From higher dimensional class field theory to a new theory of motives

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Classical CFT

Consider

$$\mathrm{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}) \cong \{\pm 1\}, \quad \mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) \cong \{\pm 1\}$$

These do not tell us arithmetic difference of $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{2})$.
On the other hand, from the fact $\mathbb{Q}(\sqrt{-1}) \subset \mathbb{Q}(\zeta_4)$, we get

$$\mathrm{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}) \cong (\mathbb{Z}/4\mathbb{Z})^\times \tag{1}$$

Similarly, from the fact $\mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\zeta_8)$, we get

$$\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) \cong (\mathbb{Z}/8\mathbb{Z})^\times/\{\pm 1\} \tag{2}$$

The latter isomorphisms imply the following

Theorem 0.1 (Fermat)

For an odd prime p, we have equivalences:

$$\exists x, y \in \mathbb{Z},\ p = x^2 + y^2 \Leftrightarrow p \equiv 1 \mod 4,$$
$$\exists x, y \in \mathbb{Z},\ p = x^2 - 2y^2 \Leftrightarrow p \equiv \pm 1 \mod 8.$$

(1) and (2) follow from the isomorphism (which follows from the irreducibility of cyclotomic polynomials)

$$c_n : (\mathbb{Z}/n\mathbb{Z})^\times \xrightarrow{\simeq} \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})\ ;\ a \to \sigma_a \tag{3}$$

where $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ is determined by $\sigma_a(\zeta_n) = \zeta_n^a$.

Let P_Q be the set of all prime numbers and (dn) P_Q be the subset of those $p|n$. By an easy observation, (3) is rephrased by an exact sequence

$$Q(n)^+ \ ! \ ^{div} \ \bigoplus_{p \in P_Q \ (n)} Z \ ! \ \mathrm{Gal}(Q(\ _n)/Q) \ ! \ 0; \qquad (4)$$

where sends $1 \in Z$ at p-component to $_p$ and

$$Q(n)^+ := \{ a \in Q \mid a \equiv 1 \mod n1 \};$$

$$\mathrm{div}(a) = \sum_{p \in P_Q \ (n)} v_p(a) \ (a \in Q(n)^+);$$

where $v_p(a)$ is the p-adic valution of $a \in Q$ and

$$a \equiv 1 \mod n1, \quad a > 0; \ ; a = 1 + \frac{nc}{b}; \ b; c \in Z; \ (b; n) = 1:$$

In order to describe $\pi_1^{ab}(U)$ for non-proper U, Kerz-Saito introduced a new theory of Chow groups called *Chow groups with modulus*. For a smooth variety U over a field k, we consider pairs (X, D) where X is normal and proper over k and D is an effective Cartier divisor on X such that $U = X \setminus |D|$. Such a (X, D) is called *a modulus pair with interior* U. The Chow group of zero-cycles of a modulus pair (X, D) is defined as

$$\mathrm{CH}_0(X|D) = \mathrm{Coker}\left(\bigoplus_{C\subset U} k(C)_D \xrightarrow{\delta} Z_0(U) = \bigoplus_{x\in U_{(0)}} \mathbb{Z}\right), \qquad (9)$$

where the direct sum ranges over integral curves on U and $k(C)_D$ is a subgroup of $k(C)^\times$ of rational functions f on C such that

$$f \equiv 1 \mod D \times_X \tilde{C},$$

where $\tilde{C}$ is the normalization of the closure of C in X.

Theorem 0.3 (Kerz-S (2016), Binda-Krishna-S (2022))

Assume k is finite. Then, the Frobenius map $\tilde{\rho}_U$ from (7) induces an isomorphism of finite groups

$$\mathrm{CH}_0(X|D)^0 \xrightarrow{\simeq} \pi_1^{ab}(X,D)^0,$$

where $\pi_1^{ab}(X,D)$ is the quotient of $\pi_1^{ab}(U)$ classifying the coverings $V \to U$ whose Artin conductor is bounded by D. In particular we have an isomorphism of pro-finite topological groups

$$\varprojlim_{(X,D)} \mathrm{CH}_0(X|D)^0 \xrightarrow{\simeq} \pi_1^{ab}(U)^0,$$

where the limit is over all modulus pairs with interior U, or over all effective Cartier divisors such that $X \setminus U = |D|$ for a fixed X.

Cohomology theory of modulus pairs

In algebraic geometry, commonly considered sheaf theories on categories $\mathbf{Sch} \supset \mathbf{Sm}$ (over some fixed base). Motivated by Kerz-Saito's theorem, Kahn-Miyazaki-Yamazaki-S started in 2018 a project to construct general cohomology theories of modulus pairs (X, D): X are proper scheme over a field k and D are effective Cartier divisors such that $X \setminus |D| \in \mathbf{Sm}$ (the interiors are not fixed as before). We defined what morphisms between modulus pairs are so that they form a category $\mathbf{MSm}$ with good properties (e,g, admits fiber products).

Miyazaki defined a Grothendieck topology MNis on $\mathbf{MSm}$ so that the category $\mathrm{Shv}_{\mathrm{MNis}}(\mathbf{MSm})$ of sheaves of abelian groups on the site $\mathbf{MSm}$ is Grothendieck abelian so that we can apply homological algebra to define *cohomology of* (X, D) *with coefficient* $F \in \mathrm{Shv}_{\mathrm{MNis}}(\mathbf{MSm})$.

Application to ramification theory

A remarkable feature of the new theory is its relation to ramification theory. To explain this, we need to introduce *reciprocity sheaves*.

We have an easy functor $\omega^* : \mathrm{Shv}_{\mathrm{Nis}}(\mathbf{Sm}) \to \mathrm{Shv}_{\mathrm{MNis}}(\mathbf{MSm})$:

$$\omega^* F(X, D) = F(X \setminus D) \quad \text{for } F \in \mathrm{Shv}_{\mathrm{Nis}}(\mathbf{Sm}) \text{ and } (X, D) \in \mathbf{MSm},$$

KMSY defined a mysterious functor
$\psi : \mathrm{Shv}_{\mathrm{Nis}}(\mathbf{Sm}) \to \mathrm{Shv}_{\mathrm{MNis}}(\mathbf{MSm})$ as

$$\psi F := \text{the maximal } \overline{\square}\text{-invariant object of } \omega^* F,$$

$\overline{\square} = (\mathbb{P}^1, \infty)$, G is $\overline{\square}$-invariant if $G(X, D) = G((X, D) \otimes \overline{\square})$.

Remark 0.4

It is impossible to compute ψF just from the definition.

For $U \in \mathbf{Sm}$, $\{\psi F(X, D)\}_{(X,D)}$ with (X, D) ranging over modulus pairs with interior U, gives a filtration on $F(U)$.

Definition 0.1

$F \in \mathrm{Shv}_{\mathrm{Nis}}(\mathbf{Sm})$ is called reciprocity sheaves if the filtration is exhaustive and admits transfers (push-forward for finite maps).

Let $\mathbf{RSC}_{\mathrm{Nis}}$ be the category of reciprocity sheaves.

Example 0.5

The following are reciprocity sheaves.

1. $\mathbb{A}^1$-invariant sheaves with transfers.
2. Ω^j, $W_n\Omega^j$.
3. $F(X) = \{\text{rank 1-connections over } X\}/\simeq$.
4. G finite abelian group, $F(X) = \{\text{étale } G\text{-torsors over } X\}/\simeq$.
5. G finite flat group scheme, $F(X) = \{\text{flat } G\text{-torsors over } X\}/\simeq$.

Theorem 0.6 (Rülling-S (2022))

For $F \in \mathbf{RSC}_{\mathrm{Nis}} \subset \mathrm{Shv}_{\mathrm{Nis}}(\mathbf{Sm})$ *and* $(X, D) \in \mathbf{MSm}$ *such that* X *smooth projective and* $|D|$ *is a SNCD,*

$$\psi F(X, D) = F^{\mathrm{AS}}(X, D) := \{a \in F(U) |\ p_1^*(a) - p_2^*(a) \in F(P_X^{(D)})\},$$

where $U = X \setminus D$ *and* $p_1, p_2 : U \times U \to X$ *are projections, and* $P_X^{(D)}$ *is the dilatation of* X *along* D*: The blow-up of* $X \times X$ *in* D *embedded diagonally into* $X \times X$ *and with the strict transforms of* $X \times D$ *and* $D \times X$ *removed. Note the open immersion* $U \times U \to X \times X$ *lifts to* $U \times U \to P_X^{(D)}$*.*

AS stands for Abbes and T.Saito who studied ramification theory for G-torsor (even for non-abelian case) using dilatations.

Using the theorem, one can compute ψF for some examples, giving a unified interpretation to different phenomena such as poles of differential forms, Artin conductors, and irregularities of connections.

Example 0.7

(1) $\psi\Omega^j(X,D) = \Omega^j_X(\log|D|)(D-|D|)$ if $\mathrm{ch}(k)=0$.
A formula exists also in case $\mathrm{ch}(k)=p>0$ but omitted.

(2) For $F(U) = \mathrm{Hom}_{cont}(\pi_1^{ab}(U), \mathbb{Q}/\mathbb{Z})$,

$$\psi F(X,D) = \mathrm{Hom}_{cont}(\pi_1^{ab}(X,D), \mathbb{Q}/\mathbb{Z}) \subset F(U).$$

Recall $\pi_1^{ab}(X,D)$ is defined by Artin conductors.

(3) For $F(U) = \{\text{line bundles on } U \text{ with connections}\}/\simeq$,
$\psi F(X,D)$ is computed by using the irregularities of connections.