LEFSCHETZ THEOREM FOR ABELIAN FUNDAMENTAL GROUP WITH MODULUS

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Abstract. We prove a Lefschetz hypersurface theorem for abelian fundamental groups allowing wild ramification along some divisor. In fact, we show that isomorphism holds if the degree of the hypersurface is large relative to the ramification along the divisor.

1. Statement of main results

Let X be a normal variety over a perfect field k and $U \subset X$ be an open subset such that $X \setminus U$ is the support of an effective Cartier divisor on X. Let D be an effective Cartier on X with support in $X \setminus U$. We introduce the abelian fundamental group $\pi_1^{ab}(X, D)$ as a quotient of $\pi_1^{ab}(U)$ classifying abelian étale coverings of U with ramification bounded by D. More precisely, for an integral curve $Z \subset U$, let Z^N be the normalization of the closure of Z in X with $\psi_Z : Z^N \to X$, the natural map. Let $Z_\infty \subset Z^N$ be the finite set of points x such that $\phi_Z(x) \notin U$. Then $\pi_1^{ab}(X, D)$ is defined as the Pontryagin dual of the group $\mathrm{fil}_D H^1(U)$ of continuous characters $\chi : \pi_1^{ab}(U) \to \mathbb{Q}/\mathbb{Z}$ such that for any integral curve $Z \subset U$, its restriction $\chi|_Z : \pi_1^{ab}(Z) \to \mathbb{Q}/\mathbb{Z}$ satisfies the following equality of Cartier divisors on Z^N:

$$\sum_{y \in Z_\infty} \mathrm{art}_y(\chi|_Z)[y] \ \leq\ \phi_Z^* D,$$

where $\mathrm{art}_y(\chi|_Z) \in \mathbb{Z}_{\geq 0}$ is the Artin conductor of $\chi|_Z$ at $y \in Z_\infty$ and $\phi_Z^* D$ is the pullback of D by the natural map $\psi_Z : Z^N \to X$.

Such a global measure of ramification in terms of curves has first considered by Deligne and Laumon, see [La].

Now assume that X is smooth projective over k (we fix a projective embedding) and that $C = X \setminus U$ is a simple normal crossing divisor. Let Y be a smooth hypersurface section such that $Y \times_X C$ is a reduced simple normal crossing divisor on Y and write $\deg(Y)$ for the degree of Y with respect to the fixed projective embedding of X. Set $E = Y \times_X D$. Then one sees from the definition that the map $Y \cap U \to U$ induces a natural map

$$\iota_{Y/X} : \pi_1^{ab}(Y, E) \to \pi_1^{ab}(X, D).$$

Our main theorem says:

Theorem 1.1. *Assume that Y is* sufficiently ample *with respect (X, D) (see Definition 3.1). If $d := \dim(X) \geq 3$, $\iota_{Y/X}$ is an isomorphism. If $d = 2$, $\iota_{Y/X}$ is surjective.*

Below we see that Y is sufficiently ample if $\deg(Y) \gg 0$.

Corollary 1.2. *Let X be a normal proper variety over a finite field k. Then $\pi_1^{ab}(X, D)^0$ is finite, where*

$$\pi_1^{ab}(X, D)^0 = \mathrm{Ker}\big(\pi_1^{ab}(X, D) \to \pi_1^{ab}(\mathrm{Spec}(k))\big).$$

Proof. In case X and $X \setminus U$ satisfy the assumption of Theorem 1.1, the corollary follows from the corresponding statement for curves. The finiteness in the curves case is a consequence of class field theory. For the general case, one can take by [dJ] an alteration $f : X' \to X$ such that X' and $X' \setminus U'$ with $U' = f^{-1}(U)$ satisfies the assumption of Theorem 1.1. Then the assertion follows from the fact that the map $f_* : \pi_1^{ab}(U') \to \pi_1^{ab}(U)$ has a finite cokernel. □

Corollary 1.2 can also be deduced from [Ra, Thm. 6.2]. It has recently been generalized to the non-commutative setting by Deligne, see [EK].

Theorem 1.1 is a central ingredient in our paper [KeS].

2. Review of ramification theory

First we review local ramification theory. Let K denotes a henselian discrete valuation field of $\mathrm{ch}(K) = p > 0$ with the ring $\mathcal{O}_K$ of integers and residue field κ. Let π be a prime element of $\mathcal{O}_K$ and $\mathfrak{m}_K = (\pi) \subset \mathcal{O}_K$ be the maximal ideal. By the Artin-Schreier-Witt theory, we have a natural isomorphism for $s \in \mathbb{Z}_{\geq 1}$,

$$\delta_s : W_s(K)/(1-F)W_s(K) \xrightarrow{\cong} H^1(K, \mathbb{Z}/p^s\mathbb{Z}), \tag{2.1}$$

where $W_s(K)$ is the ring of Witt vectors of length s and F is the Frobenius. We have the Brylinski-Kato filtration indexed by integers $m \geq 0$

$$\mathrm{fil}_m^{\log} W_s(K) = \{(a_{s-1}, \dots, a_1, a_0) \in W_s(K) \mid p^i v_K(a_i) \geq -m\},$$

where v_K is the normalized valuation of K. In this paper we use its non-log version introduced by Matsuda [Ma]:

$$\mathrm{fil}_m W_s(K) = \mathrm{fil}_{m-1}^{\log} W_s(K) + V^{s-s'} \mathrm{fil}_m^{\log} W_{s'}(K),$$

where $s' = \min\{s, \mathrm{ord}_p(m+1)\}$. We define ramification filtrations on $H^1(K) := H^1(K, \mathbb{Q}/\mathbb{Z})$ as

$$\mathrm{fil}_m^{\log} H^1(K) = H^1(K)\{p'\} \oplus \bigcup_{s \geq 1} \delta_s(\mathrm{fil}_m^{\log} W_s(K)) \qquad (m \geq 0),$$
$$\mathrm{fil}_m H^1(K) = H^1(K)\{p'\} \oplus \bigcup_{s \geq 1} \delta_s(\mathrm{fil}_m W_s(K)) \qquad (m \geq 1),$$

where $H^1(K)\{p'\}$ is the prime-to-p part of $H^1(K)$. We note that this filtration is shifted by one from Matsuda's filtration [Ma, Def.3.1.1]. We also let $\mathrm{fil}_0 H^1(K)$ be the subgroup of all unramified characters.

Definition 2.1. For $\chi \in H^1(K)$ we denote the minimal m with $\chi \in \mathrm{fil}_m H^1(K)$ by $\mathrm{art}_K(\chi)$ and call it the Artin conductor of χ.

We have the following fact (cf. [Ka] and [Ma]).

Lemma 2.2.

1. $\mathrm{fil}_1 H^1(K)$ *is the subgroup of tamely ramified characters.*
2. $\mathrm{fil}_m H^1(K_\lambda) \subset \mathrm{fil}_m^{\log} H^1(K) \subset \mathrm{fil}_{m+1} H^1(K)$.
3. $\mathrm{fil}_m H^1(K) = \mathrm{fil}_{m-1}^{\log} H^1(K)$ *if* $(m, p) = 1$.

The structure of graded quotients:

$$\mathrm{gr}_m H^1(K) = \mathrm{fil}_m H^1(K)/\mathrm{fil}_{m-1} H^1(K) \quad (m > 1)$$

are described as follows. Let Ω_K^1 be the absolute Kähler differential module and put

$$\mathrm{fil}_m \Omega_K^1 = \mathfrak{m}_K^{-m} \otimes_{\mathcal{O}_K} \Omega_{\mathcal{O}_K}^1.$$

We have an isomorphism

$$\mathrm{gr}_m \Omega_K^1 = \mathrm{fil}_m \Omega_K^1/\mathrm{fil}_{m-1} \Omega_K^1 \simeq \mathfrak{m}_K^{-m} \Omega_{\mathcal{O}_K}^1 \otimes_{\mathcal{O}_K} \kappa. \tag{2.2}$$

We have the maps

$$F^s d : W_s(K) \to \Omega^1_K \; ; \; (a_{s-1}, \dots, a_1, a_0) \to \sum_{i=0}^{s-1} a_i^{p^i - 1} da_i.$$

and one can check $F^s d(\mathrm{fil}_n W_s(K)) \subset \mathrm{fil}_n \Omega^1_K$.

Theorem 2.3. *([Ma]) The maps $F^s d$ factor through δ_s and induce a natural map*

$$\mathrm{fil}_n H^1(K) \to \mathrm{fil}_n \Omega^1_K$$

which induces for $m > 1$ an injective map (called the refined Artin conductor for K)

$$\mathrm{art}_K : \mathrm{gr}_n H^1(K) \hookrightarrow \mathrm{gr}_n \Omega^1_K. \tag{2.3}$$

Next we review global ramification theory. Let X, C be as in the introduction and fix a Cartier divisor D with $|D| \subset C$. We recall the definition of $\pi_1^{ab}(X, D)$. We write $H^1(U)$ for the étale cohomology group $H^1(U, \mathbb{Q}/\mathbb{Z})$ which is identified with the group of continuous characters $\pi_1^{ab}(U) \to \mathbb{Q}/\mathbb{Z}$.

Definition 2.4. We define $\mathrm{fil}_D H^1(U)$ to be the subgroup of $\chi \in H^1(U)$ satisfying the condition: for all integral curves $Z \subset X$ not contained in C, its restriction $\chi|_Z : \pi_1^{ab}(Z) \to \mathbb{Q}/\mathbb{Z}$ satisfies the following equality of Cartier divisors on Z^N:

$$\sum_{y \in Z_\infty} \mathrm{art}_y(\chi|_Z)[y] \;\le\; \phi_Z^* D,$$

where $\mathrm{art}_y(\chi|_Z) \in \mathbb{Z}_{\ge 0}$ is the Artin conductor of $\chi|_Z$ at $y \in Z_\infty$ and $\phi_Z^* D$ is the pullback of D by the natural map $\phi_Z : Z^N \to X$. Define

$$\pi_1^{ab}(X, D) = \mathrm{Hom}(\mathrm{fil}_D H^1(U), \mathbb{Q}/\mathbb{Z}), \tag{2.4}$$

endowed with the usual pro-finite topology of the dual.

For the rest of this section we assume that X is smooth and C is simple normal crossing. Let I be the set of generic points of C and let $C_\lambda = \overline{\{\lambda\}}$ for $\lambda \in I$. Write

$$D = \sum_{\lambda \in I} m_\lambda C_\lambda. \tag{2.5}$$

For $\lambda \in I$ let K_λ be the henselization of $K = k(X)$ at λ. Note that K_λ is a henselian discrete valuation field with residue field $k(C_\lambda)$.

Proposition 2.5. *We have*

$$\mathrm{fil}_D H^1(U) = \mathrm{Ker}\big(H^1(U) \to \bigoplus_{\lambda \in I} H^1(K_\lambda)/\mathrm{fil}_{m_\lambda} H^1(K_\lambda)\big).$$

Proof. This is a consequence of ramification theory developed in [Ka] and [Ma]. See [KeS, Cor.2.7] for a proof. □

Proposition 2.6. *Fix $\lambda \in I$ such that $m_\lambda > 1$ in (2.5). The refined Artin conductor art_{K_λ} (cf. Theorem 2.3) induces a natural injective map*

$$\mathrm{art}_{C_\lambda} : \mathrm{fil}_D H^1(U)/\mathrm{fil}_{D - C_\lambda} H^1(U) \hookrightarrow H^0(C_\lambda, \Omega^1_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C_\lambda})$$

which is compatible with pullback along maps $f : X' \to X$ of smooth varieties with the property that $f^{-1}(C)$ is a reduced simple normal crossing divisor.

Proof. This follows from the integrality result [Ma, 4.2.2] of the refined Artin conductor. □

Proposition 2.6 motivates us to introduce the following log-variant of $\mathrm{fil}_D H^1(U)$.

Definition 2.7. We define $\mathrm{fil}^{\log}_D H^1(U)$ as

$$\mathrm{fil}^{\log}_D H^1(U) = \mathrm{Ker}\big(H^1(U) \to \bigoplus_{\lambda\in I} H^1(K_\lambda)/\mathrm{fil}^{\log}_{m_\lambda} H^1(K_\lambda)\big).$$

Lemma 2.8.

(1) $\mathrm{fil}_C H^1(U)$ *is the subgroup of tamely ramified characters.*
(2) $\mathrm{fil}_D H^1(U) \subset \mathrm{fil}^{\log}_D H^1(U) \subset \mathrm{fil}_{D+C} H^1(U)$.
(3) $\mathrm{fil}_D H^1(U) = \mathrm{fil}^{\log}_{D-C} H^1(U)$ *if* $(m_\lambda, p) = 1$ *for all* $\lambda \in I$.

Proof. This is a direct consequence of Lemma 2.2. □

3. Proof of main theorem

Let X be a smooth projective variety over a perfect field of characteristic $p > 0$ and $C \subset X$ be a reduced simple normal crossing divisor on X. Let $j : U = X\backslash C \subset X$ be the open immersion. Take an effective Cartier divisor

$$D = \sum_{\lambda\in I} m_\lambda C_\lambda \quad \text{with } m_\lambda \geq 0.$$

Let $I' = \{\lambda \in I \mid p|m_\lambda\}$ and put

$$D' = \sum_{\lambda\in I'} (m_\lambda + 1)C_\lambda \; + \sum_{\lambda\in I\backslash I'} m_\lambda C_\lambda.$$

Let Y be a smooth hypersurface section such that $Y \times_X C$ is a reduced simple normal crossing divisor on Y.

Definition 3.1.

(1) Assuming $\dim(X) \geq 3$, we say that Y is sufficiently ample for (X, D) if the following conditions hold:
  $(A1)$ $H^i(X, \Omega^d_X(-\Xi + Y)) = 0$ for any effective Cartier divisor $\Xi \leq D$ and for $i = d, d-1, d-2$.
  $(A2)$ For any $\lambda \in I'$, we have

$$H^0(C_\lambda, \Omega^1_X(D' - Y) \otimes \mathcal{O}_{C_\lambda}) = H^0(C_\lambda, \mathcal{O}_{C_\lambda}(D' - Y)) = H^1(C_\lambda, \mathcal{O}_{C_\lambda}(D' - 2Y)) = 0.$$

(2) Assuming $\dim(X) = 2$, we say that Y is sufficiently ample for (X, D) if the following condition holds:
  (B) $H^i(X, \Omega^d_X(-\Xi + Y)) = 0$ for any effective Cartier divisor $\Xi \leq D$ and for $i = 1, 2$.

We remark that there is an integer N such that any smooth Y of degree$\geq N$ is sufficiently ample for (X, D).

Theorem 1.1 is a direct consequence of the following.

Theorem 3.2. *Let Y be sufficiently ample for (X, D). Write $E = Y \times_X D$.*

(1) *Assuming $d := \dim(X) \geq 3$, we have isomorphisms*

$$\mathrm{fil}_D H^1(U) \xrightarrow{\cong} \mathrm{fil}_E H^1(U \cap Y) \quad \textit{and} \quad \mathrm{fil}^{\log}_D H^1(U) \xrightarrow{\cong} \mathrm{fil}^{\log}_E H^1(U \cap Y).$$

(2) *Assuming $d = 2$, we have injections*

$$\mathrm{fil}_D H^1(U) \hookrightarrow \mathrm{fil}_E H^1(U \cap Y) \quad \textit{and} \quad \mathrm{fil}^{\log}_D H^1(U) \hookrightarrow \mathrm{fil}^{\log}_E H^1(U \cap Y).$$

For an abelian group M, we let $M\{p'\}$ denote the prime-to-p torsion part of M.

Lemma 3.3. (1) *Assuming* $d := \dim(X) \geq 3$, *we have an isomorphism*

$$\mathrm{fil}_D H^1(U)\{p'\} \xrightarrow{\cong} \mathrm{fil}_E H^1(U \cap Y)\{p'\}$$

and the same isomorphism for $\mathrm{fil}_D^{\log}$.

(2) *Assuming* $d = 2$, *we have an injection*

$$\mathrm{fil}_D H^1(U)\{p'\} \hookrightarrow \mathrm{fil}_E H^1(U \cap Y)\{p'\}$$

and the same injection for $\mathrm{fil}_D^{\log}$.

Proof. Noting

$$\mathrm{fil}_D H^1(U)\{p'\} = \mathrm{fil}_C H^1(U)\{p'\} = \mathrm{fil}_C^{\log} H^1(U)\{p'\} = \mathrm{fil}_D^{\log} H^1(U)\{p'\},$$

this follows from the tame case of Theorem 1.1 due to [SS]. □

By the above lemma, Theorem 3.2 is reduced to the following.

Theorem 3.4. *Let the assumption be as in Theorem 3.2. Take an integer* $n > 0$.

(1) *Assuming* $d := \dim(X) \geq 3$, *we have isomorphisms*

$$\mathrm{fil}_D H^1(U)[p^n] \xrightarrow{\cong} \mathrm{fil}_E H^1(U \cap Y)[p^n]$$

and the same isomorphism for $\mathrm{fil}_D^{\log}$.

(2) *Assuming* $d = 2$, *we have an injection*

$$\mathrm{fil}_D H^1(U)[p^n] \hookrightarrow \mathrm{fil}_E H^1(U \cap Y)[p^n]$$

and the same injection for $\mathrm{fil}_D^{\log}$.

In what follows we consider an effective Cartier divisor with $\mathbb{Z}[1/p]$-coefficient:

$$D = \sum_{\lambda \in I} m_\lambda C_\lambda, \quad m_\lambda \in \mathbb{Z}[1/p]_{\geq 0}.$$

We put

$$[D] = \sum_{\lambda \in I} [m_\lambda] C_\lambda \quad \text{with } [m_\lambda] = \max\{i \in \mathbb{Z} \mid i \leq m_\lambda\}.$$

and $\mathcal{F}(\pm D) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\pm[D])$ for an $\mathcal{O}_X$-module. For D as above, let $\mathrm{fil}_D^{\log} W_n\mathcal{O}_X$ be the subsheaf of $j_* W_n \mathcal{O}_U$ of local sections

$$\underline{a} \in W_n\mathcal{O}_U \quad \text{such that } \underline{a} \in \mathrm{fil}_{m_\lambda}^{\log} W_n(K_\lambda) \quad \text{for any } \lambda \in I,$$

where $\mathrm{fil}_{m_\lambda}^{\log} W_n(K_\lambda) := \mathrm{fil}_{[m_\lambda]}^{\log} W_n(K_\lambda)$ is defined in §2 for the henselization K_λ of $K = k(X)$ at λ. We note

$$\mathcal{O}_X(D) = \mathrm{fil}_D^{\log} W_n\mathcal{O}_X \quad \text{for } n = 1.$$

The following facts are easily checked:

- The Frobenius F induces $F : \mathrm{fil}_{D/p}^{\log} W_n\mathcal{O}_X \to \mathrm{fil}_D^{\log} W_n\mathcal{O}_X$.
- The Verschiebung V induces $V : \mathrm{fil}_D^{\log} W_{n-1}\mathcal{O}_X \to \mathrm{fil}_D^{\log} W_n\mathcal{O}_X$.
- The restriction R induces $R : \mathrm{fil}_D^{\log} W_n\mathcal{O}_X \to \mathrm{fil}_{D/p}^{\log} W_{n-1}\mathcal{O}_X$.
- The following sequence is exact:

$$0 \to \mathcal{O}_X(D) \xrightarrow{V^{n-1}} \mathrm{fil}_D^{\log} W_n\mathcal{O}_X \xrightarrow{R} \mathrm{fil}_{D/p}^{\log} W_{n-1}\mathcal{O}_X \to 0. \tag{3.1}$$

We define an object $(\mathbb{Z}/p^n\mathbb{Z})_{X|D}$ of the derived category $D^b(X)$ of bounded complexes of étale sheaves on X:

$$(\mathbb{Z}/p^n\mathbb{Z})_{X|D} = \mathrm{Cone}\big(\mathrm{fil}^{\log}_{D/p} W_n\mathcal{O}_X \xrightarrow{1-F} \mathrm{fil}^{\log}_{D} W_n\mathcal{O}_X\big)[-1].$$

We have a distinguished triangle in $D^b(X)$:

$$(\mathbb{Z}/p^n\mathbb{Z})_{X|D} \to \mathrm{fil}^{\log}_{D/p} W_n\mathcal{O}_X \xrightarrow{1-F} \mathrm{fil}^{\log}_{D} W_n\mathcal{O}_X \xrightarrow{+} . \tag{3.2}$$

Lemma 3.5. *There is a distinguished triangle*

$$(\mathbb{Z}/p\mathbb{Z})_{X|D} \to (\mathbb{Z}/p^n\mathbb{Z})_{X|D} \to (\mathbb{Z}/p^{n-1}\mathbb{Z})_{X|D/p} \xrightarrow{+} .$$

Proof. The lemma follows from the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(D/p) \arrow[r, "V^{n-1}"] \arrow[d, "1-F"] & \mathrm{fil}^{\log}_{D/p} W_n\mathcal{O}_X \arrow[r, "R"] \arrow[d, "1-F"] & \mathrm{fil}^{\log}_{D/p^2} W_{n-1}\mathcal{O}_X \arrow[r] \arrow[d, "1-F"] & 0 \\
0 \arrow[r] & \mathcal{O}_X(D) \arrow[r, "V^{n-1}"] & \mathrm{fil}^{\log}_{D} W_n\mathcal{O}_X \arrow[r, "R"] & \mathrm{fil}^{\log}_{D/p} W_{n-1}\mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

□

Lemma 3.6. *There is a canonical isomorphism*

$$\mathrm{fil}^{\log}_D H^1(U)[p^n] \simeq H^1(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}).$$

Proof. Noting that the restriction of $(\mathbb{Z}/p^n\mathbb{Z})_{X|D}$ to U is $\mathbb{Z}/p^n\mathbb{Z}$ on U, we have the localization exact sequence

$$H^1(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}) \to H^1(U, \mathbb{Z}/p^n\mathbb{Z}) \to H^2_C(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}). \tag{3.3}$$

For the generic point λ of C_λ, (3.2) gives us an exact sequence

$$H^1_\lambda(X, \mathrm{fil}^{\log}_{D/p} W_n\mathcal{O}_X) \xrightarrow{1-F} H^1_\lambda(X, \mathrm{fil}^{\log}_{D} W_n\mathcal{O}_X) \to H^2_\lambda(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}) \to H^2_\lambda(X, \mathrm{fil}^{\log}_{D/p} W_n\mathcal{O}_X).$$

By [Gr, Cor.3.10] and (3.1) we have

$$H^i_\lambda(X, \mathrm{fil}^{\log}_{D/p} W_n\mathcal{O}_X) = H^i_\lambda(X, \mathrm{fil}^{\log}_{D} W_n\mathcal{O}_X) = 0 \quad \text{for } i \geq 2$$

and

$$H^1_\lambda(X, \mathrm{fil}^{\log}_{D/p} W_n\mathcal{O}_X) \simeq W_n(K_\lambda)/\mathrm{fil}^{\log}_{m_\lambda/p} W_n(K_\lambda),$$

$$H^1_\lambda(X, \mathrm{fil}^{\log}_{D} W_n\mathcal{O}_X) \simeq W_n(K_\lambda)/\mathrm{fil}^{\log}_{m_\lambda} W_n(K_\lambda).$$

Thus we get

$$H^2_\lambda(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}) \simeq H^1(K_\lambda)[p^n]/\mathrm{fil}^{\log}_{m_\lambda} H^1(K_\lambda)[p^n].$$

Hence Lemma 3.6 follows from (3.3) and the injectivity of

$$H^2_C(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}) \to \bigoplus_{\lambda \in I} H^2_\lambda(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}).$$

This injectivity is a consequence of

Claim 3.7. *For $x \in C$ with $\dim(\mathcal{O}_{X,x}) \geq 2$ we have*

$$H^2_x(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}) = 0.$$

By Lemma 3.5 it suffices to show Claim 3.7 in case $n = 1$. Triangle (3.2) gives us an exact sequence

$$H^1_x(X, \mathcal{O}_X(D)) \to H^2_x(X, (\mathbb{Z}/p\mathbb{Z})_{X|D}) \to H^2_x(X, \mathcal{O}_X(D/p)) \xrightarrow{1-F} H^2_x(X, \mathcal{O}_X(D)).$$

If $\dim(\mathcal{O}_{X,x}) > 2$, $H^1_x(X, \mathcal{O}_X(D)) = 0$ and $H^2_x(X, \mathcal{O}_X(D/p)) = 0$ by [Gr, Cor.3.10], which implies $H^2_x(X, (\mathbb{Z}/p\mathbb{Z})_{X|D}) = 0$ as desired.

We now assume $\dim(\mathcal{O}_{X,x}) = 2$. Let $(\mathbb{Z}/p\mathbb{Z})_X$ denote the constant sheaf $\mathbb{Z}/p\mathbb{Z}$ on X and put

$$\mathcal{F}_{X|D} = \mathrm{Coker}\big(\mathcal{O}_X(D/p) \xrightarrow{1-F} \mathcal{O}_X(D)\big).$$

Note that $\mathcal{F}_{X|D} = 0$ for $D = 0$. By definition we have a distinguished triangle

$$(\mathbb{Z}/p\mathbb{Z})_X \to (\mathbb{Z}/p\mathbb{Z})_{X|D} \to \mathcal{F}_{X|D} \xrightarrow{+} .$$

By [SGA1, X, Theorem 3.1], we have $H^2_x(X, (\mathbb{Z}/p\mathbb{Z})_X) = 0$. Hence we are reduced to showing

$$H^2_x(X, \mathcal{F}_{X|D}) = 0. \tag{3.4}$$

Without loss of generality we can assume that D has integral coefficients. We prove (3.4) by induction on multiplicities of D reducing to the case $D = 0$. Fix an irreducible component C_λ of C with the multiplicity $m_\lambda \geq 1$ in D and put $D' = D - C_\lambda$. We have a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & (\mathbb{Z}/p\mathbb{Z})_X \arrow[d] & (\mathbb{Z}/p\mathbb{Z})_X \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_X(D'/p) \arrow[r] \arrow[d, "1-F"] & \mathcal{O}_X(D/p) \arrow[r] \arrow[d, "1-F"] & \mathcal{L} \arrow[r] \arrow[d, "F"] & 0 \\
0 \arrow[r] & \mathcal{O}_X(D') \arrow[r] & \mathcal{O}_X(D) \arrow[r] & \mathcal{O}_{C_\lambda}(D) \arrow[r] & 0 .
\end{tikzcd}$$

Here $\mathcal{O}_{C_\lambda}(D) = \mathcal{O}_X(D) \otimes \mathcal{O}_{C_\lambda}$, and $\mathcal{L} = \mathcal{O}_{C_\lambda}(D/p)$ if $p | m_\lambda$, and $\mathcal{L} = 0$ otherwise. Thus we get short exact sequences

$$0 \to \mathcal{F}_{X|D'} \to \mathcal{F}_{X|D} \to \mathcal{O}_{C_\lambda}(D) \to 0 \quad \text{if } p \nmid m_\lambda,$$

$$0 \to \mathcal{F}_{X|D'} \to \mathcal{F}_{X|D} \to \mathcal{O}_{C_\lambda}(D)/\mathcal{O}_{C_\lambda}(D/p)^p \to 0 \quad \text{if } p | m_\lambda.$$

We may assume $H^2_x(X, \mathcal{F}_{X|D'}) = 0$ by the induction hypothesis. Hence (3.4) follows from

$$H^2_x(C_\lambda, \mathcal{O}_{C_\lambda}(D)) = 0, \tag{3.5}$$

$$H^2_x(C_\lambda, \mathcal{O}_{C_\lambda}(D)/\mathcal{O}_{C_\lambda}(E)^p) = 0, \tag{3.6}$$

where we put $E = [D/p]$. We may assume $x \in C_\lambda$ so that $\dim(\mathcal{O}_{C_\lambda,x}) = 1$ by the assumption $\dim(\mathcal{O}_{X,x}) = 2$. (3.5) is a consequence of [Gr, Cor.3.10]. In view of an exact sequence

$$0 \to \mathcal{O}_{C_\lambda}(pE)/\mathcal{O}_{C_\lambda}(E)^p \to \mathcal{O}_{C_\lambda}(D)/\mathcal{O}_{C_\lambda}(E)^p \to \mathcal{O}_{C_\lambda}(D)/\mathcal{O}_{C_\lambda}(pE) \to 0 ,$$

(3.6) follows from

$$H^2_x(C_\lambda, \mathcal{O}_{C_\lambda}(pE)/\mathcal{O}_{C_\lambda}(E)^p) = 0 \quad \text{and} \quad H^2_x(C_\lambda, \mathcal{O}_{C_\lambda}(D)/\mathcal{O}_{C_\lambda}(pE)) = 0.$$

The first assertion follows from [Gr, Cor.3.10] noting that $\mathcal{O}_{C_\lambda}(pE)/\mathcal{O}_{C_\lambda}(E)^p$ is a locally free $\mathcal{O}^p_{C_\lambda}$-module. The second assertion holds since $\mathcal{O}_{C_\lambda}(D)/\mathcal{O}_{C_\lambda}(pE)$ is supported in a proper closed subscheme T of C_λ and x is a generic point of T if $x \in T$. This complete the proof of Lemma 3.6. □ □

In view of the above results, the assertions for $\mathrm{fil}^{\log}_D$ of Theorem 3.4(1) and (2) follows from the following.

Theorem 3.8. *Let the assumption be as in Theorem 3.2. The natural map*

$$H^1(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}) \to H^1(Y, (\mathbb{Z}/p^n\mathbb{Z})_{Y|D})$$

is an isomorphism for $d := \dim(X) \geq 3$, *and it is injective for* $d = 2$.

Proof. By Lemma 3.5 we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
H^1(X,(\mathbb{Z}/p\mathbb{Z})_{X|D}) \arrow[r] \arrow[d] & H^1(Y,(\mathbb{Z}/p\mathbb{Z})_{Y|E}) \arrow[d] \\
H^1(X,(\mathbb{Z}/p^n\mathbb{Z})_{X|D}) \arrow[r] \arrow[d] & H^1(Y,(\mathbb{Z}/p^n\mathbb{Z})_{Y|D}) \arrow[d] \\
H^1(X,(\mathbb{Z}/p^{n-1}\mathbb{Z})_{X|D/p}) \arrow[r] \arrow[d] & H^1(Y,(\mathbb{Z}/p^{n-1}\mathbb{Z})_{Y|E/p}) \arrow[d] \\
H^2(X,(\mathbb{Z}/p\mathbb{Z})_{X|D}) \arrow[r] & H^2(Y,(\mathbb{Z}/p\mathbb{Z})_{Y|E})
\end{tikzcd}$$

The theorem follows by the induction on n from the following. □

Lemma 3.9. *Let the assumption be as Theorem 3.2.*

(1) *Assuming $d \geq 3$, the natural map*

$$H^i(X,(\mathbb{Z}/p\mathbb{Z})_{X|D}) \to H^i(Y,(\mathbb{Z}/p\mathbb{Z})_{Y|E})$$

is an isomorphism for $i = 1$ and injective for $i = 2$.

(2) *Assuming $d = 2$, the natural map*

$$H^1(X,(\mathbb{Z}/p\mathbb{Z})_{X|D}) \to H^1(Y,(\mathbb{Z}/p\mathbb{Z})_{Y|E})$$

is injective.

Proof. We define an object $\mathcal{K}$ of $D^b(X)$:

$$\mathcal{K} = \mathrm{Cone}\big(\mathcal{O}_X(D/p - Y) \xrightarrow{1-F} \mathcal{O}_X(D - Y)\big)[-1].$$

By the commutative diagram with exact horizontal sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(D/p - Y) \arrow[r] \arrow[d, "1-F"] & \mathcal{O}_X(D/p) \arrow[r] \arrow[d, "1-F"] & \mathcal{O}_Y(E/p) \arrow[r] \arrow[d, "1-F"] & 0 \\
0 \arrow[r] & \mathcal{O}_X(D - Y) \arrow[r] & \mathcal{O}_X(D) \arrow[r] & \mathcal{O}_Y(E) \arrow[r] & 0
\end{tikzcd}$$

we have a distinguished triangle in $D^b(X)$:

$$\mathcal{K} \to (\mathbb{Z}/p\mathbb{Z})_{X|D} \to (\mathbb{Z}/p\mathbb{Z})_{Y|E} \xrightarrow{+} .$$

Hence it suffices to show $H^i(X,\mathcal{K}) = 0$ for $i = 1, 2$ in case $d \geq 3$ and $H^1(X,\mathcal{K}) = 0$ in case $d = 2$. We have an exact sequence

$$\begin{aligned} H^0(\mathcal{O}_X(D-Y)) \to H^1(X,\mathcal{K}) \to H^1(\mathcal{O}_X(D/p-Y)) \\ \to H^1(\mathcal{O}_X(D-Y)) \to H^2(X,\mathcal{K}) \to H^2(\mathcal{O}_X(D/p-Y)) \end{aligned}$$

By Serre duality, for a divisor Ξ on X, we have

$$H^i(X,\mathcal{O}_X(\Xi - Y)) = H^{d-i}(X,\Omega^d_X(-\Xi + Y))^\vee.$$

Thus the desired assertion follows from Definition 3.1($A1$) and (B). □

It remains to deduce the assertions for fil_D of Theorem 3.4(1) and (2) from that for $\mathrm{fil}_D^{\log}$. Let D' be as in the beginning of this section and $E' = D' \times_X Y$. Noting that the multiplicities of D' are prime to p, we have by Lemma 2.8(3)

$$\mathrm{fil}_{D'}H^1(U) = \mathrm{fil}^{\log}_{D'-C}H^1(U) \quad \text{and} \quad \mathrm{fil}_{E'}H^1(U\cap Y) = \mathrm{fil}^{\log}_{E'-C\cap Y}H^1(U\cap Y).$$

Thus the assertions for $\mathrm{fil}^{\log}_{D'-C}$ of Theorem 3.4 implies that for $\mathrm{fil}_{D'}$. Since $\mathrm{fil}_D \subset \mathrm{fil}_{D'}$, it immediately implies the injectivity of

$$\mathrm{fil}_D H^1(U) \to \mathrm{fil}_E H^1(U\cap Y).$$

It remains to deduce its surjectivity from that of

$$\mathrm{fil}_{D'}H^1(U) \to \mathrm{fil}_{E'}H^1(U\cap Y)$$

assuming $d \geq 3$. For this it suffices to show the injectivity of

$$\mathrm{fil}_{D'}H^1(U)/\mathrm{fil}_D H^1(U) \to \mathrm{fil}_{E'}H^1(U\cap Y)/\mathrm{fil}_E H^1(U\cap Y).$$

By Proposition 2.6 we have a commutative diagram

$$\begin{tikzcd}
\mathrm{fil}_{D'}H^1(U)/\mathrm{fil}_D H^1(U) \arrow[r, hook] \arrow[d] & \bigoplus_{\lambda\in I'} H^0(C_\lambda, \Omega^1_X(D')\otimes_{\mathcal{O}_X}\mathcal{O}_{C_\lambda}) \arrow[d] \\
\mathrm{fil}_{E'}H^1(U\cap Y)/\mathrm{fil}_E H^1(U\cap Y) \arrow[r, hook] & \bigoplus_{\lambda\in I'} H^0(C_\lambda\cap Y, \Omega^1_Y(D')\otimes_{\mathcal{O}_Y}\mathcal{O}_{C_\lambda\cap Y})
\end{tikzcd}$$

Thus we are reduced to showing the injectivity of the right vertical map. Putting $\mathcal{L} = \mathrm{Ker}(\Omega^1_X \to i_*\Omega^1_Y)$ where $i : Y \subset X$, the assertion follows from

$$H^0(C_\lambda, \mathcal{L}(D')\otimes_{\mathcal{O}_X}\mathcal{O}_{C_\lambda}) = 0.$$

Note that we used the fact that Y and C_λ intersect transversally. We have an exact sequence

$$0 \to \Omega^1_X(-Y) \to \mathcal{L} \to \mathcal{O}_X(-Y)\otimes\mathcal{O}_Y \to 0.$$

From this we get an exact sequence

$$0 \to \Omega^1_X(D'-Y)\otimes_{\mathcal{O}_X}\mathcal{O}_{C_\lambda} \to \mathcal{L}(D')\otimes_{\mathcal{O}_X}\mathcal{O}_{C_\lambda} \to \mathcal{O}_{C_\lambda}(D'-Y)\otimes\mathcal{O}_{C_\lambda\cap Y} \to 0.$$

We also have an exact sequence

$$0 \to \mathcal{O}_{C_\lambda}(D'-2Y) \to \mathcal{O}_{C_\lambda}(D'-Y) \to \mathcal{O}_{C_\lambda}(D'-Y)\otimes\mathcal{O}_{C_\lambda\cap Y} \to 0.$$

Therefore the desired assertion follows from Definition 3.1($A2$). This completes the proof of Theorem 3.4. □

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