatorname{coker}(\mathbb{Q} \to h_0^{\log}(\mathbf{P}^1, [0] + [\infty])),$$

given for example by [BPØ20, 8.2.4] together with [MVW06, Theorem 7.16], we see that $\omega^*_{\log}\mathbf{G}_m$ is a compact object in **logCI**, hence in $\mathbf{Shv}^{\mathrm{ltr}}$. By Lemma 5.4 we have an isomorphism

(5.5.1)
$$\omega^*_{\log}\mathbf{G}_m \otimes_{\mathbf{logCI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}F) \cong \varinjlim_i \omega^*_{\log}\big(\mathbf{G}_m \otimes_{\mathbf{HI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F_i^{\mathbf{A}^1})\big).$$

Moreover, since $\omega_\sharp^{\log}$ commutes with (filtered) colimits, we have that

(5.5.2)
$$\omega_\sharp^{\log}\big(\omega^*_{\log}\mathbf{G}_m \otimes_{\mathbf{logCI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}F)\big) = \varinjlim \big(\mathbf{G}_m \otimes_{\mathbf{HI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F_i^{\mathbf{A}^1})\big).$$

In particular, since $\mathbf{HI}_{\text{ét}}$ is closed under colimits we have that for exery $F \in \mathbf{RSC}_{\text{ét},\leq 1}$,

$$\omega_\sharp^{\log}\big(\omega^*_{\log}\mathbf{G}_m \otimes_{\mathbf{logCI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}F)\big) \in \mathbf{HI}_{\text{ét}}\,.$$

Lemma 5.6. *Let $F \in \mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q})$, then:*

$$H^j_{\mathrm{dNis}}(X, \omega^*_{\log}\mathbf{G}_m \otimes_{\mathbf{logCI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}F)) = 0 \quad \text{for } j > 1. \tag{5.6.1}$$

Proof. Recall that for a torus T, we have that

$$H^j_{\mathrm{Nis}}(X^\circ, T) = H^j_{\mathrm{Zar}}(X^\circ, T) = 0 \quad \text{for } j > 1. \tag{5.6.2}$$

By Proposition 4.24, let $F = \varinjlim_i F_i$ with $F_i \in \mathbf{RSC}^\star_{\text{ét},\leq 1}$. Since dNis-cohomology commutes with filtered colimits, the left hand side of (5.6.1) is isomorphic to

$$\varinjlim H^j_{\mathrm{dNis}}(X, \omega^*_{\log}\big(\mathbf{G}_m \otimes_{\mathbf{HI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F_i^{\mathbf{A}^1})\big)).$$

The sheaf $\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F_i^{\mathbf{A}^1})$ is a 0-motivic homotopy invariant sheaf by [ABV09, Corollary 1.3.9], hence $T_i := \mathbf{G}_m \otimes_{\mathbf{HI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F_i^{\mathbf{A}^1})$ is a torus. Then since $\omega^*_{\log}$ is exact and preserves injective sheaves, we have by (5.6.2) that

$$H^j_{\mathrm{dNis}}(X, \omega^*_{\log}T_i) = H^j_{\mathrm{Nis}}(X^\circ, T_i)) = 0 \quad \text{for } j > 1.$$

□

For a smooth scheme X, let $\mathrm{NS}^r(X)$ be the group of codimension r-cycles modulo algebraic equivalence.

Lemma 5.7. *Let $F \in \mathbf{RSC}_{\text{ét},\leq 1}$ and let $\mathfrak{X} = (X, D)$ be a proper modulus pair such that $X^\circ = X - |D|$ is affine and $\mathrm{NS}^1(X^\circ_{\bar{k}}) = 0$. Then we have an isomorphism*

$$\mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét}}}(\mathrm{Alb}_{\mathfrak{X}}, \omega_\sharp^{\log}(\omega^*_{\log}\mathbf{G}_m \otimes_{\mathbf{logCI}} \underline{\mathrm{Hom}}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}F))) \simeq H^1_{\mathrm{dNis}}(X, \omega^*_{\log}\mathbf{G}_m \otimes_{\mathbf{logCI}} \underline{\mathrm{Hom}}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}(F))).$$

Proof. Let $F = \varinjlim F_i$ with $F_i \in \mathbf{RSC}^\star_{\text{ét},\leq 1}$ and $T = \omega^*_{\log}\mathbf{G}_m \otimes_{\mathbf{logCI}} \underline{\mathrm{Hom}}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}F)$ and $T_i = \omega^*_{\log}\mathbf{G}_m \otimes_{\mathbf{logCI}} \underline{\mathrm{Hom}}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}F_i)$. By Remark 4.26, $\mathrm{Alb}_{\mathfrak{X}}$ is a compact object in $\mathbf{RSC}_{\text{ét}}$, and filtered colimits are exact and $\mathbf{RSC}_{\text{ét},\leq 1} \subseteq \mathbf{RSC}_{\text{ét}}$ is closed under extensions, hence by (5.5.2) we have that

$$\mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét}}}(\mathrm{Alb}_{\mathfrak{X}}, \omega_\sharp^{\log}T) \cong \varinjlim_i \mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\leq 1}}(\mathrm{Alb}_{\mathfrak{X}}, T_i).$$

On the other hand, by Lemma 5.4 we have

$$H^1_{\mathrm{dNis}}(X,T)\cong \varinjlim_i H^1_{\mathrm{dNis}}(X,\omega^*_{\log}(\mathbf{G}_m\otimes_{\mathbf{HI}}\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m,F_i^{\mathbf{A}^1}))).$$

Moreover, since $\omega^*_{\log}$ is exact and preserves injective sheaves we have that

$$H^1_{\mathrm{dNis}}(X,\omega^*_{\log}(\mathbf{G}_m\otimes_{\mathbf{HI}}\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m,F_i^{\mathbf{A}^1})))\cong H^1_{\mathrm{Nis}}(X^\circ,\mathbf{G}_m\otimes_{\mathbf{HI}}\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m,F_i^{\mathbf{A}^1})).$$

Finally, $\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m,F_i^{\mathbf{A}^1})$ is a lattice by [ABV09, Corollary 1.3.9], hence it is enough to show that for every lattice L we have an isomorphism

$$\mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\le 1}}(\mathrm{Alb}_{\mathfrak{X}},\mathbf{G}_m\otimes_{\mathbf{HI}}L)\cong H^1_{\mathrm{Nis}}(X^\circ,\mathbf{G}_m\otimes_{\mathbf{HI}}L).$$

By [ABV09, Lemma 2.4.5], since X° is affine and $\mathrm{NS}^1(X^\circ_{\overline{k}})=0$, we have that

$$H^1_{\mathrm{Nis}}(X^\circ,\mathbf{G}_m\otimes_{\mathbf{HI}}L)\cong \mathrm{Ext}^1_{\mathbf{HI}_{\text{ét},\le 1}}(\mathrm{Alb}_{X^\circ},\mathbf{G}_m\otimes_{\mathbf{HI}}L),$$

so it is enough to show that the canonical map $\mathbf{Alb}_{\mathfrak{X}}\to\mathbf{Alb}_{X^\circ}$ induces an isomorphism

$$\mathrm{Ext}^1_{\mathbf{HI}_{\text{ét},\le 1}}(\mathbf{Alb}_{X^\circ},\mathbf{G}_m\otimes_{\mathbf{HI}}L)\cong \mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\le 1}}(\mathbf{Alb}_{\mathfrak{X}},\mathbf{G}_m\otimes_{\mathbf{HI}}L).$$

If k is algebraically closed, we have $L\cong\mathbb{Q}^r$, hence the above isomorphism comes from Lemma 5.3. A Galois descent argument (see [ABV09, Lemma 2.4.5, Step 1]) allows us to deduce the isomorphism above for any k. □

5.2. **Deriving the Albanese functor.** We are ready to prove Theorem 5.1. The categories $\mathbf{RSC}_{\text{ét},\le 1}$ and $\mathbf{Shv}^{\mathrm{ltr}}$ are Grothendieck abelian categories and the functor $\omega^{\mathbf{CI}}_{\log}$ from (5.0.1) is exact and commutes with filtered colimits.

The derived ∞-category $\mathcal{D}(\mathbf{Shv}^{\mathrm{ltr}})$ is equivalent by classical reason to the ∞-category underlying the model category $\mathbf{Cpx}(\mathbf{PSh}^{\mathrm{ltr}}(k,\Lambda))$ with the dNis-local model structure considered in [BM21]. In particular, by [BM21, Lemma 2.15], the functor $i_{\mathrm{ltr}}\colon\mathcal{D}(\mathbf{Shv}^{\mathrm{ltr}})\to \mathrm{Ch}_{\mathrm{dg}}(\mathbf{Shv}^{\mathrm{ltr}})$ preserves filtered colimtis. In particular, the commutative square of ∞-categories:

$$\begin{array}{ccc}\mathrm{Ch}_{\mathrm{dg}}(\mathbf{RSC}_{\text{ét},\le 1}) & \xrightarrow{\mathrm{Ch}_{\mathrm{dg}}(\omega^{\mathbf{CI}}_{\log})} & \mathrm{Ch}_{\mathrm{dg}}(\mathbf{Shv}^{\mathrm{ltr}})\\ \downarrow{\scriptstyle L_{\le 1}} & & \downarrow{\scriptstyle L_{\mathrm{ltr}}}\\ \mathcal{D}(\mathbf{RSC}_{\text{ét},\le 1}) & \xrightarrow{\mathcal{D}(\omega^{\mathbf{CI}}_{\log})} & \mathcal{D}(\mathbf{Shv}^{\mathrm{ltr}})\end{array}$$

satisfies the hypotheses of A.4, so that $\mathcal{D}(\omega^{\mathbf{CI}}_{\log})$ has a pro-left adjoint $L\,\mathrm{Alb}^{\log}$. We consider the BC-admissibility with respect to this diagram. Recall from Lemma A.14 that a compact object $P\in\mathbf{Shv}^{\mathrm{ltr}}$ is BC-admissible as an object of $\mathrm{Ch}_{\mathrm{dg}}(\mathbf{Shv}^{\mathrm{ltr}})$ if and only if for every injective object $I\in\mathbf{RSC}_{\text{ét},\le 1}$ we have

$$\mathrm{Ext}^n_{\mathbf{Shv}^{\mathrm{ltr}}}(P,\omega^{\mathbf{CI}}_{\log}(I))=0 \text{ for } i\neq 0.$$

We make the following definition (see [ABV09, Def. 2.4.2]):

Definition 5.8. $X\in\mathbf{SmlSm}(k)$ is $\mathrm{Alb}^{\log}$-*trivial* if X° is affine, $\mathrm{NS}^1(X^\circ_{\overline{k}})=0$ and

$$H^j_{\mathrm{Zar}}(\underline{X},\mathcal{O}_{\underline{X}})=0\quad\text{ for } j>0.$$

Remark 5.9. If $\underline{X}=\mathrm{Spec}(R)$ is affine and ∂X is supported on a principal divisor with global equation f, then $X^\circ=\mathrm{Spec}(R[\frac{1}{f}])$ is affine, in particular if $\mathrm{NS}^1(X^\circ_{\overline{k}})=0$ we have that X is $\mathrm{Alb}^{\log}$-trivial and X° satisfies the hypotheses of [ABV09, Proposition 2.4.4].

The main technical input of the proof of Theorem 5.1 is the following result:

Proposition 5.10. *Let $X\in\mathbf{SmlSm}(k)$ be $\mathrm{Alb}^{\log}$-trivial, then the complex $\mathbb{Q}_{\mathrm{ltr}}(X)[0]$ is BC-admissible.*

Proof. We follow (with some modifications) the path of the proof of [ABV09, Proposition 2.4.4]. Since $\mathbb{Q}_{\mathrm{ltr}}(X)[0]$ is a compact object, it is enough to prove that

$$\mathrm{Ext}^i_{\mathbf{Shv}^{ltr}}(\mathbb{Q}_{\mathrm{ltr}}(X), \omega^{\mathbf{CI}}_{\log}(I)) = 0, \quad \text{for } i > 0 \text{ and for every } I \in \mathbf{RSC}_{\text{ét},\leq 1} \text{ injective.}$$

By [BPØ20, Proposition 4.3.2] the Ext groups in $\mathbf{Shv}^{\mathrm{ltr}}$ can be computed as cohomology groups:

$$\mathrm{Ext}^i_{\mathbf{Shv}^{\mathrm{ltr}}}(\mathbb{Q}_{\mathrm{ltr}}(X), \omega^{\mathbf{CI}}_{\log}(I)) = H^i_{\mathrm{dNis}}(X, \omega^{\mathbf{CI}}_{\log}(I)),$$

so we need to check that $H^i_{\mathrm{dNis}}(X, \omega^{\mathbf{CI}}_{\log}(I)) = 0$ for $i > 0$. In order to control this cohomology, we look then at the adjunction map

$$\omega^*_{\log}\mathbf{G}_m \otimes_{\mathbf{logCI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}(I)) \to \omega^{\mathbf{CI}}_{\log}(I) \tag{5.10.1}$$

We will make the following claim:

Claim 5.11. *Let N (resp. Q) be the kernel and the cokernel (computed in $\mathbf{Shv}^{\mathrm{ltr}}$) of the morphism* (5.10.1)*. Then for $j > 1$ we have the following vanishing:*

$$H^j_{\mathrm{dNis}}(X, N) = H^j_{\mathrm{dNis}}(X, Q) = 0. \tag{5.11.1}$$

Combining Claim 5.11, and Lemma 5.6, we get that $H^j_{\mathrm{dNis}}(X, \omega^{\mathbf{CI}}_{\log}(I)) = 0$ for $j > 1$ and that we have a surjection

$$H^1_{\mathrm{dNis}}(X, \omega^*_{\log}\mathbf{G}_m \otimes_{\mathbf{logCI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}(I))) \to H^1_{\mathrm{dNis}}(X, \omega^{\mathbf{CI}}_{\log}(I)) \to 0. \tag{5.11.2}$$

We are then left to show that the displayed morphism in (5.11.2) is the zero map.

For every modulus pair $\mathfrak{X} \in \mathbf{Comp}(X)$, the canonical map $\mathbb{Q}_{tr}(X) \to \omega^{\mathbf{CI}}_{\log}\mathrm{Alb}_{\mathfrak{X}}$ gives for any $F \in \mathbf{RSC}_{\text{ét},\leq 1}$ a natural map (again we are using the fact that $\omega^{\mathbf{CI}}_{\log}$ is exact):

$$\mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\leq 1}}(\mathrm{Alb}_{\mathfrak{X}}, F) \to \mathrm{Ext}^1_{\mathbf{Shv}^{ltr}}(\omega^{\mathbf{CI}}_{\log}\mathrm{Alb}_{\mathfrak{X}}, \omega^{\mathbf{CI}}_{\log}F) \to H^1_{\mathrm{dNis}}(X, \omega^{\mathbf{CI}}_{\log}F),$$

hence, from (5.11.2) we get a commutative diagram (cfr. with the proof of [ABV09, 2.4.4])
(5.11.3)

$$\begin{array}{ccc}
\mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\leq 1}}(\mathrm{Alb}_{\mathfrak{X}}, \omega^{\log}_{\sharp}(\omega^*_{\log}\mathbf{G}_m \otimes_{\mathbf{logCI}} \underline{\mathrm{Hom}}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}I))) & \longrightarrow & \mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\leq 1}}(\mathrm{Alb}_{\mathfrak{X}}, I) \\
\downarrow & & \downarrow \\
H^1_{\mathrm{dNis}}(X, \omega^*_{\log}\mathbf{G}_m \otimes_{\mathbf{logCI}} \underline{\mathrm{Hom}}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}(I))) & \longrightarrow & H^1_{\mathrm{dNis}}(X, \omega^{\mathbf{CI}}_{\log}(I)).
\end{array}$$

Since I is injective in $\mathbf{RSC}_{\text{ét},\leq 1}$, the term $\mathrm{Ext}^1_{\mathbf{RSC}_{\text{ét},\leq 1}}(\mathrm{Alb}_{\mathfrak{X}}, I)$ is zero. On the other hand, the left vertical map is an isomorphism by Lemma 5.7, which implies that (5.11.2) is indeed the zero map. This finishes the reduction of the proof of 5.10 to Claim 5.11.

Proof of Claim 5.11. Since cohomology commutes with filtered colimits, let $I = \varinjlim F_i$ with F_i finitely generated 1-reciprocity sheaves. For all i, let K_i and N_i be the kernel and the cokernel of the adjunction map

$$\omega^*_{\log}\mathbf{G}_m \otimes_{\mathbf{logCI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}F_i) \to \omega^{\mathbf{CI}}_{\log}F_i. \tag{5.11.4}$$

As observed in Remark 5.5, since filtered colimits are exact we have that $K = \varinjlim K_i$ and $Q = \varinjlim Q_i$, hence it is enough to show that for all i:

$$H^j_{\mathrm{dNis}}(X, N_i) = 0, \quad j > 0 \tag{5.11.5}$$

$$H^j_{\mathrm{dNis}}(X, Q_i) = 0, \quad j > 0. \tag{5.11.6}$$

By Lemma 5.4 we have that the left hand side of (5.11.4) is isomorphic to $\omega^*_{\log}\big(\mathbf{G}_m \otimes_{\mathbf{HI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F_i^{\mathbf{A}^1})\big)$. Let K_i and R_i be the kernel and the cokernel of the adjunction map

$$\psi_i : \mathbf{G}_m \otimes_{\mathbf{HI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F_i^{\mathbf{A}^1}) \to F_i^{\mathbf{A}^1}.$$

We have the following diagram:
(5.11.7)

$$\begin{tikzcd}
0 \arrow[r] & N_i \arrow[r] \arrow[d, hook] & \omega^*_{\log}\mathbf{G}_m \otimes_{\mathbf{logCI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{ltr}}}(\omega^*_{\log}\mathbf{G}_m, \omega^{\mathbf{CI}}_{\log}F_i) \arrow[r, "\varphi_i"] \arrow[d, "\simeq"] & \omega^{\mathbf{CI}}_{\log}F_i \arrow[r, two heads] \arrow[d] & Q_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \omega^*_{\log}K_i \arrow[r] & \omega^*_{\log}\big(\mathbf{G}_m \otimes_{\mathbf{HI}} \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, F_i^{\mathbf{A}^1})\big) \arrow[r, "\omega^*\psi_i"] & \omega^*_{\log}F_i^{\mathbf{A}^1} \arrow[r] & \omega^*_{\log}R_i \arrow[r] & 0.
\end{tikzcd}$$

By the cancellation theorem [Voe10], $\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, \mathbf{G}_m \otimes_{\mathbf{HI}} M) \cong M$ for $M \in \mathbf{HI}$, and $\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, \_)$ is exact as an endo-functor on $\mathbf{HI}$. Hence we get $\underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, K_i) = \underline{\mathrm{Hom}}_{\mathbf{Shv}^{\mathrm{tr}}}(\mathbf{G}_m, R_i) = 0$, in particular the sheaves K_i and R_i are birational sheaves in the sense of [KS17] (see [KS17, Proposition 2.5.2]). In particular, since $\omega^*_{\log}$ is exact and preserves injectives, by [MVW06, Proposition 14.23] and [KS17, Proposition 2.3.3] that

$$H^j_{\mathrm{dNis}}(X, \omega^*_{\log}K_i) = H^j_{\mathrm{Nis}}(X^\circ, K_i) = 0, \quad j > 0 \tag{5.11.8}$$

$$H^j_{\mathrm{dNis}}(X, \omega^*_{\log}R_i) = H^j_{\mathrm{Nis}}(X^\circ, R_i) = 0, \quad j > 0. \tag{5.11.9}$$

Since N_i is a subsheaf of $\omega^*_{\log}K_i$, $\omega^{\log}_\sharp N_i$ is a subsheaf of K_i, so it is a birational sheaf, in particular it is an object of $\mathbf{HI}$ by [KS17, Proposition 2.3.3 (a)] so $N_i \cong \omega^*_{\log}\omega^{\log}_\sharp N_i$ by (2.12.2). Therefore the same argument gives the vanishing (5.11.5).

Let H_i be the kernel of the map $F_i \to F_i^{\mathbf{A}^1}$. By a snake lemma argument on (4.25.2), there exists a lattice L'_i and $r_i \geq 0$ such that $\mathbf{G}_a^{r_i}/L'_i \cong H_i$. By the exactness of $\omega^{\mathbf{CI}}_{\log}$, we have that $\omega^{\mathbf{CI}}_{\log}\mathbf{G}_a^{r_i}/\omega^*L' \cong \omega^{\mathbf{CI}}H_i$. Since L' is a lattice, $H^j_{\mathrm{dNis}}(X, \omega^*L') = H^j_{\mathrm{Nis}}(X^\circ, L') = 0$ for $j > 0$ and by [RS21, Corollary 6.8] with $q = 0$ combined with [BM21, (7.6.1)] we have that (see (2.1.2)):

$$H^j_{\mathrm{dNis}}(X, \omega^{\mathbf{CI}}_{\log}\mathbf{G}_a) = \varinjlim_{Y \in X^{\mathbf{Sm}}_{\mathrm{div}}} H^j_{\mathrm{Nis}}(\underline{Y}, \mathcal{O}_{\underline{Y}}). \tag{5.11.10}$$

By the comparison of Zariski cohomology with Nisnevich cohomology for coherent sheaves we have that for all $Y \in X^{\mathbf{Sm}}_{\mathrm{div}}$:

$$H^j_{\mathrm{Nis}}(\underline{Y}, \mathcal{O}_{\underline{Y}}) \cong H^j_{\mathrm{Zar}}(\underline{Y}, \mathcal{O}_{\underline{Y}}).$$

By definition the map $\underline{Y} \to \underline{X}$ is the composition of blowups in smooth centers, hence the well know blow-up formula (see e.g. [Gro85, Corollary IV.1.1.11]) implies:

$$H^j_{\mathrm{Zar}}(\underline{Y}, \mathcal{O}_{\underline{Y}}) \cong H^j_{\mathrm{Zar}}(\underline{X}, \mathcal{O}_{\underline{X}}) = 0, \ \ j > 0,$$

where the last vanishing comes from the fact that X was taken $\mathrm{Alb}^{\log}$-trivial. In particular, we conclude that

$$H^i_{\mathrm{dNis}}(X, \omega^{\mathbf{CI}}_{\log}H_i) = 0 \ \text{ for } i \neq 0. \tag{5.11.11}$$

From the diagram (5.11.7) and a snake lemma argument, we get the following short exact sequence:

$$0 \to \omega^*K_i/N_i \to \omega^{\mathbf{CI}}_{\log}H_i \to \ker(Q_i \to \omega^*R_i) \to 0. \tag{5.11.12}$$

Now by (5.11.8) and (5.11.5) we have that $H^j_{\mathrm{dNis}}(X, \omega^*K_i/N_i) = 0$ for $j > 0$, so by (5.11.11), (5.11.9) and (5.11.12) we deduce (5.11.6). ☐

Given Proposition 5.10, we can show the following

Lemma 5.12. *The category* $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k, \mathbb{Q})$ *is generated by the set of* $\mathrm{Alb}^{\log}$*-trivial objects of* $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k, \mathbb{Q})$.

Proof. The category $\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k, \mathbb{Q})$ is compactly generated, and a set of compact generators is given by $\mathbb{Q}_{\mathrm{ltr}}(X)[i]$, for $X \in \mathbf{SmlSm}$ and $i \in \mathbb{Z}$. By Proposition 5.10, it is then enough to show that any $X \in \mathbf{SmlSm}$ can be covered (even Zariski-locally) by $X_i \in \mathbf{SmlSm}$ which are $\mathrm{Alb}^{\log}$-trivial. Let $(\underline{U_i}, f_i)$ be a Zariski cover of $\underline{X}$ such that $|\partial X|_{|\underline{U_i}}$ is principal. By [ABV09, Corollary 2.4.6]), we can cover each $\underline{U_i}$ by affine $\underline{U_{ij}}$ such that $\mathrm{NS}^1((\underline{U_{ij}})_{\overline{k}}) = 0$, and since $|\partial X|_{\underline{U_i}}$ is principal, $|\partial X|_{\underline{U_{ij}}}$ is again principal. Considering the log schemes $U_{ij} := (\underline{U_{ij}}, \partial X_{|U_{ij}})$, we have that $\mathrm{NS}^1((\underline{U_{ij}})_{\overline{k}}) \to \mathrm{NS}^1((U_{ij}^{\circ})_{\overline{k}})$ is surjective by [Ful98, Example 10.3.4], hence $\mathrm{NS}^1(U_{ij}^{\circ}) = 0$. We conclude that $\{U_{ij}\}$ is a Zariski cover of X by $\mathrm{Alb}^{\log}$-trivial log schemes. □

Proof of Theorem 5.1. From Lemma 5.12 and Proposition 5.10, we have that the ∞-category $\mathcal{D}(\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k, \mathbb{Q}))$) is generated by a set of compact BC-admissible compact objects concentrated in degree zero. The existence of the derived log Albanese functor $L\,\mathrm{Alb}^{\log}$ as pro-left derived functor of $\mathrm{Alb}^{\log}$ follows then from Theorem A.13, and by constrution it is equivalent to the pro-left adjoint of the functor $\mathcal{D}(\omega^{\mathbf{CI}}_{\log})$.

We are left to show that the functor $L\,\mathrm{Alb}^{\log}$ factors through the localization

$$L_{\overline{\square}} \colon \mathcal{D}(\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k, \mathbb{Q})) \to \mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k, \mathbb{Q}).$$

Recall that $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k, \mathbb{Q})$ is obtained as localization of $\mathcal{D}(\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k, \mathbb{Q}))$ with respect to the class of maps:

$$\mathbb{Q}_{\mathrm{ltr}}(X \times \overline{\square})[n] \to \mathbb{Q}_{\mathrm{ltr}}(X)[n] \tag{CI}$$

for X in $\mathbf{SmlSm}(k)$. From the proof of Lemma 5.12, we can suppose that X is $\mathrm{Alb}^{\log}$-trivial (this is the exact analogue of [ABV09, 2.4.1]). We are then left to show that $L\,\mathrm{Alb}^{\log}(\mathbb{Q}_{\mathrm{ltr}}(X \times \overline{\square}) \to \mathbb{Q}_{\mathrm{ltr}}(X))$ is contractible for X an $\mathrm{Alb}^{\log}$-trivial object. Since $\mathbb{Q}_{\mathrm{ltr}}(X)$ is BC-admissible by Proposition 5.10, we have by Remark A.9 that

$$L\,\mathrm{Alb}^{\log}(\mathbb{Q}_{\mathrm{tr}}(X)[n]) = \text{“}\varprojlim_i\text{”}\mathbf{Alb}_{\mathfrak{X}^{(i)}}[n], \quad \text{for any choice of } \mathfrak{X} \in \mathbf{Comp}(X).$$

Note also that if X is $\mathrm{Alb}^{\log}$-trivial, so is $X \times \overline{\square}$. Indeed, $(X \times \overline{\square})^{\circ} = X^{\circ} \times \mathbf{A}^1$ is affine if X° is affine, $\mathrm{NS}^1((X^{\circ} \times \mathbf{A}^1)_{\overline{k}}) \cong \mathrm{NS}^1(X^{\circ}_{\overline{k}})$ and $H^i_{\mathrm{Zar}}(\underline{X} \times \mathbf{P}^1, \mathcal{O}_{\underline{X} \times \mathbf{P}^1}) \cong H^i_{\mathrm{Zar}}(\underline{X}, \mathcal{O}_{\underline{X}})$ for all i. Therefore

$$L\,\mathrm{Alb}^{\log}(\mathbb{Q}_{\mathrm{ltr}}(X \times \overline{\square})[n]) = \text{“}\varprojlim_i\text{”}\mathbf{Alb}_{\mathfrak{X}^{(i)} \times \overline{\square}}, \quad \text{for any choice of } \mathfrak{X} \in \mathbf{Comp}(X).$$

On the other hand, by construction we have

$$\mathbf{Alb}_{\mathfrak{X}^{(i)} \times \overline{\square}} \cong \mathbf{Alb}_{\mathfrak{X}^{(i)}},$$

proving the factorization. The pro-adjunction now is formal since for $X \in \mathcal{D}(\mathbf{RSC}_{\text{ét},\leq 1})$, we have that $\mathcal{D}(\omega^{\mathbf{CI}}_{\log})(X) \in \mathcal{D}(\mathbf{logCI})$, hence it is $(\mathrm{dNis}, \overline{\square})$-local by [BM21, Corollary 5.5], so:

$$\mathcal{D}(\omega^{\mathbf{CI}}_{\log})(X) \simeq i_{\overline{\square}} L_{\overline{\square}} \mathcal{D}(\omega^{\mathbf{CI}}_{\log})(X) \simeq i_{\overline{\square}} \omega^{\mathbf{logDM}^{\mathrm{eff}}}_{\leq 1}(X),$$

hence for any $Y \in \mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k, \mathbb{Q})$

$$\begin{aligned}\mathrm{Map}_{\mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k,\mathbb{Q})}(Y, \omega^{\mathbf{logDM}^{\mathrm{eff}}}_{\leq 1}(X)) &\simeq \mathrm{Map}_{\mathcal{D}(\mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{d\acute{e}t}}(k,\mathbb{Q}))}(i_{\overline{\square}} Y, \mathcal{D}(\omega^{\mathbf{CI}}_{\log})(X)) \\ &\simeq \mathrm{Map}_{\mathrm{Pro}\text{-}\mathcal{D}(\mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q}))}(L\,\mathrm{Alb}^{\log}(Y), X)\end{aligned}$$

as required. □

5.3. **Logarithmic 1-motivic complexes.** For any perfect field k and any commutative ring Λ, recall the stable ∞-category of 1-motivic complexes $\mathcal{DM}^{\mathrm{eff}}_{\leq 1}(k, \Lambda)$, i.e. the full stable ∞-subcategory of $\mathcal{DM}^{\mathrm{eff}}(k, \Lambda)$ generated by $M(X)$, with $\dim(X) \leq 1$. If $\Lambda = \mathbb{Q}$, by [ABV09, Theorem 2.4.1], the composition

$$L\,\mathrm{Alb}_{\leq 1} \colon \mathcal{DM}^{\mathrm{eff}}_{\leq 1}(k, \mathbb{Q}) \hookrightarrow \mathcal{DM}^{\mathrm{eff}}(k, \mathbb{Q}) \xrightarrow{L\,\mathrm{Alb}} \mathcal{D}(\mathbf{HI}_{\leq 1}(k, \mathbb{Q}))$$

is an equivalence. Let

$$\omega^* : \mathcal{DM}^{\mathrm{eff}}(k, \mathbb{Q}) \to \mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k, \mathbb{Q})$$

be the functor from (2.6.1). We give the following definition:

Definition 5.13. For any perfect field k and any commutative ring Λ, we let $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}_{\leq 1}(k,\Lambda)$ be the full stable ∞-subcategory of $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k,\Lambda)$ generated by $\omega^*\mathcal{DM}^{\mathrm{eff}}_{\leq 1}$ and $\omega^{\mathbf{CI}}_{\log}(U)[n]$ for a unipotent group scheme U. We will call it the stable ∞-category of *log* 1*-motives*.

Remark 5.14. By [BPØ20, Theorem 7.6.7], if k satisfies (RS) the ∞-category $\omega^*\mathcal{DM}^{\mathrm{eff}}_{\leq 1}$ is equivalent to the ∞-subcategory of $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k,\Lambda)$ generated by $M(X)$ with $\underline{X}$ a proper smooth curve. Moreover, if $\mathrm{ch}(k)=0$, every unipotent group scheme splits as a direct sum of $\mathbf{G}_a$, hence the categroy $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}_{\leq 1}(k,\Lambda)$ is generated by $\omega^*\mathcal{DM}^{\mathrm{eff}}_{\leq 1}$ and $\omega^{\mathbf{CI}}\mathbf{G}_a[n]$. Moreover, if Λ is a $\mathbb{Q}$-algebra, the functor $\omega^{\mathbf{logDM}^{\mathrm{eff}}}_{\leq 1}$ factors through $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}_{\leq 1}(k,\mathbb{Q})$: indeed the category $D(\mathbf{RSC}_{\leq 1}(k,\mathbb{Q}))$ is generated by $h_0(\mathfrak{C})[n]$ with $\mathfrak{C}=(\overline{C},C_\infty)$ a proper modulus pair of dimension 1 such that $\overline{C}-|C_\infty|$ is affine, and there is a fiber sequence

$$h_0(\mathfrak{C}_{red})[n-1]\to \mathbf{G}_a^{\oplus r}[n]\to h_0(\mathfrak{C})[n].$$

Let $C\in\mathbf{SmlSm}(k)$ be the log scheme $(\overline{C},\partial C)$ with $|\partial C|=|C_\infty|$, then

$$\omega^{\mathbf{logDM}^{\mathrm{eff}}}_{\leq 1}h_0(\mathfrak{C}_{red})[m]=\omega^*h_0^{\mathbf{A}^1}(\overline{C}-C_\infty)[m]=\omega^*(h_0^{\mathbf{A}^1}(\overline{C}-C_\infty))[m]=\omega^*M(\overline{C}-C_\infty)[m]$$

and by construction $\omega^{\mathbf{CI}}\mathbf{G}_a[n]=\omega^{\mathbf{logDM}^{\mathrm{eff}}}_{\leq 1}\mathbf{G}_a[n]$. By repeating this argument backwards we conclude that the functor $\omega^{\mathbf{logDM}^{\mathrm{eff}}}_{\leq 1}$ is also essentially surjecrive on $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}_{\leq 1}(k,\Lambda)$.

From now on, we consider again $\mathrm{ch}(k)=0$ and $\Lambda=\mathbb{Q}$. We have the following generalization of [ABV09, Theorem 2.4.1]:

Theorem 5.15. *The composition $L\,\mathrm{Alb}^{\log}\circ\omega^{\mathbf{logDM}^{\mathrm{eff}}}_{\leq 1}$ is equivalent to the constant pro-object functor. In particular, the functor $\omega^{\mathbf{logDM}^{\mathrm{eff}}}_{\leq 1}$ is fully faithful and induces an equivalence*

$$\omega^{\mathbf{logDM}^{\mathrm{eff}}}_{\leq 1}\colon \mathcal{D}(\mathbf{RSC}_{\leq 1}(k,\mathbb{Q}))\xrightarrow{\sim}\mathbf{log}\mathcal{DM}^{\mathrm{eff}}_{\leq 1}(k,\mathbb{Q})\colon \varprojlim L\,\mathrm{Alb}^{\log}.$$

Proof. By construction, the proof follows from Proposition 5.16 and Lemma 5.18 below. □

Proposition 5.16. *There is a commutative diagram of stable ∞-cateogries:*

$$\begin{tikzcd}
 & \mathcal{D}(\mathbf{HI}_{\leq 1}(k,\mathbb{Q})) \arrow[rr, "j"] & & \text{Pro-}\mathcal{D}(\mathbf{RSC}_{\leq 1}(k,\mathbb{Q})) \\
\mathcal{DM}^{\mathrm{eff}}_{\leq 1}(k,\mathbb{Q}) \arrow[ur, bend left, "L\,\mathrm{Alb}_{\leq 1}"] \arrow[r, hook] & \mathcal{DM}^{\mathrm{eff}}(k,\mathbb{Q}) \arrow[rr, "\omega^*"] & & \mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k,\mathbb{Q}). \arrow[u, "L\,\mathrm{Alb}^{\log}"']
\end{tikzcd}$$

Proof. The category $\mathcal{DM}^{\mathrm{eff}}_{\leq 1}(k,\mathbb{Q})$ is generated by $M(C)$ with C affine curve, and for such C we have $\mathrm{NS}^1(C_{\overline{k}})=0$. It is then enough to show that

$$L\,\mathrm{Alb}^{\log}(\omega^*M(C))\simeq jL\,\mathrm{Alb}_{\leq 1}(M(C)).$$

Since C is affine, the right hand side is equivalent to the constant pro-object $\mathrm{Alb}(C)[0]$.

By [BPØ20, Theorem 8.2.11], we have that $\omega^*M(C)=M(\overline{C},\partial C)$ with $\overline{C}$ the smooth projective compactification of C. To conclude, we have to show that

$$\pi_iL\,\mathrm{Alb}^{\log}(\overline{C},\partial C)=\begin{cases}\mathrm{Alb}(C) & \text{if } i=0\\ 0 & \text{otherwise.}\end{cases}$$

Notice that in this case, $(\overline{C},\partial C)$ is not $\mathrm{Alb}^{\log}$-trivial since $H^1(\overline{C},\mathcal{O}_{\overline{C}})\neq 0$ if $g(\overline{C})\neq 0$. On the other hand, there exist $x,y\in C$ such that $\overline{C}-\{x\}$, $\overline{C}-\{y\}$ and $\overline{C}-\{x,y\}$ are affine. In particular, $(\overline{C}-\{x,y\},\partial C)$, $(\overline{C}-\{x\},\partial C)$ and $(\overline{C}-\{y\},\partial C)$ are $\mathrm{Alb}^{\log}$-trivial and there is a sZar-distinguished square

$$\begin{array}{ccc} (\overline{C}-\{x,y\},\partial C) & \longrightarrow & (\overline{C}-\{x\},\partial C) \\ \downarrow & & \downarrow \\ (\overline{C}-\{y\},\partial C) & \longrightarrow & (\overline{C},\partial C) \end{array}$$

which induces a fiber-cofiber sequence in $\mathbf{log}\mathcal{DM}^{\mathrm{eff}}(k,\mathbb{Q})$:

$$M(\overline{C}-\{x,y\},\partial C)\to M(\overline{C}-\{x\},\partial C)\oplus M(\overline{C}-\{y\},\partial C)\to M(\overline{C},\partial C)$$

and so a fiber-cofiber sequence in Pro-$\mathcal{D}(\mathbf{RSC}_{\leq 1}(k,\mathbb{Q}))$:

$$L\,\mathrm{Alb}^{\log}(\overline{C}-\{x,y\},\partial C)\to L\,\mathrm{Alb}^{\log}(\overline{C}-\{x\},\partial C)\oplus L\,\mathrm{Alb}^{\log}(\overline{C}-\{y\},\partial C)\to L\,\mathrm{Alb}^{\log}(\overline{C},\partial C).$$

Since $(\overline{C}-\{x,y\},\partial C)$, $(\overline{C}-\{x\},\partial C)$ and $(\overline{C}-\{y\},\partial C)$ are $\mathrm{Alb}^{\log}$-trivial, by Proposition 5.10 and Remark A.9 we conclude that

$$\pi_i L\,\mathrm{Alb}^{\log}(\overline{C},\partial C)=\begin{cases}\mathrm{Alb}^{\log}(\overline{C},\partial C) & \text{if } i=0\\ \ker\Big(\mathrm{Alb}^{\log}(\overline{C}-\{x,y\},\partial C)\to {\scriptstyle \mathrm{Alb}^{\log}(\overline{C}-\{x\},\partial C)\oplus \atop \mathrm{Alb}^{\log}(\overline{C}-\{y\},\partial C)}\Big) & \text{if } i=1\\ 0 & \text{otherwise.}\end{cases}$$

By construction, we have that $\mathrm{Alb}^{\log}(\overline{C},\partial C)$ is the constant pro-object $\mathrm{Alb}(C)$, so to conclude, we need to show that the map in pro-$\mathbf{RSC}_{\leq 1,\text{ét}}(k,\mathbb{Q})$

$$\mathrm{Alb}^{\log}(\overline{C}-\{x,y\},\partial C)\to \mathrm{Alb}^{\log}(\overline{C}-\{x\},\partial C)\oplus\mathrm{Alb}^{\log}(\overline{C}-\{y\},\partial C)$$

is a monomorphism. It is an easy consequence of [Isa02, Proposition 3.2] that a map of of pro-objects is a monomorphism if and only if it is injective levelwise on a coinitial system. In particular, it is enough to show that for $n\gg 0$ the map

$$\mathbf{Alb}_{(\overline{C},|\partial C|+n(\{x\}+\{y\})}\to \mathbf{Alb}_{(\overline{C},|\partial C|+n\{x\})}\oplus\mathbf{Alb}_{(\overline{C},|\partial C|+n\{y\})}$$

is an injective map of étale reciprocity sheaves. By Proposition 3.20 and the proof of Proposition 3.14, we have $\mathbf{Alb}_{\mathfrak{C}}=\underline{\mathrm{Pic}}(\overline{C},C_\infty)$ for any $\mathfrak{C}=(\overline{C},C_\infty)$, hence by [Sai20, Theorem 0.2], it is enough to show that for every function field K over k, the map

$$\mathrm{Pic}(\overline{C}_K,|\partial C|_K+n(\{x\}_K+\{y\}_K))\to\mathrm{Pic}(\overline{C}_K,(|\partial C|_K+n\{x\}_K)\oplus\mathrm{Pic}(\overline{C}_K,(|\partial C|_K+n\{y\}_K)$$

is injective, which follows by Lemma 5.17 below. ◻

Lemma 5.17. *Let X be a proper smooth curve over a field and D,E be effective Cartier divisors on X such that $|D|\neq\emptyset$ and $|D|\cap|E|\neq\emptyset$. Then there exist an exact sequence*

$$0\to H^0(X,\mathcal{O}_E^\times)\to\mathrm{Pic}(X,D+E)\to\mathrm{Pic}(X,D)\to 0.$$

If $E=E_1+E_2$ with $|E_1|\cap|E_2|=\emptyset$, then the map

$$\mathrm{Pic}(X,D+E)\to\mathrm{Pic}(X,D+E_1)\oplus\mathrm{Pic}(X,D+E_2)$$

is injective.

Proof. The second assertion follows immediately from the first. To show the first, recall an isomorphism

$$\mathrm{Pic}(X,D+E)\cong H^1(X,\mathcal{O}^\times_{X|D+E})\ \text{ with } \mathcal{O}^\times_{X|D+E}=\mathrm{Ker}(\mathcal{O}_X^\times\to\mathcal{O}^\times_{D+E}).$$

and a similar isomorphism with $D+E$ replaced by D. By the assumption $|D|\cap|E|\neq\emptyset$, we have an exact sequence

$$0\to\mathcal{O}^\times_{X|D+E}\to\mathcal{O}^\times_{X|D}\to\mathcal{O}^\times_E\to 0,$$

which induces the desired exact sequence noting $H^0(X,\mathcal{O}^\times_{X|D})=0$ if $|D|\neq\emptyset$. ◻

Lemma 5.18. *For all $i > 0$, $L_i \operatorname{Alb}^{\log}(\omega_{\leq 1}^{\mathbf{logDM}^{\mathrm{eff}}} \mathbf{G}_a) = 0$.*

Proof. By purity and étale descent, it is enough to show the vanishing over every algebraically closed field extension K/k. Let $X \in \mathbf{SmlSm}(K)$ and $(\overline{X}, D)$ be a Cartier compactification of X. By the construction of $\mathcal{L}og$ in [Sai21, §6] and [RS21, Cor. 6.8(1)], we have

$$\omega_{\log}^{\mathbf{CI}}\mathbf{G}_a(X) = \varinjlim_n \Gamma(\overline{X}, \mathcal{O}_{\overline{X}}(nD)) = \Gamma(X, \mathcal{O}_X). \tag{5.18.1}$$

Hence, there is an explicit resolution (the normalized Bar construction, see [Bre69]):

$$\mathbb{Q}_{\mathrm{ltr}}\widetilde{(\mathbf{A}^1, \mathrm{triv})}^{\times \bullet} \to \omega_{\leq 1}^{\mathbf{logDM}^{\mathrm{eff}}} \mathbf{G}_a. \tag{5.18.2}$$

where $\mathbb{Q}_{\mathrm{ltr}}\widetilde{(\mathbf{A}^1, \mathrm{triv})}$ is the complement of the retraction map $\mathbb{Q} \xrightarrow{i_0} \mathbb{Q}_{\mathrm{ltr}}(\mathbf{A}^1, \mathrm{triv}) \to \mathbb{Q}$, and the augumentation $\mathbb{Q}_{\mathrm{ltr}}\widetilde{(\mathbf{A}^1, \mathrm{triv})} \to \omega^{\mathbf{CI}}\mathbf{G}_a(X)$ is induced by

$$\mathbb{Z}(\mathbf{A}^1) = \mathbb{Z}(\mathrm{Hom}_{\mathbf{Sch}_k}(\_, \mathbf{A}^1)) \to \mathrm{Hom}_{\mathbf{Sch}_k}(\_, \mathbf{A}^1) = \mathbf{G}_a$$

See [Bre69, §1-2]. The map factors through $\mathbb{Q}_{\mathrm{tr}}(\mathbf{A}^1)$ and by (5.18.1) gives naturally a map in $\mathbf{Shv}^{\mathrm{ltr}}$. The log scheme $(\mathbf{A}^1, \mathrm{triv})$ is Alb-admissible by Proposition 5.10 and Alb-admissible objects are stable under direct summands, so we get the following explicit construction:

$$L \operatorname{Alb}(\omega_{\leq 1}^{\mathbf{logDM}^{\mathrm{eff}}} \mathbf{G}_a) = \operatorname{Alb}(\mathbb{Q}_{\mathrm{ltr}}\widetilde{(\mathbf{A}^1, \mathrm{triv})})^{\times \bullet} = \text{“}\varprojlim_m\text{”}(\underline{\mathrm{Pic}}^0(\mathbf{P}^1, m\infty) \otimes \mathbb{Q})^{\times \bullet}.$$

Let K as above: we have that

$$(\underline{\mathrm{Pic}}^0(\mathbf{P}^1, m\infty) \otimes \mathbb{Q})(K) = \mathrm{Pic}^0(\mathbf{P}^1_K, m\infty)$$

where the right hand side is the group of vector bundles of degree 0 with a trivialization at $m\infty$, which is already a $\mathbb{Q}$-vector space since $\mathrm{ch}(K) = 0$. It is then enough to show that for $m, n \geq 1$, $\pi_n(\mathrm{Pic}^0(\mathbf{P}^1_K, m\infty)^{\times \bullet}) = 0$. Let us fix m. We compute the boundary maps δ_n. For $n \geq 1$ have the following square:

$$\begin{tikzcd} Z_0(\mathbf{A}^n_K)_{\mathbb{Q}} \arrow[r, "d_n^B"] \arrow[d, "(\mathrm{Alb}_{(\mathbf{P}^1, m\infty)})^{\times n}"] & Z_0(\mathbf{A}^{n-1}_K)_{\mathbb{Q}} \arrow[d, "(\mathrm{Alb}_{(\mathbf{P}^1, m\infty)})^{\times n-1}"] \\ \mathrm{Pic}^0(\mathbf{P}^1_K, m\infty)^{\times n} \arrow[r, "\delta_n"] & \mathrm{Pic}^0(\mathbf{P}^1_K, m\infty)^{\times n-1} \end{tikzcd} \tag{5.18.3}$$

where $Z_0(\mathbf{A}^n_K)_{\mathbb{Q}} = \mathbb{Q}_{\mathrm{ltr}}(\mathbf{A}^n, \mathrm{triv})(K)$ is the group of zero-cycles with rational coefficients and the map d_n^B is the boundary maps for the complex (5.18.2) evaluated on K, defined on generators as follows (see [Bre69]): let $(\{a_1\}, \ldots, \{a_n\}) \in Z_0(\mathbf{A}^n_K)_{\mathbb{Q}}$ be a generator, then

$$\begin{aligned} d_n^B(K) \colon (\{a_1\}, \ldots, \{a_n\}) \mapsto & (\{a_2\}, \ldots, \{a_n\}) \\ & + \sum_{i=1}^{n-1} (\ldots, \{a_i + a_{i+1}\}, \ldots) + (-1)^n(\{a_1\}, \ldots, \{a_{n-1}\}). \end{aligned} \tag{5.18.4}$$

On the other hand, the Albanese map sends $(\{a_1\}, \ldots, \{a_n\})$ to the n-uple

$$\Big(\Big(\mathcal{O}, \frac{T_1^{(1)} - a_1 T_0^{(1)}}{T_1^{(1)} - T_0^{(1)}}\Big), \ldots, \Big(\mathcal{O}, \frac{T_1^{(n)} - a_n T_0^{(n)}}{T_1^{(n)} - T_0^{(n)}}\Big) \Big)$$

where $[T_0^{(i)} : T_1^{(i)}]$ is the coordinate of the i-th copy of $\mathbf{P}^1_K$. Consider $z_i := \frac{T_1^{(i)}}{T_0^{(i)}}$, we have for every i an isomorphism that on generators is defined as follows:

$$\mathrm{Pic}^0(\mathbf{P}^1_K, m\infty) \cong 1 + z_iK[z_i]/(z_i)^m \quad \Big(\mathcal{O}, \frac{T_1^{(i)} - a_iT_0^{(i)}}{T_1^{(i)} - T_0^{(i)}}\Big) \mapsto \frac{1 - a_iz_i}{1 - z_i} \tag{5.18.5}$$

where the operation on $1 + z_iK[z_i]/(z_i)^m$ is the multiplication of polynomials. To ease the notation, for $a \in K$ we denote by $[a]$ the element $1 - az_i$ in $1 + z_iK[z_i]/(z_i)^m$ and by $[a] + [b]$ the element $(1 - az_i)(1 - bz_i)$, hence we will denote $\frac{1-az_i}{1-z_i}$ by $[a] - [1]$. From (5.18.3), (5.18.4) and (5.18.5) we conclude that the differential δ_n is computed as follows: the i-th component of $\delta_n([a_1] - [1], \ldots, [a_n] - [1])$ is

$$\Big(\sum_{q=0}^{i-1}(-1)^q([a_{i+1}] - [1])\Big) + (-1)^i([a_{i+1} + a_i] - [1]) + \Big(\sum_{q=i+1}^{n-1}(-1)^q([a_i] - [1])\Big).$$

On the other hand, we have that $([1], \ldots, [1]) = \mathrm{Alb}(\{0\}, \ldots, \{0\})$, hence

$$\delta_n([1], \ldots, [1]) = \mathrm{Alb}(d_n^B(\{0\}, \ldots, \{0\})) = \begin{cases} 0 & \text{if } n \equiv_2 1 \\ ([1], \ldots, [1]) & \text{if } n \equiv_2 0. \end{cases}$$

Finally we can compute $\delta_n([a_1], \ldots, [a_n]) = \delta_n([a_1] - [1], \ldots, [a_n] - [1]) + \delta_n([1], \ldots, [1])$. We let $(\_)_i$ denote the i-th component, there are four cases:

$$\delta_n([a_1^r], \ldots, [a_n^r])_i = \begin{cases} [a_{i+1} + a_i] & \text{if } n \equiv_2 i \equiv_2 0 \\ [a_i] + [a_{i+1}] - [a_{i+1} + a_i] & \text{if } n \equiv_2 0, i \equiv_2 1 \\ [a_i + a_{i+1}] - [a_i] & \text{if } n \equiv_2 1, i \equiv_2 0 \\ [a_{i+1}] - [a_i + a_{i+1}] & \text{if } n \equiv_2 i \equiv_2 1. \end{cases}$$

The group $1 + z_iK[z_i]/(z_i)^m$ has a bounded filtration where for $1 \leq r \leq m$:

$$\mathrm{Fil}_r^{(i)} = 1 + z_i^rK[z_i]/(z_i)^m.$$

For each r, we let $[a]_r = 1 + az_i^r$. We make the following claim:

Claim 5.19. *Let $r \geq 1$ and let $([a_1^r]_r, \ldots, [a_n^r]_r) \in \mathrm{Fil}_r^{(1)} \oplus \ldots \mathrm{Fil}_r^{(n)}$. Then modulo mod $\mathrm{Fil}_{r+1}^{(i)}$ we have:*

$$\delta_n([a_1^r]_r, \ldots, [a_n^r]_r)_i = \begin{cases} [(a_{i+1} + a_i)^r]_r & \textit{if } n \equiv_2 i \equiv_2 0 \\ [a_i^r + a_{i+1}^r - (a_{i+1} + a_i)^r]_r & \textit{if } n \equiv_2 0, i \equiv_2 1 \\ [(a_i + a_{i+1})^r - a_i^r]_r & \textit{if } n \equiv_2 1, i \equiv_2 0 \\ [a_{i+1}^r - (a_i + a_{i+1})^r]_r & \textit{if } n \equiv_2 i \equiv_2 1. \end{cases}$$

Admitting Claim 5.19, since K is algebraically closed we conclude that for each r, we have that

$$\delta_n\big(\mathrm{Fil}_r^{(1)} \oplus \ldots \oplus \mathrm{Fil}_r^{(n)}\big) \subseteq \big(\mathrm{Fil}_r^{(1)} \oplus \ldots \oplus \mathrm{Fil}_r^{(n-1)}\big).$$

In particular, the complex $(\mathrm{Pic}^0(\mathbf{P}^1_K, m\infty)^\bullet, \delta_\bullet)$ has a bounded filtration as follows:

$$(\mathrm{Pic}^0(\mathbf{P}^1_K, m\infty)^\bullet, \delta_\bullet) = \bigoplus_{i=1}^{\bullet}(\mathrm{Fil}_1^{(i)}, \delta_\bullet) \supseteq \ldots \bigoplus_{i=1}^{\bullet}(\mathrm{Fil}_r^{(i)}, \delta_\bullet) \supseteq \bigoplus_{i=1}^{\bullet}(\mathrm{Fil}_m^{(i)}, \delta_\bullet) = 0.$$

Moreover, for all r we have an evident isomorphism (see also (5.19.2) and (5.19.3) below)

$$\mathrm{gr}^r\big(\mathrm{Pic}^0(\mathbf{P}^1_K, m\infty)^n\big) \cong K^n \quad ([a_1]_r, \ldots, [a_n]_r) \mapsto (a_1, \ldots a_n)$$

and the boundary map $\mathrm{gr}^r(\delta_n)\colon K^n \to K^{n-1}$ is computed as in Claim 5.19. Using the spectral sequence of the filtered complex (notice that the filtration is bounded so it always abrupts), we are reduced to show that for all $n \geq 1$ and $1 \leq r \leq m$,

$$\pi_n(\mathrm{gr}^r\big(\mathrm{Pic}^0(\mathbf{P}^1_K, m\infty)^\bullet\big), \mathrm{gr}^r(\delta_\bullet)).$$

Let $(a_1 \dots a_n) \in K^n$ such that $\mathrm{gr}^r(\delta_n)(a_1 \dots a_n) = 0$, we need to find $(b_1, \dots, b_{n+1})$ such that $a_i = \mathrm{gr}^r(\delta_{n+1})(b_1, \dots, b_{n+1})_i$. Since K is algebraically closed we can suppose that $a_i = x_i^r$. By Claim 5.19 we get that

$$\mathrm{gr}^r(\delta_n)(x_1^r \dots x_n^r)_i = 0 \Leftrightarrow (x_i + x_{i+1})^r = \begin{cases} 0 & \text{if } n \equiv_2 i \equiv_2 0 \\ x_i^r + x_{i+1}^r & \text{if } n \equiv_2 0, i \equiv_2 1 \\ x_i^r & \text{if } n \equiv_2 1, i \equiv_2 0 \\ x_{i+1}^r & \text{if } n \equiv_2 i \equiv_2 1. \end{cases} \tag{5.19.1}$$

Let $i \geq 2$ and suppose that we have constructed $b_1, \dots, b_i$ such that for all $\ell \leq i-1$,

$$x_\ell^r = \mathrm{gr}^r(\delta_{n+1})(b_1, \dots, b_i, 0 \dots, 0)_\ell$$

then we have a polynomial relation on b_{i+1} by imposing the condition

$$x_i^r = \mathrm{gr}^r(\delta_{n+1})(b_1, \dots, b_i, b_{i+1}, 0 \dots, 0)_i$$

and deduce the existence of b_{n+1} as root of such polynomial. Hence it remains to construct b_1 and b_2 in the case $i = 1$. If $n \equiv_2 0$, we choose $b_1 = (-x_1)^r$ and $b_2 = (x_1 + x_2)^r$: from the second case of(5.19.1) we have $(x_1 + x_2)^r = x_1^r + x_2^r$, so we get that

$$\mathrm{gr}^r(\delta_{n+1})(b_1, b_2, 0 \dots, 0)_1 = (x_1 + x_2)^r - (x_1 + x_2 - x_1)^r = x_1^r + x_2^r - x_2^r = x_1^r.$$

If $n \equiv_2 1$, we choose $b_1 = x_1^r$ and $b_2 = x_2^r$: from the last case of (5.19.1) we have $(x_1 + x_2)^r = x_2^r$, so we get that

$$\mathrm{gr}^r(\delta_{n+1})(b_1, b_2, 0 \dots, 0)_1 = x_1^r + x_2^r - (x_1 + x_2)^r = x_1^r.$$

This implies that $\pi_n(\mathrm{gr}^r\big(\mathrm{Pic}^0(\mathbf{P}^1_K, m\infty)^\bullet\big), \mathrm{gr}^r(\delta_\bullet)) = 0$ for all $n \geq 1$ and reduces the proof of Lemma 5.18 to Claim 5.19.

Proof of Claim 5.19. Let $\delta_{n,i} := \delta_n([a_1^r]_r, \dots [a_n^r]_r)_1$. Notice that for all $b \in K$:

$$-[b]_r = (1 - bz_i^r)^{-1} = 1 + bz_i^r + O(z_i^{2r}) = [-b]_r \bmod \mathrm{Fil}^{(i)}_{r+1} \tag{5.19.2}$$

and for all $b, c \in K$;

$$[b]_r + [c]_r = (1 - bz_i^r)(1 - cz_i^r) = 1 - (b+c)z_i^r + O(z_i^{r+1}) = [b+c]_r \bmod \mathrm{Fil}^{(i)}_{r+1}. \tag{5.19.3}$$

Let ζ_r be a primitive r-th root of unity in K, then

$$[a_i^r]_r = 1 - a_i^r z_i^r = \prod_{q=0}^{r-1}(1 - \zeta_r^q a_i z_i) = \sum_{q=1}^{r-1}[\zeta_r^q a_i],$$

thus $\delta_{n,i} = \sum_{q=1}^{r-1} \delta_n([\zeta_r^q a_1], \dots, [\zeta_r^q a_n])_i$. We check the four cases:

(1) $n \equiv_2 i \equiv_2 0$: $\delta_{n,i} = \sum_{q=1}^{r-1}[\zeta_r^q a_{i+1} + \zeta_r^q a_i] = \sum_{q=1}^{r-1}[\zeta_r^q(a_i + a_{i+1})] = [(a_i + a_{i+1})^r]_r$
(2) $n \equiv_2 0, i \equiv_2 1$: $\delta_{n,i} = \sum_{q=1}^{r-1}[\zeta_r^q a_{i+1}] + [\zeta_r^q a_i] - [\zeta_r^q a_{i+1} + \zeta_r^q a_i]$
$= [a_{i+1}^r]_r + [a_i^r]_r - [(a_i + a_{i+1})^r]_r$
(3) $n \equiv_2 1, i \equiv_2 0$: $\delta_{n,i} = \sum_{q=1}^{r-1}[\zeta_r^q a_{i+1} + \zeta_r^q a_i] - [\zeta_r^q a_i] = [(a_i + a_{i+1})^r]_r - [a_i^r]_r$
(4) $n \equiv_2 i \equiv_2 1$: $\delta_{n,i} = \sum_{q=1}^{r-1}[\zeta_r^q a_{i+1}] - [\zeta_r^q a_{i+1} + \zeta_r^q a_i] = [a_i^r]_r - [(a_i + a_{i+1})^r]_r$.

Here all congruences follow from (5.19.2) and (5.19.3). □

6. Some computations

Recall from [BVK16, Theorem 9.2.3] (see also [Par21, Theorem 1.1] for a statement in a language more similar to ours) that

$$L\operatorname{Alb}(\omega(X)) \simeq \operatorname{Alb}(\omega(X)) \oplus \mathrm{NS}^*(\omega(X))[1], \tag{6.0.1}$$

where $\mathrm{NS}^*(\omega(X))$ is the torus dual to the Néron-Severi. The goal of this section is to give an explicit description of $L\operatorname{Alb}^{\log}(X)$: for $X = (\underline{X}, \partial X)$, recall that $\omega(X)$ is defined as $\underline{X} - |\partial X|$. We will prove the following result:

Theorem 6.1. *Let $X \in \mathbf{SmlSm}$ geometrically connected and $(\overline{X}, D)$ a Cartier compactification of X. Then we have that*

$$L_i \operatorname{Alb}^{\log}(X) \cong \begin{cases} \text{“}\varprojlim\text{”}(H^i(\overline{X}, \mathcal{O}_{\overline{X}}(nD))^\vee) \otimes_k \mathbf{G}_a & \text{for } 2 \le i \le \dim(X) \\ \text{“}\varprojlim\text{”}\Big((H^1(\overline{X}, \mathcal{O}_{\overline{X}}(nD))/H^1(\overline{X}, \mathcal{O}_{\overline{X}}))^\vee\Big) \otimes_k \mathbf{G}_a \oplus \mathrm{NS}^*(\omega(X)) & \text{for } i = 1 \\ \operatorname{Alb}^{\log}(X) & \text{for } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the canonical map $L\operatorname{Alb}^{\log}(X) \to \oplus_{i=0}^{\dim(X)} L_i \operatorname{Alb}^{\log}(X)[i]$ is an equivalence.

We start with the following observation:

Proposition 6.2. *The inclusion $\mathcal{D}(i_{\mathbf{A}^1})\colon \mathcal{D}(\mathbf{HI}_{\text{ét},\le 1}) \to \mathcal{D}(\mathbf{RSC}_{\text{ét},\le 1})$ has a left adjoint $L^{\le 1}_{\mathbf{A}^1}$ such that $\pi_i(L^{\le 1}_{\mathbf{A}^1})(F) = h_i^{\mathbf{A}^1}(F)$ (the Suslin hyperhomology).*

Proof. By Proposition 4.31, the inclusion $\mathcal{D}(\mathbf{RSC}_{\text{ét},\le 1}) \to \mathcal{D}(\mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\mathbb{Q}))$ is fully faithful, then for $F \in \mathcal{D}(\mathbf{RSC}_{\text{ét},\le 1})$ and $H \in \mathcal{D}(\mathbf{HI}_{\text{ét},\le 1})$:

$$\begin{aligned} &\operatorname{Map}_{\mathcal{D}(\mathbf{RSC}_{\text{ét},\le 1})}(F, \mathcal{D}(i_{\mathbf{A}^1})(H)) \simeq \operatorname{Map}_{\mathcal{D}(\mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\mathbb{Q}))}(F, \mathcal{D}(i_{\mathbf{A}^1})(H)) \\ &\simeq \operatorname{Map}_{\mathcal{DM}^{\mathrm{eff}}(k,\mathbb{Q}))}(C_*^{\mathbf{A}^1}(F), H) \simeq \operatorname{Map}_{\mathcal{D}(\mathbf{HI}_{\text{ét},\le 1})}(L\operatorname{Alb} C_*^{\mathbf{A}^1}(F), H) \end{aligned}$$

which implies that $L^{\le 1}_{\mathbf{A}^1}$ exists and it coincides with $L\operatorname{Alb} C_*^{\mathbf{A}^1}$. We can write $F = \operatorname{hocolim}_{i,n} F_i[n]$ with $F_i \in \mathbf{RSC}^\star_{\le 1}$. Since $C_*^{\mathbf{A}^1}\colon \mathcal{D}(\mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\mathbb{Q})) \to \mathcal{DM}^{\mathrm{eff}}(k,\mathbb{Q}))$ commutes with all (homotopy) colimits as a left adjoint, we have

$$C_*^{\mathbf{A}^1}(F) \simeq \operatorname{hocolim} C_*^{\mathbf{A}^1}(F_i[n]) \overset{(*)}{\simeq} \operatorname{hocolim}(h_0^{\mathbf{A}^1}(F_i)[n]),$$

where $(*)$ follows from Remark 4.25. Hence the homotopy groups of $C_*^{\mathbf{A}^1}(F)$ are 1-motivic so that $\pi_i L^{\le 1}_{\mathbf{A}^1}(F) = \pi_i L\operatorname{Alb} C_*^{\mathbf{A}^1}(F) = \pi_i C_*^{\mathbf{A}^1}(F) = h_i^{\mathbf{A}^1}(F)$. □

Lemma 6.3. *Let $\text{Pro-}L^{\le 1}_{\mathbf{A}^1}\colon \text{Pro-}\mathcal{D}(\mathbf{RSC}_{\text{ét},\le 1}) \to \text{Pro-}\mathcal{D}(\mathbf{HI}_{\text{ét},\le 1})$. For $X \in \mathbf{SmlSm}(k)$ geometrically connected, $\text{Pro-}L^{\le 1}_{\mathbf{A}^1} L\operatorname{Alb}^{\log}(X)$ is a constant pro-object and*

$$\varprojlim \text{pro-}L^{\le 1}_{\mathbf{A}^1} L\operatorname{Alb}^{\log}(X) \simeq L\operatorname{Alb}(\omega(X)).$$

Proof. Let $\mathbb{Q}_{\mathrm{ltr}}(Y_\bullet) \to \mathbb{Q}_{\mathrm{ltr}}(X)$ be a resolution in $\mathbf{Shv}^{\mathrm{ltr}}_{\text{dét}}(k,\mathbb{Q})$ by Alb-trivial objects. Then $L_i \operatorname{Alb}^{\log}(X) = \pi_i(\operatorname{Alb}^{\log}(Y_\bullet))$. On the other hand, by construction, $\mathbb{Q}_{\mathrm{tr}}(\omega(Y_\bullet)) \to \mathbb{Q}_{\mathrm{tr}}(\omega(X))$ is a resolution in $\mathbf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\mathbb{Q})$ by affine NS^1-local objects in the sense of [ABV09], so $L_i \operatorname{Alb}(\omega(X)) = \pi_i \operatorname{Alb}(\omega(Y_\bullet))$. By Proposition 3.19 and (5.2.1), for $\mathfrak{Y}_\bullet$ Cartier compactifications of $Y_\bullet$, we have a fiber-cofiber sequence:

$$\text{“}\varprojlim_n\text{”} U(\mathfrak{Y}_\bullet^{(n)}) \otimes_k \mathbf{G}_a \to \operatorname{Alb}^{\log}(Y_\bullet) \to c(\operatorname{Alb}(\omega(Y_\bullet))), \tag{6.3.1}$$

where $U(\mathfrak{Y}_\bullet^{(n)})$ are the k-vector spaces from Definition 3.5. The complex $\operatorname{Alb}(\omega(Y_\bullet))$ is $\mathbf{A}^1$-local, so we have that

$$\text{Pro-}L^{\le 1}_{\mathbf{A}^1} c(\operatorname{Alb}(\omega(Y_\bullet))) \simeq c(\operatorname{Alb}(\omega(Y_\bullet))) = c(L\operatorname{Alb}(\omega(X)))$$

and for all i, n, there exist $r_{i,n}$ such that $U(\mathfrak{Y}_i^{(n)}) = \mathbf{G}_a^{r_{i,n}}$, so we conclude since:

$$\text{Pro-}L_{\mathbf{A}^1}^{\leq 1}(\text{“}\varprojlim_n\text{”}(\mathfrak{Y}_\bullet^{(n)})) = \text{“}\varprojlim_n\text{”}L_{\mathbf{A}^1}^{\leq 1}(U(\mathfrak{Y}_\bullet^{(n)})) = 0.$$

□

The lemma above gives a natural map $L\,\mathrm{Alb}^{\log}(X) \to cL\,\mathrm{Alb}(\omega(X))$. Let

$$\mathrm{Fib}(X) := \mathrm{hofib}(L\,\mathrm{Alb}^{\log}(X) \to cL\,\mathrm{Alb}(\omega(X))). \tag{6.3.2}$$

In view of (6.3.1), we have that

$$\mathrm{Fib}(X) \simeq \text{“}\varprojlim_n\text{”}U(\mathfrak{Y}_\bullet^{(n)}). \tag{6.3.3}$$

since for $\mathbb{Q}_{\mathrm{ltr}}(Y_\bullet) \to \mathbb{Q}_{\mathrm{ltr}}(X)$ an Alb-trivial resolution and $\mathfrak{Y}_\bullet$ a Cartier compactification of $Y_\bullet$, the map $\mathbf{Alb}(\mathfrak{Y}_i^{(n)}) \to \mathrm{Alb}(\omega(Y_i))$ is surjective with kernel $U(\mathfrak{Y}_i^{(n)})$ for all i.

Definition 6.4. Let $\langle \mathbf{G}_a \rangle \subseteq \mathcal{D}(\mathbf{RSC}_{\text{ét},\leq 1})$ be the stable ∞-subcategory generated by direct sums of $\mathbf{G}_a$ and let $\langle \mathbf{G}_a \rangle^\omega$ be the full subcategory of compact objects. In particular, $F \in \langle \mathbf{G}_a \rangle^\omega$ if and only if there exist $s \leq t \in \mathbb{Z}$ and $r_s, \dots, r_t \geq 0$ such that

$$F \simeq \bigoplus_{i=s}^{t} \mathbf{G}_a^{r_i}[i].$$

Remark 6.5. Let $\text{Pro-}(\langle \mathbf{G}_a \rangle^\omega) \subseteq \text{Pro-}\mathcal{D}(\mathbf{RSC}_{\text{ét},\leq 1})$, then for all X as above $\mathrm{Fib}(X) \in \text{Pro-}(\langle \mathbf{G}_a \rangle^\omega)$

Remark 6.6. Let $\mathbf{Vect}_{\mathrm{fd}}$ be the category of finite dimensional k-vector spaces. By Proposition 4.31 and [Ser75, VII]:

$$\mathrm{Map}_{\mathcal{D}(\mathbf{RSC}_{\text{ét},\leq 1})}(\mathbf{G}_a^r, \mathbf{G}_a^s[i]) \simeq \mathrm{Hom}_{\mathbf{Vect}_{\mathrm{fd}}}(k^r, k^s)[i] \simeq \mathrm{Map}_{\mathcal{D}(\mathbf{Vect}_{\mathrm{fd}})}(k^r, k^s[i]).$$

In particular, the functor $V \mapsto \mathrm{Spec}(k[V^\vee])$ induces an equivalence of ∞-categories

$$\_ \otimes_k \mathbf{G}_a \colon \mathcal{D}(\mathbf{Vect}_{\mathrm{fd}}) \to \langle \mathbf{G}_a \rangle^\omega$$

with quasi inverse given by $R\Gamma(\mathrm{Spec}(k), \_)$.

Let $(\_)^\vee \colon \mathbf{Vect}_{\mathrm{fd}} \to \mathbf{Vect}_{\mathrm{fd}}^{\mathrm{op}}$ denote the equivalence given by the dual vector space. It induces an equivalence:

$$(\_)^\vee \colon \text{Pro-}\mathcal{D}(\mathbf{Vect}_{\mathrm{fd}}) \to \text{Ind-}\mathcal{D}(\mathbf{Vect}_{\mathrm{fd}}^{\mathrm{op}}).$$

Remark 6.7. There is a commutative diagram of stable ∞-category:

$$\begin{tikzcd} \text{Pro-}\langle \mathbf{G}_a \rangle^\omega \arrow[r, "\text{Pro-}R\Gamma(\mathrm{Spec}(k),\_)"'] \arrow[rr, bend left, "\mathrm{Map}(\_,\mathbf{G}_a)"] & \text{Pro-}\mathcal{D}(\mathbf{Vect}_{\mathrm{fd}}) \arrow[r, "(\_)^\vee"'] & \text{Ind-}\mathcal{D}(\mathbf{Vect}_{\mathrm{fd}}^{\mathrm{op}}) \end{tikzcd}$$

where Map denotes the mapping space in $\text{Pro-}\mathcal{D}(\mathbf{RSC}_{\text{ét},\leq 1})$. This easily follows from the fact that any map of sheaves $f \colon \mathbf{G}_a^r \to \mathbf{G}_a^s$ is indeed a map of vector groups, hence since $\mathbf{G}_a = k \otimes_k \mathbf{G}_a$, we have a commutative diagram in $\mathcal{D}(\mathbf{Vect}_{\mathrm{fd}})$:

$$\begin{tikzcd} \mathrm{Map}(\mathbf{G}_a^s, \mathbf{G}_a) \arrow[r, "\simeq"] \arrow[d, "f^*"] & R\Gamma(\mathrm{Spec}(k), \mathbf{G}_a^s)^\vee \arrow[d, "f(k)^t"] \\ \mathrm{Map}(\mathbf{G}_a^r, \mathbf{G}_a) \arrow[r, "\simeq"] & R\Gamma(\mathrm{Spec}(k), \mathbf{G}_a^r)^\vee. \end{tikzcd}$$

Proposition 6.8. *Let $X \in \mathbf{SmlSm}(k)$ geometrically connected. Then for any $(\overline{X}, D)$ Cartier compactification of $\underline{X}$ and $i \geq 2$, we have that*

$$L_i\,\mathrm{Alb}^{\log}(X) \simeq \text{“}\varprojlim\text{”}(H^i(\overline{X}, \mathcal{O}_{\overline{X}}(nD))^\vee) \otimes_k \mathbf{G}_a.$$

In particular, $L_i\,\mathrm{Alb}^{\log}(X) = 0$ for $i \geq \max(\dim(X), 2)$.

Proof. By (6.0.1), for X as above we have that for $i \geq 2$

$$L_i \operatorname{Alb}^{\log}(X) \simeq \pi_i \mathrm{Fib}(X).$$

By Remark 6.7 we have that

$$\mathrm{Fib}(X) = R\Gamma(\mathrm{Spec}(k), \mathrm{Fib}(X)) \otimes_k \mathbf{G}_a \simeq \mathrm{Map}(\mathrm{Fib}(X), \mathbf{G}_a)^{\vee} \otimes_k \mathbf{G}_a. \tag{6.8.1}$$

By Theorem 5.1 and [BPØ20, Theorem 9.7.1] and (5.18.1).

$$\mathrm{Map}_{\mathrm{Pro}\text{-}\mathcal{D}(\mathbf{RSC}_{\acute{e}t,\leq 1})}(L\operatorname{Alb}^{\log}(X), \mathbf{G}_a) \simeq \mathrm{Map}_{\mathbf{log}\mathcal{DM}^{\mathrm{eff}}}(M(X), \omega^{\mathbf{CI}}_{\log}\mathbf{G}_a) \simeq R\Gamma(\underline{X}, \mathcal{O}_{\underline{X}}). \tag{6.8.2}$$

Let now $(\overline{X}, D)$ be a Cartier compactification of $\underline{X}$. This gives an isomorphism:

$$R\Gamma(\underline{X}, \mathcal{O}_{\underline{X}}) \simeq \varinjlim_n R\Gamma(\overline{X}, \mathcal{O}_{\overline{X}}(nD)).$$

and the right hand side is in Ind-$\mathcal{D}(\mathbf{Vect}^{\mathrm{op}}_{\mathrm{fd}})$, which completes the proof. □

Since $L\operatorname{Alb}^{\log}(X)$ is bounded, there are maps

$$L\operatorname{Alb}^{\log}(X) \to \tau_{\leq i} L\operatorname{Alb}^{\log}(X) \leftarrow \tau_{\geq i}\tau_{\leq i} L\operatorname{Alb}^{\log}(X) \simeq L_i \operatorname{Alb}^{\log}(X)[i]$$

which induces a zig-zag

$$L\operatorname{Alb}^{\log}(X) \to \oplus_{i=1}^{\dim(X)} \tau_{\leq i} L\operatorname{Alb}^{\log}(X) \leftarrow \oplus_{i=1}^{\dim(X)} L_i \operatorname{Alb}^{\log}(X)[i]. \tag{6.8.3}$$

Lemma 6.9. *For $X \in \mathbf{SmlSm}(k)$ geometrically connected, we have an equivalence:*

$$L\operatorname{Alb}^{\log}(X) \cong \oplus_{i=0}^{\dim(X)} L_i \operatorname{Alb}^{\log}(X)[i].$$

Proof. By Proposition 4.31, we have that for every $F \in \mathbf{RSC}_{\acute{e}t,\leq 1}$, and $i \geq 2$,

$$\mathrm{Ext}^i_{\mathrm{pro}\text{-}\mathbf{RSC}_{\acute{e}t,\leq 1}}(\pi_j \mathrm{Fib}(X), c(F)) = 0, \quad \text{resp. } \mathrm{Ext}^i_{\mathrm{pro}\text{-}\mathbf{RSC}_{\acute{e}t,\leq 1}}(\pi_j L\operatorname{Alb}(\omega(X)), c(F)) = 0,$$

so by (6.3.2), we get $\mathrm{Ext}^i_{\mathrm{pro}\text{-}\mathbf{RSC}_{\acute{e}t,\leq 1}}(L_j \operatorname{Alb}^{\log}(X), c(F)) = 0$. Then the spectral sequence

$$E_2^{ij} = \mathrm{Ext}^i_{\mathrm{pro}\text{-}\mathbf{RSC}_{\acute{e}t,\leq 1}}(L_j \operatorname{Alb}^{\log}(X), c(F)) \Rightarrow \pi_{i+j} \mathrm{Map}_{\mathrm{Pro}\text{-}\mathcal{D}(\mathbf{RSC}_{\acute{e}t,\leq 1})}(L\operatorname{Alb}^{\log}(X), c(F)[0])$$

degenerates at page 2, giving for every j a short exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathrm{pro}\text{-}\mathbf{RSC}_{\acute{e}t,\leq 1}}(L_j \operatorname{Alb}^{\log}(X), c(F)) &\to \pi_{1+j} \mathrm{Map}_{\mathrm{Pro}\text{-}\mathcal{D}(\mathbf{RSC}_{\acute{e}t,\leq 1})}(L\operatorname{Alb}^{\log}(X), c(F)) \\ &\to \mathrm{Hom}_{\mathrm{pro}\text{-}\mathbf{RSC}_{\acute{e}t,\leq 1}}(L_{1+j} \operatorname{Alb}^{\log}(X), c(F)) \to 0. \end{aligned} \tag{6.9.1}$$

On the other hand, the complex $\oplus L_i \operatorname{Alb}^{\log}(X)[i]$ also fits in short exact sequences

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathrm{pro}\text{-}\mathbf{RSC}_{\acute{e}t,\leq 1}}(L_j \operatorname{Alb}^{\log}(X), c(F)) &\to \pi_{1+j} \mathrm{Map}_{\mathrm{Pro}\text{-}\mathcal{D}(\mathbf{RSC}_{\acute{e}t,\leq 1})}(\oplus L_i \operatorname{Alb}^{\log}[i], c(F)) \\ &\to \mathrm{Hom}_{\mathrm{pro}\text{-}\mathbf{RSC}_{\acute{e}t,\leq 1}}(L_{1+j} \operatorname{Alb}^{\log}(X), c(F)) \to 0 \end{aligned}$$

so the zig-zag of (6.8.3) gives for all $F = \text{“}\varprojlim\text{”} F_i$ and all n an equivalence

$$\begin{aligned} \mathrm{Map}_{\mathrm{Pro}\text{-}\mathcal{D}(\mathbf{RSC}_{\acute{e}t,\leq 1})}(\oplus L_i \operatorname{Alb}^{\log}(X)[i], F[n]) &\simeq \varprojlim \mathrm{Map}_{\mathrm{Pro}\text{-}\mathcal{D}(\mathbf{RSC}_{\acute{e}t,\leq 1})}(\oplus L_i \operatorname{Alb}^{\log}(X)[i], c(F_i)[n]) \\ &\simeq \varprojlim \Sigma^{-n} \mathrm{Map}_{\mathrm{Pro}\text{-}\mathcal{D}(\mathbf{RSC}_{\acute{e}t,\leq 1})}(\oplus L_i \operatorname{Alb}^{\log}(X)[i], c(F_i)) \\ &\simeq \varprojlim \Sigma^{-n} \mathrm{Map}_{\mathrm{Pro}\text{-}\mathcal{D}(\mathbf{RSC}_{\acute{e}t,\leq 1})}(L\operatorname{Alb}^{\log}(X), c(F_i)) \\ &\simeq \varprojlim \mathrm{Map}_{\mathrm{Pro}\text{-}\mathcal{D}(\mathbf{RSC}_{\acute{e}t,\leq 1})}(L\operatorname{Alb}^{\log}(X), c(F_i)[n]) \\ &\simeq \mathrm{Map}_{\mathrm{Pro}\text{-}\mathcal{D}(\mathbf{RSC}_{\acute{e}t,\leq 1})}(L\operatorname{Alb}^{\log}(X), F[n]). \end{aligned}$$

Then a spectral sequence argument and the fact that $L\operatorname{Alb}^{\log}(X)$ is bounded implies that for all $C \in \mathrm{Pro}\text{-}\mathcal{D}(\mathbf{RSC}_{\acute{e}t,\leq 1})$ we have

$$\mathrm{Map}_{\mathrm{Pro}\text{-}\mathcal{D}(\mathbf{RSC}_{\acute{e}t,\leq 1})}(\oplus L_i \operatorname{Alb}^{\log}(X)[i], C) \simeq \mathrm{Map}_{\mathrm{Pro}\text{-}\mathcal{D}(\mathbf{RSC}_{\acute{e}t,\leq 1})}(L\operatorname{Alb}^{\log}(X), C)$$

which allows us to conclude. □

Proof of Theorem 6.1. The only case left is $i=1$: we consider the long exact sequence of homotopy groups arising from (6.3.2). The map $L_1\operatorname{Alb}(\omega(X))\to\pi_0(\mathrm{Fib}(X))$ is zero since by (6.0.1), $L_1\operatorname{Alb}(\omega(X))$ is a torus, so we get a short exact sequcend in pro- $\mathbf{RSC}_{\text{ét},\le 1}$:

$$0\to\pi_1\mathrm{Fib}(X)\to L_1\operatorname{Alb}^{\log}(X)\to L_1\operatorname{Alb}(\omega(X))\overset{(6.0.1)}{\cong} c(\mathrm{NS}^*(\omega(X)))\to 0.$$

Since $\pi_1\mathrm{Fib}(X)$ is a pro-vector group, this short exact sequence splits and

$$L_1\operatorname{Alb}^{\log}(X)\cong\pi_1\mathrm{Fib}(X)\oplus\mathrm{NS}^*(\omega(X)). \tag{6.9.2}$$

Moreover, we have

$$\mathrm{Map}(\pi_1\mathrm{Fib}(X),\mathbf{G}_a)\simeq\mathrm{Map}(L_1\operatorname{Alb}^{\log}(X),\mathbf{G}_a)\simeq\mathrm{Hom}_{\text{pro-}\mathbf{RSC}_{\text{ét},\le 1}}(L_1\operatorname{Alb}^{\log}(X),\mathbf{G}_a)[0],$$

where Map denotes the mapping space in Pro-$\mathcal{D}(\mathbf{RSC}_{\text{ét},\le 1})$. Hence (6.9.2) gives:

$$\pi_1 L\operatorname{Alb}^{\log}(X)\cong(\mathrm{Hom}(L_1\operatorname{Alb}^{\log}(X),\mathbf{G}_a)^\vee)\otimes_k\mathbf{G}_a\oplus\mathrm{NS}^*(\omega(X)). \tag{6.9.3}$$

Let us compute $\mathrm{Hom}(L_1\operatorname{Alb}^{\log}(X),\mathbf{G}_a)$: by (6.8.2) and (6.9.1) we have an exact sequence:

$$0\to\mathrm{Ext}^1(\operatorname{Alb}^{\log}(X),\mathbf{G}_a)\to H^1(X,\mathcal{O}_X)\to\mathrm{Hom}(L_1\operatorname{Alb}^{\log}(X),\mathbf{G}_a)\to 0. \tag{6.9.4}$$

In particular, we have a similar exact sequence for the log scheme $(\overline{X},\mathrm{triv})$, which we now inversigate. For $\overline{X}$ proper we have by construction (see Theorem 4.28) that $\operatorname{Alb}^{\log}(\overline{X})$ is the constant pro-object $\operatorname{Alb}(\overline{X})$, so there is a surjective map

$$\operatorname{Alb}^{\log}(X)\twoheadrightarrow\operatorname{Alb}^{\log}(\overline{X})$$

whose kernel is an extension of the torus $T:=\ker(\operatorname{Alb}(\omega(X))\to\operatorname{Alb}(\overline{X}))$ by the pro-vector group "$\varprojlim$"$U(\overline{X},nD)\otimes_k\mathbf{G}_a$, where $U(\overline{X},nD)$ comes from Definition 3.5. For $i\ge 1$, we have that $\varinjlim\mathrm{Ext}^i(U(\overline{X},nD),\mathbf{G}_a)=0$ and $\mathrm{Ext}^i(T,\mathbf{G}_a)=0$, so we have a surjective map:

$$\mathrm{Ext}^1(\operatorname{Alb}^{\log}(\overline{X}),\mathbf{G}_a)\twoheadrightarrow\mathrm{Ext}^1(\operatorname{Alb}^{\log}(X),\mathbf{G}_a). \tag{6.9.5}$$

Combining (6.9.4), and (6.9.5) we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1(\operatorname{Alb}^{\log}(\overline{X}),\mathbf{G}_a) \arrow[r] \arrow[d, two heads] & H^1(\overline{X},\mathcal{O}_X) \arrow[r] \arrow[d] & \mathrm{Hom}(L_1\operatorname{Alb}^{\log}(\overline{X}),\mathbf{G}_a) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1(\operatorname{Alb}^{\log}(X),\mathbf{G}_a) \arrow[r] & H^1(X,\mathcal{O}_X) \arrow[r] & \mathrm{Hom}(L_1\operatorname{Alb}^{\log}(X),\mathbf{G}_a) \arrow[r] & 0
\end{tikzcd}$$

so to conclude it is enough to show that $L_1\operatorname{Alb}^{\log}(\overline{X})\cong\mathrm{NS}^*(\overline{X})$, which implies that $\mathrm{Hom}(L_1\operatorname{Alb}^{\log}(\overline{X}),\mathbf{G}_a)=0$, so the diagram above implies

$$\mathrm{Hom}(L_1\operatorname{Alb}^{\log}(X),\mathbf{G}_a)\simeq\mathrm{coker}(H^1(\overline{X},\mathcal{O}_X)\to H^1(X,\mathcal{O}_X))$$

and we will conclude by duality and (6.9.3).

By (6.9.2), it is enough to show that $\pi_1\mathrm{Fib}(\overline{X})=0$. By (6.8.1), we have

$$\pi_1\mathrm{Fib}(\overline{X})\cong\pi_1(\mathrm{Map}(\mathrm{Fib}(\overline{X}),\mathbf{G}_a)^\vee\otimes_k\mathbf{G}_a)\cong(\pi_{-1}\mathrm{Map}(\mathrm{Fib}(\overline{X}),\mathbf{G}_a)^\vee\otimes_k\mathbf{G}_a),$$

so it is enough to show that $\pi_{-1}\mathrm{Map}(\mathrm{Fib}(\overline{X}),\mathbf{G}_a)=0$. By (6.3.2) for $\overline{X}$ we have a fiber-cofiber sequence:

$$\mathrm{Map}(L\operatorname{Alb}(\overline{X}),\mathbf{G}_a)\to\mathrm{Map}(L\operatorname{Alb}^{\log}(\overline{X}),\mathbf{G}_a)\to\mathrm{Map}(\mathrm{Fib}(\overline{X}),\mathbf{G}_a). \tag{6.9.6}$$

By (6.0.1), since $\mathrm{NS}^*(\overline{X})$ is a torus and $\operatorname{Alb}(\overline{X})$ is an abelian variety, by [Ser75, VII] we have

$$\begin{aligned}\pi_{-1}\mathrm{Map}(L\operatorname{Alb}(\overline{X}),\mathbf{G}_a)&\cong\pi_0\mathrm{Map}(\mathrm{NS}^*(\overline{X}),\mathbf{G}_a)\oplus\pi_{-1}\mathrm{Map}(\operatorname{Alb}(\overline{X}),\mathbf{G}_a)\\&\cong\mathrm{Ext}^1(\operatorname{Alb}(\overline{X}),\mathbf{G}_a)\cong H^1(\overline{X},\mathcal{O}_{\overline{X}}),\end{aligned}$$

where the last isomorphism is classical, and

$$\pi_{-2}\mathrm{Map}(L\operatorname{Alb}(\overline{X}),\mathbf{G}_a)=0.$$

Finally, $\pi_{-1}\operatorname{Map}(L\operatorname{Alb}^{\log}(\overline{X}),\mathbf{G}_a)\cong H^1(\overline{X},\mathbf{G}_a)$ by (6.8.2), so the map

$$\pi_{-1}\operatorname{Map}(L\operatorname{Alb}(\overline{X}),\mathbf{G}_a)\to\pi_{-1}\operatorname{Map}(L\operatorname{Alb}^{\log}(\overline{X}),\mathbf{G}_a)$$

in the long exact sequence of homotopy groups of (6.9.6) is an isomorphism, which implies the desired vanishing. □

Remark 6.10. We observe from Theorem 6.1 two extreme cases: if X is affine, we have that $H^i(X,\mathcal{O}_X)\simeq\varinjlim_n H^i(\overline{X},\mathcal{O}_X(nD))=0$ for $i\geq 1$, so

$$L_i\operatorname{Alb}^{\log}(X)=\begin{cases}\operatorname{Alb}^{\log}(X) & \text{if } i=0\\ \mathrm{NS}^*(X) & \text{if } i=1\\ 0 & \text{otherwise.}\end{cases}$$

In particular, $L_i\operatorname{Alb}^{\log}(X)$ is constant for $i\geq 1$. For X proper, $H^1(X,\mathcal{O}_X)\cong H^1(\overline{X},\mathcal{O}_{\overline{X}})$, so

$$L_i\operatorname{Alb}^{\log}(X)=\begin{cases}\operatorname{Alb}(X) & \text{if } i=0\\ \mathrm{NS}^*(X) & \text{if } i=1\\ (H^i(\overline{X},\mathcal{O}_X)^\vee)\otimes_k\mathbf{G}_a & \text{if } i\geq 2.\end{cases}$$

In this case, $L\operatorname{Alb}^{\log}(X)$ is a constant pro-object. This shows that Proposition 5.16 cannot be extended to the whole $\mathcal{DM}^{\mathrm{eff}}(k,\mathbb{Q})$: in general, if $M\in\mathcal{DM}^{\mathrm{eff}}(k,\mathbb{Q})$, $L\operatorname{Alb}^{\log}(\omega^*M)$ is not equal to $L\operatorname{Alb}(M)$: the difference is controlled by coherent cohomology of degree ≥ 2.

6.1. **Open questions.** We end this Section with the following observation. It seems to be an interesting question to determine under which conditions $L_i\operatorname{Alb}^{\log}$ is a constant pro-object.

This is related to the following problem: let $X'\to X$ be a desingularisation of a d-dimensional, integral variety over a field k, and let D be an effective Cartier divisor on X' covering the exceptional fibre, and assume that $\operatorname{codim}_X(\pi(D))\geq 2$. Let rD denote the r-th infinitesimal thickening of D and $F^dK_0(X',rD)$ the subgroup of the relative K-group $K_0(X',rD)$ generated by the cycle classes of closed points of $X'-|D|$, for each $r\geq 1$.

Bloch and Srinivas conjectured (see [Sri85, p. 6]) that the pro-object "$\varprojlim_n$"$F^dK_0(X',nD)$ is essentially constant and equal to $F^dK_0(X)$. The Bloch–Srinivas conjecture was proved for normal surfaces by Krishna–Srinivas [KS02, Theorem 1.1], and for $\mathrm{ch}(k)=0$ it was later extended to higher dimensional projective and affine varieties over an algebraically closed field by Krishna [Kri06, Theorem 1.1] [Kri10, Thorem 1.2] and Morrow [Mor15, Theorem 0.1, *(iii), (iv)*][7].

The proof of [Mor15] indeed relies on a natural reformulation of the Bloch–Srinivas conjecture for the Chow groups with modulus:

Theorem 6.11 (cfr. [Mor15, Theorem 0.3])**.** *Let k be an algebraically closed field of characteristic zero and $\pi\colon X'\to X$ and D be as above and assume that X is projective. Then the pro-object "$\varprojlim$"$_n\mathrm{CH}_0(X,nD)$ is constant with stable value equal to the Levine–Weibel Chow group of zero cycles $\mathrm{CH}_0^{LW}(Y)$ of [LW85] (see also [BK18]).*

By the universal property of the Albanese map, we deduce that if in the situation of Theorem 6.11 we assume that D is a simple normal crossing divisor, the pro-object "$\varprojlim$"$_n\mathbf{Alb}_{(X',nD)}$ is indeed essentially constant, so the pro-object $L_0\operatorname{Alb}^{\log}(X-\pi(|D|),\mathrm{triv})$ is essentially constant. Then the following question arises naturally:

Question 6.12. Let X be a proper variety and $U\subset X$ be a smooth open subvariety. When is the pro-reciprocity sheaf $L_i\operatorname{Alb}^{log}(M(U,\mathrm{triv}))$ essentially constant?

[7]The conjecture is indeed true in a more general class of examples: the interested reader can check [Mor15, Theorem 0.1 *(i)-(vii)*]

Notice that in general $L_i \operatorname{Alb}^{log}(M(U, \mathrm{triv}))$ is not essentially constant: let X be a proper non-singular surface, and $U = X - Y$ for some closed subscheme Y of X. Let (X', D) be the blow-up of X in Y. As observed in [Har68, p. 407], if some irreducible component of Y is a point, then $\dim(H^1(U, \mathcal{O}_U)) = \infty$. On the other hand, $\dim(H^1(X', \mathcal{O}_{X'}(nD)))$ is finite for every n, so $L_1 \operatorname{Alb}^{log}(M(U, \mathrm{triv}))$ is not constant by Theorem 1.3. At the moment, we do not know if there is a nice family of pairs $U \subseteq X$ that answers Question 6.12 positively.

7. Laumon 1-motives and compact objects

In this Section, we combine the results of [Ber14] with some arguments of [BVK16]. As before, let k be a field of characteristic zero.

7.1. **Review of Laumon 1-motives.** The following definition is adapted from [Ber14, 1].

Definition 7.1. An *effective Laumon* 1*-motive* is a two-terms complex $M = [\Gamma \xrightarrow{u} G]$, where Γ is a formal k-group and G is a connected algebraic k-group, both seen as objects of $\mathbf{Shv}_{\text{ét}}(k)$. We say that M is *étale* if Γ is a lattice.[8]. An *effective morphism*

$$M = [\Gamma \xrightarrow{u} G] \xrightarrow{(f,g)} M' = [\Gamma' \xrightarrow{u'} G']$$

is a map of complexes. We denote the category of effective (resp. étale) Laumon 1-motives by $\mathcal{M}_1^{a,\mathrm{eff}}$ (resp. $\mathcal{M}_{1,\text{ét}}^{a,\mathrm{eff}}$). An effective Laumon 1-motive is an *effective Deligne* 1*-motive* if G is semi-abelian. We write $\mathcal{M}_1^{D,\mathrm{eff}}$ for the full subcategory of effective Deligne 1-motives.

Definition 7.2. An effective morphism $(f, g)\colon M \to M'$ is *strict* if g has (smooth) connected kernel, and a *quasi-isomorphism* if g is an isogeny, f is surjective and $\ker(f) = \ker(g)$ is a finite k-group scheme.

Note that if (f, g) is strict and g is an isogeny, then g is an isomorphism of commutative algebraic groups. Write Σ for the class of quasi-isomorphisms: it admits a calculus of right fractions (see [BVK16, C.2.4] or [Ber13, Lemma 1.6]). We can now give the following Definition (see [Ber14, Definition 2] and [BVB09, Definition 1.4.4]).

Definition 7.3. The category of *étale Laumon* (resp. *Laumon*, resp *Deligne*) 1*-motives* $\mathcal{M}_{1,\text{ét}}^{a}$ (resp. $\mathcal{M}_1^a$, resp. $\mathcal{M}_1^D$) is the localization by Σ of the effective category.

Recall [BVK16, Appendix B] for the notion of $\mathcal{C} \otimes \mathbb{Q}$ for an additive category $\mathcal{C}$. The proof of the following proposition is identical to [BVK16, Corollary C.7.3]:

Proposition 7.4. *The categories $\mathcal{M}_1^D \otimes \mathbb{Q}$, $\mathcal{M}_1^a \otimes \mathbb{Q}$ and $\mathcal{M}_{1,\text{ét}}^a \otimes \mathbb{Q}$ are abelian.*

Definition 7.5 (see [Ber14]). Let $\mathcal{M}_{1,\text{ét}}^{a,\star}$ be the full subcategory of $\mathcal{M}_{1,\text{ét}}^{a}$ whose objects are 1-motives $M = [\Gamma \xrightarrow{u} G]$ with $\ker(u) = 0$.

Lemma 7.6. *The category $\mathcal{M}_{1,\text{ét}}^{a,\star} \otimes \mathbb{Q}$ is a generating subcategory of $\mathcal{M}_{1,\text{ét}}^{a} \otimes \mathbb{Q}$, and it is closed under kernels and extensions. Moreover, for every object $M \in \mathcal{M}_{1,\text{ét}}^{a} \otimes \mathbb{Q}$ there exists a monomorphism $f : M' \to M''$ in $\mathcal{M}_{1,\text{ét}}^{a,\star} \otimes \mathbb{Q}$ such that $M = \mathrm{coker}(f)$.*

Proof. This is essentially [Ber14, Lemma 4]. ☐

Remark 7.7. The reader might wonder if there are interesting examples of étale Laumon 1-motives which are not Deligne 1-motives. The prototype of such example is given by the 1-motive $M^\natural = [\Gamma \to \mathbf{G}^\natural]$ which is the universal $\mathbf{G}_a$-extension of the Deligne 1-motive $[\Gamma \to G]$. Starting from the motive $M^\natural$ it is possible to construct the universal *sharp*

[8] Note that this definition is different from the one given in [BVB09, 1.4], where the authors require in addition that $U(G) = 0$

extension $M^\sharp$ of M, as discussed in [BVB09]. Note however that the cateogry $\mathcal{M}^a_{1,\text{ét}}$ is not closed under $\sharp$-extensions: as remarked in [BVB09, 3.1.5], $[0 \to \mathbf{G}_a]^\sharp = [\widehat{\mathbf{G}}_a \to \mathbf{G}_a^2]$, which is clearly not étale.

Remark 7.8. The category of Deligne 1-motives has an interesting self-duality, induced by the classical Cartier duality for algebraic groups. This extends to Laumon 1-motives, see [BVB09]. Note that while the Cartier dual of a Deligne 1-motive is again a Deligne 1-motive, the dual of an étale Laumon 1-motive is in general not étale. For example, if A is an Abelian variety (see as 1-motive $[0 \to A]$), its universal $\mathbf{G}_a$ extension is the étale Laumon 1-motive $A^\natural = [0 \to A^\natural]$, which is not a Deligne 1-motive. Its Cartier dual $(A^\natural)^*$ is the 1-motive $[\widehat{A'} \to A']$, where A' is the dual Abelian variety of A and $\widehat{A'}$ is the formal completion of A' along the identity. Clearly, $(A^\natural)^*$ is a Laumon 1-motive that is not étale.

Remark 7.9. Consider the functor

$$\rho^{\text{eff}}\colon \mathcal{M}^{a,\text{eff}}_{1,\text{ét}} \to \mathbf{Shv}_{\text{ét}}(k,\mathbb{Q}) \quad [L \xrightarrow{u} G] \mapsto \operatorname{coker}(u) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

If $[L_1 \xrightarrow{u_1} G_1] \to [L_2 \xrightarrow{u_2} G_2] \in \Sigma$, then by definition $\operatorname{coker}(u_1) \otimes \mathbb{Q} \cong \operatorname{coker}(u_2) \otimes \mathbb{Q}$. This together with [BVK16, Lemma B.1.2] implies that ρ^{eff} induces:

$$\rho\colon \mathcal{M}^a_{1,\text{ét}} \otimes \mathbb{Q} \to \mathbf{Shv}_{\text{ét}}(k,\mathbb{Q}).$$

Lemma 7.10. *The restriction of ρ to $\mathcal{M}^{a,\star}_{1,\text{ét}} \otimes \mathbb{Q}$ induces an equivalence (cf. Def. 4.23):*

$$\rho^\star\colon \mathcal{M}^{a,\star}_{1,\text{ét}} \otimes \mathbb{Q} \xrightarrow{\cong} \mathbf{RSC}^\star_{\text{ét},\leq 1}.$$

Proof. By definition, for $[L \to G] \in \mathcal{M}^{a,\star}_{1,\text{ét}} \otimes \mathbb{Q}$, $\rho([L \to G]) \in \mathbf{RSC}^\star_{\text{ét},\leq 1}$, and by Proposition 2.10 for every morphism f we have that $\rho(f)$ is a map in $\mathbf{RSC}^\star_{\text{ét},\leq 1}$, hence $\rho^\star$ is well defined. The presentation of Proposition 4.24 gives then a quasi-inverse of $\rho^\star$. $\square$

Remark 7.11. For a category $\mathcal{C}$, we write $\operatorname{Ind}(\mathcal{C})$ for the Ind-category of $\mathcal{C}$ as in e.g. [KS06]. By [KS06, Prop. 6.3.4] and Remark 4.26 the functor $\operatorname{Ind}(\mathbf{RSC}^\star_{\text{ét},\leq 1}) \to \mathbf{RSC}_{\text{ét},\leq 1}$ induced by filtered colimits is fully faithful. It is also essentially surjective by Proposition 4.24, hence it is an equivalence. Combining this with Lemma 7.6 and 7.10, we have a functor

$$T\colon \mathcal{M}^a_{1,\text{ét}} \otimes \mathbb{Q} \to \operatorname{Ind}(\mathcal{M}^a_{1,\text{ét}} \otimes \mathbb{Q}) \overset{(*)}{\simeq} \operatorname{Ind}(\mathcal{M}^{a,\star}_{1,\text{ét}} \otimes \mathbb{Q}) \simeq \mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q}) \tag{7.11.1}$$

where $(*)$ follows from the fact that $\mathcal{M}^{a,\star}_{1,\text{ét}} \otimes \mathbb{Q}$ is a generating subcategory of $\mathcal{M}^a_{1,\text{ét}} \otimes \mathbb{Q}$. Since $\mathcal{M}^a_{1,\text{ét}} \otimes \mathbb{Q}$ is abelian by Lemma 7.4, it is idempotent-complete, hence following the steps of [KS06, Exercise 6.1] the functor (7.11.1) is fully faithful and it identifies $\mathcal{M}^a_{1,\text{ét}} \otimes \mathbb{Q}$ with a set of compact generators of $\mathbf{RSC}_{\text{ét},\leq 1}$. Moreover, by [KS06, Proposition 8.6.11], the category $T(\mathcal{M}^a_{1,\text{ét}} \otimes \mathbb{Q})$ is closed under extensions in $\mathbf{RSC}_{\text{ét},\leq 1}$.

7.2. **The derived category of étale Laumon 1-motives.** By [KS06, Proposition 8.6.11] and Remark 7.11, the image of the functor T of (7.11.1) is a Serre subcategory of $\mathbf{RSC}_{\leq 1,\text{ét}}(k,\mathbb{Q})$, hence we can consider the triangulated category $D^b_{\mathcal{M}^a_{1,\text{ét}}}(\mathbf{RSC}_{\leq 1,\text{ét}}(k,\mathbb{Q}))$ of bounded complexes of $\mathbf{RSC}_{\leq 1}(k,\mathbb{Q})$ such that $H_n(C) = T([L_n \to G_n])$ for $[L_n \to G_n] \in \mathcal{M}^a_{1,\text{ét}} \otimes \mathbb{Q}$.

Remark 7.12. The functor T of (7.11.1) induces an equivalence of triangulated categories:

$$D^b(\mathcal{M}^a_{1,\text{ét}} \otimes \mathbb{Q}) \simeq D^b_{\mathcal{M}^a_{1,\text{ét}}}(\mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q}))$$

where the latter is the triangulated derived category , since by Proposition 4.31, every object of $\mathbf{RSC}^\star_{\text{ét},\leq 1}$ is of projective dimension at most 1 in the sense of [Kel99], in particular the image of T satisfies [Kel99, 1.21 Lemma (c2)], hence the equivalence comes from [Kel99, 1.21 Lemma (c)].

Definition 7.13. Let $\mathcal{D}^b(\mathcal{M}^a_{1,\text{ét}}\otimes\mathbb{Q})$ be the full ∞-subcategory of $\mathcal{D}(\mathbf{RSC}_{\text{ét},\leq 1})$ spanned by bounded complexes $C\in\mathcal{D}^b(\mathbf{RSC}_{\text{ét},\leq 1})$ such that $\pi_n C = T([L_n\to G_n])$ for $[L_n\to G_n]\in\mathcal{M}^a_{1,\text{ét}}\otimes\mathbb{Q}$.

Remark 7.14. Notice that since the category $\mathcal{M}^a_{1,\text{ét}}\otimes\mathbb{Q}$ does not have enough injective nor projective objects, we cannot use [Lur17, 1.3] to construct $\mathcal{D}^b(\mathcal{M}^a_{1,\text{ét}}\otimes\mathbb{Q})$ directly.

Lemma 7.15. *There is an equivalence of ∞-categories:*

$$\mathcal{D}(\mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q}))^{\omega}\simeq\mathcal{D}^b(\mathcal{M}^a_{1,\text{ét}}\otimes\mathbb{Q})$$

where the left hand side denotes the subcategory of compact objects as in [Lur09, Notation 5.3.4.6]*.*

Proof. Since the set of objects of $\mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q})$ lying in the image of (7.11.1) is a set of compact generators, by [Sta16, Lemma 094B] we have an equivalence

$$\mathcal{D}(\mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q}))^{\omega}\simeq\mathrm{Idem}(\mathcal{D}^b(\mathcal{M}^a_{1,\text{ét}}\otimes\mathbb{Q}))$$

where the right hand side is the idempotent completion of $\mathcal{D}^b(\mathcal{M}^a_{1,\text{ét}}\otimes\mathbb{Q})$, see [Lur09, Definition 5.1.4.1]. On the other hand, the category $\mathcal{D}^b(\mathcal{M}^a_{1,\text{ét}}\otimes\mathbb{Q})$ is idempotent-complete since the image of (7.11.1) is idempotent complete (it is an abelian subcategory), hence every object of $\mathcal{D}(\mathbf{RSC}_{\text{ét},\leq 1}(k,\mathbb{Q}))^{\omega}$ lies in $\mathcal{D}^b(\mathcal{M}^a_{1,\text{ét}}\otimes\mathbb{Q})$. The other inclusion is clear. □

Theorem 7.16. *Let $\mathbf{log}\mathcal{DM}^{\text{eff}}_{\leq 1,\text{gm}}(k,\mathbb{Q}) := \mathbf{log}\mathcal{DM}^{\text{eff}}_{\leq 1}(k,\mathbb{Q})^{\omega}$. The functor $\omega^{\mathbf{logDM}^{\text{eff}}}_{\leq 1}$ preserves compact objects and it induces an equivalence*

$$\mathcal{D}^b(\mathcal{M}^a_{1,\text{ét}}\otimes\mathbb{Q})\xrightarrow{\sim}\mathbf{log}\mathcal{DM}^{\text{eff}}_{\leq 1,\text{gm}}(k,\mathbb{Q}).$$

Proof. By Lemma 7.15, if $C\in\mathcal{D}(\mathbf{RSC}_{\leq 1})$ is compact, then it is a bounded complex such that $\pi_i(C)=T(M_i)$ for $M_i\in\mathcal{M}^a_{1,\text{ét}}\otimes\mathbb{Q}$. In particular, there exists n such that $C=\tau_{\geq n}C$, so we get a fiber-cofiber sequence in $\mathcal{D}(\mathbf{RSC}_{\text{ét},\leq 1})$:

$$T(M_n)[n]\to C\to\tau_{\geq n-1}C.$$

If C is compact, then $\tau_{\geq n-1}C$ is compact, so by induction on the length of the bounded complex it is enough to show that for $M\in\mathcal{M}^a_{1,\text{ét}}\otimes\mathbb{Q}$, the object $\omega^{\mathbf{logDM}^{\text{eff}}}_{\leq 1}(T(M)[n])$ is compact. As observed in Lemma 7.6, there is an exact sequence in $\mathcal{M}^a_{1,\text{ét}}\otimes\mathbb{Q}$:

$$0\to M'\to M''\to M\to 0$$

with $M',M''\in\mathcal{M}^{a,\star}_{1,\text{ét}}$, and since T is exact we have

$$\omega^{\mathbf{logDM}^{\text{eff}}}_{\leq 1}(T(M)[n])=\mathrm{cofib}(\omega^{\mathbf{logDM}^{\text{eff}}}_{\leq 1}(T(M')[n])\to\omega^{\mathbf{logDM}^{\text{eff}}}_{\leq 1}(T(M'')[n]).$$

Thus, it is enough to show that $\omega^{\mathbf{logDM}^{\text{eff}}}_{\leq 1}(T(M)[n])$ is compact for $M=[L\overset{u}{\hookrightarrow}G]$. In this case, we have that $T(M)=\mathrm{coker}(u)$, so we have a cofiber sequence

$$\omega^{\mathbf{logDM}^{\text{eff}}}_{\leq 1}(L[n])\to\omega^{\mathbf{logDM}^{\text{eff}}}_{\leq 1}(G[n])\to\omega^{\mathbf{logDM}^{\text{eff}}}_{\leq 1}(T(M)[n]).$$

We conclude since $\omega^{\mathbf{logDM}^{\text{eff}}}_{\leq 1}(L)[n]=\omega^*L[n]$ and $\omega^{\mathbf{logDM}^{\text{eff}}}_{\leq 1}(G[n])=\omega^{\mathbf{CI}}_{\log}(G)[n]$ are compact. The equivalence then follows from Theorem 5.15 and Lemma 7.15. □

Appendix A. Pro-left derived functors

In this appendix we generalize to pro-left adjoints the results discussed in [ABV09, 2.1] and [KS06, 14.3] for left adjoints. We use in an essential way the formalism of (stable) ∞-categories of [Lur17] and [Lur09].

A.1. We consider the following commutative square:

$$\text{(A.0.1)}\qquad \begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\ \downarrow{\scriptstyle L_{\mathcal{C}}} & & \downarrow{\scriptstyle L_{\mathcal{D}}} \\ \mathcal{C}' & \xrightarrow{G'} & \mathcal{D}' \end{array}$$

where $\mathcal{C}, \mathcal{C}'$, $\mathcal{D}$ and $\mathcal{D}'$ are ∞-categories, $L_{\mathcal{C}}$ (resp. $L_{\mathcal{D}}$) has a fully faithful right adjoint $i_{\mathcal{C}}$ (resp. $i_{\mathcal{D}}$). Let $\alpha : L_{\mathcal{D}}G \to G'L_{\mathcal{C}}$ be the equivalence that makes the diagram commute. As observed in [Lur17, Definition 4.7.4.13], α induces a natural transformation (the *Beck–Chevalley* map)

$$BC(\alpha) : Gi_{\mathcal{C}} \to i_{\mathcal{D}}G'.$$

Definition A.1. In the situation of A.1, an object X of $\mathcal{D}$ is $BC(\alpha)$-admissible if and only if for any $Y \in \mathcal{C}'$ the Beck-Chevalley map induces an equivalence:

$$\text{(A.1.1)}\qquad \mathrm{Map}_{\mathcal{D}}(X, i_{\mathcal{D}}G'(Y)) \simeq \mathrm{Map}_{\mathcal{D}}(X, Gi_{\mathcal{C}}(Y)).$$

Assume that G and G' have left adjoints F and F', then by adjunction α induces a map

$$\beta\colon F'L_{\mathcal{D}} \to L_{\mathcal{C}}F.$$

A.2. Let $\mathcal{C}$ be an (arbitrary) ∞-category. We consider as in [Hoy18, Section 2] the ∞-category of pro-objects Pro-$\mathcal{C}$ together with the "constant pro-object" embedding $c\colon \mathcal{C} \to$ Pro-$(\mathcal{C})$ such that $\mathrm{Map}_{\text{Pro-}\mathcal{C}}(\_, c(Y))$ commmutes with cofiltered limits. Every element in Pro-$\mathcal{C}$ is corepresented by a diagram $I \to \mathcal{C}$ for I (the nerve of) a small cofiltered poset. We will often denote an object of Pro-$\mathcal{C}$ as "$\varprojlim_{i\in I}$"X_i for a diagram $I \to \mathcal{C}$. By construction we have that

$$\mathrm{Map}_{\text{Pro-}\mathcal{C}}(\text{“}\varprojlim_{i\in I_1}\text{”}X_i, \text{“}\varprojlim_{j\in I_2}\text{”}Y_j) \simeq \varprojlim_{j}\varinjlim_{i}\mathrm{Map}_{\mathcal{C}}(X_i, Y_j)$$

where the limits and colimits are computed in $\mathcal{S}$.

Remark A.2. The functors $L_{\mathcal{C}}$ and $i_{\mathcal{C}}$ extend levelwise to an adjunction (Pro-$L_{\mathcal{C}}$, Pro-$i_{\mathcal{C}}$) on Pro-$\mathcal{C}$ and Pro-$\mathcal{C}'$ with the same properties. The verification is immediate. In particular, if $\mathcal{D}$ and $\mathcal{D}'$ have all limits, G and G' give the following commutative diagram:

$$\begin{array}{ccc} \text{Pro-}\mathcal{C} & \xrightarrow{\text{Pro-}G} & \mathcal{D} \\ \downarrow{\scriptstyle \text{Pro-}L_{\mathcal{C}}} & & \downarrow{\scriptstyle L_{\mathcal{D}}} \\ \text{Pro-}\mathcal{C}' & \xrightarrow{\text{Pro-}G'} & \mathcal{D}' \end{array}$$

which satisfies the hypotheses of Situation A.1 with equivalence

$$\alpha^{pro} : L_{\mathcal{D}}\text{Pro-}G \to \text{Pro-}G'\text{Pro-}L_{\mathcal{C}}.$$

In particular, since $i_{\mathcal{D}}$ commutes with all limits being a right adjoint, it is immediate that $X \in \mathcal{D}$ is $BC(\alpha)$-admissible if and only if it is $BC(\alpha^{pro})$-admissible.

Remark A.3. If $\mathcal{C}$ is an accessible stable ∞-category equipped a t-structure $(\mathcal{C}_{\leq 0}, \mathcal{C}_{\geq 0})$ with heart $\mathcal{C}^{\heartsuit}$, the ∞-category Pro-$\mathcal{C}$ is also stable (see e.g. [KST19, Lemma 2.5]) and it comes equipped with a t-structure such that $(\text{Pro-}\mathcal{C})_{\leq 0}$ (resp. $(\text{Pro-}\mathcal{C})_{\geq 0}$) is the full subcategory of objects which are formal limits of objects in $\mathcal{C}_{\leq 0}$ (resp $\mathcal{C}_{\geq 0}$).

A.3. Let $\mathbf{A}$ be a Grothendieck abelian category. Write $\mathrm{Ch}(\mathbf{A})$ for the model category of chain complexes with the injective model structure. Let W be the class of quasi isomorphisms. We consider the ∞-categories (see [Lur17, 1.3.5]) $\mathrm{Ch}_{\mathrm{dg}}(\mathbf{A}) = N_{\mathrm{dg}}(\mathrm{Ch}(\mathbf{A}))$ and $\mathcal{D}(\mathbf{A}) = N_{\mathrm{dg}}(\mathrm{Ch}(\mathbf{A}))[W^{-1}]$. An exact functor $G\colon \mathbf{A} \to \mathbf{B}$ between Grothendieck abelian categories induces a dg-functor $\mathrm{Ch}(G) : \mathrm{Ch}(\mathbf{A}) \to \mathrm{Ch}(\mathbf{B})$ which preserves W, so by taking the dg-nerve it induces a functor $\mathrm{Ch}_{\mathrm{dg}}(G)\colon \mathrm{Ch}_{\mathrm{dg}}(\mathbf{A}) \to \mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})$ such that $\mathrm{Ch}_{\mathrm{dg}}(G)(C) = \mathrm{Ch}(G)(C)$, and by e.g. [Hin20, Proposition 4.3], it induces $\mathcal{D}(G)\colon \mathcal{D}(\mathbf{A}) \to \mathcal{D}(\mathbf{B})$ on the localizations. Note that both functors are clearly stable (i.e. they commute with shifts). By construction, we have a commutative square of ∞-categories:

$$\begin{tikzcd} \mathrm{Ch}_{\mathrm{dg}}(\mathbf{A}) \arrow[r, "\mathrm{Ch}_{\mathrm{dg}}(G)"] \arrow[d, "L_{\mathbf{A}}"] & \mathrm{Ch}_{\mathrm{dg}}(\mathbf{B}) \arrow[d, "L_{\mathbf{B}}"'] \\ \mathcal{D}(\mathbf{A}) \arrow[r, "\mathcal{D}(G)"] & \mathcal{D}(\mathbf{B}) \end{tikzcd} \tag{A.3.1}$$

where $L_{\mathbf{A}}$ and $L_{\mathbf{B}}$ have fully faithful right adjoints $i_{\mathbf{A}}$ and $i_{\mathbf{B}}$. We will fix α that makes (A.3.1) commute and just say that an object is BC-admissible.

Remark A.4. Note that, since G is exact, we can identify the Beck-Chevalley transformation $\mathrm{Ch}_{\mathrm{dg}}(G)i_{\mathbf{A}} \to i_{\mathbf{B}}\mathcal{D}(G)$ as follows. For any object $I \in \mathcal{D}(\mathbf{A})$ (i.e. a fibrant complex in $\mathrm{Ch}(\mathbf{A})$ for the injective model structure), the object $\mathcal{D}(G)(I)$ is a fibrant replacement of $\mathrm{Ch}_{\mathrm{dg}}(G)(i_{\mathcal{A}}(I))$. The map $\mathrm{Ch}_{\mathrm{dg}}(G)(i_{\mathbf{A}}I) \to i_{\mathbf{B}}\mathcal{D}(G)(I)$ in $\mathrm{Ch}(\mathbf{B})$ is thus given by the functorial fibrant replacement. In particular, if $\mathrm{Ch}(G)$ is a right Quillen functor, the map $\mathrm{Ch}_{\mathrm{dg}}(G)(i_{\mathbf{A}}I) \to i_{\mathbf{B}}\mathcal{D}(G)(I)$ is an equivalence in $\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})$. On the other hand, the functors condidered here are not necessarily right Quillen.

Remark A.5. If the functor G has a left adjoint F, then X is BC-admissible if and only if it is F-admissible in the sense of [ABV09, Definition 2.1.5].

A.4. **Pro-left derived functors.** Fix $\mathbf{A}$, $\mathbf{B}$ and G as above and we assume that G and $i_{\mathbf{B}}$ commute with filtered colimits. Since G is exact, it preserves finite limits, so it has a pro-left adjoint $F\colon \mathbf{B} \to \text{pro-}\mathbf{A}$. The functor $\mathcal{D}(G)$ is then an accessible functor between presentable ∞-categories that preserves finite limits, hence it has a pro-left adjoint $LF\colon \mathcal{D}(\mathbf{B}) \to \text{Pro-}\mathcal{D}(\mathbf{A})$ (see e.g. [Hoy18, Remark 2.2]).

For any chain complex C, let $\sigma_{\leq n}C$ and $\sigma_{\geq n}C$ denote the stupid truncations (see [Sta16, Tag 0118]) [9]. We have an equivalence in $\mathrm{Ch}_{\mathrm{dg}}$:

$$C \cong \varinjlim_{n}(\varprojlim_{m} \sigma_{\geq -m}\sigma_{\leq n}C). \tag{A.5.1}$$

Definition A.6. We say that a chain complex C is *strictly bounded* if there exists m, n such that $C = \sigma_{\leq n}C = \sigma_{\geq -m}C$.

Remark A.7. Notice that if $C \in \mathrm{Ch}(\mathbf{B})$, the object $\mathrm{Ch}(F)(C)$ a priori lives in $\mathrm{Ch}_{\mathrm{dg}}(\text{pro-}\mathbf{A})$, which contains strictly $\text{Pro-}\mathrm{Ch}_{\mathrm{dg}}(\mathbf{A})$. If C is a strictly bounded complex, let m, n such that $C = \sigma_{\leq n}C = \sigma_{\geq -m}C$, then for $r \in [-m, n]$ let $F^{\heartsuit}(C_r) = \text{“}\varprojlim\text{”}_{i \in I_r}(X_r)_i$. Then one can find a cofinal set $I \subseteq I_r$ for all r such that

$$\mathrm{Ch}(F)(C) = \text{“}\varprojlim_{i\in I}\text{”}(\ldots \to (X_r)_i \to (X_{r-1})_i \to \ldots).$$

In particular, $\mathrm{Ch}(F)(C) \in \text{Pro-}(\mathrm{Ch}_{\mathrm{dg}}(\mathbf{A}))$.

Proposition A.8. *For all $C \in \mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})$, there is an equivalence in $\mathcal{S}$:*

$$\mathrm{Map}_{\text{Pro-}\mathrm{Ch}_{\mathrm{dg}}(\mathbf{A})}(\mathrm{Ch}(F)(C)[0], Y) \simeq \mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})}(X[0], \mathrm{Ch}_{\mathrm{dg}}(G)(Y)).$$

Proof. Let $C_n = 0$ for $n \notin [-r, s]$. The cofiber of the map $\sigma_{\leq s-1}C \to C$ is equivalent to $C_s[s]$, hence by induction on $r+s$ we are reduced to the case where $C = C_s[s]$ with $C_s \in \mathcal{B}$, and clearly $\mathrm{Ch}(F)(C_s[s]) = F(C_s)[s]$. Since $\text{Pro-}\mathrm{Ch}_{\mathrm{dg}}(\mathbf{A})$ is pointed, by [Lur17, Remark 1.1.2.8] it is enough to show that for all m,

$$\pi_0 \mathrm{Map}_{\text{Pro-}\mathrm{Ch}_{\mathrm{dg}}(\mathbf{A})}(F(X)[m], Y) \simeq \pi_0 \mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})}(X[m], \mathrm{Ch}_{\mathrm{dg}}(G)(Y)).$$

Let $F(X) = \text{“}\varprojlim\text{”}T_i$, then for all m we have an isomorphism of abelian groups:

$$\mathrm{Hom}_{\mathrm{Ch}(\mathbf{B})}(X[m], \mathrm{Ch}(G)(Y)) = \varinjlim \mathrm{Hom}_{\mathrm{Ch}(\mathbf{A})}(T_i[m], Y),$$

[9]Notice that we chose the convention for *chain* complexes, which is different from [ABV09, Lemma 2.1.10]: there the convention is for *cochain* complexes

and since π_0 commutes with filtered colimits in $\mathcal{S}$ we have by [Lur17, Remark 1.3.1.5, Remark 1.3.1.11 and Definition 1.3.2.1]:

$$\begin{aligned}\pi_0 \operatorname{Map}_{\text{Pro-}\mathrm{Ch}_{\mathrm{dg}}(\mathbf{A})}(F(X)[n], Y) &\cong \varinjlim \pi_0 \operatorname{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{A})}(T_i[n], Y)\\ &\cong \operatorname{coker}(\varinjlim \operatorname{Hom}_{\mathrm{Ch}(\mathbf{A})}(T_i[n+1], Y) \to \varinjlim \operatorname{Hom}_{\mathrm{Ch}(\mathbf{A})}(T_i[n], Y)).\\ &\cong \operatorname{coker}(\operatorname{Hom}_{\mathrm{Ch}(\mathbf{B})}(X[n+1], \mathrm{Ch}(G)(Y)) \to \operatorname{Hom}_{\mathrm{Ch}(\mathbf{A})}(X[n], \mathrm{Ch}(G)(Y)))\\ &\cong \pi_0 \operatorname{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})}(X[n], \mathrm{Ch}_{\mathrm{dg}}(G)(Y)).\end{aligned}$$

□

Remark A.9. A priori, there is no relation between F and LF, but if X is strictly bounded and BC-admissible, then Proposition A.8 implies that

$$LF(L_{\mathbf{B}}(X)) \simeq L_{\text{Pro-}\mathbf{A}} \mathrm{Ch}(F)(X)[0].$$

In particular for $X \in \mathbf{B}$ such that $X[0]$ is BC-admissible,

$$\pi_n(LF(L_{\mathbf{B}}(X[0]))) = \begin{cases} F(X) & \text{if } n = 0 \\ 0 & \text{otherwise.}\end{cases}$$

Remark A.9 motivates the following definition:

Definition A.10. In the situation of A.4, LF is said to be a *pro-left derived functor* of F if for every $X \in \mathbf{B}$

$$\pi_0 LF(X[0]) \cong F(X).$$

A.5. ***BC*-admissible resolution.** We will fix the setting of A.4. By abuse of notation, we will say that $P \in \mathbf{B}$ is BC-admissible if $P[0] \in \mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})$ is.

Proposition A.11. *Let $P_\bullet \in \mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})$ be a strictly bounded complex (see Definition A.6) such that P_n is BC-admissible for all n. Then $P_\bullet$ is BC-admissible.*

Proof. Up to shift, we can suppose that $P_\bullet = \sigma_{\geq 0} P_\bullet = \sigma_{\leq n} P_\bullet$ for some $n \geq 0$: we proceed by induction on n. If $n = 0$, then $P_\bullet = P_0[0]$, and it is BC-admissible by assumption. Let $n > 0$ and consider the fiber-cofiber sequence in $\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})$:

$$\sigma_{\leq n-1} P_\bullet \to P_\bullet \to P_n[n].$$

For all $I \in \mathcal{D}(\mathbf{A})$, we conclude by the following diagram in $\mathcal{S}$ where the left and right vertical maps are equivalences by induction:

$$\begin{tikzcd} \operatorname{Map}(P_n[0], \mathrm{Ch}_{\mathrm{dg}}(G)(i_{\mathbf{A}} I[-n])) \arrow[r] \arrow[d, "\simeq"] & \operatorname{Map}(P_\bullet, \mathrm{Ch}_{\mathrm{dg}}(G)(i_{\mathbf{A}} I)) \arrow[r] \arrow[d] & \operatorname{Map}(\sigma_{\leq n-1} P_\bullet, \mathrm{Ch}_{\mathrm{dg}}(G)(i_{\mathbf{A}} I)) \arrow[d, "\simeq"] \\ \operatorname{Map}(P_n[0], i_{\mathbf{B}} \mathcal{D}(G)(I[-n])) \arrow[r] & \operatorname{Map}(P_\bullet, i_{\mathbf{B}} \mathcal{D}(G)(I)) \arrow[r] & \operatorname{Map}(\sigma_{\leq n-1} P_\bullet, i_{\mathbf{B}} \mathcal{D}(G)(I)). \end{tikzcd}$$

□

Recall that $\mathbf{B}$ is said to be generated by a set of objects E if and only if E is closed under direct sums and for every $X \in \mathbf{B}$ there exists a surjective map

(A.11.1) $$P_0 \to X \to 0$$

with $P_0 \in E$. Suppose that that $\mathbf{B}$ is generated by a set of objects E which are BC-admissible. Then let K be the kernel of (A.11.1), so there exists $P_1 \in E$ together with a surjective map $P_1 \twoheadrightarrow K$, hence we have an exact sequence:

(A.11.2) $$P_1 \to P_0 \to X \to 0.$$

By iterating (A.11.2) one can construct a resolution $P_\bullet \to X[0]$ where $P_n \in E$ and $P_n = 0$ for $n < 0$. We will call this a *connective BC-admissible resolution*.

Lemma A.12. *Suppose that $\mathbf{B}$ is generated by a set of objects which are BC-admissible. For any $X \in \mathbf{B}$ and any connective BC-admissible resolution $P_\bullet \to X[0]$, we have that*

$$LF(L_{\mathbf{B}}(X[0]) \simeq \varinjlim_n L_{\text{Pro-}\mathbf{A}}(\mathrm{Ch}(F)(\sigma_{\leq n}P_\bullet)).$$

Proof. Since $P_\bullet \to X[0]$ is a resolution, we have $L_{\mathbf{B}}(P_\bullet) \simeq L_{\mathbf{B}}(X[0])$. Moreover, since $P_\bullet$ is connective, we have that $P_\bullet = \varinjlim \sigma_{\leq n}P_\bullet$, and $\sigma_{\leq n}P_\bullet$ are BC-admissible by Proposition A.11. Since LF and $L_{\mathbf{B}}$ commute with all colimits, by Remark A.9 we conclude that

$$LF(L_{\mathbf{B}}X[0]) \simeq \varinjlim LF(L_{\mathbf{B}}\sigma_{\leq n}P_\bullet) \simeq \varinjlim_n L_{\text{Pro-}\mathbf{A}}(\mathrm{Ch}(F)(\sigma_{\leq n}P_\bullet)),$$

$\square$

We can now prove the main theorem of this appendix:

Theorem A.13. *In the situation of A.4, suppose that $\mathbf{B}$ is generated by a set objects which are BC-admissible. Then the functor LF is a pro-left derived functor of F.*

Proof. Let $X \in \mathbf{B}$ and $P_\bullet \to X[0]$ a BC-admissible resolution, in particular $P_\bullet \in \mathcal{D}(\mathbf{B})_{\geq 0}$. Since $L_{\mathbf{B}}X[0] \in \mathcal{D}(\mathbf{B})_{\geq 0}$ and LF is right t-exact, $LF(L_{\mathbf{B}}X[0]) \in \text{Pro-}\mathcal{D}(\mathbf{A})_{\geq 0}$, hence

$$\pi_0 LF(L_{\mathbf{B}}X[0])[0] \simeq \tau_{\leq 0}LF(L_{\mathbf{B}}X[0]).$$

Since $\tau_{\leq 0}$ is a left adjoint, it commutes with colimits, so by Lemma A.12 we have

$$\tau_{\leq 0}LF(X[0]) \simeq \tau_{\leq 0}\varinjlim_n L_{\text{Pro-}\mathbf{A}}\,\mathrm{Ch}(F)(\sigma_{\leq n}P_\bullet) \simeq \varinjlim_n \tau_{\leq 0}L_{\text{Pro-}\mathbf{A}}\,\mathrm{Ch}(F)(\sigma_{\leq n}P_\bullet)$$

where the last colimit is computed in $\text{Pro-}\mathcal{D}(\mathbf{A})_{\leq 0}$. On the other hand, by definition of the t-structure on $\text{Pro-}\mathcal{D}(\mathbf{A})$ we have

$$\tau_{\leq 0}L_{\text{Pro-}\mathbf{A}}\,\mathrm{Ch}(F)(\sigma_{\leq n}P_\bullet) \simeq L_{\text{Pro-}\mathbf{A}}\tau_{\leq 0}\,\mathrm{Ch}(F)(\sigma_{\leq n}P_\bullet).$$

For $n \geq 1$, we have that $\tau_{\leq 0}\,\mathrm{Ch}(F)(\sigma_{\leq n}P_\bullet) = \mathrm{coker}(F(P_1) \to F(P_0))[0]$, and since F is a left adjoint, it preserves cokernels, so

$$\mathrm{coker}(F(P_1) \to F(P_0)) = F(\mathrm{coker}(P_1 \to P_0)) = F(X).$$

We conclude that in $\text{Pro-}\mathcal{D}(\mathbf{A})_{\leq 0}$ we have

$$\varinjlim_n \tau_{\leq 0}L_{\text{Pro-}\mathbf{A}}\,\mathrm{Ch}(F)(\sigma_{\leq n}P_\bullet) \simeq \varinjlim_n L_{\text{Pro-}\mathbf{A}}F(X)[0] \simeq L_{\text{Pro-}\mathbf{A}}F(X)[0]$$

since $L_{\text{Pro-}\mathbf{A}}F(X)[0] \in \text{Pro-}\mathcal{D}(\mathbf{A})^\heartsuit$ we conclude that

$$\pi_0 LF(X[0]) \simeq \pi_0(\varinjlim_n \tau_{\leq 0}L_{\text{Pro-}\mathbf{A}}F(\sigma_{\leq n}P_\bullet)) \simeq F(X).$$

$\square$

A.6. We end this appendix with a criterion of BC-admissibility.

Lemma A.14 (See [ABV09, Lemma 2.1.10])**.** *In the situation of A.4, let $P \in \mathbf{B}$ such that $P[0]$ is compact in $\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})$. Then P is BC-admissible if and only if for any injective object $I_0 \in \mathbf{A}$, $\mathrm{Ext}^i_{\mathbf{B}}(P, G(I_0)) = 0$ for $i \neq 0$.*

Proof. Suppose first that P is BC-admissible. If $I_0 \in \mathbf{A}$ is injective, then $I_0[0]$ is fibrant, so $I_0[0] = i_{\mathbf{A}}L_{\mathbf{A}}(I_0[0])$. Let $I = L_{\mathcal{B}}(I_0[0])$, then:

$$\mathrm{Ext}^n_{\mathbf{B}}(P, G(I_0)) = \pi_0\,\mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})}(P[0], i_{\mathcal{B}}\mathcal{D}(G)(I[-n])).$$

Since $P[0]$ is BC-admissible, we have:

$$\begin{aligned}\pi_0\,\mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})}(P[0], i_{\mathbf{B}}\mathcal{D}(G)(I[-n])) &= \pi_0\,\mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})}(P[0], \mathrm{Ch}_{\mathrm{dg}}(G)(i_{\mathbf{A}}I[-n]))) \\ &= \pi_0\,\mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})}(P[0], G(I_0)[-n])).\end{aligned}$$

The last term is zero for $n \neq 0$, hence $\mathrm{Ext}^n_{\mathcal{B}}(P, G(I_0)) = 0$ if $n \neq 0$.

Let us now show the converse implication. We need to show that for every $Y \in \mathcal{D}(\mathbf{A})$ the Beck-Chevalley map induces an equivalence:

$$\mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathcal{B})}(P[0], i_{\mathbf{B}}\mathcal{D}(G)(Y)) \simeq \mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathcal{B})}(P[0], \mathrm{Ch}_{\mathrm{dg}}(G)(i_{\mathbf{A}}Y)).$$

Let $I := i_{\mathbf{A}}(Y)$, so $\sigma_{\leq -n}\sigma_{\geq m}I$ is a strictly bounded complex of injectives; since $P[0]$ is compact and the colimit in (A.5.1) is filtered, Lemma A.15 below implies that:

(A.14.1)

$$\begin{aligned}\mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})}(P[0], \mathrm{Ch}_{\mathrm{dg}}(G)(I)) &\simeq \varinjlim_n(\varprojlim_m \mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})}(P[0], \sigma_{\leq -n}\sigma_{\geq m}\,\mathrm{Ch}_{\mathrm{dg}}(G)(I)))\\ &\simeq \varinjlim_n(\varprojlim_m \mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})}(P[0], \mathrm{Ch}_{\mathrm{dg}}(G)(\sigma_{\leq -n}\sigma_{\geq m}I)))\\ &\simeq \varinjlim_n(\varprojlim_m \mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})}(P[0], i_{\mathbf{B}}\mathcal{D}(G)(L_{\mathbf{A}}\sigma_{\leq -n}\sigma_{\geq m}I))\\ &\simeq \mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})}(P[0], \varinjlim_n \varprojlim_m i_{\mathbf{B}}\mathcal{D}(G)(L_{\mathbf{A}}\sigma_{\leq -n}\sigma_{\geq m}I)).\end{aligned}$$

Next, observe that, for any n and m, the map $\mathrm{Ch}_{\mathrm{dg}}(G)(\sigma_{\leq -n}\sigma_{\geq m}I) \to i_{\mathbf{B}}\mathcal{D}(G)(L_{\mathbf{A}}\sigma_{\leq -n}\sigma_{\geq m}I)$ is a fibrant replacement (see Remark A.4) of bounded complexes, which is given by the total complex of injective resolutions of each $G(I_r) \to J_r^\bullet$. So, we have that

(A.14.2)

$$\varprojlim_m i_{\mathbf{B}}\mathcal{D}(G)(L_{\mathbf{A}}\sigma_{\leq -n}\sigma_{\geq m}I) \simeq \varprojlim_m \mathrm{Tot}_{r\in[-n,m]}(J_r^\bullet) = \mathrm{Tot}_{r\geq -n}(J_r^\bullet) \simeq i_{\mathbf{B}}\mathcal{D}(G)(L_{\mathbf{A}}\sigma_{\leq -n}I).$$

Since $i_{\mathbf{B}}$ commutes with filtered colimits by assumption, (A.14.1) and (A.14.2) imply:

$$\mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})}(P[0], \mathrm{Ch}_{\mathrm{dg}}(G)(i_{\mathbf{A}}Y)) \simeq \mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})}(P[0], i_{\mathbf{B}}\varinjlim_n \mathcal{D}(G)(L_{\mathbf{A}}\sigma_{\leq -n}I)). \tag{A.14.3}$$

For every $q \in \mathbb{Z}$, we have that for $n \gg q$:

$$\pi_q\mathcal{D}(G)(Y) \cong G(\pi_q Y) \cong G(\pi_q\sigma_{\leq -n}I) \cong \pi_q\mathcal{D}(G)(L_{\mathbf{A}}\sigma_{\leq -n}I),$$

so since homotopy groups commute with filtered colimits:

$$\pi_q \varinjlim_n \mathcal{D}(G)(L_{\mathbf{A}}\sigma_{\leq -n}I) \cong \varinjlim_n \pi_q\mathcal{D}(G)(L_{\mathbf{A}}\sigma_{\leq -n}I) \cong \pi_q\mathcal{D}(G)(Y) \tag{A.14.4}$$

The proof follows then from (A.14.3) and (A.14.4). □

Lemma A.15. *Let $P \in \mathbf{B}$ such that for any injective object $I_0 \in \mathcal{A}$, $\mathrm{Ext}^i_{\mathcal{B}}(P, G(I_0)) = 0$ for $i \neq 0$. Then for any strictly bounded complex $I^b \in \mathrm{Ch}(\mathbf{A})$ of injective objects of $\mathbf{A}$:*

$$\mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})}(P[0], \mathrm{Ch}_{\mathrm{dg}}(G)(I^b)) \simeq \mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(\mathbf{B})}(P[0], i_{\mathbf{B}}\mathcal{D}(G)(L_{\mathbf{A}}I^b)).$$

Proof. Let $I^b_n = 0$ for $n \notin [-r, s]$. The cofiber of $\sigma_{\leq s-1}I^b \to I^b$ is equivalent to $I^b_s[s]$: by induction on $r+s$ we reduce to $I^b = I_s[s]$ with I_s an injective object of $\mathcal{B}$. We conclude:

$$\pi_n \mathrm{Map}_{\mathrm{Ch}_{\mathrm{dg}}(B)}(P[0], i_{\mathbf{B}}\mathcal{D}(G)(L_{\mathbf{A}}I_s[s])) \cong \mathrm{Ext}^{s-n}_{\mathcal{B}}(P, G(I_s)) = \begin{cases} 0 & \text{if } n \neq s \\ \mathrm{Hom}_{\mathcal{B}}(P, G(I_s)) & \text{if } n = s.\end{cases}$$

□

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