

RECIPROCITY SHEAVES AND THEIR RAMIFICATION FILTRATIONS

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ABSTRACT. We define a motivic conductor for any presheaf with transfers F using the categorical framework developed for the theory of motives with modulus by Kahn-Miyazaki-Saito-Yamazaki. If F is a reciprocity sheaf this conductor yields an increasing and exhaustive filtration on $F(L)$, where L is any henselian discrete valuation field of geometric type over the perfect ground field. We show if F is a smooth group scheme, then the motivic conductor extends the Rosenlicht-Serre conductor; if F assigns to X the group of finite characters on the abelianized étale fundamental group of X , then the motivic conductor agrees with the Artin conductor defined by Kato-Matsuda; if F assigns to X the group of integrable rank one connections (in characteristic zero), then it agrees with the irregularity. We also show that this machinery gives rise to a conductor for torsors under finite flat group schemes over the base field, which we believe to be new. We introduce a general notion of conductors on presheaves with transfers and show that on a reciprocity sheaf the motivic conductor is minimal and any conductor which is defined only for henselian discrete valuation fields of geometric type with *perfect* residue field can be uniquely extended to all such fields without any restriction on the residue field. For example the Kato-Matsuda Artin conductor is characterized as the canonical extension of the classical Artin conductor defined in the perfect residue field case.

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1. INTRODUCTION

Fix a perfect field k and let \mathbf{Sm} be the category of separated smooth k -schemes. Let \mathbf{Cor} be the category of finite correspondences: \mathbf{Cor} has the same objects as \mathbf{Sm} and morphisms in \mathbf{Cor} are finite correspondences (see 2.1 for a precise definition). Let \mathbf{PST} be the category of additive presheaves of abelian groups on \mathbf{Cor} , called presheaves with transfers. In this note we give a construction which associates to each $F \in \mathbf{PST}$ a collection of functions

$$c^F = \{c_L^F : F(L) \rightarrow \mathbb{N} \cup \{\infty\}\}_{L \in \Phi},$$

where \mathbb{N} is the set of non-negative integers, Φ is the collection of henselian discrete valuation fields which are the fraction fields of the henselization $\mathcal{O}_{X,x}^h$ of $X \in \mathbf{Sm}$ at points x of codimension one in X , and

$$F(L) = \varinjlim_V F(V - D_x),$$

where $V \rightarrow X$ ranges over étale neighborhoods of x and D_x is the closure of x in V . We call c^F *the motivic conductor for F* . Our main aim is to convince the reader that our construction deserves such a pretentious terminology. Indeed, it gives a unified way to understand different conductors such as the Artin conductor of a character of the abelian fundamental group $\pi_1^{\text{ab}}(X)$ with $X \in \mathbf{Sm}$ along a boundary of X , the Rosenlicht-Serre conductor of a morphism from a curve to

a commutative algebraic k -group, and the irregularity of a line bundle with connections on $X \in \mathbf{Sm}$ along a boundary of X . It also gives rise to a new conductor for G -torsors with G a finite flat k -group scheme. The latter conductor specializes to the classical Artin conductor in case G is constant.

Our construction of the motivic conductors is rather simple once we have the new categorical framework introduced in [KMSY21a], [KMSY21b] at our disposal (see (1.0.1) below). The main aim of *loc. cit.* is to develop a theory of *motives with modulus* generalizing Voevodsky's theory of motives in order to capture non- \mathbf{A}^1 -invariant phenomena and objects. The basic principle is that the category \mathbf{Cor} should be replaced by the larger category of *modulus pairs*, $\underline{\mathbf{MCor}}$: Objects are pairs $\mathcal{X} = (\overline{X}, X_\infty)$ consisting of a separated k -scheme of finite type \overline{X} and an effective (possibly empty) Cartier divisor X_∞ on it such that the complement $\overline{X} \setminus X_\infty$ is smooth. Morphisms are given by finite correspondences between the smooth complements satisfying certain admissibility conditions (see §3 for the precise definition). Let $\mathbf{MCor} \subset \underline{\mathbf{MCor}}$ be the full subcategory consisting of objects (\overline{X}, X_∞) with \overline{X} proper over k . We then define $\underline{\mathbf{MPST}}$ (resp. \mathbf{MPST}) as the category of additive presheaves of abelian groups on $\underline{\mathbf{MCor}}$ (resp. \mathbf{MCor}). We have a functor

$$\omega : \underline{\mathbf{MCor}} \rightarrow \mathbf{Cor}, \quad (\overline{X}, X_\infty) \mapsto \overline{X} - |X_\infty|,$$

and two pairs of adjunctions

$$\mathbf{MPST} \begin{array}{c} \xleftarrow{\tau^*} \\ \xrightarrow{\tau_!} \end{array} \underline{\mathbf{MPST}}, \quad \mathbf{MPST} \begin{array}{c} \xleftarrow{\omega^*} \\ \xrightarrow{\omega_!} \end{array} \mathbf{PST},$$

where τ^* is induced by the inclusion $\tau : \mathbf{MCor} \rightarrow \underline{\mathbf{MCor}}$ and $\tau_!$ is its left Kan extension, and ω^* is induced by ω and $\omega_!$ is its left Kan extension (see 3.3 for more concrete descriptions of these functors). A basic notion is the \square -invariance, where $\square = (\mathbf{P}^1, \infty) \in \mathbf{MCor}$: $F \in \mathbf{MPST}$ is called \square -invariant if $F(\mathcal{X}) \simeq F(\mathcal{X} \otimes \square)$ for all $\mathcal{X} \in \mathbf{MCor}$ (see 3.1 for the tensor product \otimes in \mathbf{MCor}). It is an analogue of the \mathbf{A}^1 -invariance¹ exploited by Voevodsky in his theory of motives. We write \mathbf{CI} for the full subcategory of \mathbf{MPST} consisting of \square -invariant objects. We know ([KSY, Lem 2.1.7]) that the inclusion $\mathbf{CI} \rightarrow \mathbf{MPST}$ admits a right adjoint h_{\square}^0 which associates to $F \in \mathbf{MPST}$ the maximal \square -invariant subobject of F . We define the functor

$$\omega^{\mathbf{CI}} : \mathbf{PST} \xrightarrow{\omega^*} \mathbf{MPST} \xrightarrow{h_{\square}^0} \mathbf{CI},$$

¹Recall $F \in \mathbf{PST}$ is \mathbf{A}^1 -invariant if $F(X) \simeq F(X \times \mathbf{A}^1)$ for all $X \in \mathbf{Sm}$.

and write $\tilde{F} = \tau_1 \omega^{\mathbf{CI}} F \in \mathbf{MPST}$, for $F \in \mathbf{PST}$. Then the motivic conductor c^F for $F \in \mathbf{PST}$ is defined by

$$(1.0.1) \quad c_L^F(a) = \min\{n \mid a \in \tilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-n})\}, \quad \text{for } a \in F(L).$$

Here, for $G \in \mathbf{MPST}$, $L = \text{Frac}(\mathcal{O}_{X,x}^h) \in \Phi$, and $n \in \mathbb{Z}_{\geq 1}$, we put

$$G(\mathcal{O}_L, \mathfrak{m}_L^{-n}) = \varinjlim_V G(V, nD_x),$$

where $V \rightarrow X$ ranges over étale neighborhoods of x and D_x is the closure of x in V and nD_x is its n -th thickening in V . By convention,

$$G(\mathcal{O}_L, \mathfrak{m}_L^{-n}) = G(\mathcal{O}_L) = \varinjlim_V G(V), \quad \text{for } n = 0.$$

For $G = \tilde{F}$ there are natural inclusions $\tilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-n}) \hookrightarrow F(L)$, which are used to define (1.0.1). It turns out that $\{\tilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-n})\}_{n \in \mathbb{Z}_{\geq 0}}$ induces an increasing filtration on $F(L)$ which is exhaustive if $F \in \mathbf{RSC}$. Here \mathbf{RSC} is the full subcategory of \mathbf{PST} consisting of the objects belonging to the essential image of \mathbf{CI} under ω_1 . Objects of \mathbf{RSC} are called *reciprocity presheaves* and play a key role in this note. We know (see [KSY, Cor 2.3.4]) that \mathbf{RSC} contains all \mathbf{A}^1 -invariant objects in \mathbf{PST} . Moreover it contains many interesting objects F which are not \mathbf{A}^1 -invariant. In this note we consider in particular the following examples (X runs over objects of \mathbf{Sm}):

- (i) $F(X) = \text{Hom}_{\mathbf{Sm}}(X, \Gamma)$, where Γ is a smooth commutative algebraic k -group which may have non-trivial unipotent part (for example $\Gamma = \mathbf{G}_a$).
- (ii) $F(X) = H_{\text{ét}}^1(X, \mathbb{Q}/\mathbb{Z}) = \text{Hom}_{\text{cont}}(\pi_1(X)^{\text{ab}}, \mathbb{Q}/\mathbb{Z})$.
- (iii) $F(X) = \text{Conn}^1(X)$ (resp. $\text{Conn}_{\text{int}}^1(X)$) the group of isomorphism classes of (resp. integrable) rank 1 connections on X . Here we assume $\text{ch}(k) = 0$.
- (iv) $F(X) = H_{\text{fppf}}^1(X, \Gamma)$, where Γ is a finite flat k -group.

We prove the following (see Theorems 5.2, 7.20, 8.8, and 6.11 for the precise statements).

- Theorem 1.** (1) *In case (i), c_L^F agrees with the conductor of Rosenlicht-Serre ([Ser84]) if L has perfect residue field. If $\text{ch}(k) = p$ is positive and $F = W_n$ is the group scheme of p -typical Witt vectors of length n , then c_L^F agrees with a conductor defined by Kato-Russell in [KR10] for any L .*
- (2) *In case (ii), c_L^F agrees with the Artin conductor Art_L of Kato-Matsuda (see §7.1)².*

²It coincides with the classical Artin conductor if L has perfect residue field.

(3) In case (iii), c^F agrees with the irregularity of connections.

As far as we know, the motivic conductor c^F in the case (iv) is new and we give an explicit description only in case the infinitesimal unipotent part of G is α_p , where $p = \text{ch}(k)$ (see Theorem 9.12).

An amusing application of the motivic conductor c^F is to give an explicit description of the maximal \mathbf{A}^1 -invariant part of F : Let $\mathbf{HI} \subset \mathbf{PST}$ be the full subcategory of \mathbf{A}^1 -invariant objects. The inclusion $\mathbf{HI} \rightarrow \mathbf{PST}$ admits a right adjoint $h_{\mathbf{A}^1}^0$ which associates to $F \in \mathbf{PST}$ the maximal \mathbf{A}^1 -invariant subobject of F (see 4.30 for an explicit description of $h_{\mathbf{A}^1}^0$). Let $\mathbf{NST} \subset \mathbf{PST}$ be the full subcategory of Nisnevich sheaves, i.e., those objects $F \in \mathbf{PST}$ whose restrictions to $\mathbf{Sm} \subset \mathbf{Cor}$ are sheaves with respect to the Nisnevich topology.

Theorem 2. For $F \in \mathbf{RSC} \cap \mathbf{NST}$ and $X \in \mathbf{Sm}$, we have

$$h_{\mathbf{A}^1}^0(F)(X) = \bigcap_{\rho} \{a \in F(X) \mid c^F(\rho^*a) \leq 1\},$$

where ρ ranges over all morphisms $\text{Spec } L \rightarrow X$ with $L \in \Phi$.

In case $F = H_{\text{ét}}^1(-, \mathbb{Q}/\mathbb{Z})$ from (ii) (resp. $F = \text{Conn}_{\text{int}}^1$ from (iii)), Theorem 2 asserts that the maximal \mathbf{A}^1 -invariant part of F is precisely the subsheaf of tame characters (resp. regular singular connections).

In what follows we fix $F \in \mathbf{RSC} \cap \mathbf{NST}$ and introduce a class of collections of functions

$$c = \{c_L : F(L) \rightarrow \mathbb{N}\}_{L \in \Phi}$$

which may be called *conductors for F* . Let $\text{Func}_{\Phi}(F, \mathbb{N})$ be the partially ordered set consisting of collections of functions with partial order given by $c \leq c'$, if $c_L(a) \leq c'_L(a)$ for all $L \in \Phi$ and $a \in F(L)$. Let $\mathbf{CI}(F)$ be the partially ordered set consisting of subobjects G of $\omega^{\mathbf{CI}F}$ such that the induced maps $\omega_!G \rightarrow \omega_!\omega^{\mathbf{CI}F}$ are isomorphisms and with partial order given by inclusion. Then every $G \in \mathbf{CI}(F)$ gives rise to an exhaustive increasing filtration $\{\tau_!G(\mathcal{O}_L, \mathfrak{m}_L^{-n})\}_{n \geq 0}$ on $F(L)$ and we define $c^G \in \text{Func}_{\Phi}(F, \mathbb{N})$ by

$$c_L^G(a) = \min\{n \mid a \in \tau_!G(\mathcal{O}_L, \mathfrak{m}_L^{-n})\}, \quad \text{for } a \in F(L).$$

By definition the motivic conductor c^F of F is $c^{\omega^{\mathbf{CI}F}}$ and $c^F \leq c^G$, for all $G \in \mathbf{CI}(F)$. Now a question is whether there is a simple characterization of the poset $\{c^G \mid G \in \mathbf{CI}(F)\}$ in $\text{Func}_{\Phi}(F, \mathbb{N})$. We answer it in the following refined form. Let n be a positive integer or ∞ . Let $\Phi_{\leq n} \subset \Phi$ be the collection of such L that $\text{trdeg}_k(L) \leq n$. (Note that in positive characteristic $\Phi_{\leq 1}$ consists precisely of those $L \in \Phi$ that have

a perfect residue field.) Let $\text{Func}_\Phi(F, \mathbb{N})_{\leq n}$ be the poset consisting of collections of functions

$$c = \{c_L : F(L) \rightarrow \mathbb{N}\}_{L \in \Phi_{\leq n}}$$

with partial order defined in the same manner as $\text{Func}_\Phi(F, \mathbb{N})$. There is an obvious restriction functor

$$(1.0.2) \quad \text{Func}_\Phi(F, \mathbb{N}) \rightarrow \text{Func}_\Phi(F, \mathbb{N})_{\leq n}, \quad c \mapsto c^{\leq n}.$$

We then introduce six axioms (c1) through (c6) for $\text{Func}_\Phi(F, \mathbb{N})_{\leq n}$ (cf. Definitions 4.3 and 4.22) and call those elements satisfying the axioms *semi-continuous conductors of level n* . Let $\text{Cond}(F)_{\leq n}^{sc}$ be the subposet of $\text{Func}_\Phi(F, \mathbb{N})_{\leq n}$ consisting of such objects. Write $\text{Cond}(F)^{sc}$ for $\text{Cond}(F)_{\leq n}^{sc}$ with $n = \infty$.³ For example, for $F = H_{\text{ét}}^1(-, \mathbb{Q}/\mathbb{Z})$ from (ii), the classical Artin conductor $\{\text{Art}_L\}_{L \in \Phi_{\leq 1}}$ is an element of $\text{Cond}(F)_{\leq 1}^{sc}$ and the Kato-Matsuda conductor $\{\text{Art}_L\}_{L \in \Phi}$ is an element of $\text{Cond}(F)^{sc}$. We show the following (see Theorem 4.25).

Theorem 3. (1) $c^G \in \text{Cond}(F)^{sc}$ for every $G \in \mathbf{CI}(F)$.
 (2) There exists an order reversing map

$$\text{Cond}(F)_{\leq n}^{sc} \rightarrow \mathbf{CI}(F), \quad c \mapsto \hat{F}_c$$

such that $c = (c^{\hat{F}_c})^{\leq n}$. For $\mathcal{X} = (\bar{X}, X_\infty) \in \mathbf{MCor}$ with $X = \bar{X} - |X_\infty|$ we have

$$\hat{F}_c(\mathcal{X}) = \{a \in F(X) \mid c_{\bar{X}}(a) \leq X_\infty\},$$

where $c_{\bar{X}}(a) \leq X_\infty$ means that for any $L \in \Phi_{\leq n}$ and any morphism $\rho : \text{Spec } \mathcal{O}_L \rightarrow \bar{X}$ such that $\rho(\text{Spec } L) \in X$, $c_L(\rho^*a)$ is not more than the multiplicity of the pullback of X_∞ along ρ .

As a consequence, we obtain the following (see Theorem 4.25(4)).

Corollary 1. There exists a unique map

$$\text{Cond}(F)_{\leq n}^{sc} \rightarrow \text{Cond}(F)^{sc}, \quad c \mapsto c^\infty,$$

such that $\hat{F}_c = \hat{F}_{c^\infty}$ and that $c = (c^\infty)^{\leq n}$.

We call c^∞ the *canonical extension of c* . For example, the Kato-Matsuda Artin conductor is the canonical extension of the classical Artin conductor. We say F has level n , if $(c^F)^{\leq n} \in \text{Cond}(F)_{\leq n}^{sc}$; in this case c^F is the canonical extension of $(c^F)^{\leq n}$, by Theorem 4.25(5). We show that $F = H_{\text{ét}}^1(-, \mathbb{Q}/\mathbb{Z})$ in (ii) is of level 1 (see Theorem 8.8), $F = \text{Conn}^1$ (resp. $F = \text{Conn}_{\text{int}}^1$) from (iii) is of level 2 (resp. 1)

³There is one axiom (c4) which is not preserved by the functor (1.0.2). So it does not induce $\text{Cond}(F)^{sc} \rightarrow \text{Cond}(F)_{\leq n}^{sc}$.

(see Theorem 6.11), and $F = H_{\text{fppf}}^1(-, \Gamma)$ from (iv) is of level 1 if the infinitesimal unipotent part of Γ is trivial and else is of level 2 (see Theorem 9.12).

We give a description of the content of each section: In section 2 we explain how to extend a presheaf with transfers to the category of regular schemes over k which are pro-smooth; this is well-known and we include it only for lack of reference. In section 3 we recall the necessary constructions and results from the theory of motives with modulus as developed in [KMSY21a], [KMSY21b], [KSY16], [KSY], and [Sai20]. Then we introduce in section 4 the notion of (semi-continuous) conductors and prove Theorems 3 and 2. We close the section with a discussion of the relation between the motivic conductor of a reciprocity sheaf with certain vanishing properties of its associated symbol. This is needed in order to prove in the later sections that a certain conductor is equal to the motivic one; the main point being Corollary 4.40. In the second part we consider various conductors which are mostly classical and show that they are motivic in our sense. Kähler differentials and rank one connections are considered in section 6, where $\text{ch}(k) = 0$. In the following sections we assume $\text{ch}(k) = p > 0$. In section 7 it is shown that one of the conductors defined by Kato-Russell for W_n is motivic. We use this in section 8 to show that the Kato-Matsuda conductor for characters is motivic, which yields also a description of the motivic conductor for lisse $\bar{\mathbb{Q}}_\ell$ -sheaves of rank 1. Finally, in section 9 we define and investigate a conductor for torsors under finite flat k -groups, which we believe to be new. The general pattern of these computations is always the same: First we show that the collection $c = \{c_L\}$ defined in the various situations defines a semi-continuous conductor (of a certain level) in the sense of Definitions 4.3 and 4.14, then we do a symbol computation to show that this conductor is actually motivic. Note however, that the actual computations in the various cases differ quite a bit.

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Conventions 1.1. We work over a perfect field k . If K/k is a field extension, then by a K -scheme we will always mean a scheme which is separated and of finite type over K . In contrast, the phrase *scheme over K* refers to any scheme morphism $X \rightarrow \text{Spec } K$. By a *smooth K -scheme* we mean a K -scheme which is smooth over K . We denote by \mathbf{Sm}_K the category of such schemes and set $\mathbf{Sm} = \mathbf{Sm}_k$. For k -schemes X and Y we write $X \times Y$ instead of $X \times_k Y$. For any scheme X we denote by $X^{(i)}$ the set of i -codimensional points of X .

Part 1. The general theory

2. PRESHEAVES WITH TRANSFERS ON PRO-SMOOTH SCHEMES

The material in this section is well-known, we give some details for lack of reference.

2.1. Denote by \mathbf{Cor} the category of finite correspondences of Suslin-Voevodsky. Recall that the objects are the smooth k -schemes and morphisms are given by correspondences, i.e., $\mathbf{Cor}(X, Y)$ is the free abelian group generated by prime correspondences, i.e., integral closed subschemes $V \subset X \times Y$ which are finite and surjective over a connected component of X . Given two prime correspondences $V \in \mathbf{Cor}(X, Y)$ and $W \in \mathbf{Cor}(Y, Z)$ their composition is given by the intersection product (see e.g. [Ser65, V, C])

$$(2.1.1) \quad W \circ V = p_{13*}(p_{12}^*V \cdot p_{23}^*W),$$

where p_{ij} denotes the projection from $X \times Y \times Z$ to the factor (i, j) .

Denote by \mathbf{ProCor} the pro-category of \mathbf{Cor} , i.e., objects are functors $I^\circ \rightarrow \mathbf{Sm}$, $i \mapsto X_i$, where I is a filtered essentially small category, and the morphisms between two pro-objects $(X_i)_{i \in I}$ and $(Y_j)_{j \in J}$ are given by

$$\mathbf{ProCor}((X_i), (Y_j)) = \varprojlim_{j \in J} \varinjlim_{i \in I} \mathbf{Cor}(X_i, Y_j).$$

Definition 2.2. We define the category $\mathbf{Cor}^{\text{pro}}$ as follows: The objects are the noetherian regular schemes over k of the form

$$(2.2.1) \quad X = \varprojlim_{i \in I} X_i,$$

where $(X_i)_{i \in I}$ is a projective system of smooth k -schemes indexed by a partially ordered set and with affine transition maps $X_i \rightarrow X_j$, $i \geq j$. If X and Y are two objects in $\mathbf{Cor}^{\text{pro}}$, then $\mathbf{Cor}^{\text{pro}}(X, Y) = \mathbf{Cor}(X, Y)$ is the free abelian group generated by prime correspondences in the sense of 2.1. The composition is defined in the same way as in the case of \mathbf{Cor} . (Note that this still makes sense by [Ser65, V, B, 3., Thm 1].)

Remarks 2.3. (1) All objects in $\mathbf{Cor}^{\text{pro}}$ are separated, noetherian, and regular schemes over k . Any affine, noetherian, and regular scheme over k defines an object in $\mathbf{Cor}^{\text{pro}}$, by [Pop86, (1.8) Thm] and [SGA 4_I, Exp I, Prop 8.1.6].

(2) Note, that for $X, Y \in \mathbf{Cor}^{\text{pro}}$ the cartesian product $X \times Y$ does not need to be noetherian; but if $Y \in \mathbf{Sm}$ and $X \in \mathbf{Cor}^{\text{pro}}$, then $X \times Y \in \mathbf{Cor}^{\text{pro}}$.

Lemma 2.4. *Let A be a k -algebra which is noetherian, regular, and is a directed limit $A = \varinjlim_{i \in I} A_i$, where the A_i are smooth and of finite type over k and the transition maps $A_i \rightarrow A_j$, $j \geq i$ are flat. Let X be a regular quasi-projective A -scheme. Then $X \in \mathbf{Cor}^{\text{pro}}$.*

Proof. Set $S_i = \text{Spec } A_i$ and $S = \text{Spec } A = \varprojlim_i S_i$. Choose an S -embedding $X \subset \mathbf{P}_S^n$. We find an i_0 and a subscheme $X_{i_0} \subset \mathbf{P}_{S_{i_0}}^n$ such that $X = X_{i_0} \times_{S_{i_0}} S$. Set $X_i := X_{i_0} \times_{S_{i_0}} S_i$, for $i \geq i_0$. Then the transition maps $X_j \rightarrow X_i$, $j \geq i \geq i_0$, are affine and flat, hence so is the projection $\tau_i : X = \varprojlim_i X_i \rightarrow X_{i_0}$. Since X is regular, there exists an open neighborhood $U_{i_0} \subset X_{i_0}$ containing $\tau_{i_0}(X)$ which is regular (see [EGA IV₂, Cor (6.5.2)]). Since U_{i_0} is of finite type over the perfect field k , it is even smooth. Set $U_i = U_{i_0} \times_{S_{i_0}} S_i$. Then the transition maps $U_j \rightarrow U_i$, $j \geq i \geq i_0$, are affine and flat, each U_i is smooth, and we have $X = \varprojlim_i U_i$; hence $X \in \mathbf{Cor}^{\text{pro}}$. \square

Lemma 2.5. *There is a (up to isomorphism) canonical and faithful functor*

$$\mathbf{Cor}^{\text{pro}} \rightarrow \mathbf{ProCor}, \quad \varprojlim_i X_i \mapsto (X_i).$$

Proof. For any $X \in \mathbf{Cor}^{\text{pro}}$ we choose once and for all a projective system $(X_i)_{i \in I}$ as in (2.2.1). In particular, $(X_i) \in \mathbf{ProCor}$. Note, if $X = \varprojlim_{j \in J} X'_j$, then $(X_i) \cong (X'_j)$ in \mathbf{ProSm} . Take $X = \varprojlim_{i \in I} X_i$ and $Y = \varprojlim_{j \in J} Y_j$ in $\mathbf{Cor}^{\text{pro}}$ and let $V \subset X \times Y$ be a prime correspondence. For any scheme S over k we denote by

$$\rho_i : X \times S \rightarrow X_i \times S, \quad \rho_{i',i} : X_{i'} \times S \rightarrow X_i \times S, \quad i' \geq i,$$

and by

$$\sigma_j : Y \times S \rightarrow Y_j \times S, \quad \sigma_{j',j} : Y_{j'} \times S \rightarrow Y_j \times S, \quad j' \geq j,$$

the projection and transition maps of $(X_i \times S)$ and $(Y_j \times S)$, respectively. By assumption all these maps are affine. For all j , the morphism $V \rightarrow X \times Y_j$ induced by σ_j is a morphism of finite type X -schemes. Since V is finite over X , its image $\sigma_j(V) \subset X \times Y_j$ is proper over X . Hence $V \rightarrow \sigma_j(V)$ is proper and affine, hence finite. Since X is noetherian $\sigma_j(V)$ is finite over X , hence we obtain a well defined correspondence $\sigma_{j*}V \in \mathbf{Cor}(X, Y_j)$ with the property

$$(2.5.1) \quad \sigma_{j*}V = 0 \iff V = 0.$$

Furthermore, since $X \times Y_j$ is noetherian, we find an index i (depending on j) and a correspondence $V_{i,j} \in \mathbf{Cor}(X_i, Y_j)$ such that

$$\sigma_{j*}V = \rho_i^*V_{i,j}.$$

If we find i' and $V'_{i',j}$ with $\rho_{i'}^* V'_{i',j} = \sigma_{j*} V$, then clearly $V_{i,j} = V'_{i',j}$ in $\varinjlim_i \mathbf{Cor}(X_i, Y_j)$. Therefore we obtain a well-defined element V_j

$$\mathbf{Cor}(X_i, Y_j) \rightarrow \varinjlim_i \mathbf{Cor}(X_i, Y_j), \quad V_{i,j} \mapsto V_j.$$

By the base change formula (see (2.5.6) below) we obtain $\sigma_{j',j*} V_{j'} = V_j$. We obtain a morphism

$$(2.5.2) \quad \mathbf{Cor}^{\text{pro}}(X, Y) \rightarrow \mathbf{ProCor}((X_i), (Y_j)), \quad V \mapsto (V_j)_j.$$

It is injective by (2.5.1). Finally we have to check that (2.5.2) is compatible with composition. Take $Z = \varprojlim_{l \in L} Z_l \in \mathbf{Cor}^{\text{pro}}$. For any scheme S over k denote by

$$\tau_l : Z \times S \rightarrow Z_l \times S$$

the projection map. Take prime correspondences $V \in \mathbf{Cor}^{\text{pro}}(X, Y)$ and $W \in \mathbf{Cor}^{\text{pro}}(Y, Z)$. For any $l \in L$ we find an index $j(l) \in J$ and a correspondence $W_{j(l),l} \in \mathbf{Cor}(Y_{j(l)}, Z_l)$ such that $\tau_{l*} W = \sigma_{j(l)*} W_{j(l),l}$. For any $j(l)$ we find an index $i(j(l)) \in I$ and a correspondence $V_{i(j(l)),j(l)} \in \mathbf{Cor}(X_{i(j(l))}, Y_{j(l)})$ such that $\sigma_{j(l)*} V = \rho_{i(j(l))}^* V_{i(j(l)),j(l)}$. Then the compatibility of (2.5.2) will hold if we can show

$$(2.5.3) \quad \tau_{l*}(W \circ V) = \rho_{i(j(l))}^*(W_{j(l),l} \circ V_{i(j(l)),j(l)}), \quad \text{for all } l \in L.$$

To this end we recall some well-known formulas. Assume we are given the following diagram of schemes over k which are in $\mathbf{Cor}^{\text{pro}}$,

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ h' \downarrow & & \downarrow h \\ X & \xrightarrow{f} & Y \xrightarrow{g} Z, \end{array}$$

and assume the square is cartesian and tor-independent. Then for cycles $\alpha, \beta, \beta', \gamma$ on X, Y, Z , respectively, the following relations hold as soon as both sides of the equation are defined (see [Ser65, V, C]):

$$(2.5.4) \quad f^* g^* \gamma = (g \circ f)^* \gamma.$$

$$(2.5.5) \quad g_* f_* \alpha = (g \circ f)_* \alpha.$$

$$(2.5.6) \quad h^* f_* \alpha = f'_* h'^* \alpha.$$

$$(2.5.7) \quad f^*(\beta \cdot \beta') = f^*(\beta) \cdot f^*(\beta').$$

$$(2.5.8) \quad f_*(\alpha \cdot f^*(\beta)) = f_*(\alpha) \cdot \beta.$$

Using these formulas it is straightforward but a bit longish to check, that (2.5.3) holds. Indeed, since all cycles involved are always finite over some scheme over k it will be clear that the formulas in question

are defined; the base change formula (2.5.6) will be only applied in cases where one of the maps f or h is flat, hence the tor-independence condition will be automatic. (But note that h might not be flat so there might appear higher Tor's in the computation of h^* and h'^* , respectively.) This finishes the proof. \square

2.6. A presheaf with transfers in the sense of Suslin-Voevodsky is a functor $F : \mathbf{Cor}^o \rightarrow \mathbf{Ab}$; they form the category \mathbf{PST} . We extend it to a functor $F : \mathbf{ProCor}^o \rightarrow \mathbf{Ab}$ by the formula

$$F((X_i)_{i \in I}) := \varinjlim_i F(X_i).$$

Precomposing F with the functor from Lemma 2.5 we obtain presheaves on $\mathbf{Cor}^{\text{pro}}$, which we again denote by F ,

$$F : (\mathbf{Cor}^{\text{pro}})^o \rightarrow \mathbf{Ab}.$$

For $\alpha \in \mathbf{Cor}^{\text{pro}}(X, Y)$ we denote by $\alpha^* = F(\alpha) : F(Y) \rightarrow F(X)$, the induced map. If $f : X \rightarrow Y$ is a morphism with graph $\Gamma_f \subset X \times Y$ between k -schemes which are objects in $\mathbf{Cor}^{\text{pro}}$, then we set

$$(2.6.1) \quad f^* := \Gamma_f^* : F(Y) \rightarrow F(X);$$

if f is a finite morphism and $\Gamma_f^t \subset Y \times X$ is the transposed of the graph of f we set

$$(2.6.2) \quad f_* := (\Gamma_f^t)^* : F(X) \rightarrow F(Y).$$

3. REVIEW OF RECIPROCITY SHEAVES

In this section we collect some definitions, notations and results from [KMSY21a], [KMSY21b], [KSY], and [Sai20].

3.1. A *modulus pair* $\mathcal{X} = (\overline{X}, X_\infty)$ consists of a separated and finite type k -scheme \overline{X} and an effective Cartier divisor $X_\infty \geq 0$ such that the open complement $X := \overline{X} \setminus |X_\infty|$ is smooth. We say \mathcal{X} is a *proper modulus pair* if \overline{X} is proper over k . A basic example is the cube

$$\overline{\square} := (\mathbf{P}_k^1, \infty).$$

Let $\mathcal{X} = (\overline{X}, X_\infty)$ and $\mathcal{Y} = (\overline{Y}, Y_\infty)$ be two modulus pairs with corresponding opens $X = \overline{X} \setminus |X_\infty|$ and $Y = \overline{Y} \setminus |Y_\infty|$, respectively. The modulus pair $\mathcal{X} \otimes \mathcal{Y}$ is defined by

$$(3.1.1) \quad \mathcal{X} \otimes \mathcal{Y} := (\overline{X} \times \overline{Y}, X_\infty \times \overline{Y} + \overline{X} \times Y_\infty).$$

An *admissible prime correspondence from \mathcal{X} to \mathcal{Y}* is a prime correspondence $V \in \mathbf{Cor}(X, Y)$ satisfying the following condition

$$(3.1.2) \quad X_{\infty|V^N} \geq Y_{\infty|V^N},$$

where $\bar{V}^N \rightarrow \bar{V} \subset \bar{X} \times \bar{Y}$ is the normalization of the closure of V . We denote by $\mathbf{Cor}_{\text{adm}}(\mathcal{X}, \mathcal{Y}) \subset \mathbf{Cor}(X, Y)$ the subgroup generated by admissible correspondences. Assume \mathcal{X} is a *proper* modulus pair. Recall from [KSY, Lem 2.2.2], that the presheaf with transfers $h_0(\mathcal{X}) \in \mathbf{PST}$ is defined by

$$h_0(\mathcal{X})(S) = \text{Coker} \left(\mathbf{Cor}_{\text{adm}}(\bar{\square} \otimes S, \mathcal{X}) \xrightarrow{i_0^* - i_1^*} \mathbf{Cor}(S, X) \right),$$

where we write S instead of (S, \emptyset) and $i_\varepsilon : S \hookrightarrow \mathbf{A}_S^1$ is the ε -section, $\varepsilon \in \{0, 1\}$. We have a natural quotient map $\mathbb{Z}_{\text{tr}}(X) \rightarrow h_0(\mathcal{X})$, where $\mathbb{Z}_{\text{tr}}(X)$ is the presheaf with transfers representing X , i.e., $\mathbb{Z}_{\text{tr}}(X)(S) = \mathbf{Cor}(S, X)$.

Definition 3.2 ([KSY, Def 2.2.4]). Let $F \in \mathbf{PST}$, $X \in \mathbf{Sm}$ and $a \in F(X)$. We say a has *SC-modulus* (or *just modulus*) \mathcal{X} , if $\mathcal{X} = (\bar{X}, X_\infty)$ is a proper modulus pair with $X = \bar{X} \setminus |X_\infty|$ and the Yoneda map $a : \mathbb{Z}_{\text{tr}}(X) \rightarrow F$, factors via

$$\begin{array}{ccc} \mathbb{Z}_{\text{tr}}(X) & \xrightarrow{a} & F, \\ & \searrow & \nearrow \exists \\ & h_0(\mathcal{X}) & \end{array}$$

i.e., for any $S \in \mathbf{Sm}$ and any correspondence $\gamma \in \mathbf{Cor}_{\text{adm}}(\bar{\square} \times S, \mathcal{X}) \subset \mathbf{Cor}(\mathbf{A}^1 \times S, X)$ we have $i_0^* \gamma^* a = i_1^* \gamma^* a$.

We say F has *SC-reciprocity*, if for all $X \in \mathbf{Sm}$ any $a \in F(X)$ has a modulus. We denote by $\mathbf{RSC} \subset \mathbf{PST}$ the full subcategory consisting of presheaves with transfers which have SC-reciprocity. Further we set

$$\mathbf{RSC}_{\text{Nis}} = \mathbf{RSC} \cap \mathbf{NST},$$

where $\mathbf{NST} \subset \mathbf{PST}$ is the full subcategory of Nisnevich sheaves with transfers.

3.3. It is shown in [KSY] that the presheaves in \mathbf{RSC} are in fact induced by presheaves on modulus pairs in the following way: Let $\mathcal{X} = (\bar{X}, X_\infty)$ and $\mathcal{Y} = (\bar{Y}, Y_\infty)$ be modulus pairs with corresponding opens X and Y , respectively. An admissible correspondence from \mathcal{X} to \mathcal{Y} (see 3.1.1) is called *left proper*, if the closure in $\bar{X} \times \bar{Y}$ of all its irreducible components is proper over \bar{X} . We denote by $\underline{\mathbf{MCor}}(\mathcal{X}, \mathcal{Y}) \subset \mathbf{Cor}(X, Y)$ the subgroup of all left proper admissible correspondences. This subgroup is stable under composition of correspondences (see [KMSY21a, Prop 1.2.3]). Hence we can define the category $\underline{\mathbf{MCor}}$ with objects the modulus pairs and morphisms given by admissible left proper correspondences. We denote by \mathbf{MCor} the full subcategory

with objects the proper modulus pairs. We denote by $\underline{\mathbf{MPST}}$ the category of presheaves on $\underline{\mathbf{MCor}}$ and by \mathbf{MPST} the category of presheaves on \mathbf{MCor} . By [KMSY21a, Prop 2.2.1, Prop 2.3.1, Prop 2.4.1] there are three pairs of adjoint functors $(\omega_!, \omega^*)$, $(\underline{\omega}_!, \underline{\omega}^*)$ and $(\tau_!, \tau^*)$:

$$\mathbf{PST} \begin{array}{c} \xleftarrow{\omega_!} \\ \xrightarrow{\omega^*} \end{array} \underline{\mathbf{MPST}} \begin{array}{c} \xleftarrow{\tau_!} \\ \xrightarrow{\tau^*} \end{array} \mathbf{MPST} \begin{array}{c} \xrightarrow{\omega_!} \\ \xleftarrow{\omega^*} \end{array} \mathbf{PST},$$

which are given by

$$(3.3.1) \quad \underline{\omega}^* F(\overline{X}, X_\infty) = F(\overline{X} \setminus |X_\infty|), \quad \underline{\omega}_! H(X) = H(X, \emptyset),$$

$$(3.3.2) \quad \omega^* F(\overline{X}, X_\infty) = F(\overline{X} \setminus |X_\infty|), \quad \omega_! G(X) \cong \varinjlim_{\mathcal{X} \in \mathbf{MSm}(X)} G(\mathcal{X}),$$

$$(3.3.3) \quad \tau^* F(\mathcal{X}) = F(\mathcal{X}), \quad \tau_! G(\mathcal{U}) \cong \varinjlim_{\mathcal{X} \in \mathbf{Comp}(\mathcal{U})} G(\mathcal{X}),$$

where $\mathbf{MSm}(X)$ is the subcategory of \mathbf{MCor} with objects the proper modulus pairs with corresponding opens X and only those morphism which map to the identity in $\mathbf{Cor}(X, X)$, and $\mathbf{Comp}(\mathcal{U})$ is the category of compactifications of $\mathcal{U} = (\overline{U}, U_\infty)$, i.e., objects are proper modulus pairs $\mathcal{X} = (\overline{X}, \overline{U}_\infty + \Sigma)$, where \overline{U}_∞ and Σ are effective Cartier divisors such that $\overline{X} \setminus |\Sigma| = \overline{U}$ and $\overline{U}_{\infty|\overline{U}} = U_\infty$, and the morphisms are those which map to the identity in $\underline{\mathbf{MCor}}(\mathcal{U}, \mathcal{U})$, see [KMSY21a, Lem 2.4.2]. The functors $\omega_!$, $\underline{\omega}_!$, $\tau_!$ are exact and we have $\omega_! = \underline{\omega}_! \tau_!$.

We denote by \mathbf{CI} the full subcategory of \mathbf{MPST} of *cube invariant* objects, i.e., those $F \in \mathbf{MPST}$, which satisfy that for any proper modulus pair \mathcal{X} the pullback along $\mathcal{X} \otimes \overline{\square} \rightarrow \mathcal{X}$ induces an isomorphism

$$F(\mathcal{X}) \cong F(\mathcal{X} \otimes \overline{\square}).$$

By [KSY, Prop 2.3.7] we have $\omega_!(\mathbf{CI}) = \mathbf{RSC}$ and there is a fully faithful left exact functor $\omega^{\mathbf{CI}} : \mathbf{RSC} \rightarrow \mathbf{CI}$ given by

$$(3.3.4) \quad \omega^{\mathbf{CI}}(F)(\overline{X}, X_\infty) = \{a \in F(\overline{X} \setminus X_\infty) \mid a \text{ has moduls } (\overline{X}, X_\infty)\}.$$

We have

$$(3.3.5) \quad \underline{\omega}_! \tau_! \omega^{\mathbf{CI}}(F) \cong \omega_! \omega^{\mathbf{CI}}(F) \cong F.$$

3.4. We recall some more definitions and results from [KMSY21a], [KMSY21b], and [Sai20] related to Nisnevich sheaves.

For $F \in \underline{\mathbf{MPST}}$ and $\mathcal{X} = (\overline{X}, X_\infty) \in \underline{\mathbf{MCor}}$ we denote by $F_{\mathcal{X}}$ the presheaf on $\overline{X}_{\text{ét}}$ defined by

$$(3.4.1) \quad (U \xrightarrow{u} \overline{X}) \mapsto F_{\mathcal{X}}(U) := F(U, u^* X_\infty).$$

We denote by $\underline{\mathbf{MNST}}$ the full subcategory of $\underline{\mathbf{MPST}}$ consisting of those F such that $F_{\mathcal{X}}$ is a Nisnevich sheaf on \overline{X} , for any $\mathcal{X} = (\overline{X}, X_\infty) \in$

MCor. Further, \mathbf{MNST} is the full subcategory of \mathbf{MPST} consisting of F such that $\tau_! F \in \underline{\mathbf{MNST}}$. By [KMSY21a, Thm 4.5.5] and [KMSY21b, Thm 4.2.4] there are exact sheafification functors (i.e., left adjoints to the natural inclusions)

$$\underline{a}_{\mathrm{Nis}} : \mathbf{MPST} \rightarrow \underline{\mathbf{MNST}}, \quad a_{\mathrm{Nis}} : \mathbf{MPST} \rightarrow \mathbf{MNST},$$

such that

- (1) $(\underline{a}_{\mathrm{Nis}} F)(\mathcal{X}) = \varinjlim_{f: Y \xrightarrow{\sim} \overline{X}} F_{\mathcal{X}, \mathrm{Nis}}(Y, f^* X_\infty)$, where $\mathcal{X} = (\overline{X}, X_\infty) \in \underline{\mathbf{MCor}}$, $F_{\mathcal{X}, \mathrm{Nis}}$ denotes the Nisnevich sheafification of the presheaf $F_{\mathcal{X}}$ on $\overline{X}_{\acute{e}t}$, and the limit is over all proper birational morphisms $f : Y \rightarrow \overline{X}$ which restrict to an isomorphism $Y \setminus |f^* X_\infty| \xrightarrow{\sim} \overline{X} \setminus |X_\infty|$;
- (2) $\tau_!$ restricts to an exact functor $\tau_! : \mathbf{MNST} \rightarrow \underline{\mathbf{MNST}}$ and satisfies

$$(3.4.2) \quad \underline{a}_{\mathrm{Nis}} \tau_! F = \tau_! a_{\mathrm{Nis}} F, \quad \text{for all } F \in \mathbf{MPST}.$$

It follows that $a_{\mathrm{Nis}} = \tau^* \underline{a}_{\mathrm{Nis}} \tau_!$ and

$$\underline{a}_{\mathrm{Nis}}|_{\underline{\mathbf{MNST}}} = \mathrm{id}_{\underline{\mathbf{MNST}}}, \quad a_{\mathrm{Nis}}|_{\mathbf{MNST}} = \mathrm{id}_{\mathbf{MNST}}.$$

By [KMSY21b, Prop 6.2.1],

$$\underline{a}_{\mathrm{Nis}} \underline{\omega}^* = \underline{\omega}^* a_{\mathrm{Nis}}^V, \quad a_{\mathrm{Nis}} \omega^* = \omega^* a_{\mathrm{Nis}}^V,$$

where $a_{\mathrm{Nis}}^V : \mathbf{PST} \rightarrow \mathbf{NST}$ is Voevodsky's Nisnevich sheafification functor (see [Voe00b, Lem 3.1.6]), and we obtain induced functors

$$\underline{\omega}^* : \mathbf{NST} \rightarrow \underline{\mathbf{MNST}}, \quad \omega^* : \mathbf{NST} \rightarrow \mathbf{MNST}.$$

Lemma 3.5. *For $F \in \mathbf{RSC}_{\mathrm{Nis}}$ we have $\omega^{\mathbf{CI}} F \subset a_{\mathrm{Nis}} \omega^{\mathbf{CI}} F \subset \omega^* F$ in \mathbf{MPST} (see Definition 3.2 and (3.3.4) for notation). Here the first inclusion is given by the unit of adjunction.*

Proof. By definition $\omega^{\mathbf{CI}} F \subset \omega^* F$. We obtain the following commutative diagram

$$\begin{array}{ccc} a_{\mathrm{Nis}} \omega^{\mathbf{CI}} F & \hookrightarrow & a_{\mathrm{Nis}} \omega^* F \\ \uparrow & & \parallel \\ \omega^{\mathbf{CI}} F & \hookrightarrow & \omega^* F, \end{array}$$

in which the vertical maps are induced by adjunction. The vertical map on the right is an isomorphism since $\omega^* F \in \mathbf{MNST}$, the top horizontal map is an inclusion since a_{Nis} is exact. This gives the statement. \square

Remark 3.6. It follows from Corollary 4.16 below that the first inclusion in Lemma 3.5 is actually an equality.

3.7. We define the category $\underline{\mathbf{MCor}}^{\text{pro}}$ as follows: The objects are pairs $\mathcal{X} = (\overline{X}, X_\infty)$, where

- (1) \overline{X} is a separated noetherian scheme over k of the form $\overline{X} = \varprojlim_{i \in I} \overline{X}_i$, with $(\overline{X}_i)_{i \in I}$ a projective system of separated finite type k -schemes indexed by a partially ordered set with affine transition maps $\tau_{i,j} : \overline{X}_i \rightarrow \overline{X}_j$, $i \geq j$,
- (2) $X_\infty = \varprojlim_{i \in I} X_{i,\infty}$, with $X_{i,\infty}$ an effective Cartier divisor on \overline{X}_i , such that $\overline{X}_i \setminus |X_{i,\infty}|$ is smooth, for all i , and $\tau_{i,j}^* X_{j,\infty} = X_{i,\infty}$, $i \geq j$,
- (3) $\overline{X} \setminus |X_\infty|$ is regular.

The morphisms are given by the admissible left proper correspondences which are verbatim defined as in 3.3. That the composition of correspondences in $\mathbf{Cor}^{\text{pro}}$ induces a well-defined composition in $\underline{\mathbf{MCor}}^{\text{pro}}$ is shown in the same way as in [KMSY21a, Prop 1.2.3].

Lemma 3.8. *There is a (up to isomorphism) canonical and faithful functor*

$$\underline{\mathbf{MCor}}^{\text{pro}} \rightarrow \mathbf{ProMCor}, \quad \varprojlim_i (\overline{X}_i, X_{i,\infty}) \mapsto (\overline{X}, X_\infty)_i.$$

Proof. Let $\mathcal{X} = (\overline{X}, X_\infty)$, $\mathcal{Y} = (\overline{Y}, Y_\infty) \in \mathbf{ProMCor}$. We write $\mathcal{X} = \varprojlim_{i \in I} \mathcal{X}_i$ with $\mathcal{X}_i = (\overline{X}_i, X_{i,\infty})$, and similarly $\mathcal{Y} = \varprojlim_{j \in J} \mathcal{Y}_j$. Set $X = \overline{X} \setminus |X_\infty|$, etc. We have to show that the injection (2.5.2) restricts to

$$(3.8.1) \quad \underline{\mathbf{MCor}}^{\text{pro}}(\mathcal{X}, \mathcal{Y}) \rightarrow \mathbf{ProMCor}((\mathcal{X}_i), (\mathcal{Y}_j)).$$

To this end let $V \in \underline{\mathbf{MCor}}^{\text{pro}}(\mathcal{X}, \mathcal{Y})$ be a left proper admissible correspondence. For $j \in J$ denote by $\sigma_j(V)$ the image of V under the projection $X \times Y \rightarrow X \times Y_j$. Then $\sigma_j(V)$ is a finite prime correspondence as was observed in the proof of Lemma 2.5. Let $\overline{V} \subset \overline{X} \times \overline{Y}$ be the closure of V . By assumption \overline{V} is proper over X . Since $\overline{X} \times Y_j$ is separated and of finite type over \overline{X} the image of \overline{V} in $\overline{X} \times Y_j$ is closed and proper over X ; hence it is equal to the closure $\overline{\sigma_j(V)}$ of $\sigma_j(V)$. Now [KMSY21a, Lem 1.2.1] yields

$$(3.8.2) \quad X_\infty|_{\overline{\sigma_j(V)}^N} \geq Y_{j,\infty}|_{\overline{\sigma_j(V)}^N},$$

with the notation from (3.1.2). As in the proof of Lemma 2.5 we find an index $i_0 \in I$ and a finite correspondence $V_{i_0,j} \subset X_{i_0} \times Y_j$ which pulls back to $\sigma_j(V)$. We can also assume (after possibly enlarging i_0) that the closure $\overline{V}_{i_0,j} \subset \overline{X}_{i_0} \times \overline{Y}_j$ of $V_{i_0,j}$ pulls back to $\overline{\sigma_j(V)}$. We obtain

the cartesian diagram

$$\begin{array}{ccc} \overline{\sigma(V_j)} & \longrightarrow & \overline{X} = \varprojlim \overline{X}_i \\ \downarrow & & \downarrow \\ \overline{V}_{i_0,j} & \longrightarrow & \overline{X}_{i_0}. \end{array}$$

Since the upper horizontal arrow is proper, the lower horizontal arrow becomes proper after possibly enlarging i_0 , see [EGA IV₃, Thm (8.10.5), (xii)]. Hence by our construction and (3.8.2), the scheme $V_{i_0,j} = \overline{V}_{i_0,j} \cap (X_{i_0} \times Y_j)$ is a left proper admissible correspondence from \mathcal{X}_{i_0} to \mathcal{Y}_j and gives a well-defined element

$$V_j \in \varinjlim_{i \in I} \mathbf{ProMCor}(\mathcal{X}_i, \mathcal{Y}_j).$$

This shows that (2.5.2) restricts to (3.8.1). \square

3.9. Let $F \in \mathbf{MPST}$. Using Lemma 3.8 we can extend F to a presheaf on $\mathbf{MCor}^{\text{pro}}$ by the formula

$$F(\mathcal{X}) = \varinjlim_i F(\mathcal{X}_i), \quad \mathcal{X} = \varprojlim_i \mathcal{X}_i \in \mathbf{MCor}^{\text{pro}}.$$

4. CONDUCTORS FOR PRESHEAVES WITH TRANSFERS

Definition 4.1. (1) We say that L is a *henselian discrete valuation field of geometric type (over k)* (or short that L is a *henselian dvf*) if L is a discrete valuation field and its ring of integers is equal to the henselization of the local ring of a smooth k -scheme U in a 1-codimensional point $x \in U^{(1)}$, i.e., $\mathcal{O}_L = \mathcal{O}_{U,x}^h$. For $n \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ we set

$$\Phi = \{L \text{ henselian dvf}\}, \quad \Phi_{\leq n} = \{L \in \Phi \mid \text{trdeg}(L/k) \leq n\}.$$

Note that in positive characteristic $\Phi_{\leq 1}$ consists precisely of the henselian dvf's with perfect residue field.

- (2) Let X be a smooth k -scheme. A *henselian dvf point* of X is a k -morphism $\text{Spec } L \rightarrow X$, with $L \in \Phi$.
- (3) Let $\mathcal{X} = (\overline{X}, X_\infty)$ be a modulus pair with $X = \overline{X} \setminus |X_\infty|$. A *henselian dvf point* of \mathcal{X} is a henselian dvf point $\rho : \text{Spec } L \rightarrow X$ extending to $\text{Spec } \mathcal{O}_L \rightarrow \overline{X}$. Note, if it exists, such an extension is unique, and if \mathcal{X} is proper, then there always exists an extension. We will denote this extension also by ρ . We will also write $\rho : \text{Spec } L \rightarrow \mathcal{X}$ for the henselian dvf point of \mathcal{X} defined by ρ .

Notation 4.2. (1) Let $F \in \mathbf{PST}$ and $X \in \mathbf{Sm}$. A henselian dvf point $\rho : \eta = \text{Spec } L \rightarrow X$ is a morphism in $\mathbf{Cor}^{\text{pro}}$ (see 2.2). Hence we get a morphism (see 2.6)

$$\rho^* : F(X) \rightarrow F(\eta) =: F(L), \quad a \mapsto \rho^* a.$$

Also $\eta = \text{Spec } L \rightarrow \text{Spec } \mathcal{O}_L = \bar{\eta}$ is in $\mathbf{Cor}^{\text{pro}}$ and we get an induced map $F(\mathcal{O}_L) := F(\bar{\eta}) \rightarrow F(L)$.

(2) Let $\mathcal{X} = (\bar{X}, X_\infty)$ be a modulus pair with $X = \bar{X} \setminus |X_\infty|$ and $\rho : \text{Spec } L \rightarrow \mathcal{X}$ a henselian dvf point. Then we denote by

$$v_L(X_\infty) = v(\rho^* X_\infty) \in \mathbb{N}_0$$

the multiplicity of X_∞ pulled back along $\bar{\rho}$.

Definition 4.3. Let $F \in \mathbf{PST}$ and $n \in [1, \infty]$. A *conductor of level n* for F is a collection of set maps

$$c = \{c_L : F(L) \rightarrow \mathbb{N}_0 \mid L \in \Phi_{\leq n}\}$$

satisfying the following properties for all $L \in \Phi_{\leq n}$ and all $X \in \mathbf{Sm}$:

- (c1) $c_L(a) = 0 \Rightarrow a \in \text{Im}(F(\mathcal{O}_L) \rightarrow F(L))$.
- (c2) $c_L(a + b) \leq \max\{c_L(a), c_L(b)\}$.
- (c3) $c_L(f_* a) \leq \lceil \frac{c_{L'}(a)}{e(L'/L)} \rceil$, for any finite morphism $f : \text{Spec } L' \rightarrow \text{Spec } L$ and any $a \in F(L')$. Here $e(L'/L)$ denotes the ramification index of L'/L and $\lceil - \rceil$ is the round up.
- (c4) Assume $a \in F(\mathbf{A}_X^1)$ satisfies $c_{k(x)(t)_\infty}(\rho_x^* a) \leq 1$, for all $x \in X$ with $\text{trdeg}(k(x)/k) \leq n - 1$, where $k(x)(t)_\infty := \text{Frac}(\mathcal{O}_{\mathbf{P}_x^1, \infty}^h)$ and $\rho_x : \text{Spec } k(x)(t)_\infty \rightarrow \mathbf{A}_X^1$ is the natural map. Then $a \in \pi^* F(X)$, with $\pi : \mathbf{A}_X^1 \rightarrow X$ the projection.
- (c5) For any $a \in F(X)$ there exists a proper modulus pair $\mathcal{X} = (\bar{X}, X_\infty)$ with $X = \bar{X} \setminus |X_\infty|$, such that for all $\rho : \text{Spec } L \rightarrow \mathcal{X}$ we have

$$c_L(\rho^* a) \leq v_L(X_\infty).$$

A conductor of level ∞ will be simply called *conductor*.

Remarks 4.4. (1) If F is homotopy invariant, then setting $c_L(a) = 0$, if $a \in \text{Im}(F(\mathcal{O}_L) \rightarrow F(L))$, and $c_L(a) = 1$ else, defines a conductor (of any level).

(2) If $c = \{c_L\}$ is a conductor for F . Then for any L we have

$$(4.4.1) \quad a \in \text{Im}(F(\mathcal{O}_L) \rightarrow F(L)) \iff c_L(a) = 0.$$

Indeed, if $a \in \text{Im}(F(\mathcal{O}_L) \rightarrow F(L))$, then we find a smooth k -scheme U , a 1-codimensional point $x \in U^{(1)}$, a k -morphism $\text{Spec } \mathcal{O}_L \rightarrow \text{Spec } \mathcal{O}_{U,x} \rightarrow U$ and an element $\tilde{a} \in F(U)$ such that

$\rho^* \tilde{a} = a \in F(L)$, where $\rho : \text{Spec } L \rightarrow \text{Spec } \mathcal{O}_L \rightarrow U$. The vanishing of $c_L(a)$ hence follows directly from 4.3(c5).

- (3) Let $c = \{c_L\}$ be a conductor. Then $c^{\leq n} := \{c_L \mid \text{trdeg}(L/k) \leq n\}$ is a conductor if and only if $c^{\leq n}$ satisfies (c4).

Definition 4.5. Let $F \in \mathbf{PST}$ and let $c = \{c_L\}$ be a conductor of level n for F . Let $\mathcal{X} = (\overline{X}, X_\infty)$ be a modulus pair with $X = \overline{X} \setminus |X_\infty|$. For $a \in F(X)$, we write

$$c_{\overline{X}}(a) \leq X_\infty$$

to mean $c_L(\rho^* a) \leq v_L(X_\infty)$, for all henselian dvf points $\rho : \text{Spec } L \rightarrow \mathcal{X}$ with $\text{trdeg}(L/k) \leq n$ (see Definition 4.1).

Lemma 4.6. Let c be a conductor of some level for $F \in \mathbf{PST}$, $X \in \mathbf{Sm}$, and $a \in F(X)$. Let $\mathcal{X} = (\overline{X}, X_\infty)$ be any proper modulus pair with $X = \overline{X} \setminus |X_\infty|$. Then there exists a natural number $n \geq 1$ such that $c_{\overline{X}}(a) \leq n \cdot X_\infty$.

Proof. By 4.3(c5), there exists a proper modulus pair $\mathcal{X}_1 = (\overline{X}_1, X_{1,\infty})$ with corresponding open X and such that $c_L(\rho^* a) \leq v_L(X_{1,\infty})$, for all ρ . We find a proper normal k -scheme \overline{X}_2 with k -morphisms $f : \overline{X}_2 \rightarrow \overline{X}$, $f_1 : \overline{X}_2 \rightarrow \overline{X}_1$ such that $\overline{X}_2 \setminus |f^* X_\infty| = X = \overline{X}_2 \setminus |f_1^* X_{1,\infty}|$. Take $n \geq 1$ with $f_1^* X_{1,\infty} \leq n \cdot f^* X_\infty$. Then for $\rho : \text{Spec } L \rightarrow X$

$$c_L(\rho^* a) \leq v_L(X_{1,\infty}) = v_L(f_1^* X_{1,\infty}) \leq v_L(n \cdot f^* X_\infty) = v_L(n \cdot X_\infty).$$

Hence the statement. \square

Proposition 4.7. Let $F \in \mathbf{PST}$ and let c be a conductor of level n for F . Then

$$\mathbf{MCor} \ni \mathcal{X} = (\overline{X}, X_\infty) \mapsto F_c(\mathcal{X}) := \{a \in F(\overline{X} \setminus |X_\infty|) \mid c_{\overline{X}}(a) \leq X_\infty\}$$

defines an object in \mathbf{MPST} . Furthermore (see 3.3 for notations):

- (1) For any $\mathcal{X} \in \mathbf{MCor}$ the pullback along the projection map $\mathcal{X} \otimes \overline{\square} \rightarrow \mathcal{X}$ induces an isomorphism $F_c(\mathcal{X}) \cong F_c(\mathcal{X} \otimes \overline{\square})$. In particular, $\tau^* F_c \in \mathbf{CI}$.
- (2) $\omega_{\dagger} \tau^* F_c \cong F$.
- (3) $F \in \mathbf{NST} \Rightarrow F_c \in \mathbf{MNST}$.

Proof. We start by showing $F_c \in \mathbf{MPST}$. Let $\mathcal{X} = (\overline{X}, X_\infty)$ and $\mathcal{Y} = (\overline{Y}, Y_\infty)$ be two modulus pairs with corresponding opens X and Y , respectively. We have to show that a left proper admissible prime correspondence $V \in \mathbf{MCor}(\mathcal{X}, \mathcal{Y}) \subset \mathbf{Cor}(X, Y)$ sends the subgroup $F_c(\mathcal{Y}) \subset F(Y)$ to $F_c(\mathcal{X}) \subset F(X)$. Take $a \in F_c(\mathcal{Y})$ and a henselian dvf point $\rho : \eta = \text{Spec } L \rightarrow \mathcal{X}$ with $\text{trdeg}(L/k) \leq n$. We have to show

$$(4.7.1) \quad c_L(\rho^* V^* a) \leq v(\overline{\rho}^* X_\infty).$$

Since $V \rightarrow X$ is finite, $(\eta \times_X V)_{\text{red}}$ is a disjoint union of points $\eta_i = \text{Spec } L_i$, with $L_i \in \Phi_{\leq n}$. Thus

$$V \circ \rho = \sum_i m_i \cdot \eta_i \in \mathbf{Cor}(\eta, Y),$$

with some multiplicities $m_i \in \mathbb{N}$. For each i we get a commutative diagram

$$\begin{array}{ccccc} & & \rho_i & & \\ & & \curvearrowright & & \\ \eta_i & \longrightarrow & V & \longrightarrow & Y \\ f_i \downarrow & & \downarrow & & \\ \eta & \xrightarrow{\rho} & X, & & \end{array}$$

where ρ_i is a henselian dvf point of Y and f_i is finite. We have $\eta_i = \Gamma_{\rho_i} \circ \Gamma_{f_i}^t$ in $\mathbf{Cor}(\eta, Y)$ (see 2.6 for the notation). Thus

$$(4.7.2) \quad \rho^* V^* = \sum_i m_i \cdot f_{i*} \rho_i^* : F(Y) \rightarrow F(\eta).$$

Since the closure \bar{V} of V in $\bar{X} \times \bar{Y}$ is proper over \bar{X} and ρ extends to $\bar{\rho}$, we see that ρ_i extends to $\bar{\rho}_i$ as in the diagram

$$\begin{array}{ccccc} & & \bar{\rho}_i & & \\ & & \curvearrowright & & \\ \text{Spec } \mathcal{O}_{L_i} & \longrightarrow & \bar{V} & \longrightarrow & \bar{Y} \\ \downarrow & & \downarrow & & \\ \text{Spec } \mathcal{O}_L & \xrightarrow{\bar{\rho}} & \bar{X}. & & \end{array}$$

Since V satisfies the modulus condition (3.1.2) we get

$$(4.7.3) \quad v_{L_i}(X_\infty) \geq v_{L_i}(Y_\infty).$$

Indeed, let B be the local ring of \bar{V} at the image of the closed point of \mathcal{O}_{L_i} , x and $y \in B$ the local equations for $X_{\infty|\bar{V}}$ and $Y_{\infty|\bar{V}}$, respectively, and \bar{x} and \bar{y} their images in $\mathcal{O}_{L_i} \setminus \{0\}$. Then (3.1.2), says that $x/y \in \text{Frac}(B)$ is a root of a monic polynomial $P(T) \in B[T]$. It follows that $\bar{x}/\bar{y} \in L_i$ is a root of the image of $P(T)$ under $B[T] \rightarrow \mathcal{O}_{L_i}[T]$, i.e., \bar{x}/\bar{y} is integral over \mathcal{O}_{L_i} , i.e., $v_{L_i}(\bar{x}) \geq v_{L_i}(\bar{y})$.

Let j be an index with $c_L(f_{j*}\rho_j^*a) = \max_i\{c_L(f_{i*}\rho_i^*a)\}$. We obtain

$$\begin{aligned}
(4.7.4) \quad c_L(\rho^*V^*a) &= c_L\left(\sum_i m_i \cdot f_{i*}\rho_i^*a\right), && \text{by (4.7.2)} \\
&\leq c_L(f_{j*}\rho_j^*a), && \text{by 4.3(c2)} \\
&\leq \left\lceil \frac{c_{L_j}(\rho_j^*a)}{e(L_j/L)} \right\rceil, && \text{by 4.3(c3)} \\
&\leq \left\lceil \frac{v_{L_j}(Y_\infty)}{e(L_j/L)} \right\rceil, && a \in F_c(\mathcal{Y}) \\
&\leq \left\lceil \frac{v_{L_j}(X_\infty)}{e(L_j/L)} \right\rceil, && \text{by (4.7.3)} \\
&= v_L(X_\infty),
\end{aligned}$$

where the last equality follows from $v_{L_j}(X_\infty) = e(L_j/L)v_L(X_\infty)$. This proves (4.7.1) and hence that F_c is in **MPST**.

Next, we prove (1). Let $\mathcal{X} = (\overline{X}, X_\infty)$ be a modulus pair with $X = \overline{X} \setminus |X_\infty|$. Denote by $\pi : X \times \mathbf{A}_k^1 \rightarrow X$ the projection and by $i_0 : X \hookrightarrow X \times \mathbf{A}_k^1$ the zero section. These define morphisms $\pi \in \mathbf{MCor}(\mathcal{X} \otimes \overline{\square}, \mathcal{X})$ and $i_0 \in \mathbf{MCor}(\mathcal{X}, \mathcal{X} \otimes \overline{\square})$. We have to show that $\pi^* : F_c(\mathcal{X}) \rightarrow F_c(\mathcal{X} \otimes \overline{\square})$ is an isomorphism. Since $i_0^*\pi^* = \text{id}_{F_c(\mathcal{X})}$, it suffices to show that π^* is surjective. Take $a \in F_c(\mathcal{X} \otimes \overline{\square})$. For any henselian dvf point $\rho : \text{Spec } L \rightarrow (\mathbf{P}_X^1, \{\infty\}_X)$, with $\text{trdeg}(L/k) \leq n$, we have

$$c_L(\rho^*a) \leq v_L(X_\infty \times \mathbf{P}^1 + \overline{X} \times \{\infty\}) = v_L(X \times \{\infty\}).$$

Hence by 4.3(c4), there exists an element $b \in F(X)$ with $\pi^*(b) = a$. We have to check that $b \in F_c(\mathcal{X})$. Take $\rho : \text{Spec } L \rightarrow \mathcal{X}$ a henselian dvf point with $\text{trdeg}(L/k) \leq n$. Then $i_0 \circ \rho : \text{Spec } L \rightarrow \mathcal{X} \otimes \overline{\square}$ is a henselian dvf point and thus

$$\begin{aligned}
c_L(\rho^*b) &= c_L(\rho^*i_0^*\pi^*b) = c_L((i_0 \circ \rho)^*a) \\
&\leq v_L(X_\infty \times \mathbf{P}^1 + \overline{X} \times \{\infty\}) = v_L(X_\infty).
\end{aligned}$$

Hence $b \in F_c(\mathcal{X})$. Statement (2) follows directly from (3.3.2) and 4.3(c5). Finally (3). For $\mathcal{X} = (\overline{X}, X_\infty)$, the presheaf $F_{c,\mathcal{X}}$ on $\overline{X}_{\text{ét}}$ (see (3.4.1)) is given by

$$(U \xrightarrow{u} \overline{X}) \mapsto \{a \in F(U \setminus |u^*X_\infty|) \mid c_U(a) \leq u^*X_\infty\}.$$

We have to show that this is a Nisnevich sheaf. Since F is a Nisnevich sheaf it suffices to show the following: Let $u : U \rightarrow \overline{X}$ be an étale map, $a \in F(U \setminus |u^*X_\infty|)$ and assume there is a Nisnevich cover $\sqcup_i U_i \xrightarrow{\sqcup u_i} U$ so that $c_{U_i}(u_i^*a) \leq u_i^*u^*X_\infty$, all i . Then we have to show $c_U(a) \leq u^*X_\infty$.

To this end, observe that if $\rho : \text{Spec } L \rightarrow (U, u^*X_\infty)$ is a henselian dvf point with $\text{trdeg}(L/k) \leq n$ and $x \in U$ is the image point of the closed point of $\text{Spec } \mathcal{O}_L$, then by the functoriality of henselization $\bar{\rho}$ factors via $\text{Spec } \mathcal{O}_L \rightarrow \text{Spec } \mathcal{O}_{U,x}^h \rightarrow U$. Hence there is an i such that $\bar{\rho}$ factors via $\text{Spec } \mathcal{O}_L \rightarrow U_i \xrightarrow{u_i} U$. Thus $c_L(\rho^*a) \leq v_L(u_i^*v^*X_\infty) = v_L(v^*X_\infty)$. This completes the proof. \square

4.8. Let $F \in \mathbf{PST}$ and let c be a conductor of some level for F . Let $F_c \in \mathbf{MPST}$ be as in Proposition 4.7. We set (see 3.3 for notation)

$$\tilde{F}_c := \tau_! \tau^* F_c \in \mathbf{MPST}.$$

By adjunction we have a natural map

$$\tilde{F}_c \rightarrow F_c$$

which is injective. Indeed, on $\mathcal{X} = (\bar{X}, X_\infty) \in \mathbf{MCor}$ it is given by the inclusion inside $F(\bar{X} \setminus |X_\infty|)$

$$\tilde{F}_c(\mathcal{X}) = \varinjlim_{\mathcal{Y} \in \mathbf{Comp}(\mathcal{X})} F_c(\mathcal{Y}) \rightarrow F_c(\mathcal{X}).$$

By Proposition 4.7 and [KMSY21b, Lem 4.2.5] (or a similar argument as in the proof of 4.7(3)) we have

$$(4.8.1) \quad F \in \mathbf{NST} \Rightarrow \tilde{F}_c \in \mathbf{MNST}.$$

4.9. Let $F \in \mathbf{RSC}$. Denote by $\mathbf{CI}(F)$ the partially ordered set consisting of those subobjects $G \subset \omega^{\mathbf{CI}} F$ in \mathbf{MPST} , such that the induced map $\omega_! G \rightarrow \omega_! \omega^{\mathbf{CI}} F = F$ is an isomorphism, and where the partial order is given by inclusion $G_1 \subset G_2$. We set

$$\mathbf{CI}(F)_{\text{Nis}} := \mathbf{CI}(F) \cap \mathbf{MNST}.$$

Lemma 4.10. *Let $F \in \mathbf{RSC}$ and $G \in \mathbf{CI}(F)$. Then $G_1 = \tau_! G \in \mathbf{MPST}$ has the following properties:*

- (1) *the unit $G_1 \hookrightarrow \omega^* \omega_! G_1$ of the adjunction $(\omega_!, \omega^*)$ is injective;*
- (2) *the counit $\tau_! \tau^* G_1 \xrightarrow{\cong} G_1$ of the adjunction $(\tau_!, \tau^*)$ is an isomorphism;*
- (3) *for all $\mathcal{X} \in \mathbf{MCor}$ the pullback $G_1(\mathcal{X}) \xrightarrow{\cong} G_1(\mathcal{X} \otimes \square)$ is an isomorphism.*

Proof. Note that (2) follows directly from $\tau^* \tau_! = \text{id}$. We show (1) and (3). The inclusion $G \hookrightarrow \omega^{\mathbf{CI}} F$ yields a commutative diagram

$$\begin{array}{ccc} G_1 & \hookrightarrow & \tau_! \omega^{\mathbf{CI}} F \\ \downarrow & & \downarrow \\ \omega^* \omega_! G_1 & \longrightarrow & \omega^* \omega_! \tau_! \omega^{\mathbf{CI}} F = \omega^* F. \end{array}$$

Here the top horizontal row is injective by the exactness of $\tau_!$, the vertical maps are induced by adjunction, the vertical map on the right is injective by (3.3.4). It follows that the vertical map on the left is injective; furthermore the injectivity of the top horizontal map and [Sai20, Lem 1.15, 1.16] imply that G_1 is $\overline{\square}$ -invariant. \square

Remark 4.11. The above lemma says that $\tau_! \mathbf{CI}(F) \subset {}^\tau \mathbf{CI}^{sp}$, in the notation of [Sai20].

Lemma 4.12. *Let $F \in \mathbf{PST}$ and let c be a conductor of some level for F . Then $\tau^* \tilde{F}_c = \tau^* F_c \in \mathbf{CI}(F)$ (see 4.8 for notation). If $F \in \mathbf{NST}$, then $\tau^* F_c \in \mathbf{CI}(F)_{\text{Nis}}$.*

Proof. By Proposition 4.7(2), it suffices to show that there is an inclusion $\tau^* F_c \hookrightarrow \omega^{\mathbf{CI}} F$ inside $\omega^* F$. For \mathcal{X} a proper modulus pair set $\mathbb{Z}_{\text{tr}}(\mathcal{X}) := \mathbf{MCor}(-, \mathcal{X})$, and

$$h_0^{\overline{\square}}(\mathcal{X}) = \text{Coker} \left(\mathbb{Z}_{\text{tr}}(\mathcal{X})(\overline{\square} \otimes -) \xrightarrow{i_0^* - i_1^*} \mathbb{Z}_{\text{tr}}(\mathcal{X}) \right).$$

By [KMSY21a, Lem 1.1.3] and [KSY, Lem 2.2.2] we have (see 3.1 and 3.3 for notation)

$$\omega_! \mathbb{Z}_{\text{tr}}(\mathcal{X}) = \mathbb{Z}_{\text{tr}}(X), \quad \omega_! h_0^{\overline{\square}}(\mathcal{X}) = h_0(\mathcal{X}),$$

where $X = \overline{X} \setminus |X_\infty|$. Take $a \in F_c(\mathcal{X}) \subset F(X)$. Since F_c is cube invariant, by Proposition 4.7, the Yoneda map $a : \mathbb{Z}_{\text{tr}}(\mathcal{X}) \rightarrow \tau^* F_c$ factors via the quotient map $\mathbb{Z}_{\text{tr}}(\mathcal{X}) \rightarrow h_0^{\overline{\square}}(\mathcal{X})$. Applying $\omega_! = \omega_! \tau_!$ we see that the Yoneda map $a : \mathbb{Z}_{\text{tr}}(X) \rightarrow F$ in \mathbf{PST} defined by $a \in F(X)$ factors via $\mathbb{Z}_{\text{tr}}(X) \rightarrow h^0(\mathcal{X})$, i.e., $a \in \omega^{\mathbf{CI}} F(\mathcal{X})$. This proves the lemma. \square

Notation 4.13. Let $L \in \Phi$. Denote by $s \in S := \text{Spec } \mathcal{O}_L$ the closed point. For all $n \geq 1$ we have $(S, n \cdot s) \in \underline{\mathbf{MCor}}^{\text{pro}}$ (see 3.7). Let $G \in \underline{\mathbf{MPST}}$; we extend it to a presheaf on $\underline{\mathbf{MCor}}^{\text{pro}}$. For $n \geq 0$ we introduce the following notation:

$$G(\mathcal{O}_L, \mathbf{m}_L^{-n}) := \begin{cases} \omega_! G(S) = G(S, \emptyset) & \text{if } n = 0 \\ G(S, n \cdot s) & \text{if } n \geq 1. \end{cases}$$

Definition 4.14. Let $F \in \mathbf{RSC}_{\text{Nis}}$ and $G \in \mathbf{CI}(F)$ (see 4.9). We denote by $c^G = \{c_L^G\}$ the family of maps $c_L^G : F(L) \rightarrow \mathbb{N}_0$, $L \in \Phi$, defined as follows

$$c_L^G(a) := \min\{n \geq 0 \mid a \in \tau_! G(\mathcal{O}_L, \mathbf{m}_L^{-n})\}.$$

This is well-defined since

$$F(L) = \underline{\omega}_! \tau_!(G)(L) = \tau_!(G)(L, \emptyset) = \bigcup_n \tau_! G(\mathcal{O}_L, \mathfrak{m}^{-n}).$$

In case $G = \omega^{\mathbf{CI}} F$ we write

$$(4.14.1) \quad c^F := c^{\omega^{\mathbf{CI}} F},$$

and call c^F the *motivic conductor* of F .

Theorem 4.15. *Let F be a presheaf with transfers.*

- (1) *If F has a conductor c of some level, then $F \in \mathbf{RSC}$.*
- (2) *If $F \in \mathbf{RSC}_{\text{Nis}}$ and $G \in \mathbf{CI}(F)$ (see 4.9), then the family $c^G = \{c_L^G\}$ (see Definition 4.14) is a conductor for F in the sense of Definition 4.3. In particular, c^F is a conductor for F .*
- (3) *Let $F \in \mathbf{RSC}_{\text{Nis}}$ and $G \in \mathbf{CI}(F)$. Then in $\underline{\mathbf{MPST}}$*

$$\tau_! G \subset \tilde{F}_{c^G}$$

and for all $L \in \Phi$ and $n \geq 0$, we have

$$\tau_! G(\mathcal{O}_L, \mathfrak{m}_L^{-n}) = \tilde{F}_{c^G}(\mathcal{O}_L, \mathfrak{m}_L^{-n}).$$

- (4) *Let $F \in \mathbf{RSC}_{\text{Nis}}$ and let c be a conductor for F (of some level). Then*

$$\tilde{F}_c \subset \tau_! \omega^{\mathbf{CI}} F = \tilde{F}_{c^F},$$

where c^F is the motivic conductor, see (4.14.1).

In particular,

$$F \in \mathbf{RSC}_{\text{Nis}} \iff F \in \mathbf{NST} \text{ and } F \text{ has a conductor (of some level).}$$

Proof. (1). We have $F = \omega_! \tau^* F_c \in \omega_!(\mathbf{CI}) \subset \mathbf{RSC}$, by Proposition 4.7 and [KSY, Prop 2.3.7]. Next (2). We check the properties from Definition 4.3. Set $G_1 := \tau_! G$. (c1) follows from $\omega_! G_1(\mathcal{O}_L) = \omega_! G(\mathcal{O}_L) = F(\mathcal{O}_L)$; (c2) is obvious. As for (c3), let L'/L be a finite extension of henselian dvf's with ramification index e . The induced finite morphism $f : \text{Spec } \mathcal{O}_{L'} \rightarrow \text{Spec } \mathcal{O}_L$ induces a morphism in $\underline{\mathbf{MCor}}^{\text{pro}}$:

$$(\text{Spec } \mathcal{O}_{L'}, n \cdot s_L) \rightarrow (\text{Spec } \mathcal{O}_{L'}, en \cdot s_{L'}),$$

where s_L (resp. $s_{L'}$) are the closed points. This yields the commutative diagram

$$\begin{array}{ccccc} G_1(\mathcal{O}_{L'}, \emptyset) & \longrightarrow & G_1(\mathcal{O}_{L'}, \mathfrak{m}_{L'}^{-en}) & \longrightarrow & \underline{\omega}^* \underline{\omega}_! G_1(\mathcal{O}_{L'}, \mathfrak{m}_{L'}^{-en}) = F(L') \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ G_1(\mathcal{O}_L, \emptyset) & \longrightarrow & G_1(\mathcal{O}_L, \mathfrak{m}_L^{-n}) & \longrightarrow & \underline{\omega}^* \underline{\omega}_! G_1(\mathcal{O}_L, \mathfrak{m}_L^{-n}) = F(L). \end{array}$$

Hence, we obtain the following inequality which implies (c3):

$$c_L^G(f_*a) \leq \min\{n \mid a \in G_1(\mathcal{O}_L, \mathfrak{m}_L^{-en})\} = \min\{n \mid c_L^G(a) \leq en\}.$$

The following claim clearly implies (c4):

Claim 4.15.1. Let $X \in \mathbf{Sm}$ and $a \in F(\mathbf{A}_X^1)$. Assume X connected with function field K . Set $K(t)_\infty := \text{Frac}(\mathcal{O}_{\mathbf{P}_{K,\infty}^1}^h)$ inducing the henselian dvf point $\text{Spec } K(t)_\infty \rightarrow (\mathbf{P}_K^1, \infty)$. Assume $c_{K(t)_\infty}^G(a_K) \leq 1$, where $a_K \in F(\mathbf{A}_K^1)$ is the restriction of a . Then $a \in F(X)$.

Proof of Claim 4.15.1. The restriction map $F(\mathbf{A}_X^1) \rightarrow F(\mathbf{A}_K^1)$ is injective, by [KSY16, Thm 6] and [KSY, Cor 3.2.3]; thus it suffices to show $a_K \in F(K)$. Set $G_{1,\text{Nis}} := \underline{a}_{\text{Nis}}(G_1)$ (see 3.4). Consider the Nisnevich localization exact sequence

$$G_{1,\text{Nis}}(\mathbf{P}_K^1, \infty) \rightarrow G_{1,\text{Nis}}(\mathbf{A}_K^1, \emptyset) \rightarrow G_1(K(t)_\infty, \emptyset)/G_1(\mathcal{O}_{K(t)_\infty}, \infty).$$

By [Sai20, Thm 4.1], we have $G_{1,\text{Nis}}(\mathbf{A}_K^1, \emptyset) = G_1(\mathbf{A}_K^1, \emptyset) = F(\mathbf{A}_K^1)$. Hence our assumption implies a_K comes from $G_{1,\text{Nis}}(\mathbf{P}_K^1, \infty)$ and the desired assertion follows from the cube invariance of $G_{1,\text{Nis}}$, see [Sai20, Thm 10.1] (and Remark 4.11),

$$G_{1,\text{Nis}}(\mathbf{P}_K^1, \infty) \simeq G_{1,\text{Nis}}(K, \emptyset) = G_1(K, \emptyset) \simeq F(K).$$

Next we prove (c5). Let $X \in \mathbf{Sm}$ and $a \in F(X)$. We can assume that X is not proper over k . Take any $\mathcal{X} = (\overline{X}, X_\infty) \in \mathbf{MCor}$ such that $X = \overline{X} - |X_\infty|$. We have

$$F(X) = \omega_! G(X) = \varinjlim_{n>0} G(\overline{X}, n \cdot X_\infty),$$

and hence $a \in G(\overline{X}, n \cdot X_\infty)$, for some n . Then, for any henselian dvf point $\text{Spec } L \rightarrow (\overline{X}, n \cdot X_\infty)$, we get $a \in G_1(\mathcal{O}_L, \mathfrak{m}_L^{-nv_L(X_\infty)})$ so that $c_L^F(a) \leq n \cdot v_L(X_\infty)$. This completes the proof of (2).

(3). It follows directly from the definition of F_{cG} in Proposition 4.7, that we have $\pi_! G \subset F_{cG}$; hence also $\pi_! G = \pi_! \tau^* \pi_! G \subset \pi_! \tau^* F_{cG} = \tilde{F}_{cG}$. Furthermore, the equality in the second part of the statement comes from the inclusions

$$\pi_! G(\mathcal{O}_L, \mathfrak{m}_L^{-n}) \subset \tilde{F}_{cG}(\mathcal{O}_L, \mathfrak{m}_L^{-n}) \subset \{a \in F(L) \mid a \in \pi_! G(\mathcal{O}_L, \mathfrak{m}_L^{-n})\},$$

where the first inclusion comes from the above and the second holds by definition. Finally (4). The inclusion $\tilde{F}_c \subset \pi_! \omega^{\mathbf{CI}} F$ follows from Lemma 4.12. The equality $\tilde{F}_{cF} = \pi_! \omega^{\mathbf{CI}} F$, now follows from this and (3). This completes the proof. \square

Corollary 4.16. *The functor $\omega^{\mathbf{CI}} : \mathbf{RSC} \rightarrow \mathbf{CI}$ restricts to a functor $\omega^{\mathbf{CI}} : \mathbf{RSC}_{\text{Nis}} \rightarrow \mathbf{CI}_{\text{Nis}} := \mathbf{CI} \cap \mathbf{MNST}$.*

Proof. Take $F \in \mathbf{RSC}_{\text{Nis}}$. By Theorem 4.15, Proposition 4.7(3), and (4.8.1) we have $\tau_1 \omega^{\mathbf{CI}} F = \tilde{F}_{c^F} \in \underline{\mathbf{MNST}}$. Hence $\omega^{\mathbf{CI}} F \in \mathbf{MNST}$, by definition, see 3.4. \square

Notation 4.17. Let $F \in \mathbf{RSC}_{\text{Nis}}$. In the following we will simply write

$$\tilde{F} := \tilde{F}_{c^F} = \tau_1 \omega^{\mathbf{CI}} F.$$

By Corollary 4.16 we have $\tau^* \tilde{F} \in \mathbf{CI}(F)_{\text{Nis}}$ (see 4.9).

Corollary 4.18. *Let $F \in \mathbf{RSC}_{\text{Nis}}$. Denote by $(c^F)^{\leq n}$ the restriction of the motivic conductor to $\text{trdeg} \leq n$. Assume $(c^F)^{\leq n}$ is a conductor of level n . Then*

$$\tilde{F}_{(c^F)^{\leq n}} = \tilde{F}.$$

Proof. Clearly $\tilde{F}_{c^F} \subset \tilde{F}_{(c^F)^{\leq n}}$, and ‘ \supset ’ holds by Theorem 4.15(4). \square

Proposition 4.19. *Let $F_1 \subset F_2$ be an inclusion in $\mathbf{RSC}_{\text{Nis}}$. Then the restriction of the motivic conductor of F_2 to F_1 is equal to the motivic conductor on F_1 , i.e.,*

$$c^{F_1} = (c^{F_2})|_{F_1}.$$

Proof. Let $a \in F_1(X)$. By the definition of the motivic conductor it suffices to show: a has modulus (\bar{X}, X_∞) as an element in $F_2(X)$, if and only if it has the same modulus as an element in $F_1(X)$. This is obvious, see Definition 3.2. \square

Lemma 4.20. *Let $F_1, F_2 \in \mathbf{RSC}_{\text{Nis}}$. Let $L \in \Phi$ and $a_i \in F_i(L)$. Then $c_L^{F_1 \oplus F_2}(a_1 + a_2) = \max\{c_L^{F_1}(a_1), c_L^{F_2}(a_2)\}$.*

Proof. Direct from Definition 4.14. \square

Proposition 4.21. *Let k_1/k be an algebraic (hence separable) field extension and let $F \in \mathbf{RSC}_{\text{Nis}, k_1}$ (i.e. F is a contravariant functor $\mathbf{Cor}_{k_1} \rightarrow \mathbf{Ab}$ which is a Nisnevich sheaf on \mathbf{Sm}_{k_1} and has SC-reciprocity). Denote by $R_{k_1/k} F : \mathbf{Sm} = \mathbf{Sm}_k \rightarrow \mathbf{Ab}$ the functor given by*

$$X \mapsto R_{k_1/k} F(X) := F(X_{k_1}),$$

where $X_{k_1} = X \times_{\text{Spec } k} \text{Spec } k_1$. Then $R_{k_1/k} F \in \mathbf{RSC}_{\text{Nis}}$ and its motivic conductor is given by

$$c_L^{R_{k_1/k} F}(a) = \max_i \{c_{L_i}^F(a_i)\},$$

where $L \otimes_k k_1 \cong \prod_i L_i$ and $a = (a_i) \in R_{k_1/k} F(L) = \prod_i F(L_i)$.

Proof. The first statement follows from the definition of $\mathbf{RSC}_{\text{Nis}}$; for the second observe that for $L \in \Phi$ the k_1 -algebra $L \otimes_k k_1 = \prod_i L_i$ is unramified over L , hence (see 4.17 for notation)

$$\widetilde{R_{k_1/k} F}(\mathcal{O}_L, \mathfrak{m}_L^{-n}) = \prod_i \tilde{F}(\mathcal{O}_{L_i}, \mathfrak{m}_{L_i}^{-n}).$$

This yields the statement. \square

4.1. Semi-continuous conductors.

Definition 4.22. Let $F \in \mathbf{PST}$ and let c be a conductor of level $n \in [1, \infty]$ for F . We say c is *semi-continuous* if it satisfies the following condition:

- (c6) Let $X \in \mathbf{Sm}$ with $\dim(X) \leq n$ and $Z \subset X$ a smooth prime divisor with generic point z and $K = \text{Frac}(\mathcal{O}_{X,z}^h)$. Then for any $a \in F(X \setminus Z)$ with $c_K(a_K) \leq r$ there exists a Nisnevich neighborhood $u : U \rightarrow X$ of z and a compactification $\mathcal{Y} = (\bar{Y}, Y_\infty)$ of $(U, r \cdot u^* Z)$ such that (see Definition 4.5 for notation)

$$c_{\bar{Y}}(a_U) \leq Y_\infty,$$

where a_U (resp. a_K) denotes the restriction of a to U (resp. K).

Lemma 4.23. *Let $F \in \mathbf{PST}$ and let c be a conductor of level n for F . The following statements are equivalent:*

- (1) c is semi-continuous;
- (2) $\tilde{F}_c(\mathcal{O}_L, \mathfrak{m}_L^{-r}) = \{a \in F(L) \mid c_L(a) \leq r\}$, for $L \in \Phi_{\leq n}$, $r \geq 0$.

Proof. Let $a \in F(L)$. Then $a \in \tilde{F}_c(\mathcal{O}_L, \mathfrak{m}_L^{-r})$ if and only if there exists a smooth scheme X , a smooth prime divisor Z on X with generic point z , a k -isomorphism $\mathcal{O}_L \cong \mathcal{O}_{X,z}^h$, an element $\tilde{a} \in F(X \setminus Z)$ restricting to a , and a compactification $\mathcal{Y} = (\bar{Y}, Y_\infty)$ of $(X, r \cdot Z)$, such that $c_{\bar{Y}}(\tilde{a}) \leq Y_\infty$. From this description we see that this ' \subset ' inclusion in (2) always holds, while this ' \supset ' inclusion is equivalent to the semi-continuity of c . \square

Corollary 4.24. *Let $F \in \mathbf{RSC}_{\text{Nis}}$ and let c be a semi-continuous conductor of level n for F . Then $(c^F)^{\leq n} \leq c$, i.e., for all $L \in \Phi_{\leq n}$ and all $a \in F(L)$ we have $c_L^F(a) \leq c_L(a)$.*

Proof. Follows from Theorem 4.15(4) and Lemma 4.23. \square

We can summarize part of the above as follows:

Theorem 4.25. *Let $F \in \mathbf{RSC}_{\text{Nis}}$.*

- (1) *Any $G \in \mathbf{CI}(F)$ (see 4.9) defines a semi-continuous conductor c^G (see 4.14). For $G_1 \subset G_2$ in $\mathbf{CI}(F)$ we have $c^{G_2} \leq c^{G_1}$.*

- (2) Let c be a conductor of level $n \in [0, \infty]$. Then $\tau^*F_c \in \mathbf{CI}(F)_{\text{Nis}}$. For $c_1 \leq c_2$ we have $\tau^*F_{c_2} \subset \tau^*F_{c_1}$. If furthermore c is semi-continuous, then $c = (c^{\tau^*F_c})^{\leq n}$.
- (3) Assume $G \in \mathbf{CI}(F)_{\text{Nis}}$ satisfies

$$G(\mathcal{X}) = \left\{ a \in F(X) \mid \rho^*a \in \pi_1 G(\mathcal{O}_L, \mathfrak{m}_L^{-v_L(X_\infty)}), \quad \forall \rho : \text{Spec } L \rightarrow \mathcal{X}, \right. \\ \left. \text{with } L \in \Phi \right\},$$

for all proper modulus pairs $\mathcal{X} = (\overline{X}, X_\infty)$ with $X = \overline{X} \setminus |X_\infty|$. Then $\tau^*F_{c_G} = G$.

- (4) Let c be a semi-continuous conductor of level n for F (possibly only defined on $\text{trdeg} \leq n$). Then there exists a unique semi-continuous conductor c^∞ for F with the following properties:

$$\tau^*F_c = \tau^*F_{c^\infty} \quad \text{and} \quad c = (c^\infty)^{\leq n}.$$

We call c^∞ the canonical extension of c .

- (5) Assume the restriction $(c^F)^{\leq n}$ of the motivic conductor to $\text{trdeg} \leq n$ is a conductor. Then its canonical extension is the motivic conductor, i.e., $((c^F)^{\leq n})^\infty = c^F$.

Proof. (1) holds Theorem 4.15(3) and Lemma 4.23. (2) follows from Lemma 4.12 and Lemma 4.23. (3) holds by the definitions involved. (4). Set $G := \tau^*F_c$. Then $G \in \mathbf{CI}(F)$ by (2) and it satisfies the condition from (3) by Lemma 4.23. Set $c^\infty := c^G$. Then c^∞ has the wanted properties by (3) and (2). Finally (5) follows from Corollary 4.18. \square

We finish this section with some lemmas which are needed later on.

Definition 4.26. Let $F \in \mathbf{RSC}_{\text{Nis}}$. We say F is proper if the following equivalent conditions are satisfied:

- (1) For all $X \in \mathbf{Sm}$ and any dense open $U \subset X$ the restriction map $F(X) \xrightarrow{\sim} F(U)$ is an isomorphism.
- (2) Any conductor c on F is trivial, i.e., $c_L = 0$ for all L .

(For this (2) \Rightarrow (1) implication use that (c4) implies that $F \in \mathbf{HI}_{\text{Nis}}$ and then the statement follows from Voevodsky's Gersten resolution, cf. [KY13, Lem 10.3].)

Lemma 4.27. Let $0 \rightarrow F_1 \xrightarrow{\varphi} F \xrightarrow{\psi} F_2 \rightarrow 0$ be an exact sequence in \mathbf{NST} and with $F_1, F_2 \in \mathbf{RSC}_{\text{Nis}}$ and assume F_1 is proper.

Then $F \in \mathbf{RSC}_{\text{Nis}}$. Any (semi-continuous) conductor c of level n on F_2 , induces a (semi-continuous) conductor $c\psi = \{c_L \circ \psi\}_L$ of level n on F . Furthermore, the motivic conductor of F is given by $c^F = c^{F_2}\psi$

Proof. Let c be a conductor of level n on F_2 . Then $c\psi$ clearly satisfies (c2), (c3), (c5) (and (c6) if c does). By the properness of F_1 we have an isomorphism $F(L)/F(\mathcal{O}_L) \cong F_2(L)/F_2(\mathcal{O}_L)$, which implies (c1). Assume $a \in F(\mathbf{A}_X^1)$ satisfies the assumption in (c4) for $c\psi$. Let $\pi : \mathbf{A}_X^1 \rightarrow X$ be the projection and $i : X \hookrightarrow \mathbf{A}_X^1$ the zero-section. Then $\psi(a - \pi^*i^*a) = \psi(a) - \pi^*i^*\psi(a) \in F_2(\mathbf{A}_X^1)$ satisfies the assumption from (c4) for c ; hence it lies in $\pi^*F_2(X)$, hence is zero; therefore $a - \pi^*i^*a \in F_1(\mathbf{A}_X^1) = \pi^*F_1(X)$, hence it is zero, i.e., $a = \pi^*i^*a$. This shows that $c\psi$ satisfies (c4). Therefore, $c\psi$ is a conductor of level n . Thus Theorem 4.15 yields $F \in \mathbf{RSC}_{\text{Nis}}$ and $\tilde{F}_{c^{F_2}\psi}(\mathcal{O}_L, \mathfrak{m}_L^{-n}) \subset \tilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-n})$. We have inclusions

$$\tilde{F}_{c^{F_2}\psi}(\mathcal{O}_L, \mathfrak{m}_L^{-n})/F_1(\mathcal{O}_L) \hookrightarrow \tilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-n})/F_1(\mathcal{O}_L) \hookrightarrow \tilde{F}_2(\mathcal{O}_L, \mathfrak{m}_L^{-n}),$$

where the second map is injective by the properness of F_1 . Since $\tilde{F}_2 = \tilde{F}_{2,c^{F_2}}$, the composition is an isomorphism; hence $c^F = c^{F_2}\psi$. \square

Lemma 4.28. *Let $\varphi : F \twoheadrightarrow G$ be a surjection in \mathbf{NST} . Let $c = \{c_L : F(L) \rightarrow \mathbb{N}\}_{L \in \Phi_{\leq n}}$ be a collection of maps. Define $\bar{c} = \{\bar{c}_L : G(L) \rightarrow \mathbb{N}\}_{L \in \Phi_{\leq n}}$ by*

$$\bar{c}_L(a) := \min\{c_L(\tilde{a}) \mid \tilde{a} \in F(L) \text{ with } \varphi(\tilde{a}) = a\}.$$

If c satisfies (c1) (resp. (c2), (c3), (c6)), then so does \bar{c} .

Furthermore, if φ has the following property: For all $X \in \mathbf{Sm}$ there exists a proper modulus pair (\bar{X}, X_∞) with $X = \bar{X} \setminus X_\infty$, such that for all $x \in \bar{X}$ the map φ induces a surjection

$$(4.28.1) \quad F(\bar{X}_{(x)}^h \setminus X_{\infty,(x)}) \twoheadrightarrow G(\bar{X}_{(x)}^h \setminus X_{\infty,(x)}),$$

where $\bar{X}_{(x)}^h = \text{Spec } \mathcal{O}_{\bar{X},x}^h$ and $X_{\infty,(x)}$ denotes the restriction of X_∞ to $\bar{X}_{(x)}^h$. Then \bar{c} satisfies (c5), if c does.

Proof. (c1). If $\bar{c}_L(a) = 0$, then there exists a lift $\tilde{a} \in F(L)$ with $c_L(\tilde{a}) = 0$, hence $\tilde{a} \in F(\mathcal{O}_L)$, by (c1) for c , hence $a \in G(\mathcal{O}_L)$.

(c2). Let $a, b \in G(L)$. Take lifts $\tilde{a}, \tilde{b} \in F(L)$ with $c_L(\tilde{a}) = \bar{c}_L(a)$ and $c_L(\tilde{b}) = \bar{c}_L(b)$. Then by (c2) for c

$$\bar{c}_L(a + b) \leq c_L(\tilde{a} + \tilde{b}) \leq \max\{c_L(\tilde{a}), c_L(\tilde{b})\} = \max\{\bar{c}_L(a), \bar{c}_L(b)\}.$$

(c3). Let $f : \text{Spec } L' \rightarrow \text{Spec } L$ be a finite extension with ramification index e and let $a \in G(L')$. Take a lift $\tilde{a} \in F(L')$ with $\bar{c}_{L'}(a) = c_{L'}(\tilde{a})$. Then by (c3) for c

$$\bar{c}_L(f_*a) \leq c_L(f_*\tilde{a}) \leq \left\lceil \frac{c_{L'}(\tilde{a})}{e} \right\rceil = \left\lceil \frac{\bar{c}_{L'}(a)}{e} \right\rceil.$$

(c6). Let $X, z \in Z, K$ be as in (c6) and $a \in G(X \setminus Z)$ with $\bar{c}_K(a_K) \leq r$. Let $\tilde{a}_K \in F(K)$ be a lift of a_K with $c_K(\tilde{a}_K) = \bar{c}_K(a_K)$. Since $\text{Spec } K = \text{Spec } \mathcal{O}_K \setminus Z_{\mathcal{O}_K}$, we find a Nisnevich neighborhood $U \rightarrow X$ of z and an element $\tilde{a} \in F(U \setminus Z)$ which restricts to \tilde{a}_K . After possibly shrinking U around z , we may assume that $\varphi(\tilde{a}) = a|_{U \setminus Z_U}$. By (c6) for F , we may shrink U further around z to obtain a compactification $\mathcal{Y} = (\bar{Y}, Y_\infty)$ of $(U, r \cdot Z_U)$ such that

$$\bar{c}_{\bar{Y}}(a_U) \leq c_{\bar{Y}}(\tilde{a}_U) \leq Y_\infty.$$

(c5)(assuming (4.28.1)). Let $X \in \mathbf{Sm}$ and $a \in G(X)$. Let $\mathcal{X} = (\bar{X}, X_\infty)$ be a proper modulus pair with $X = \bar{X} \setminus |X_\infty|$ as in (4.28.1). This condition implies that we find a finite Nisnevich cover $\{U_i \rightarrow \bar{X}\}_i$ and $\tilde{a}_i \in F(U_{i,X})$, such that $\varphi(\tilde{a}_i) = a|_{U_{i,X}}$ in $G(U_{i,X})$, where $\{U_{i,X} \rightarrow X\}_i$ is the induced Nisnevich cover of X . Let $\mathcal{Y}_i = (\bar{Y}_i, Y_{i,\infty})$ be a compactification of $(U_i, X_{\infty|U_i})$ which admits a morphism $\bar{Y}_i \rightarrow \bar{X}$ extending $U_i \rightarrow \bar{X}$ and inducing a morphism of proper modulus pairs $\mathcal{Y}_i \rightarrow \mathcal{X}$. By (c5) for c and (the proof of) Lemma 4.6 we find an integer $N \gg 0$, such that $c_L(\rho^* \tilde{a}_i) \leq N \cdot v_L(Y_{i,\infty})$, for all $\rho : \text{Spec } L \rightarrow U_{i,X} = \bar{Y}_i \setminus |Y_{i,\infty}|$, $L \in \Phi_{\leq n}$. Let $\rho : \text{Spec } L \rightarrow X$ be any henselian dvf point with $L \in \Phi_{\leq n}$; denote by $s \in \bar{X}$ the image of the closed point under the induced map $\bar{\rho} : \text{Spec } \mathcal{O}_L \rightarrow \bar{X}$. By the Nisnevich property, there exists an i and a point $s_i \in U_i$ such that $U_i \rightarrow \bar{X}$ induces an isomorphism $s_i \xrightarrow{\sim} s$. Hence $\bar{\rho}$ factors via $U_i \hookrightarrow \bar{Y}_i \rightarrow \bar{X}$. Thus

$$\bar{c}_L(\rho^* a) \leq c_L(\rho^* \tilde{a}_i) \leq N \cdot v_L(Y_{i,\infty}) = N \cdot v_L(X_\infty),$$

where for the equality we used $(Y_{i,\infty})|_{U_i} = (X_\infty)|_{U_i}$. Thus a satisfies (c5) for $(\bar{X}, N \cdot X_\infty)$. \square

4.2. Homotopy invariant subsheaves.

Corollary 4.29. *Let $F \in \mathbf{NST}$ be \mathbf{A}^1 -invariant (in particular $F \in \mathbf{RSC}_{\text{Nis}}$). Then the motivic conductor of F is given by*

$$c_L^F(a) = \begin{cases} 0 & \text{if } a \in F(\mathcal{O}_L) \\ 1 & \text{else.} \end{cases}$$

Proof. The right hand side defines a conductor, as already remarked in 4.4; it is clearly semi-continuous. By Corollary 4.24 we get ' \leq ' in the statement and (c1) forces it to be an equality. \square

4.30. We denote by \mathbf{HI} the category of \mathbf{A}^1 -invariant presheaves with transfers and set $\mathbf{HI}_{\text{Nis}} := \mathbf{HI} \cap \mathbf{NST}$. It follows immediately from Definition 3.2 that we have $\mathbf{HI} \subset \mathbf{RSC}$ and $\mathbf{HI}_{\text{Nis}} \subset \mathbf{RSC}_{\text{Nis}}$.

Let $F \in \mathbf{PST}$. For $X \in \mathbf{Sm}$, we denote by

$$h_{\mathbf{A}^1}^0(F)(X)$$

the subset of $F(X)$ formed by those sections $a \in F(X)$ for which the Yoneda map $a : \mathbb{Z}_{\mathrm{tr}}(X) \rightarrow F$ factors via

$$h_0^{\mathbf{A}^1}(X) = \mathrm{Coker}(\mathbb{Z}_{\mathrm{tr}}(X)(- \times \mathbf{A}_k^1) \xrightarrow{i_0^* - i_1^*} \mathbb{Z}_{\mathrm{tr}}(X)) \in \mathbf{PST}.$$

We immediately see that $X \mapsto h_{\mathbf{A}^1}^0(F)(X)$ defines a sub-presheaf with transfers of F , since $h_0^{\mathbf{A}^1}(X) \in \mathbf{HI}$ (see, e.g., [Voe00a, Prop 3.6]) we have $h_{\mathbf{A}^1}^0(F) \in \mathbf{HI}$; furthermore, it has the following universal property: any morphism $H \rightarrow F$ in \mathbf{PST} with $H \in \mathbf{HI}$ factors uniquely via a morphism $H \rightarrow h_{\mathbf{A}^1}^0(F)$ in \mathbf{HI} . Note, if $F \in \mathbf{NST}$, then $h_{\mathbf{A}^1}^0(F) \in \mathbf{HI}_{\mathrm{Nis}}$. Indeed, by [Voe00b, Thm 3.1.12] Nisnevich sheafification induces an exact functor $\mathbf{HI} \rightarrow \mathbf{HI}_{\mathrm{Nis}}$, thus we obtain natural inclusions in \mathbf{PST}

$$h_{\mathbf{A}^1}^0(F) \hookrightarrow h_{\mathbf{A}^1}^0(F)_{\mathrm{Nis}} \hookrightarrow F_{\mathrm{Nis}} = F,$$

since $h_{\mathbf{A}^1}^0(F)_{\mathrm{Nis}} \in \mathbf{HI}$ the second inclusion factors via $h_{\mathbf{A}^1}^0(F)$; hence $h_{\mathbf{A}^1}^0(F) = h_{\mathbf{A}^1}^0(F)_{\mathrm{Nis}}$.

Proposition 4.31. *Let $F \in \mathbf{PST}$ and let c be a conductor of level n for F . Then*

$$X \mapsto F^{c \leq 1}(X) := \left\{ a \in F(X) \mid c_L(\rho^* a) \leq 1, \quad \forall \rho : \mathrm{Spec} L \rightarrow X \right. \\ \left. \text{with } L \in \Phi_{\leq n} \right\}$$

defines a homotopy invariant sub-presheaf with transfers of F . If $F \in \mathbf{NST}$, then $F^{c \leq 1} \in \mathbf{HI}_{\mathrm{Nis}}$.

Proof. To show $F^{c \leq 1} \in \mathbf{PST}$ is equivalent to the following: let $V \in \mathbf{Cor}(X, Y)$ be a finite prime correspondence and $a \in F^{c \leq 1}(Y)$; then for all henselian dvf points $\rho : \mathrm{Spec} L \rightarrow X$ with $\mathrm{trdeg}(L/k) \leq n$, we have

$$c_L(\rho^* V^* a) \leq 1.$$

This follows from the calculation in (4.7.4). The \mathbf{A}^1 -invariance of $F^{c \leq 1}$ follows directly from (c4). The last statement is proven similarly as in Proposition 4.7(3). \square

Corollary 4.32. *Let $F \in \mathbf{RSC}_{\mathrm{Nis}}$ with motivic conductor c^F . Then*

$$h_{\mathbf{A}^1}^0(F) = F^{c^F \leq 1}.$$

Proof. By Proposition 4.31 we have $F^{c^F \leq 1} \subset h_{\mathbf{A}^1}^0(F)$. By Proposition 4.19 and Corollary 4.29 we have $(c^F)_{|h_{\mathbf{A}^1}^0(F)} = c^{h_{\mathbf{A}^1}^0(F)} \leq 1$; hence $h_{\mathbf{A}^1}^0(F) \subset F^{c^F \leq 1}$. \square

Corollary 4.33. *Let $F \in \mathbf{RSC}_{\text{Nis}}$. Assume for all $L \in \Phi$ we have*

$$(4.33.1) \quad \tilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-1}) = F(\mathcal{O}_L).$$

Let $X \in \mathbf{Sm}$ be proper over k and $U \subset X$ dense open. Then

$$h_{\mathbf{A}^1}^0(F)(U) = F(X).$$

In particular, if F satisfies (4.33.1), then $X \mapsto F(X)$ is a birational invariant on smooth proper schemes.

Proof. By Corollary 4.32

$$h_{\mathbf{A}^1}^0(F)(U) = F^{c^F \leq 1}(U).$$

Hence $F(X) \subset h_{\mathbf{A}^1}^0(F)(U)$ and by (4.33.1) also

$$h_{\mathbf{A}^1}^0(F)(U) \subset \bigcap_{x \in X^{(1)}} F(\mathcal{O}_{X,x}^h).$$

By [Sai20, Cor 0.3]

$$\bigcap_{x \in X^{(1)}} F(\mathcal{O}_{X,x}^h) = F(X).$$

All together yields the statement. \square

4.3. Local symbols.

4.34. We recall the notion of local symbols for reciprocity sheaves, see [Ser84, III, §1], [KSY16, Prop 5.2.1] or [IR17, 1.5] for details.

Let $F \in \mathbf{RSC}_{\text{Nis}}$. If L/K is a finite field extension of finitely generated fields over k , we denote by $\text{Tr}_{L/K} : F(L) \rightarrow F(K)$ the map induced by the transfer structure on F . For $X \in \mathbf{Cor}^{\text{pro}}$, $x \in X$, and $a \in F(X)$ we denote $a(x) \in F(x)$ the pullback of a along $x \hookrightarrow X$.

Let K be a function field over k and C a regular projective K -curve. Note that $C \in \mathbf{Cor}^{\text{pro}}$ by Lemma 2.4. For $x \in C_{(0)}$ a closed point we write v_x for the corresponding normalized discrete valuation on $K(C)^\times$, $\mathfrak{m}_x \subset \mathcal{O}_{C,x}$ for the maximal ideal, and set $U_x^{(n)} := 1 + \mathfrak{m}_x^n \subset \mathcal{O}_{C,x}^\times$, $n \geq 1$. Let $D = \sum n_x \cdot x$ be an effective Cartier divisor on C and $a \in \tilde{F}(C, D)$ (see 4.17 for the notation \tilde{F}). Then there exists a family of maps

$$\{(a, -)_{C/K,x} : K(C)^\times \rightarrow F(K)\}_{x \in C_{(0)}}$$

which is uniquely determined by the following properties:

- (LS1) $(a, -)_{C/K,x} : K(C)^\times \rightarrow F(K)$ is a group homomorphism;
- (LS2) $(a, f)_{C/K,x} = v_x(f) \text{Tr}_{K(x)/K}(a(x))$, for $x \in C \setminus |D|$;
- (LS3) $(a, U_x^{(n_x)})_{C/K,x} = 0$;
- (LS4) $\sum_{x \in C_{(0)}} (a, f)_{C/K,x} = 0$.

It follows from the uniqueness that the family $\{(a, -)_{C/K,x}\}$ does not depend on the chosen modulus D . Furthermore, from the uniqueness one can deduce the following properties:

(LS5) $(-, -)_{C/K,x} : F(K(C)) \times K(C)^\times \rightarrow F(K)$ is bilinear;

(LS6) let $h : F \rightarrow G$ be a morphism in $\mathbf{RSC}_{\text{Nis}}$, then in $G(K)$

$$h((a, f)_{C/K,x}) = (h(a), f)_{C/K,x}, \quad \text{all } a \in F(K(C)), f \in K(C)^\times.$$

Let K'/K be a finite field extension, $C'/K' \in \mathbf{Cor}^{\text{pto}}$ a projective curve, and $\pi : C' \rightarrow C$ a finite morphism over $\text{Spec } K' \rightarrow \text{Spec } K$, then:

(LS7) for $b \in F(K'(C'))$, $f \in K(C)^\times$, and $x \in C_{(0)}$ we have

$$(\pi_*(b), f)_{C/K,x} = \sum_{y/x} \text{Tr}_{K'/K}(b, \pi^* f)_{C'/K',y};$$

(LS8) for $a \in F(K(C))$, $g \in K'(C')^\times$, and $x \in C_{(0)}$ we have

$$(a, \pi_* g)_{C/K,x} = \sum_{y/x} \text{Tr}_{K'/K}(\pi^*(a), g)_{C'/K',y};$$

where in both cases the sum is over all $y \in C'$ mapping to x .

Lemma 4.35. *Let $F \in \mathbf{RSC}_{\text{Nis}}$, C be a regular projective and geometrically connected K -curve. Let K'/K be a finitely generated field extension, denote by $\tau : \text{Spec } K' \rightarrow \text{Spec } K$ the induced map, and by $\tau_C : C_{K'} = C \otimes_K K' \rightarrow C$ the projection. Then*

$$\sum_{y \in \tau_C^{-1}(x)} (\tau_C^* a, \tau_C^* f)_{C_{K'}/K',y} = \tau^*(a, f)_{C/K,x}, \quad \text{in } F(K')$$

for all $a \in F(K(C))$, $f \in K(C)^\times$, and $x \in C_{(0)}$.

Proof. Let $U \subset C$ be open with $a \in F(U)$. Using the Approximation Lemma, (LS1), and (LS3) we can assume that for a given $m \geq 1$ we have $f \in U_z^{(m)}$, for all $z \in C \setminus (U \cup \{x\})$; in particular choosing m large enough we get $(a, f)_{C/K,z} = 0$. Identifying f with the finite K -morphism $C \rightarrow \mathbf{P}_K^1$ we obtain $a \in F(f^{-1}(\mathbf{P}_K^1 \setminus \{1\}) \setminus \{x\})$ and (LS2), (LS4) yield

$$-(a, f)_{C/K,x} = (i_0^* - i_\infty^*) f_* a, \quad \text{in } F(K).$$

The formula in the statement now follows by applying τ^* to this equality, using the base change formula $\tau_{\mathbf{P}^1}^* \circ f_* = (\tau_C^* f)_* \tau_C^*$ induced by the cartesian diagram

$$\begin{array}{ccc} C_{K'} & \xrightarrow{\tau_C} & C \\ \tau_C^* f \downarrow & & \downarrow f \\ \mathbf{P}_{K'}^1 & \xrightarrow{\tau_{\mathbf{P}^1}} & \mathbf{P}_K^1, \end{array}$$

and using (LS1) - (LS4) backwards. \square

Lemma 4.36. *Let $L \in \Phi$. Let C be a regular curve over a k -function field K . Assume there exists a closed point $x \in C$ and a k -morphism $u : \text{Spec } \mathcal{O}_L \rightarrow C$ inducing an isomorphism $\mathcal{O}_{C,x}^h \cong \mathcal{O}_L$. Then there is an isomorphisms induced via pullback along u*

$$F(K(C))/F(\mathcal{O}_{C,x}) \xrightarrow{\cong} F(L)/F(\mathcal{O}_L).$$

If $\mathcal{O}_{C,x}$ has a coefficient field then we have an isomorphism

$$F(K(C))/F(\mathcal{O}_{C,x}, \mathfrak{m}) \xrightarrow{\cong} F(L)/F(\mathcal{O}_L, \mathfrak{m}),$$

where for a local ring $A \in \mathbf{Cor}^{\text{pro}}$ with maximal ideal \mathfrak{m} we set

$$F(A, \mathfrak{m}) := \text{Ker}(F(A) \rightarrow F(A/\mathfrak{m})).$$

Proof. We prove the first isomorphism. The natural map in the statement is compatible with pullbacks and pushforwards on both sides. Thus we can apply the standard trick replacing k by its maximal pro- ℓ extensions for various primes ℓ , to assume k is infinite. By Gabber's Presentation Theorem (see, e.g., [CTHK97, 3.1.2]) we find an open $U \subset C$ containing x , a k -function field E and an étale morphism $\varphi : U \rightarrow \mathbf{P}_E^1$ such that $x = \varphi^{-1}(\varphi(x))$ and φ induces an isomorphism $x \xrightarrow{\cong} \varphi(x)$. It follows from [Sai20, Lem 4.2, Lem 4.3], that $(U, n \cdot x)$ is a V -pair, for all $n \geq 1$, in the sense of [Sai20, Def 2.1]. If $v : U' \rightarrow U$ is an affine Nisnevich neighborhood of x with $v^{-1}(x) = \{x'\}$, then the pullback $v^* : F(K(U))/F(\mathcal{O}_{U,x}) \xrightarrow{\cong} F(K(U'))/F(\mathcal{O}_{U',x})$ is an isomorphism, by [Sai20, Lem 4.4, (3)]. We obtain the first isomorphism of the statement by taking the limit over all Nisnevich neighborhoods v . For the second isomorphism observe that if a coefficient field $\sigma : \kappa \hookrightarrow \mathcal{O}_{C,x}$ exists, then σ^* induces a splitting of the restriction to the closed point $F(\mathcal{O}_{C,x}) \rightarrow F(\kappa)$, in particular it is surjective. We obtain isomorphisms $F(\mathcal{O}_{C,x})/F(\mathcal{O}_{C,x}, \mathfrak{m}_x) \cong F(\kappa) \cong F(\mathcal{O}_L)/F(\mathcal{O}_L, \mathfrak{m}_L)$ which together with the first statement and the five lemma yield the second isomorphism in the statement. \square

4.37. Let $F \in \mathbf{RSC}_{\text{Nis}}$. Let $L \in \Phi$ have residue field $\kappa = \mathcal{O}_L/\mathfrak{m}_L$, and let $\sigma : K \hookrightarrow \mathcal{O}_L$ be a k -homomorphism such that the induced map $K \hookrightarrow \kappa$ is a finite field extension (e.g., σ could be a coefficient field.) We define the local symbol

$$(-, -)_{L, \sigma} : F(L) \times L^\times \rightarrow F(K),$$

as follows: We find a regular projective K -curve C and a κ -point $x \in C(\kappa)$ satisfying

$$(4.37.1) \quad L = \text{Frac}(\mathcal{O}_{C,x}^h), \quad \sigma : K \rightarrow \mathcal{O}_{C,x} \xrightarrow{\text{nat.}} \mathcal{O}_L.$$

Additionally we assume that $\mathcal{O}_{C,x}$ has a coefficient field. Denote by $u : \text{Spec } \mathcal{O}_L \rightarrow \text{Spec } \mathcal{O}_{C,x}$ the induced map. The symbol $(-, -)_{L,\sigma}$ is defined as the composition

$$\begin{aligned} (-, -)_{L,\sigma} : F(L) \times L^\times &\longrightarrow F(L)/F(\mathcal{O}_L, \mathfrak{m}_L) \times \varprojlim_n L^\times / U_L^{(n)} \\ &\stackrel{4.36}{\cong} F(K(C))/F(\mathcal{O}_{C,x}, \mathfrak{m}_x) \times \varprojlim_n K(C)^\times / U_x^{(n)} \xrightarrow{\overline{(-, -)}_{C/K,x}} F(K), \end{aligned}$$

where the last map is given by

$$\overline{(a, (f_n))}_{C/K,x} := (\tilde{a}, \tilde{f}_r)_{C/K,x}$$

with $\tilde{a} \in \tilde{F}(\mathcal{O}_{C,x}, \mathfrak{m}^{-r})$ a lift of a and $\tilde{f}_r \in K(C)^\times$ a lift of f_r ; this is well-defined and bilinear by (LS2), (LS3), and (LS5).

Lemma 4.38. *The symbol $(-, -)_{L,\sigma}$ defined in 4.37 above is independent of the choice of the presentation (4.37.1).*

Proof. Let $v : C' \rightarrow C$ be a K -morphism between regular projective K -curves, let $x \in C$ and $x' \in C'$ be closed points such that v is étale in a neighborhood of x' and induces an isomorphism $x' \xrightarrow{\cong} x$. Assume that $\mathcal{O}_{C,x}$ has a coefficient field. Let $E = K(C)$ and $E' = K(C')$ be the function fields. Then it suffices to show, that for all $a \in F(E)$ and $f \in E^\times$ we have

$$(4.38.1) \quad (a, f)_{C/K,x} = (v^*a, v^*f)_{C'/K,x'}.$$

We denote $E_x^\times := \varprojlim E^\times / U_x^{(n)}$ etc. Then the composition

$$E_x^\times \xrightarrow{1 \otimes \text{id}} (E_x \otimes_E E')^\times \cong \prod_{y/x} E'_y{}^\times \xrightarrow{\text{proj}} E'_{x'}{}^\times$$

is induced by v^* and is an isomorphism with inverse induced by the norm. Thus we can use the Approximation Lemma, (LS3), and the continuity of the norm map to choose $g \in E'^\times$ close to v^*f at x' and close to 1 at all $y \in v^{-1}(x) \setminus \{x'\}$ to obtain

- (1) $(v^*a, v^*f)_{C'/K,x'} = (v^*a, g)_{C'/K,x'}$;
- (2) $(v^*a, g)_{C'/K,y'} = 0$, for all $y' \in v^{-1}(x) \setminus \{x'\}$;
- (3) $(a, f)_{C/K,x} = (a, \text{Nm}_{E'/E}(g))_{C/K,x}$.

We obtain

$$(a, f)_{C/K,x} \stackrel{(LS8), (3)}{\cong} \sum_{y' \in v^{-1}(x)} (v^*a, g)_{C'/K,y'} \stackrel{(1),(2)}{\cong} (v^*a, v^*f)_{C'/K,x'},$$

which yields the statement. \square

Remark 4.39. Note that if the composition $K \xrightarrow{\sigma} \mathcal{O}_L \rightarrow \kappa$ is purely inseparable, then there does not need to exist a coefficient field of \mathcal{O}_L which contains K . This is why in 4.37 it does in general not suffice to consider coefficient fields. (In characteristic zero it does.) For coefficient fields $\sigma : K \hookrightarrow \mathcal{O}_L$ the symbol $(-, -)_{L, \sigma}$ will in general depend on the choice of σ .

Corollary 4.40. *Let L_1/L be an extension of henselian dvf's of ramification index e , i.e., $\mathfrak{m}_L \mathcal{O}_{L_1} = \mathfrak{m}_{L_1}^e$. (The extension L_1/L does not need be algebraic or finitely generated.) Let $\sigma_1 : K \rightarrow \mathcal{O}_{L_1}$ be a k -homomorphism inducing a finite field extension $K \hookrightarrow \mathcal{O}_{L_1}/\mathfrak{m}_{L_1}$. Let $F \in \mathbf{RSC}_{\text{Nis}}$ and $a \in \tilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-r})$, $r \geq 0$. Then*

$$(a_{L_1}, U_{L_1}^{(er)})_{L_1, \sigma_1} = 0,$$

where $a_{L_1} \in F(L_1)$ is the pullback of a .

Proof. We have $a_{L_1} \in \tilde{F}(\mathcal{O}_{L_1}, \mathfrak{m}_{L_1}^{-er})$ and hence the statement follows from the construction of the symbol in 4.37 and (LS3). \square

Lemma 4.41. *Let $F \in \mathbf{RSC}_{\text{Nis}}$. Let K/k be a function field, X a normal affine integral finite type K -scheme with function field E . Let $x_i \in X^{(1)}$, $i = 1, \dots, r$, be distinct one codimensional points. Then for all integers $n_i \geq 0$ the natural map*

$$\frac{F(E)}{\cap_{i=1}^r \tilde{F}(\mathcal{O}_{X, x_i}, \mathfrak{m}_{x_i}^{-n_i})} \xrightarrow{\cong} \prod_{i=1}^r \frac{F(E)}{\tilde{F}(\mathcal{O}_{X, x_i}, \mathfrak{m}_{x_i}^{-n_i})}$$

is an isomorphism, where $\tilde{F}(\mathcal{O}_{X, x_i}, \mathfrak{m}_{x_i}^0) := F(\mathcal{O}_{X, x_i})$.

Proof. Let A be the semi-localization of X at the points x_i and denote by $D = \sum_i n_i x_i$ the divisor on $U := \text{Spec } A$. (Note that we allow $|D| \not\subseteq \{x_1, \dots, x_r\}$.) We claim

$$(4.41.1) \quad \tilde{F}(U, D) = \cap_{i=1}^r \tilde{F}(\mathcal{O}_{X, x_i}, \mathfrak{m}_{x_i}^{-n_i}).$$

Indeed, by definition $\tilde{F}(U, D) = \tilde{F}_{(U, D)}(U)$; furthermore $\tilde{F}_{(U, D)}$ is a sheaf on U_{Nis} and is a subsheaf of the constant sheaf $F(K)$ (by [KSY16, Thm 6] and [KSY, Cor 3.2.3]); since $\text{Spec } E \sqcup_i \text{Spec } \mathcal{O}_{X, x_i} \rightarrow U$ is a Nisnevich cover the claim (4.41.1) follows.

The natural map in the statement is compatible with pullbacks and pushforwards on both sides. Thus we can apply the standard trick replacing k by its maximal pro- ℓ extensions for various primes ℓ , to assume k is infinite. By Gabber's Presentation Theorem (see, e.g., [CTHK97, 3.1.2]) we find a function field K_1/k and an essentially étale morphism $\varphi : U \rightarrow \mathbf{A}_{K_1}^1$ such that $\{x_1, \dots, x_r\} = \varphi^{-1}\varphi(\{x_1, \dots, x_r\}) \cong$

$\varphi(\{x_1, \dots, x_r\})$. By [Sai20, Lem 4.2, Lem 4.3] $(U, \sum_i m_i x_i)$ is a V-pair, for all $m_i \geq 0$. Let U^h be the henselization of U with respect to the radical in A (see [Ray70, XI, §2, Thm 2]) and set $D^h := D|_{U^h}$; by [Sai20, Lem 4.4, (2), (3)] we have an isomorphism

$$F(U \setminus \{x_1, \dots, x_r\})/\tilde{F}(U, D) \xrightarrow{\cong} F(U^h \setminus \{x_1, \dots, x_r\})/\tilde{F}(U^h, D^h).$$

Now the statement follows from $U^h = \sqcup_i \text{Spec } \mathcal{O}_{X, x_i}^h$, see [Ray70, XI, §2, Prop 1, 1)], (4.41.1), and Lemma 4.36. \square

Lemma 4.42. *Let $F \in \mathbf{RSC}_{\text{Nis}}$ and $\pi : \text{Spec } L' \rightarrow \text{Spec } L$ be a finite extension of henselian dvf's. Denote by $\sigma : K \rightarrow \mathcal{O}_L$ a k -morphism, such that the composition $L \rightarrow \mathcal{O}_L/\mathfrak{m}_L$ is a finite field extension; denote by $\sigma' : K \rightarrow \mathcal{O}_{L'}$ the induced map. Then we have*

- (1) $(\pi_* b, f)_{L, \sigma} = (b, \pi^* f)_{L', \sigma'}$, $b \in F(L')$, $f \in \mathcal{O}_L^\times$;
- (2) $(a, \pi_* g)_{L, \sigma} = (\pi^* a, g)_{L', \sigma'}$, $a \in F(L)$, $g \in \mathcal{O}_{L'}^\times$.

Proof. We can spread out the situation as follows: There exists a finite and surjective morphism $\bar{\pi} : C' \rightarrow C$ between regular and projective K -curves, with function fields $E' = K(C')$, $E = K(C)$, points $x' \in C'$ and $x = \bar{\pi}(x') \in C$, and elements $\tilde{a} \in F(E)$, $\tilde{b} \in F(E')$, $\tilde{f} \in \mathcal{O}_{C, x}^\times$, $g \in \mathcal{O}_{C', x'}^\times$ inducing π , σ , σ' , a , b , f , g , respectively. We prove (1): By Lemma 4.41 we find an element $b_1 \in F(E')$ with $\tilde{b} - b_1 \in F(\mathcal{O}_{C', x'})$, and $b_1 \in F(\mathcal{O}_{C', y})$, for all $y \in \bar{\pi}^{-1}(x) \setminus \{x'\}$. Since $\bar{\pi}^* \tilde{f} \in \mathcal{O}_{C', y}^\times$, for all y/x , we obtain

$$(4.42.1) \quad (b, \pi^* f)_{L', \sigma'} \stackrel{(LS1), (LS2)}{=} \sum_{y/x} (b_1, \bar{\pi}^* f)_{C'/K, y} \stackrel{(LS7)}{=} (\bar{\pi}_*(b_1), \tilde{f})_{C/K, x}.$$

Note $E' \otimes_E L \cong \prod_{y/x} E'_y$, where E'_y is the henselization of E' at y . Thus in $F(L)$ we have

$$\bar{\pi}_* b_1 = \sum_{y/x} \pi_{y*} b_1,$$

where $\pi_y : \text{Spec } L \rightarrow \text{Spec } E'_y$ is the natural map; in particular $\pi_{x'} = \pi$. Hence in $F(L)$

$$\bar{\pi}_* b_1 \equiv \pi_* b \pmod{F(\mathcal{O}_L)};$$

this together with (4.42.1) and (LS1) implies formula (1).

Now (2): By the Approximation Lemma we find $g_1 \in E'^\times$ such that

$$(\bar{\pi}^* \tilde{a}, g_1)_{C'/K, y} = 0, \quad y \in \bar{\pi}^{-1}(x) \setminus \{x'\},$$

and

$$(\bar{\pi}^* \tilde{a}, g_1)_{C'/K, x'} = (\pi^* a, g)_{L', \sigma'}.$$

Furthermore we have the following equality in L^\times

$$\mathrm{Nm}_{E'/E}(g_1) = \prod_{y/x} \mathrm{Nm}_{E'_y/L}(g_1).$$

If g_1 is close enough to 1 at the points $y \in \bar{\pi}^{-1}(x) \setminus \{x'\}$ we have $\mathrm{Nm}_{E'_y/L}(g_1) \in U_L^{(N)}$ for $N \gg 0$. Thus we can choose g_1 with the additional property

$$(\tilde{a}, \mathrm{Nm}_{E'/E}(g_1))_{C/K,x} = (a, \mathrm{Nm}_{L'/L}(g))_{L,\sigma}.$$

The formula (2) now follows from (LS8) and the above. \square

Part 2. Applications

5. ALGEBRAIC GROUPS AND THE LOCAL SYMBOL

In this section k is a perfect field and G is a commutative algebraic k -group. Note that as sheaves on \mathbf{Sm} we have $G = G_{\mathrm{red}}$ and hence we can always identify G with the smooth commutative k -group G_{red} . We fix an algebraic closure \bar{k} of k ; note $\mathrm{Spec} \bar{k} \in \mathbf{Cor}^{\mathrm{pro}}$.

5.1. Let G be a commutative algebraic k -group. Then $G \in \mathbf{RSC}_{\mathrm{Nis}}$, by [KSY, Cor 3.2.5]. Let $L \in \Phi_{\leq 1}$ have residue field κ . Let $\iota : \kappa \hookrightarrow \bar{k}$ be a k -embedding. We denote by L_ι^{sh} the strict henselization of L with respect to ι . Note that L_ι^{sh} is a henselian dvf of geometric type over \bar{k} . We write

$$(5.1.1) \quad (-, -)_{L_\iota^{sh}} : G(L_\iota^{sh}) \times L_\iota^{sh \times} \rightarrow G(\bar{k})$$

for the symbol $(-, -)_{L_\iota^{sh}, \sigma}$ from 4.37 with $\sigma : \bar{k} \hookrightarrow \mathcal{O}_{L_\iota^{sh}}$ the unique coefficient field; in this case this is the symbol defined by Rosenlicht-Serre, see [Ser84, III, §1]. If we choose a different k -embedding $\iota' : \kappa \hookrightarrow \bar{k}$, then we find an automorphism $\tau : \bar{k} \rightarrow \bar{k}$ with $\tau \circ \iota = \iota'$ inducing a (unique) isomorphism of \mathcal{O}_L -algebras $\tau : \mathcal{O}_{L_\iota^{sh}} \xrightarrow{\cong} \mathcal{O}_{L_{\iota'}^{sh}}$ and by (4.38.1)

$$\tau((a, f)_{L_\iota^{sh}}) = (\tau(a), \tau(f))_{L_{\iota'}^{sh}}.$$

We will usually drop the ι from the notation and write $L^{sh} = L_\iota^{sh}$. We define the Rosenlicht-Serre conductor of $a \in G(L)$ by

$$\mathrm{RoSe}_L(a) := \begin{cases} 0, & \text{if } a \in G(\mathcal{O}_L), \\ \min\{n \geq 1 \mid (a, U_{L^{sh}}^{(n)})_{L^{sh}} = 0\}, & \text{else.} \end{cases}$$

Note that it is independent of the choice of $\iota : \kappa \hookrightarrow \bar{k}$.

Theorem 5.2. *Let G be a commutative algebraic k -group.*

- (1) The Rosenlicht-Serre conductor $\text{RoSe} = \{\text{RoSe}_L\}_{\text{trdeg}(L/k)=1}$ is a semi-continuous conductor of level 1 on G (in the sense of Definitions 4.3 and 4.22).
- (2) Let c^G be the motivic conductor of G (see Definition 4.14) and denote by $(c^G)^{\leq 1}$ its restriction to $\Phi_{\leq 1}$. Then $\text{RoSe} = (c^G)^{\leq 1}$.

In particular, the motivic conductor extends the Rosenlicht-Serre conductor to henselian dvf's over k with non-perfect residue field and we have $\tilde{G} = \tilde{G}_{\text{RoSe}}$ (see 4.8 and 4.17 for notation).

Proof. The last statement follows from Corollary 4.18. For (1) we check that RoSe satisfies the properties from Definition 4.3. (c1) and (c2) are obvious. Let L'/L be a finite extension of henselian dvf's with $\text{trdeg}(L'/L) = 1$ and $a \in G(L')$. Let $\kappa \hookrightarrow \kappa'$ be the induced map on the residue fields and fix an embedding $\kappa' \subset \bar{k}$. Then L'^{sh} is finite over L^{sh} and $e(L'^{sh}/L^{sh}) = e(L'/L)$. Thus (c3) follows directly from Lemma 4.42(1). To check (c4) first observe, if $a \in G(\mathbf{A}_X^1)$ is not in $G(X)$ (via pullback), then we find a closed point $x \in X$ such that $a_{\mathbf{A}_x^1}$ is not in $G(x)$. (Since G is a finite type k -scheme and X is Jacobson.) Thus it suffices to show the following:

Claim. Let κ/k be a finite field extension and set $\kappa(t)_\infty = \text{Frac}(\mathcal{O}_{\mathbf{P}^1_\kappa, \infty})$. Assume $a \in G(\mathbf{A}_\kappa^1)$ has $\text{RoSe}_{\kappa(t)_\infty}(a) \leq 1$. Then $a \in G(\kappa)$.

Else $a \notin G(\kappa)$. Then its pullback $a_{\bar{k}} \in G(\mathbf{A}_{\bar{k}}^1)$ is not in $G(\bar{k})$ and we can thus find two points $x, y \in \mathbf{A}^1(\bar{k}) = \bar{k}$ such that $a_{\bar{k}}(x) \neq a_{\bar{k}}(y)$. Take $f = (t - x)/(t - y) \in \bar{k}(t)$. Then $f \in U_{\bar{k}(t)_\infty}^{(1)}$ and we obtain

$$0 = (a_{\bar{k}}, f)_{\bar{k}(t)_\infty} = -a_{\bar{k}}(x) + a_{\bar{k}}(y),$$

where the first equality follows from $\text{RoSe}_{\kappa(t)_\infty}(a) \leq 1$ and the second from (LS4) and (LS2). This yields a contradiction and thereby proves the claim. (c5) follows from the fact that G is a reciprocity sheaf and Corollary 4.40. Finally (c6) (semi-continuity for $n = 1$). Assume C is a smooth k -curve, $x \in C$ a closed point and $a \in G(C \setminus \{x\})$ with $\text{RoSe}_{L_x}(a_x) \leq n$, where $L_x = \text{Frac}(\mathcal{O}_{C,x}^h)$ and $a_x \in G(L_x)$ denotes the pullback of a . Let \bar{C} be the smooth compactification of C and let $C_\infty = (\bar{C} \setminus C)_{\text{red}}$. Choose N such that $\text{RoSe}_{L_y}(a_y) \leq N$, for all $y \in |C_\infty|$. Then $(\bar{C}, n \cdot \{x\} + N \cdot C_\infty)$ is a compactification of $(C, n \cdot \{x\})$ and we claim

$$(5.2.1) \quad \text{RoSe}_{\bar{C}}(a) \leq (n \cdot \{x\} + N \cdot C_\infty).$$

Indeed, let $\text{Spec } L \rightarrow C \setminus \{x\}$ be a henselian dvf point with $\text{trdeg}(L/k) = 1$. If $\text{Spec } \mathcal{O}_L$ maps to $C \setminus \{x\}$, then $\text{RoSe}_L(a_L) = 0$. Else we get a finite extension L^{sh}/L_y^{sh} , for some $y \in \{x\} \cup |C_\infty|$, say of ramification

index e . Let $u \in U_{L^{sh}}^{(n_y e)}$, where $n_x = n$ and $n_y = N$, for $y \neq x$. By Lemma 4.42(2) we have

$$(a_L, u)_{L^{sh}} = (a_{L_y}, \text{Nm}_{L^{sh}/L_y^{sh}}(u))_{L_y^{sh}},$$

which vanishes by $\text{Nm}_{L^{sh}/L_y^{sh}}(u) \in U_{L_y^{sh}}^{(n_y)}$ and $\text{RoSe}_{L_y}(a_y) \leq n_y$. This proves the claim (5.2.1), hence (c6), and finishes the proof of (1).

By Corollary 4.24 we have $c^{G,1} \leq \text{RoSe}$. Thus for (2) it suffices to show: If $a \in \tilde{G}(\mathcal{O}_L, \mathfrak{m}_L^{-r})$, for some $L \in \Phi_{\leq 1}$ and $r \geq 1$, then $\text{RoSe}_L(a) \leq r$. This follows from Corollary 4.40. \square

Remark 5.3. An extension of RoSe to dvf 's of higher transcendence degree over k was also constructed in [KR12] (char 0) and [KR10] (char $p > 0$). The construction essentially coincides with the extension from Theorem 5.2, but in *loc. cit.* the log version is considered, whereas here non-log one, c.f. Theorem 7.20 below.

6. DIFFERENTIAL FORMS AND IRREGULARITY OF RANK 1 CONNECTIONS

In this section we assume that the base field k has *characteristic* 0. We fix a ring homomorphism $R \rightarrow k$ which induces the structure of an R -scheme on any k -scheme. (Of main interest are $R = k$ or \mathbb{Z} .)

6.1. Kähler differentials.

6.1. Let $X \in \mathbf{Sm}$. We denote by $\Omega_{X/R}^\bullet$ the de Rham complex on X relative to R and by $d : \Omega_{X/R}^\bullet \rightarrow \Omega_{X/R}^{\bullet+1}$ the differential. We set $\Omega_X^\bullet := \Omega_{X/\mathbb{Z}}^\bullet$. We have an exact sequence

$$(6.1.1) \quad \Omega_R^1 \otimes_R \Omega_X^{\bullet-1} \rightarrow \Omega_X^\bullet \rightarrow \Omega_{X/R}^\bullet \rightarrow 0.$$

We denote the Nisnevich sheaf on \mathbf{Sm} given by $X \mapsto H^0(X, \Omega_{X/R}^q)$ by $\Omega_{/R}^q$ and set $\Omega^q := \Omega_{/\mathbb{Z}}^q$. By [KSY16, Thm A.6.2] and [KSY, Cor 3.2.5] we have $\Omega^q \in \mathbf{RSC}_{\text{Nis}}$. Since the action of finite correspondences on Ω^\bullet is Ω_k^\bullet -linear (a fortiori it is Ω_R^\bullet -linear), the morphism $\alpha^* : \Omega^q(Y) \rightarrow \Omega^q(X)$, $\alpha \in \mathbf{Cor}(X, Y)$, induces via (6.1.1) the structure of a Nisnevich sheaf with transfers on $\Omega_{/R}^q$ and we obtain $\Omega_{/R}^q \in \mathbf{RSC}_{\text{Nis}}$.

Lemma 6.2. *The differential $d : \Omega_{/R}^q \rightarrow \Omega_{/R}^{q+1}$ is a map in $\mathbf{RSC}_{\text{Nis}}$.*

Proof. We have to show, that if $\alpha \in \mathbf{Cor}(X, Y)$ is a finite correspondence, $X, Y \in \mathbf{Sm}$, then $\alpha^* d = d \alpha^*$ as maps $\Omega_{/R}^q(Y) \rightarrow \Omega_{/R}^q(X)$. Since the restriction $\Omega_{/R}^q(X) \rightarrow \Omega_{/R}^q(U)$ is injective for any dense open $U \subset X$ (by [KSY16, Thm 6]), it suffices to verify the equality after shrinking X

arbitrarily around its generic points. In particular we can assume, that X is connected and $\alpha = Z \subset X \times Y$ is a prime correspondence which is finite étale over X (here we use $\text{char}(k) = 0$). Denote by $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ the maps induced by projection. Then $Z^* = f_*g^*$. The compatibility of d with g^* is clear. Hence it remains to show $f_*d = df_*$ for a finite étale map $f : Z \rightarrow X$ between smooth schemes. In this case, we have $f_*\Omega_{Z/R}^q = f_*\mathcal{O}_Z \otimes_{\mathcal{O}_X} \Omega_{X/R}^q$ and $f_* = \text{Tr}_f \otimes \text{id}_{\Omega_{X/R}^q}$, by [CR11, Prop 2.2.23]. Thus the looked for compatibility is shown as in [Har75, II, Proof of Prop (2.2), case 2]. \square

6.3. Let $L \in \Phi$, with local parameter $t \in \mathfrak{m}_L \subset \mathcal{O}_L$. We denote $\Omega_{\mathcal{O}_L/R}^\bullet(\log)$ the dga of logarithmic differentials, i.e., the graded subalgebra of $\Omega_{L/R}^\bullet$ generated by $\Omega_{\mathcal{O}_L/R}^\bullet$ and $d\log t$. In particular, $\Omega_{\mathcal{O}_L/R}^0(\log) = \mathcal{O}_L$. For $q \geq 0$ and $a \in \Omega_{L/R}^q$, we define

$$c_L^{\text{dR}}(a) := \begin{cases} 0, & \text{if } a \in \Omega_{\mathcal{O}_L/R}^q, \\ \min \left\{ n \geq 1 \mid a \in \frac{1}{t^{n-1}} \cdot \Omega_{\mathcal{O}_L/R}^q(\log) \right\}, & \text{else.} \end{cases}$$

Theorem 6.4. *For all $q \geq 0$, the collection $c^{\text{dR}} = \{c_L^{\text{dR}}\}$ defined in 6.3 coincides with the motivic conductor, i.e., (see Definition 4.14)*

$$c^{\text{dR}} = c^{\Omega_{L/R}^q}.$$

Furthermore, the restriction $(c^{\text{dR}})^{\leq q+1}$ is a semi-continuous conductor.

Proof. We start by showing that c^{dR} is a semi-continuous conductor of level $q+1$. Properties (c1) and (c2) of Definition 4.3 are obvious.

(c3). Let L'/L be a finite extension of henselian dvf with ramification index $e = e(L'/L)$, and denote by $f : \text{Spec } L' \rightarrow \text{Spec } L$ the induced map. Let $a \in \Omega_{L'/R}^q$. We have to show:

$$(6.4.1) \quad c_L^{\text{dR}}(f_*a) \leq \left\lceil \frac{c_{L'}^{\text{dR}}(a)}{e} \right\rceil.$$

We know that f_* restricts to $\Omega_{\mathcal{O}_{L'/R}}^q \rightarrow \Omega_{\mathcal{O}_L/R}^q$ and by the well-known formula $f_*d\log = d\log \circ \text{Nm}_{L'/L}$ also to

$$(6.4.2) \quad f_* : \Omega_{\mathcal{O}_{L'/R}}^q(\log) \rightarrow \Omega_{\mathcal{O}_L/R}^q(\log).$$

Moreover, f_* is continuous with respect to the \mathfrak{m}_L -adic topology (which on $\Omega_{L'/R}^q$ is the same as the $\mathfrak{m}_{L'}$ -adic topology). We may therefore replace $\Omega_{L'/R}^q$ and $\Omega_{L/R}^q$ by the corresponding completed modules. Furthermore, it suffices to treat the two cases separately in which L'/L is either totally ramified or unramified.

1st case: $e = 1$. In this case a local parameter $t \in \mathcal{O}_L$ is also a local parameter of $\mathcal{O}_{L'}$ and hence (6.4.1) follows directly from (6.4.2) and the L -linearity of f_* .

2nd case: $e > 1$, L, L' complete and $\mathcal{O}_L/\mathfrak{m}_L = \mathcal{O}_{L'}/\mathfrak{m}_{L'}$. Let $K \hookrightarrow \mathcal{O}_L$ be a coefficient field; it also defines a coefficient field of $\mathcal{O}_{L'}$. Let $\tau \in \mathcal{O}_{L'}$ and $t \in \mathcal{O}_L$ be local parameters. Then we can identify $L' = K((\tau))$ and $\frac{1}{\tau^{n-1}} \cdot \widehat{\Omega}_{\mathcal{O}_{L'}/R}^q(\log)$ with the τ -adic completion of

$$\frac{1}{\tau^{n-1}} \cdot \left((K[\tau] \otimes_K \Omega_{K/R}^q) \oplus (K[\tau] \operatorname{dlog} \tau \otimes_K \Omega_{K/R}^{q-1}) \right).$$

Furthermore, observe that $\frac{1}{\tau^i} \operatorname{dlog} \tau = -\frac{1}{i} d\left(\frac{1}{\tau^i}\right)$, $i \geq 1$. Since f_* commutes with the differential (by 6.2) we are reduced to show:

$$(6.4.3) \quad f_*\left(\frac{1}{\tau^i}\right) \in \frac{1}{t^{r-1}} \mathcal{O}_L, \quad r := \lceil \frac{n}{e} \rceil, \text{ for all } i \in [1, n-1].$$

We compute for $i \in [1, n-1]$

$$\begin{aligned} \mathfrak{m}_L^r \cdot df_*\left(\frac{1}{\tau^i}\right) &= \mathfrak{m}_L^r \cdot f_*(-i\tau^{-i-1}d\tau) \\ &\subset f_*(\mathfrak{m}_{L'}^{er-i-1}d\tau) \\ &\subset f_*(\Omega_{\mathcal{O}_{L'}/R}^1) \subset \Omega_{\mathcal{O}_L/R}^1. \end{aligned}$$

This implies (6.4.3), once we observe that in characteristic zero we have $\mathfrak{m}_L^r \cdot da \in \Omega_{\mathcal{O}_L/R}^1$ if and only if $\mathfrak{m}_L^{r-1} \cdot a \in \mathcal{O}_L$, for any $a \in L = K((t))$.

(c4) for $c^{\operatorname{dR}, q+1}$ follows directly from the following facts, where A is a finite type smooth k -algebra:

- (i) $\Omega_{A[t]/R}^q = (k[t] \otimes_k \Omega_{A/R}^q) \oplus (\Omega_{A/R}^{q-1} \otimes_k \Omega_{k[t]/k}^1)$;
- (ii) for any non-zero $\alpha \in \Omega_{A/R}^q$ there exists a prime ideal $\mathfrak{p} \subset A$ with $\operatorname{trdeg}(k(\mathfrak{p})/k) = q$, where $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}$, such that the image of α in $\Omega_{k(\mathfrak{p})/R}^q$ is non-zero;
- (iii) $H^0(\mathbf{P}_k^1, \Omega_{\mathbf{P}_k^1/k}^1(\log \infty)) = 0$, $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}) = k$.

(Note, (ii) is easy for $R = k$ and follows in general from the natural map $\Omega_{A/R}^q \rightarrow \Omega_{A/k}^q$.)

For (c5) it suffices to observe that if $a \in H^0(X, \Omega_{X/R}^q \otimes_{\mathcal{O}_X} \mathcal{O}_X(D))$, for some proper modulus pair (X, D) , then $c_X^{\operatorname{dR}}(a) \leq D$.

Finally, (c6). Let $U = \operatorname{Spec} A$ be smooth affine and $Z \subset U$ a smooth divisor which we can assume to be principal $Z = \operatorname{Div}(t)$. Let

$$a = \frac{1}{t^{r-1}} a_1 + \frac{1}{t^{r-1}} a_2 \operatorname{dlog} t, \quad a_1 \in \Omega_{A/R}^q, a_2 \in \Omega_{A/R}^{q-1}, r \geq 1.$$

Let $(\overline{Y}, \overline{Z} + \Sigma)$ be a compactification of (U, Z) with $\overline{Z}|_U = Z$ and \overline{Y} normal. Let $\overline{Y} = \cup V_i$ be an open covering such that $V_i = \operatorname{Spec} B_i$, $\Sigma|_{V_i} = \operatorname{Div}(f_i)$, and $\overline{Z}|_{V_i} = \operatorname{Div}(\tau_i)$, with $\tau_i, f_i \in B_i$. Note that $\operatorname{Spec} B_i[1/f_i] \subset U$ is open, for all i . Hence, in $B_i[1/f_i]$ we can write

$t = \tau_i e_i$, with $e_i \in (B_i[1/f_i])^\times$. Let E_i be the Cartier divisor on V_i defined by e_i . We have $|E_i| \subset |\Sigma|_{V_i}|$. By Lemma 6.5 below, there exists $N_1 \gg 0$, such that $v_L(E_i) \leq N_1 v_L(\Sigma|_{V_i})$, for all $\text{Spec } L \rightarrow U$ and all i . Furthermore, there exists an $N_2 \geq 0$ such that $f_i^{N_2} a_1 \in \Omega_{B_i/R}^q$ and $f_i^{N_2} a_2 \in \Omega_{B_i/R}^{q-1}$, for all i . Choose $N \geq r \cdot N_1 + N_2$. Let $\rho : \text{Spec } L \rightarrow U$, $L \in \Phi$. Assume the closed point of $\text{Spec } \mathcal{O}_L$ maps into $|\bar{Z} + \Sigma| \cap V_i$, for some i . Then

$$\begin{aligned} c_L(\rho^* a) &\leq (r-1)v_L(\bar{Z}) + (r-1)v_L(E_i) + N_2 v_L(\Sigma) + 1 \\ &\leq (r-1)v_L(\bar{Z}) + (r-1)N_1 v_L(\Sigma) + N_2 v_L(\Sigma) + 1 \\ &\leq v_L(r \cdot \bar{Z} + N \cdot \Sigma). \end{aligned}$$

Hence $c_Y^{\text{dR}}(a) \leq (r \cdot \bar{Z} + N \cdot \Sigma)$, which proves (c6).

Thus c^{dR} is a semi-continuous conductor on $\Omega_{/R}^q$ and Theorem 4.15(3) yields for $n \geq 1$

$$\text{fil}_n := \frac{1}{t^{n-1}} \cdot \Omega_{\mathcal{O}_L/R}^q(\log) \subset \widetilde{\Omega_{/R}^q}(\mathcal{O}_L, \mathfrak{m}_L^{-n}),$$

for any $L \in \Phi$ with local parameter $t \in \mathcal{O}_L$. It remains to show the other inclusion. By Corollary 4.40 it suffices to show the following: Let $K \hookrightarrow \mathcal{O}_L$ be some coefficient field and extend it in the canonical way to $\sigma : K(x) \hookrightarrow \mathcal{O}_{L_x}$, where x is a variable and $L_x = \text{Frac}(\mathcal{O}_L[x]_{(t)})$. Assume $a \in \text{fil}_{r+1}$. Then the following implication holds

$$(6.4.4) \quad (a, 1 - xt^r)_{L_x, \sigma} = 0 \Rightarrow a \in \text{fil}_r,$$

where the local symbol on the left hand side is the one from 4.37 for $\Omega_{/R}^q$. Since the local symbol for $\Omega_{/R}^q$ is uniquely determined by (LS1) - (LS4), we see that it is given by

$$(a, 1 - xt^r)_{L_x, \sigma} = \text{Res}_t(a \, \text{dlog}(1 - xt^r)),$$

where we use the isomorphism $L_x = K(x)((t))$ defined by σ to compute the residue symbol on the right. To prove the implication (6.4.4) it suffices to consider a modulo fil_r ; we have

$$a \equiv \frac{1}{t^r} \alpha + \beta \frac{dt}{t^{r+1}} \pmod{\text{fil}_r},$$

for $\alpha \in \Omega_{K/R}^q$, $\beta \in \Omega_{K/R}^{q-1}$. We compute in $\Omega_{K(x)/R}^q$

$$\text{Res}_t(a \, \text{dlog}(1 - xt^r)) = -r\alpha + \beta dx.$$

This shows (6.4.4) and completes the proof. \square

Lemma 6.5. *Let X be a noetherian integral normal scheme, E, F two Cartier divisors on X and assume F is effective. If $|E| \subset |F|$, then there exists $N \geq 1$, such that for all maps $\text{Spec } \mathcal{O} \rightarrow X$ whose*

image is not contained in $|F|$, with \mathcal{O} a DVR with valuation v , we have $v(E) \leq N \cdot v(F)$.

Proof. The question is local on X ; hence we can assume E and F are given by functions $e, f \in k(X)^\times$. Let $\text{Div}(e), \text{Div}(f)$ be the associated Weil divisors. Since $|E| \subset |F|$ and F is effective we find $N \geq 1$, such that $\text{Div}(e) \leq N \cdot \text{Div}(f)$, which by the normality of X implies $f^N/e \in \Gamma(X, \mathcal{O}_X)$. This yields the statement. \square

Remark 6.6. The proof of Theorem 6.4 also shows that

$$c_L^{\text{dR}'}(a) = \begin{cases} 0, & \text{if } a \in \Omega_{\mathcal{O}_L/R}^q, \\ \min\{n \geq 2 \mid a \in \frac{1}{t^{n-1}} \cdot \Omega_{\mathcal{O}_L/R}^q\}, & \text{else,} \end{cases}$$

defines a semi-continuous conductor on Ω^q , but it coincides with the motivic one, only for $q = 0$.

Corollary 6.7. *Set $Z\Omega_{/R}^q = \text{Ker}(d : \Omega_{/R}^q \rightarrow \Omega_{/R}^{q+1})$. Then $Z\Omega_{/R}^q \in \mathbf{RSC}_{\text{Nis}}$ and its motivic conductor $c^{Z\Omega_{/R}^q} = (c^{\Omega_{/R}^q})|_{Z\Omega_{/R}^q}$ restricts to a conductor of level q .*

Proof. The formula for $c^{Z\Omega_{/R}^q}$ follows from Proposition 4.19. It remains to show that it has level q . Let $a \in Z\Omega_{/R}^q(\mathbf{A}_X^1)$ with $c_{k(x)(t)_\infty}^{\text{dR}}(a) \leq 1$, for all points $x \in X$ with $\text{trdeg}(k(x)/k) \leq q - 1$. This implies $a \in H^0(X, k[t] \otimes_k \Omega_{X/R}^q) \cap Z\Omega_{/R}^q(\mathbf{A}_X^1)$, cf. the proof of (c4) in Theorem 6.4. Hence $a \in Z\Omega_{/R}^q(X)$. This shows that $(c^{Z\Omega_{/R}^q})^{\leq q}$ satisfies (c4). \square

Corollary 6.8. (1) *Let $\mathcal{X} = (X, D) \in \mathbf{MCor}$ be a proper modulus pair. Then*

$$\widetilde{\Omega_{/R}^q}(\mathcal{X}) = H^0(X_1, \Omega_{X_1/R}^q(\log D_1)(D_1 - D_{1,\text{red}})),$$

where $\pi : X_1 \rightarrow X$ is any resolution of singularities which is an isomorphism over $X \setminus D$ and such that $D_1 := \pi^*D$ is supported on a simple normal crossings divisor. (See 4.17, for the notation $\widetilde{\Omega_{/R}^q}$.)

(2) *Let $h_{\mathbf{A}^1}^0(\Omega_{/R}^q)$ be the maximal \mathbf{A}^1 -invariant subsheaf of $\Omega_{/R}^q$. Then for $X \in \mathbf{Sm}$*

$$h_{\mathbf{A}^1}^0(\Omega_{/R}^q)(X) = H^0(\overline{X}, \Omega_{\overline{X}/R}^q(\log D)),$$

where \overline{X} is any smooth compactification of X with simple normal crossing divisor D at infinity.

Proof. First note, that $\widetilde{\Omega_{/R}^q}(\mathcal{X}) = \widetilde{\Omega_{/R}^q}(X_1, \pi^*D)$, where $\pi : X_1 \rightarrow X$ is any blow-up with center in D , since $(X, D) \cong (X_1, \pi^*D)$ in \mathbf{MCor} .

Let $\mathcal{X} = (X, D)$ be a proper modulus pair with D_{red} a simple normal crossings divisor. Write $D = \sum_i r_i \cdot \bar{\eta}_i$, with $\eta_i \in X^{(1)}$ and set $L_{\eta_i} := \text{Frac}(\mathcal{O}_{X, \eta_i}^h)$. Then it is direct to check that we have $c_L^{\text{dR}}(\rho^*a) \leq v_L(D)$, for all henselian dvf points $\rho : \text{Spec } L \rightarrow \mathcal{X}$ if and only if $c_{L_{\eta_i}}^{\text{dR}}(a) \leq r_i$, for all i . Thus the corollary follows from Theorem 6.4, Theorem 4.15(4), and Corollary 4.32. \square

6.2. Rank 1 connections and irregularity.

Lemma 6.9. *The homomorphism $\text{dlog} : \mathcal{O}_X^\times \rightarrow \Omega_{X/R}^1$, $X \in \mathbf{Sm}$, induces a morphism $\text{dlog} : \mathcal{O}^\times \rightarrow \Omega_{/R}^1$ in $\mathbf{RSC}_{\text{Nis}}$*

Proof. The proof is similar to the one of Lemma 6.2, except that we have to replace the formula $f_*d = df_*$ by $f_*\text{dlog} = \text{dlog Nm}_{Z/X}$, where $f : Z \rightarrow X$ is a finite étale map between smooth schemes. \square

6.10. Denote by $\text{Conn}^1(X)$ the group of isomorphism classes of rank 1 connections on $X \in \mathbf{Sm}$, and by $\text{Conn}_{\text{int}}^1(X)$ the subgroup of integrable connections. We have canonical group isomorphisms

$$\text{Conn}^1(X) \cong H^1(X_{\text{Zar}}, \mathcal{O}_X^\times \xrightarrow{\text{dlog}} \Omega_{X/k}^1) \cong H^0(X, (\Omega_{/k}^1 / \text{dlog } \mathcal{O}_X^\times)_{\text{Nis}})$$

and

$$\begin{aligned} \text{Conn}_{\text{int}}^1(X) &\cong H^1(X_{\text{Zar}}, \mathcal{O}_X^\times \xrightarrow{\text{dlog}} Z\Omega_{X/k}^1) \\ &\cong H^0(X, (Z\Omega_{/k}^1 / \text{dlog } \mathcal{O}^\times)_{\text{Nis}}). \end{aligned}$$

Indeed, the first isomorphism is well-known (use that the first Zariski cohomology can be computed as Čech cohomology); we show the second as follows: Let \bar{k}^X be the algebraic closure of k in $k(X)$; we consider it as a constant sheaf on X . We obtain the isomorphism

$$[\mathcal{O}_X^\times / (\bar{k}^X)^\times \xrightarrow{\text{dlog}} \Omega_{X/k}^1] \cong (\Omega_{/k}^1 / \text{dlog } \mathcal{O}_X^\times)_{\text{Zar}}[-1],$$

in the derived category of abelian sheaves on X_{Zar} ; similar with $Z\Omega_{/k}^1$. Observe that $\Omega_{/k}^1$ and \mathcal{O}^\times are already Nisnevich sheaves, hence

$$(\Omega_{/k}^1 / \text{dlog } \mathcal{O}_X^\times)_{\text{Zar}} = (\Omega_{/k}^1 / \text{dlog } \mathcal{O}_X^\times)_{\text{Nis}}.$$

Since $H^i(X_{\text{Zar}}, \bar{k}^X) = 0$ for all $i \geq 1$, we obtain

$$H^1(X_{\text{Zar}}, \mathcal{O}_X^\times \rightarrow \Omega_{X/k}^1) = H^1(X_{\text{Zar}}, \mathcal{O}_X^\times / (\bar{k}^X)^\times \rightarrow \Omega_{X/k}^1).$$

Similar with $Z\Omega_{/k}^1$. This yields the second isomorphisms.

By Lemma 6.9 and [Sai20, Thm 0.1] we obtain

$$\text{Conn}^1, \text{Conn}_{\text{int}}^1 \in \mathbf{RSC}_{\text{Nis}}.$$

For $E \in \text{Conn}^1(X)$ we denote by $\omega_E \in H^0(X, (\Omega_{/k}^1/\text{dlog } \mathcal{O}^\times)_{\text{Nis}})$, the element corresponding to E under the above isomorphism.

Let $L \in \Phi$ and let $t \in \mathcal{O}_L$ be a local parameter. Recall (e.g. from [Kat94, Def. 1.12]) that the irregularity of $E \in \text{Conn}^1(\text{Spec } L) \cong \Omega_{L/k}^1/\text{dlog } L^\times$ is defined by

$$\text{irr}_L(E) = \min \left\{ n \geq 0 \mid \omega_E \in \text{Im} \left(\frac{1}{t^n} \cdot \Omega_{\mathcal{O}_L/k}^1(\text{log}) \rightarrow \Omega_{L/k}^1/\text{dlog } L^\times \right) \right\}.$$

Theorem 6.11. *Notations are as in 6.10. The motivic conductor of $E \in \text{Conn}^1(L)$ is given by*

$$c_L^{\text{Conn}^1}(E) = \begin{cases} 0, & \text{if } E \text{ extends to an } \mathcal{O}_L\text{-connection,} \\ \text{irr}_L(E) + 1, & \text{else.} \end{cases}$$

Moreover, on Conn^1 the motivic conductor restricts to a level 2 conductor and on $\text{Conn}_{\text{int}}^1$ it restricts to a level 1 conductor.

Proof. Set $\mathcal{H}^1 := (\Omega_{/k}^1/\text{dlog } \mathcal{O}^\times)_{\text{Nis}}$, $\mathcal{H}_{\text{int}}^1 := (Z\Omega_{/k}^1/\text{dlog } \mathcal{O}^\times)_{\text{Nis}}$. For $a \in \mathcal{H}^1(L)$ we define

$$c_L^{\text{irr}}(a) := \min \{ c_L^{\text{dR}}(\tilde{a}) \mid \tilde{a} \in \Omega_{L/k}^1 \text{ lift of } a \},$$

see 6.3 for the definition of c^{dR} . It suffices to prove the following identity for the motivic conductor of \mathcal{H}^1

$$(6.11.1) \quad c^{\mathcal{H}^1} = c^{\text{irr}},$$

and that $(c^{\text{irr}})^{\leq 2}$ and $(c^{\text{irr}})^{\leq 1}_{|\mathcal{H}_{\text{int}}^1}$ satisfy (c4). It follows directly from Theorem 6.4 and Lemma 4.28, that c^{irr} satisfies (c1)-(c6) except maybe (c4) and (c5). For (c5), note that given $X \in \mathbf{Sm}$ we find by resolution of singularities a compactification $\mathcal{X} = (\bar{X}, X_\infty)$ with $\bar{X} \in \mathbf{Sm}$. In particular, for all $x \in \bar{X}$ the local ring $\mathcal{O}_{\bar{X},x}^h$ is factorial and hence so is any of its localizations. Therefore, it follows from the exact sequence

$$H^0(Y, \Omega_{Y/k}^1) \rightarrow \mathcal{H}^1(Y) \rightarrow \text{Pic}(Y),$$

for any integral scheme Y over k , that the condition (4.28.1) from Lemma 4.28 is satisfied; hence c^{irr} satisfies (c5). Next (c4). Take $a \in \mathcal{H}^1(\mathbf{A}_X^1)$ with

$$(6.11.2) \quad c_{k(x)(t)_\infty}^{\text{irr}}(a_x) \leq 1, \quad \text{for all } x \in X \text{ with } \text{trdeg}(k(x)/k) \leq 1,$$

where a_x is the restriction of a to $k(x)(t)_\infty$. Consider the exact sequence (using the \mathbf{A}^1 -invariance of $X \mapsto H^i(X, \mathcal{O}_X^\times)$)

$$(6.11.3) \quad H^0(X, \mathcal{O}_X^\times) \xrightarrow{\text{dlog}} H^0(\mathbf{A}_X^1, \Omega_{\mathbf{A}_X^1/k}^1) \rightarrow \mathcal{H}^1(\mathbf{A}_X^1) \rightarrow H^1(X, \mathcal{O}_X^\times).$$

Let $\pi : \mathbf{A}_X^1 \rightarrow X$ be the projection and $i : X \hookrightarrow \mathbf{A}_X^1$ a section. By (6.11.3) there exists an $\tilde{a} \in H^0(\mathbf{A}_X^1, \Omega_{\mathbf{A}_X^1/k}^1)$ mapping to $a - \pi^* i^* a$ and any such lift satisfies (6.11.2) with c^{irr} replaced by $c^{\Omega/k}$. Thus $\tilde{a} \in H^0(X, \Omega_{X/k}^1)$, by (c4) for $(c^{\Omega/k})^{\leq 2}$; hence $(c^{\text{irr}})^{\leq 2}$ satisfies (c4). Similarly, one proves (c4) and (c5) for $(c^{\text{irr}})_{|\mathcal{H}_{\text{int}}^1}^{\leq 1}$.

Hence c^{irr} is a semi-continuous conductor and we obtain $c^{\mathcal{H}^1} \leq c^{\text{irr}}$. We show the other inequality. Let $L \in \Phi$ and let $\sigma : K \hookrightarrow \mathcal{O}_L$ be a coefficient field. Denote by $\overline{\text{fil}}_n \subset \mathcal{H}^1(L)$ the image of $\text{fil}_n = \frac{1}{t^{n-1}} \Omega_{\mathcal{O}_L/k}^1(\log)$. Take $a \in \overline{\text{fil}}_{r+1}$. Similar as in the proof of Theorem 6.4 (around (6.4.4), and with the notation from there) it suffices to show the implication

$$(6.11.4) \quad (a, 1 - xt^r)_{L_{x,\sigma}} = 0 \text{ in } \mathcal{H}^1(K(x)) \Rightarrow a \in \overline{\text{fil}}_r.$$

Let $\tilde{a} \in \text{fil}_{r+1}$ be a lift of a ; write

$$\tilde{a} = \frac{1}{t^r} \alpha + \beta \frac{dt}{t^{r+1}} \quad \text{mod } \text{fil}_r$$

with $\alpha \in \Omega_{K/k}^1$ and $\beta \in K$. Then the left hand side of (6.11.4) is equivalent to

$$\text{Res}_t(\tilde{a} \, \text{dlog}(1 - xt^r)) = \text{dlog } f, \quad \text{for some } f \in K(x)^\times.$$

Computing the residue symbol yields

$$(6.11.5) \quad -r x \alpha + \beta dx = \text{dlog } f \quad \text{in } \Omega_{K(x)/k}^1.$$

We claim this can only happen if $\alpha = \beta = 0$. Indeed, first observe that if $h \in K((x))^\times$ is a formal Laurent series such that there exists a form $\gamma \in \Omega_{K/k}^1$ with $\text{dlog}(h) = x \cdot \gamma$ in $\widehat{\Omega}_{K((x))/k}^1$, then $\gamma = 0 = \text{dlog}(h)$. Thus (6.11.5) implies that $\text{dlog}(f \cdot \exp(-\beta x)) = 0$ in $\widehat{\Omega}_{K((x))/k}^1$. Hence there exists an element $\lambda \in k_1$ the algebraic closure of k in K such that

$$\lambda \cdot \exp(\beta x) = f \in K(x)^\times,$$

which is only possible if $\beta = 0$; it follows $\alpha = 0$. Thus $a \in \overline{\text{fil}}_r$, which proves (6.11.4) and completes the proof. \square

Corollary 6.12. *Let $X \in \mathbf{Sm}$. Then $h_{\mathbf{A}^1}^0(\text{Conn}_{\text{int}}^1)(X)$ is the group of isomorphism classes of regular singular rank 1 connections on X (see 4.30 for notation).*

Proof. Let $E \in \text{Conn}_{\text{int}}^1(X)$. Then by definition (see [Del70, II, Def 4.5]) E is regular singular if and only if $\text{irr}(\rho^* E) = 0$, for all henselian dvf points $\rho : \text{Spec } L \rightarrow X$ with $\text{trdeg}(L/k) = 1$. By Theorem 6.11 and Corollary 4.18, this is equivalent $c_L^{\text{Conn}_{\text{int}}^1}(\rho^* a) \leq 1$, for all L . Thus the statement follows from Corollary 4.32. \square

7. WITT VECTORS OF FINITE LENGTH

In this section we assume that k is a perfect field of characteristic $p > 0$. Denote by W_n the ring scheme of p -typical Witt vectors of length n . We will denote by $W_n \mathcal{O}_X$ the (Zariski-, Nisnevich-, étale-) sheaf on X defined by W_n . The restriction of W_n to k -schemes, which - by abuse of notation - we will again denote by W_n , is in particular a smooth commutative group over k . Hence $W_n \in \mathbf{RSC}_{\text{Nis}}$ (see 5.1).

7.1. Let A be a ring. Recall, that there is an isomorphism of groups

$$(7.1.1) \quad W_n(A) \xrightarrow{\cong} (1 + TA[[T]])^\times / \left\{ \prod_{s \notin \{1, p, \dots, p^{n-1}\}} (1 - b_s T^s) \mid b_s \in A \right\},$$

$$(a_0, \dots, a_{n-1}) \mapsto \prod_{i=0}^{n-1} (1 - a_i T^{p^i}).$$

Assume A is normal and we have an inclusion of rings $A \hookrightarrow B$ making B a finite A -module. Then $B[[T]]$ is finite over the normal ring $A[[T]]$ and hence the norm map, $\text{Nm} : B[[T]]^\times \rightarrow A[[T]]^\times$ induces a trace $\text{Tr} : W_n(B) \rightarrow W_n(A)$, see e.g. [Rül07b, Prop A.9].

Now assume $f : Y \rightarrow X$ is a finite and surjective k -morphism, where X is a normal k -scheme. Then the local traces above glue to give

$$\text{Tr}_f : W_n(Y) \rightarrow W_n(X).$$

Lemma 7.2. *In the situation above, Tr_f equals $f_* : W_n(Y) \rightarrow W_n(X)$, the map used to define the transfer structure on the group scheme W_n .*

Proof. Let $a \in W_n(Y)$ and $d = \deg(f)$. Recall the element $f_*(a)$ is defined by the composition

$$X \rightarrow \text{Sym}_X^d Y \xrightarrow{\sum^d a} W_n.$$

It suffices to check that $\text{Tr}_f(a)$ and $f_*(a)$ coincide on a dense open subset. Thus we can assume that X is affine integral and $f : Y \rightarrow X$ is finite free. Furthermore W_n is a direct factor of the scheme of big Witt vectors \mathbb{W}_{p^n} and Tr and f_* extend to the big Witt vectors. Thus it suffices to show the equality on the big Witt vectors \mathbb{W}_r , for $r \geq 1$. Let $S_r = \text{Spec } k[t]/(t^{r+1})$ and denote by $\varepsilon : S = \text{Spec } k \hookrightarrow S_r$ the S -section. We have the following isomorphism of S -group schemes (cf. (7.1.1))

$$\mathbb{W}_r \cong \text{Ker}(\text{Res}_{S_r/S}(\mathbf{G}_m) \xrightarrow{\varepsilon^*} \mathbf{G}_m),$$

where $\text{Res}_{S_r/S}(\mathbf{G}_m)$ denotes the Weil restriction. Denote by $f_r : Y_r \rightarrow X_r$ the base change of f along $S_r \rightarrow S$. Let $b \in \mathbb{W}_r(Y)$ which we can

view as an element in $\text{Res}_{S_r/S}(\mathbf{G}_m)(Y)$. Then the image of $f_*(b)$ in $\mathbb{W}_r(X) \subset \text{Res}_{S_r/S}(\mathbf{G}_m)(X)$ is equal to the S_r -morphism

$$f_{r*}(b) : X_r \rightarrow \text{Sym}_{X_r}^d Y_r = X_r \times_X \text{Sym}_X^d(Y) \xrightarrow{\Pi^{d,b}} \mathbf{G}_{m,S_r}.$$

Now the statement follows from the fact that $f_* = \text{Nm}$ on \mathbf{G}_m , see [SGA 4₃, Exp. XVII, Ex 6.3.18]. \square

7.3. Let $L \in \Phi$. Denote by $\text{fil}_j^{\log} W_n(L)$, $j \geq 0$, the Brylinski-Kato filtration (see [Bry83], [Kat89]), i.e.,

$$\begin{aligned} \text{fil}_j^{\log} W_n(L) &= \{a \in W_n(L) \mid [x] \cdot F^{n-1}(a) \in W_n(\mathcal{O}_L), \text{ all } x \in \mathfrak{m}_L^j\} \\ &= \{(a_0, \dots, a_{n-1}) \in W_n(L) \mid p^{n-1-i}v(a_i) \geq -j, \text{ all } i\}, \end{aligned}$$

where $[x]$ denotes the Teichmüller lift of x and $F : W_n(L) \rightarrow W_n(L)$ is the Frobenius, which by contravariant functoriality is induced by the Frobenius of L (or by covariant functoriality by the base change over $\text{Spec } k$ of the Frobenius on the $\text{Spec}(\mathbf{F}_p)$ -ring scheme W_n). We observe that for $s \geq 0$ we have

$$(7.3.1) \quad V^s(\text{fil}_j^{\log} W_n(L)) \subset \text{fil}_j^{\log} W_{n+s}(L),$$

where V is the Verschiebung on the Witt vectors. The non-log version introduced by Matsuda in [Mat97, 3.1], is given by (with the conventions from [KS16, 2.1])

$$\text{fil}_j W_n(L) = \text{fil}_{j-1}^{\log} W_n(L) + V^{n-r}(\text{fil}_j^{\log} W_r(L)), \quad j \geq 1,$$

where $r = \min\{n, \text{ord}_p(j)\}$. (This is equal to Matsuda's $\text{fil}'_{j-1} W_n(L)$.) Assume $r = \text{ord}_p(j) < n$, then $(a_0, \dots, a_{n-1}) \in \text{fil}_j W_n(L)$

$$\iff p^{n-1-i}v(a_i) \begin{cases} \geq -j, & \text{if } i \neq n-1-r, \\ > -j, & \text{else.} \end{cases}$$

This is the description given in [KR10, 4.7]. (They denote by ${}^b\text{fil}_j W_n(L)$ what we call $\text{fil}_j W_n(L)$.) One directly checks that

$$(7.3.2) \quad F^{n-1} d(\text{fil}_j W_n(L)) \subset \mathfrak{m}_L^{-j} \cdot \Omega_{\mathcal{O}_L}^1,$$

where $F^{n-1} d$ is the map

$$(7.3.3) \quad F^{n-1} d : W_n(L) \rightarrow \Omega_L^1, \quad (a_0, \dots, a_{n-1}) \mapsto \sum_{i=0}^{n-1} a_i^{p^{n-1-i}-1} da_i.$$

7.4. Let $L \in \Phi$. The F -saturation of $\text{fil}_j^{\log} W_n(L)$ and $\text{fil}_j W_n(L)$ is introduced in [KR10]:

$$(7.4.1) \quad \text{fil}_j^{\log, F} W_n(L) = \sum_{r \geq 0} F^r(\text{fil}_j^{\log} W_n(L)), \quad j \geq 0,$$

and

$$(7.4.2) \quad \mathrm{fil}_j^F W_n(L) = \sum_{r \geq 0} F^r(\mathrm{fil}_j W_n(L)), \quad j \geq 1.$$

Let κ be the residue field of \mathcal{O}_L . Denote by $\kappa[F]$ the non-commutative polynomial ring in the variable F and with coefficients in κ with relation $Fa = a^p F$ in $\kappa[F]$, for $a \in \kappa$. By [KR10, 4.7], there is an injective homomorphism for $j \geq 1$

$$(7.4.3) \quad \bar{\theta}_j : \frac{\mathrm{fil}_j^{\log, F} W_n(L)}{\mathrm{fil}_j^F W_n(L)} \hookrightarrow \kappa[F] \otimes_{\kappa} \left(\frac{\Omega_{\mathcal{O}_L}^1(\log) \otimes_{\mathcal{O}_L} \mathfrak{m}_L^{-j} / \mathfrak{m}_L^{-j+1}}{\Omega_{\mathcal{O}_L}^1 \otimes_{\mathcal{O}_L} \mathfrak{m}_L^{-j} / \mathfrak{m}_L^{-j+1}} \right)$$

induced by (cf. 7.3.2)

$$\sum_{r \geq 0} F^r(a_r) \mapsto \sum_{r \geq 0} (F^r \otimes F^{n-1} da_r).$$

For $a \in W_n(L)$, we define the Brylinski-Kato-Matsuda-Russell conductor $\gamma_{n,L}(a)$ (cf. [KR10, Thm 8.7]) by

$$\gamma_{n,L}(a) := \begin{cases} 0, & \text{if } a \in W_n(\mathcal{O}_L), \\ \min\{j \geq 1 \mid a \in \mathrm{fil}_j^F W_n(L)\}, & \text{else.} \end{cases}$$

Note that $\mathrm{fil}_1^F W_n(L) = W_n(\mathcal{O}_L)$. Thus $\gamma_{n,L}(a) = 0$ or ≥ 2 .

Proposition 7.5. *The collection*

$$\gamma_n = \{\gamma_{n,L} : W_n(L) \rightarrow \mathbb{N}_0 \mid L \in \Phi\}$$

is a semi-continuous conductor on W_n , as is its restriction $\gamma_n^{\leq 1}$.

Proof. Set $\gamma := \gamma_n$. Conditions (c1) and (c2) of Definition 4.3 are clear. (c3). Let L'/L be a finite extension of henselian dvf's. Let $e = e(L'/L)$ be the ramification index. Let $a \in W_n(L')$ and set $r := \gamma_{L'}(a)$. We have to show

$$(7.5.1) \quad \mathrm{Tr}(a) \in \mathrm{fil}_s^F W_n(L), \quad \text{with } s := \left\lceil \frac{r}{e} \right\rceil,$$

where $\mathrm{Tr} = \mathrm{Tr}_{L'/L}$, see Lemma 7.2. This is immediate if $r = 0$. Thus we can assume $r \geq 2$ and write $a = \sum_{j \geq 0} F^j(a_j)$, with $a_j \in \mathrm{fil}_r W_n(L')$. We have $\mathrm{Tr}(a_j) \in \mathrm{fil}_s^{\log} W_n(L)$. Indeed, this follows from

$$\begin{aligned} [\mathfrak{m}_L^s] \cdot F^{n-1}(\mathrm{Tr}(a_j)) &\subset \mathrm{Tr}([\mathfrak{m}_{L'}^{se}] \cdot F^{n-1}(a_j)) \\ &\subset \mathrm{Tr}([\mathfrak{m}_{L'}^r] \cdot F^{n-1}(a_j)) \\ &\subset \mathrm{Tr}(W_n(\mathcal{O}_{L'})) \subset W_n(\mathcal{O}_L), \end{aligned}$$

where for $b \in W_n(L)$ we denote $[\mathfrak{m}_L^j] \cdot b := \{[x] \cdot b \mid x \in \mathfrak{m}_L^j\}$. Hence

$$\mathrm{Tr}(a) = \sum_j F^j(\mathrm{Tr}(a_j)) \in \mathrm{fil}_s^{\log, F} W_n(L).$$

By the injectivity of $\bar{\theta}_s$ in (7.4.3) it suffices to show

$$(7.5.2) \quad \mathfrak{m}_L^s \cdot F^{n-1} d \mathrm{Tr}(a_j) \in \Omega_{\mathcal{O}_L}^1, \quad \text{all } j \geq 0.$$

By [Rül07b, Thm 2.6] the trace $\mathrm{Tr} : W_n \Omega_{L'} \rightarrow W_n \Omega_L$ which is compatible with the differential and Frobenius, is $W_n \Omega_L$ -linear, and equals the classical trace on Kähler differentials for $n = 1$. We obtain

$$\begin{aligned} \mathfrak{m}_L^s \cdot F^{n-1} d \mathrm{Tr}(a) &= \mathfrak{m}_L^s \cdot \mathrm{Tr}(F^{n-1} da) \\ &\subset \mathfrak{m}_L^s \cdot \mathrm{Tr}(\mathfrak{m}_{L'}^{-r} \cdot \Omega_{\mathcal{O}_{L'}}^1), \quad a \in \mathrm{fil}_r W_n(L'), \text{ see (7.3.2)} \\ &\subset \mathrm{Tr}(\mathfrak{m}_{L'}^{es-r} \cdot \Omega_{\mathcal{O}_{L'}}^1) \\ &\subset \mathrm{Tr}(\Omega_{\mathcal{O}_{L'}}^1) \subset \Omega_{\mathcal{O}_L}^1. \end{aligned}$$

This completes the proof of (c3).

Next we show that the restriction of γ to $\Phi_{\leq 1}$ satisfies (c4). Let $X \in \mathbf{Sm}$ and $a \in W_n(\mathbf{A}_X^1)$ with

$$(7.5.3) \quad \gamma_{k(x)(t)_\infty}(\rho_x^* a) \leq 1,$$

for closed points $x \in X$, where $k(x)(t)_\infty = \mathrm{Frac}(\mathcal{O}_{\mathbf{P}_x^1, \infty}^h)$. We have to show $a \in W_n(X)$. We may assume $X = \mathrm{Spec} A$, and thus $a \in W_n(A[t])$. If a is not constant, then we find a closed point $x \in X$ such that the image of a in $W_n(k(x)[t])$ is not constant. Hence $a_{k(x)(t)_\infty} \notin W_n(\mathcal{O}_{k(x)(t)_\infty})$, i.e., $\gamma(a_{k(x)(t)_\infty}) \geq 2$, contradicting our assumption (7.5.3).

(c5). Let $X \in \mathbf{Sm}$ and $a \in W_n(X) = H^0(X, W_n \mathcal{O}_X)$. Let $\mathcal{X} = (\bar{X}, X_\infty)$ be a proper modulus pair with $X = \bar{X} \setminus |X_\infty|$. For an effective Cartier divisor E on \bar{X} denote by $W_n \mathcal{O}_{\bar{X}}(E)$ the invertible subsheaf of $j_* W_n \mathcal{O}_{\bar{X} \setminus |E|}$ corresponding to the image of $[\mathcal{O}_{\bar{X}}(E)] \in H_{\text{ét}}^1(\bar{X}, \mathcal{O}_{\bar{X}}^\times)$ in $H_{\text{ét}}^1(\bar{X}, W_n \mathcal{O}_{\bar{X}}^\times)$ under the map induced by the Teichmüller lift. If e is an equation for E at $x \in \bar{X}$, then $W_n \mathcal{O}_{\bar{X}, x}(E) = W_n \mathcal{O}_{\bar{X}, x} \cdot \frac{1}{[e]}$. There exists an integer N such that $a \in H^0(\bar{X}, W_n \mathcal{O}_{\bar{X}}(N \cdot X_\infty))$.

Claim 7.5.1. (\bar{X}, rX_∞) satisfies (c5) for any $r > p^{n-1}N$.

Indeed, let $\rho : \mathrm{Spec} L \rightarrow \mathcal{X}$ be a henselian dvf point. Assume that the closed point $s \in S_L$ maps into X_∞ and let $f \in \mathcal{O}_{\bar{X}, \bar{\rho}(s)}$ be a local equation for X_∞ . Let $m = v_L(f)$. For $r > p^{n-1}N$ we find $[\mathfrak{m}_L^{rm-1}] \cdot F^{n-1}(a) \in W_n(\mathcal{O}_L)$; hence (see 7.3)

$$a \in \mathrm{fil}_{rm-1}^{\log} W_n(L) \subset \mathrm{fil}_{rm} W_n(L) \subset \mathrm{fil}_{rm}^F W_n(L),$$

i.e., $\gamma_L(\rho^*a) \leq rm = v_L(r \cdot X_\infty)$, proving Claim 7.5.1.

Finally, (c6). Let $X \in \mathbf{Sm}$ and $Z \subset X$ a smooth prime divisor with generic point z . Set $K = \text{Frac}(\mathcal{O}_{X,z}^h)$. Let $a \in W_n(X \setminus Z)$. Assume $a_K \in \text{fil}_j^F W_n(K)$, $j \geq 2$. Then there exists an affine Nisnevich neighborhood $U = \text{Spec } A \rightarrow X$ of z such that $Z_U = \text{div}(t)$ on U and $a_U = \sum_{s \geq 0} F^s(a_s + V^{n-r}(b_s))$, where $r = \min\{\text{ord}_p(j), n\}$ and $a_s \in W_n(A[1/t])$, $b_s \in W_r(A[1/t])$ with

$$(7.5.4) \quad [t]^{j-1} \cdot F^{n-1}(a_s) \in W_n(A), \quad [t]^j \cdot F^{r-1}(b_s) \in W_r(A).$$

Let $(\bar{Y}, \bar{Z} + \Sigma)$ be a compactification of (U, Z) with $\bar{Z}|_U = Z$ and \bar{Y} normal. Let $\bar{Y} = \cup V_i$ be an open covering such that $V_i = \text{Spec } B_i$, $\Sigma|_{V_i} = \text{Div}(f_i)$, and $\bar{Z}|_{V_i} = \text{Div}(\tau_i)$, with $\tau_i, f_i \in B_i$. Note that $\text{Spec } B_i[1/f_i] \subset U$ is open, for all i . Hence, in $B_i[1/f_i]$ we can write $t = \tau_i e_i$, with $e_i \in (B_i[1/f_i])^\times$. Let E_i be the Cartier divisor on V_i defined by e_i . We have $|E_i| \subset |\Sigma|_{V_i}|$. By Lemma 6.5, there exists $N_1 \geq 0$, such that $f_i^{N_1}/e_i \in B_i$, for all i . By (7.5.4), there exists an $N_2 \geq 0$ such that for all i and all s

$$[f_i]^{N_2} [t]^{j-1} \cdot F^{n-1}(a_s) \in W_n(B_i), \quad [f_i]^{N_2} [t]^j \cdot F^{r-1}(b_s) \in W_r(B_i).$$

Choose $N \geq j \cdot N_1 + N_2$, such that $p^n \mid N$. We obtain for all i

$$[\tau_i]^{j-1} [f_i]^{N-1} \cdot F^{n-1}(a_s) \in W_n(B_i), \quad [\tau_i]^j [f_i]^N \cdot F^{r-1}(b_s) \in W_r(B_i).$$

Let $\rho : \text{Spec } L \rightarrow U$, $L \in \Phi$. Assume the closed point of $\text{Spec } \mathcal{O}_L$ maps into $|\bar{Z} + \Sigma|$. Then it follows from the above formula that

$$\rho^* a_s \in \text{fil}_{v_L((j-1) \cdot \bar{Z} + (N-1) \cdot \Sigma)}^{\log} W_n(K) \subset \text{fil}_{v_L(j \cdot \bar{Z} + N \cdot \Sigma)} W_n(K)$$

and

$$\rho^* b_s \in \text{fil}_{v_L(j \cdot \bar{Z} + N \cdot \Sigma)}^{\log} W_r(K).$$

By the choice of N we have

$$r_0 := \min\{\text{ord}_p(v_L(j \cdot \bar{Z} + N \cdot \Sigma)), n\} \geq r = \min\{\text{ord}_p(j), n\};$$

hence

$$V^{n-r}(\rho^* b_s) \in V^{n-r_0} \text{fil}_{v_L(j \cdot \bar{Z} + N \cdot \Sigma)}^{\log} W_{r_0}(K) \subset \text{fil}_{v_L(j \cdot \bar{Z} + N \cdot \Sigma)} W_n(K).$$

Running over all $\rho : \text{Spec } L \rightarrow U$ yields

$$\gamma_{\bar{Y}}(a) \leq j \cdot \bar{Z} + N \cdot \Sigma.$$

This proves (c6) and completes the proof of the proposition. \square

The above proposition gives $c^{W_n} \leq \gamma_n$ by Corollary 4.24. We show in Theorem 7.20 below, that equality holds using symbol computations. If we restrict to $\text{trdeg}(L/k) = 1$ and k is infinite, this follows, e.g., from [KR10, Prop 6.4, (3)]. To handle the case of higher transcendence

degree we need some preparations. We start by identifying the local symbol for W_n on regular projective curves over function fields.

7.6. Let $X \in \mathbf{Sm}$. We denote by $W_n\Omega_X^\bullet$ the de Rham-Witt complex of length n on X (see [Ill79]). By [KSY, Cor 3.2.5] we have $W_n\Omega^q \in \mathbf{RSC}_{\text{Nis}}$. See also [Gro85] and [CR12] for details on how to define the transfers structure. If $f : X \rightarrow Y$ is a morphism in \mathbf{Sm} , then the morphism

$$\Gamma_f^* = f^* : W_n\Omega^q(Y) \rightarrow W_n\Omega^q(X)$$

induced by its graph $\Gamma_f \in \mathbf{Cor}(X, Y)$, is the natural pullback morphism induced by the functoriality of the de Rham-Witt complex. If f is finite and surjective, then the transpose of the graph defines an element $\Gamma_f^t \in \mathbf{Cor}(Y, X)$ and $\Gamma_f^{t*} = f_*$, where f_* is the pushforward defined using duality theory.

Lemma 7.7. (1) *The restriction, Verschiebung, Frobenius, and the differential (which are part of the structure of the de Rham-Witt complex) define morphisms in $\mathbf{RSC}_{\text{Nis}}$*

$$R : W_{n+1}\Omega^q \rightarrow W_n\Omega^q, \quad V : W_n\Omega^q \rightarrow W_{n+1}\Omega^q,$$

$$F : W_{n+1}\Omega^q \rightarrow W_n\Omega^q, \quad d : W_n\Omega^q \rightarrow W_n\Omega^{q+1}.$$

(2) *Let W_n be the algebraic group of Witt vectors of length n considered as a presheaf on \mathbf{Sm} . Then there is a unique structure of presheaf with transfers on W_n , for all n , which is unique with the following properties*

- (a) *the restriction $R : W_{n+1} \rightarrow W_n$ is compatible with the transfer structure, for all n ;*
- (b) *if $f : X \rightarrow Y$ is a morphism in \mathbf{Sm} with graph $\Gamma_f \in \mathbf{Cor}(X, Y)$, then $\Gamma_f^* : W_n(Y) \rightarrow W_n(X)$ is the pullback from the presheaf structure.*

In particular, the Nisnevich sheaf with transfers $W_n\Omega^0 = W_n\mathcal{O}$ from 7.6 coincides with the Nisnevich sheaf with transfers defined by the algebraic group W_n (see [KSY, Cor 3.2.5]).

Proof. (1). We have to show, that if $\alpha \in \mathbf{Cor}(X, Y)$ is a finite correspondence, then the following morphisms are equal on $H^0(Y, W_n\Omega_Y^q)$

$$\alpha^*R = R\alpha^*, \quad \alpha^*V = V\alpha^*, \quad \alpha^*F = F\alpha^*, \quad \alpha^*d = d\alpha^*.$$

This follows from [CR12, Proof of Prop 3.5.4].

(2). The existence of such a transfer structure follows, e.g., from 7.6. The last part of the statement follows since the two transfer structures satisfy (2)a, (2)b.

It remains to prove the uniqueness. Assume we have two transfer actions on W_n with (2)a, (2)b. For $\alpha \in \mathbf{Cor}(X, Y)$ a finite correspondence denote by $\alpha^*, \alpha^\star : W_n(Y) \rightarrow W_n(X)$ the two actions. We have to show they are equal. Let $f : X \rightarrow Y$ be a morphism. By assumption we have $\Gamma_f^* = \Gamma_f^\star =: f^*$; if f is finite and surjective we set $f_* := (\Gamma_f^t)^*$ and $f_\star := (\Gamma_f^t)^\star$. In general for α as above we want to show $\alpha^* = \alpha^\star$. It suffices to check this after shrinking X around its generic points. Hence we can assume, that X is connected and $\alpha = Z \subset X \times Y$ with Z smooth, integral, and finite free over X . Denote by $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ the maps induced by the projections. Then $\alpha^\star = f_\star g^*$ and $\alpha^* = f_* g^*$. It remains to show $f_\star = f_*$. We may shrink X further and hence assume that $f : Z = \text{Spec } L \rightarrow X = \text{Spec } K$ is induced by a finite field extension L/K of function fields over k . By transitivity it suffices to consider the two cases where L/K is either separable or purely inseparable of degree p .

1st case: L/K separable. Let K'/K be a Galois hull of L/K and set $X' = \text{Spec } K'$. We obtain the cartesian diagram

$$\begin{array}{ccc} \coprod_{i=1}^n X' & \xrightarrow{\coprod_i \sigma_i} & Z \\ \downarrow & & \downarrow f \\ X' & \xrightarrow{u} & X, \end{array}$$

where the vertical map on the left is induced by the universal property of the coproduct from the identity on X' , u is induced by the inclusion $K \hookrightarrow K'$, and the $\sigma_i : X' \rightarrow Z$, $i = 1, \dots, n$, are induced by all the K -embeddings $L \hookrightarrow K'$. For $a \in W_n(L)$ we obtain

$$u^* f_* a = (\Gamma_f^t \circ \Gamma_u)^* = \sum_i \Gamma_{\sigma_i}^*$$

and similar with $u^* f_\star$. Thus $u^* f_* = u^* f_\star$ and since $u^* : W_n(K) \hookrightarrow W_n(K')$ is injective we have proven the claim in this case.

2nd case: L/K purely inseparable of degree p . In this case we have

$$(7.7.1) \quad f_* f^*(-) = [L : K] \cdot (-) = p \cdot (-) = f_\star f^\star(-) \quad \text{on } W_n(X).$$

Let $\underline{p} : W_n \rightarrow W_{n+1}$ be the map *lift-and-multiply-by- p* ; thus it sends a Witt vector (a_0, \dots, a_{n-1}) in $W_n(A)$, where A is some \mathbf{F}_p -algebra, to $(0, a_0^p, \dots, a_{n-1}^p)$. Let $b \in W_n(L)$. Clearly we find an element $a \in W_{n+1}(K)$ such that $f^* a = \underline{p}(b)$. We obtain

$$\underline{p}(f_* b) \stackrel{(2)a}{=} f_* \underline{p}(b) = f_*(f^* a) \stackrel{(7.7.1)}{=} p \cdot a = \underline{p}R(a).$$

The same computation works for $f_{\star}b$. Thus $\underline{p}(f_{\star}b) = \underline{p}(f_{\star}b)$, and the claim follows the injectivity of \underline{p} . \square

Lemma 7.8. *Let $f : Y \rightarrow X$ be a finite and surjective morphism in **Sm**. Then for all $u \in H^0(Y, \mathcal{O}_Y^{\times})$ and all $n \geq 1$ we have*

$$f_{\star} \operatorname{dlog}[u] = \operatorname{dlog}[\operatorname{Nm}_{Y/X}(u)] \quad \text{in } H^0(X, W_n \Omega_X^1),$$

where $\operatorname{Nm}_{Y/X} : f_{\star} \mathcal{O}_Y^{\times} \rightarrow \mathcal{O}_X^{\times}$ is the usual norm.

Proof. Note that f is flat by [Mat89, Thm 23.1], hence also finite locally free, so that $\operatorname{Nm}_{Y/X}$ is defined. It suffices to prove the equality after shrinking X around its generic points. Thus we can assume that f corresponds to a finite field extension L/K . By transitivity it suffices to consider the cases where L/K is separable or purely inseparable of degree p .

1st case: L/K finite separable. We have $W_n \Omega_L^q = W_n(L) \otimes_{W_n(K)} W_n \Omega_K^q$ (see [Ill79, I, Prop 1.14]). By the projection formula and Lemma 7.7(2), we have $f_{\star} = \operatorname{Tr}_{L/K} \otimes \operatorname{id}$. Let K^{sep} be a separable closure of K . Note that $W_n(K) \rightarrow W_n(K^{\operatorname{sep}})$ is faithfully flat (since it is ind-étale and $\operatorname{Spec} W_n(K)$ is one point). Hence by étale base change and fppf descent the natural map $W_n \Omega_K^1 \rightarrow W_n \Omega_{K^{\operatorname{sep}}}^1$ is injective. Thus it suffices to check the equality in $W_n \Omega_{K^{\operatorname{sep}}}^1$. Let $\sigma_1, \dots, \sigma_r : L \hookrightarrow K^{\operatorname{sep}}$ be all K -embeddings, then by the above we have in $W_n \Omega_{K^{\operatorname{sep}}}^1$

$$f_{\star} \operatorname{dlog}[u] = \sum_{i=1}^r \sigma_i(\operatorname{dlog}[u]) = \operatorname{dlog} \left[\prod_{i=1}^r \sigma_i(u) \right] = \operatorname{dlog}[\operatorname{Nm}_{L/K}(u)].$$

2nd case: L/K is purely inseparable of degree p . We have $\operatorname{Nm}_{L/K}(u) = u^p \in K$. Since the map *lift-and-multiply-by- p* , $\underline{p} : W_n \Omega_K^1 \rightarrow W_{n+1} \Omega_K^1$ is injective by [Ill79, I, Prop 3.4] and commutes with f_{\star} the statement follows from the following equality in $W_{n+1} \Omega_K^1$:

$$\begin{aligned} \underline{p}(f_{\star} \operatorname{dlog}[u]_n) &= f_{\star} \operatorname{dlog}[u^p]_{n+1} = f_{\star}(1) \cdot \operatorname{dlog}[u^p]_{n+1} \\ &= \underline{p} \operatorname{dlog}[\operatorname{Nm}_{L/K}(u)]_n. \end{aligned}$$

This completes the proof of the lemma. \square

7.9. Let A be a ring of characteristic p and set $B := A[[t]][\frac{1}{t}]$. Recall from [Kat80, §2.2, Prop 3] and [Rül07b, Prop 2.12] that there is a residuum map

$$(7.9.1) \quad \operatorname{Res}_t : W_n \Omega_B^* \rightarrow W_n \Omega_A^{*-1}$$

which is $W_n \Omega_A^*$ -linear (where we consider the left-module structures), commutes with R, F, V , and d , is zero on $W_n \Omega_{A[[t]]}^*$, and satisfies the equality $\operatorname{Res}_t(\alpha \operatorname{dlog}[t]) = \alpha(0)$, for $\alpha \in W_n \Omega_{A[[t]]}^*$.

Let K be a function field over k and C a regular projective connected curve over K with function field $E = K(C)$. Recall from [Rül07a, Def-Prop 1] that the residue map

$$\mathrm{Res}_{C/K,x} : W_n \Omega_E^* \rightarrow W_n \Omega_K^{*-1}$$

at a closed point $x \in C$ is defined as follows: by a result of Hübel-Kunz we find an integer $m_0 \geq 0$ such that for all $m \geq m_0$ the curve $C_m := \mathrm{Spec}(\mathcal{O}_C \cap K(E^{p^m}))$ is smooth over K and, if x_m denotes the image of x under the finite homeomorphism $C \rightarrow C_m$, then the residue field $K_m := K(x_m)$ is separable over K . Hence \mathcal{O}_{C_m, x_m}^h has a unique coefficient field containing K , which we identify with K_m . Set $E_m := K(C_m) = K(E^{p^m})$. The choice of a local parameter $t \in \mathcal{O}_{C_m, x_m}$ yields a canonical inclusion $E_m \hookrightarrow K_m((t))$. We define $\mathrm{Res}_{C/K,x}$ as the composition

$$W_n \Omega_E^* \xrightarrow{\mathrm{Tr}_{E/E_m}} W_n \Omega_{E_m}^* \hookrightarrow W_n \Omega_{K_m((t))}^* \xrightarrow{(7.9.1)} W_n \Omega_{K_m}^* \xrightarrow{\mathrm{Tr}_{K_m/K}} W_n \Omega_K^{*-1}.$$

(Here we should observe that if $\pi : \mathrm{Spec} L \rightarrow \mathrm{Spec} K$ is a finite extension, then the trace $\mathrm{Tr}_{L/K} : W_n \Omega_L^q \rightarrow W_n \Omega_K^q$ from [Rül07b, Thm 2.6] is equal to the pushforward π_* from 7.6. Indeed in the case $q = 0$ this follows from Lemma 7.7(2) and Lemma 7.2; by transitivity, the general case is reduced to a simple extension $L = K[a]$ in which case it follows from the fact that both maps commute with V , F , d , satisfy a projection formula, and the equality $[a]^{i-1}d[a] = i_0^{-1}F^e d[a]^{i_0}$, where $i = p^e i_0 \geq 1$ with $(i_0, p) = 1$.)

Remark 7.10. In [Rül07b, 2.] and [Rül07a], where the trace and the residue symbol mentioned above are constructed it is always assumed that the characteristic is not 2. The reason for this that the structure theorem by Hesselholt and Madsen which in *loc. cit.* is cited as Theorem 2.1 was only known for $\mathbb{Z}_{(p)}$ -algebras, with p odd at that time. This theorem is used in Proposition 2.4 and Lemma 2.9 of *loc. cit.* which are needed to define the trace and the formal residue symbol, respectively. However, the Theorem 2.1 of *loc. cit.* is also available for $\mathbb{Z}_{(2)}$ -algebras by [Cos08, 4.2] hence all the results from *loc. cit.* extend to the case $p = 2$.

Lemma 7.11. *Let C/K and $x \in C$ be as in 7.9. Then the corresponding local symbol of $W_n \Omega^q$ (see 4.34) is given by*

$$(a, f)_{C/K,x} = \mathrm{Res}_{C/K,x}(\alpha \cdot \mathrm{dlog}[f]), \quad \alpha \in W_n \Omega_{K(C)}^q, f \in K(C)^\times,$$

where $[f] = (f, 0, \dots, 0) \in W_n(K(C))$.

In particular, if $L \in \Phi$ with coefficient field $\sigma : K \hookrightarrow \mathcal{O}_L$ and local parameter $t \in \mathcal{O}_L$, then the local symbol $(-, -)_{L,\sigma} : W_n \Omega_L^q \times L^\times \rightarrow$

$W_n\Omega_K^q$ (see 4.37) is given by the composition

$$W_n\Omega_L^q \times L^\times \xrightarrow{\hat{\sigma} \wedge \text{dlog} \circ [-] \circ \hat{\sigma}} W_n\widehat{\Omega}_{K((t))}^{q+1} \xrightarrow{\text{Res}_t} W_n\Omega_K^q,$$

where we denote by $\hat{\sigma} : L \hookrightarrow K((t))$ the canonical inclusion.

Proof. We have to show that the family of maps $\{\text{Res}_{C/K,x}(-\cdot \text{dlog}[-])\}_x$ with x running through all the closed points of C , satisfies the properties (LS1) - (LS4) from 4.34. (LS1) (linearity) is clear and since we can choose the modulus D for (LS3) as large as we want this condition is clear from Lemma 7.13 below; (LS4) (the reciprocity law) holds by [Rül07a, Thm 2] (see also Remark 7.10). It remains to show (LS2), i.e.,

$$\text{Res}_{C/K,x}(\alpha \text{dlog}(f)) = v_x(f) \text{Tr}_{K(x)/K}(\alpha(x)), \quad \alpha \in W_n\Omega_{C,x}^q.$$

To this end choose m as in 7.9 above. Then $K(x)/K(x_m)$ is purely inseparable of degree, say, p^s and we can write

$$[E : E_m] = p^{s+e},$$

where p^e is the ramification index of x/x_m . Denote by $\underline{p}^s : W_n\Omega^q \rightarrow W_{n+s}\Omega^q$ the map *lifting-and-multiplying by p^s* ; it is injective, by [III79, I, Prop 3.4]. Denote by $\sigma : K_m := K(x_m) \hookrightarrow \mathcal{O}_{C_m,x_m}^h \hookrightarrow \mathcal{O}_{C,x}^h$ the inclusion of the coefficient field. By [Rül07b, Thm 2.6(iii)] there exists a $\beta \in W_{n+s}\Omega_{K_m}^q$ mapping to $\underline{p}^s\alpha(x) \in W_{n+s}\Omega_{K(x)}^q$ and we have

$$(7.11.1) \quad \text{Tr}_{K(x)/K_m}(\alpha(x)) = R^s(\beta).$$

By the choice of β , we have

$$(7.11.2) \quad \underline{p}^s(\alpha) - \sigma(\beta) \in \text{Ker}(W_{n+s}\Omega_{\mathcal{O}_{C,x}^h}^q \rightarrow W_{n+s}\Omega_{K(x)}^q).$$

Since the kernel is the differential graded ideal generated by $W_{n+s}(\mathfrak{m}_x)$ we obtain in $W_{n+s}\Omega_K^q$

$$\begin{aligned} \underline{p}^s \text{Res}_{C/K,x}(\alpha \text{dlog}[f]) &= \text{Res}_{C/K,x}(\underline{p}^s(\alpha \text{dlog}[f])) \\ &= \text{Res}_{C/K,x}(\sigma(\beta) \text{dlog}[f]), & (7.11.2) \\ &= \text{Res}_{C_m/K,x_m}(\text{Tr}_{E/E_m}(\sigma(\beta) \text{dlog}[f])), & \text{defn.} \\ &= \text{Res}_{C_m/K,x_m}(\beta \text{dlog Nm}_{E/E_m}[f]), & 7.8 \\ &= v_{x_m}(\text{Nm}_{E/E_m}(f)) \cdot \text{Tr}_{K_m/K}(\beta), & \text{defn.} \\ &= [K(x) : K(x_m)] \cdot v_x(f) \cdot \text{Tr}_{K_m/K}(\beta) \\ &= v_x(f) \cdot \underline{p}^s \text{Tr}_{K_m/K}(R^s(\beta)) \\ &= v_x(f) \cdot \underline{p}^s \text{Tr}_{K(x)/K}(\alpha(x)), & (7.11.1). \end{aligned}$$

Here the first equality follows from the fact that $\text{Res}_{C/K,x}$ commutes with the restriction R . (This follows from the definition and the fact

that Res_t from (7.9.1) and Tr commute with R , for the latter see, e.g., Lemma 7.7(1).) The statement follows from the injectivity of \underline{p}^s . \square

7.12. Let A be a $\mathbb{Z}_{(p)}$ -algebra. For an A -algebra B we denote by $W_n\Omega_{B/A}^\bullet$ the relative de Rham-Witt complex of Langer-Zink (see [LZ04]). It is equipped with R, F, V, d as usual. If $B[x]$ is the polynomial ring with coefficients in B , we denote by $I_r \subset W_n\Omega_{B[x]/A}^\bullet$ the differential graded ideal generated by $W_n(x^r B[x])$. We define the x -adic completion of $W_n\Omega_{B[x]/A}^\bullet$ to be

$$W_n\widehat{\Omega}_{B[[x]]/A}^\bullet := \varprojlim_r W_n\Omega_{B[x]/A}^\bullet / I_r.$$

Note that $W_n\Omega_{B[x]/A}^\bullet / I_r = W_n\Omega_{(B[x]/(x^r))/A}^\bullet$ (see [GH06, Lem 2.4]). In particular, $W_n\widehat{\Omega}_{B[[x]]/A}^\bullet$ is a $W_n(B[[x]]) = \varprojlim_r W_n(B[x]/(x^r))$ -module.

Lemma 7.13. *The following equalities hold in $W_n\widehat{\Omega}_{\mathbb{Z}_{(p)}[[x]]/\mathbb{Z}_{(p)}}^1$:*

$$-\text{dlog}[1-x] = \sum_{i \geq 0} [x]^i d[x] + \sum_{s=1}^{n-1} \sum_{(j,p)=1} \frac{1}{j} dV^s([x]^j).$$

Proof. We prove this by induction over n . The case $n = 1$ is clear. Assume $n \geq 2$. By [LZ04, Cor 2.13] we find unique elements $a_i \in W_n(\mathbb{Z}_{(p)})$ and $b_{s,j} \in W_{n-s}(\mathbb{Z}_{(p)})$ such that

$$-\text{dlog}[1-x] = \sum_{i \geq 0} a_i [x]^i d[x] + \sum_{s=1}^{n-1} \sum_{(j,p)=1} dV^s(b_{s,j}[x]^j).$$

Applying F^{n-1} we obtain in $\widehat{\Omega}_{\mathbb{Z}_{(p)}[[x]]/\mathbb{Z}_{(p)}}^1$

$$\begin{aligned} -\text{dlog}(1-x) &= \sum_{k \geq 0} x^k dx \\ &= \sum_{i \geq 0} F^{n-1}(a_i) x^{(i+1)p^{n-1}-1} dx + \sum_{s=1}^{n-1} \sum_{(j,p)=1} F^{n-1-s}(b_{s,j}) j x^{jp^{n-1-s}-1} dx. \end{aligned}$$

By induction hypothesis we have for all i, j , and for $s = 1, \dots, n-2$

$$a_i = 1 + V^{n-1}(e_i), \quad b_{s,j} = \frac{1}{j} + V^{n-s-1}(f_{s,j}),$$

with $e_i, f_{s,j} \in \mathbb{Z}_{(p)}$. Comparing coefficients we obtain in $\mathbb{Z}_{(p)}$

$$1 = F^{n-1}(a_i) = 1 + p^{n-1} e_i,$$

and for $s = 1, \dots, n-2$

$$\frac{1}{j} = F^{n-s-1}(b_{s,j}) = \frac{1}{j} + p^{n-s-1} f_{s,j},$$

hence $e_i = f_{s,j} = 0$; further we find $b_{n-1,j} = 1/j \in W_1(\mathbb{Z}_{(p)})$. \square

7.14. Let K be a field and $\text{Res}_t : W_n \Omega_{K((t))}^* \rightarrow W_n \Omega_{K((t))}^{*-1}$ the residue map from 7.9.1. Then for all $r, s \geq 0$, $i, j \in \mathbb{Z}$, $a \in W_{n-r}(K)$ and $b \in W_{n-s}(K)$ the following equality holds in $W_n(K)$

$$\text{Res}_t(V^r([a][t]^i)dV^s([b][t]^j)) = \begin{cases} \text{sgn}(j)\text{gcd}(i, j) V^{r+s-c}([a]^{p^{s-c}}[b]^{p^{r-c}}), & \text{if } jp^r + ip^s = 0, \\ 0, & \text{else,} \end{cases}$$

where $\text{sgn}(j) := j/|j|$, if $j \neq 0$, and $\text{sgn}(0) := 0$, and $c = \min\{r, s\}$ (see [Rül07b, Prop 2.12])

Lemma 7.15. *Let $L \in \Phi$ and let $\sigma : K \hookrightarrow \mathcal{O}_L$ be a coefficient field. Let $t \in \mathcal{O}_L$ be a local parameter, and $c \in K$.*

(1) *Let $r \geq 1$ and write $r = p^e r_0$, with $(r_0, p) = 1$, $e \geq 0$. Then*

$$([t]^{-r_0}, 1 - t^r c)_{L, \sigma} = -r_0 V^e([c]), \quad \text{in } W_{e+1}(K).$$

(2) *Let $r \geq 1$ with $(r, p) = 1$ and $m = p^u m_0$, with $(m_0, p) = 1$, $u \geq 1$. Assume $r > m_0$. Then for all $n \geq 1$*

$$([t]^{-m}, 1 - t^r c)_{L, \sigma} = 0, \quad \text{in } W_n(K).$$

Proof. (1). By the Lemmas 7.11 and 7.13 we have

$$\begin{aligned} ([t]^{-r_0}, 1 - t^r c)_{L, \sigma} &= \text{Res}_t([t]^{-r_0} d\log[1 - t^r c]) \\ &= - \sum_{i \geq 0} \text{Res}_t([c]^i [t]^{ir-r_0} d[t^r c]) \\ &\quad - \sum_{s=1}^e \sum_{(j,p)=1} \frac{1}{j} \text{Res}_t([t]^{-r_0} dV^s([c]^j [t]^{jr})). \end{aligned}$$

Now the claim follows from 7.14. The proof of (2) is similar. \square

Lemma 7.16. *Let $L \in \Phi$ and let $t \in \mathcal{O}_L$ be a local parameter. Let $K \hookrightarrow \mathcal{O}_L$ be a coefficient field. Then, for $r \geq 1$, any element $a \in \text{fil}_r^{\log} W_n(L)/W_n(\mathcal{O}_L)$ can be written uniquely in the following way*

$$a = \sum_{0 > ip^{n-1} \geq -r} a_i [t]^i + \sum_{s=1}^{n-1} \sum_{\substack{0 > jp^{n-1-s} \geq -r \\ (j,p)=1}} V^s(b_{s,j} [t]^j),$$

where $a_i \in W_n(K)$ and $b_{s,j} \in W_{n-s}(K)$.

Proof. We can assume L is complete and hence have $L = K((t))$. By [HM04, Lem 4.1.1] (see also [Rü107b, Lem 2.9]) we can write any element a in $W_n(K((t)))/W_n(K[[t]])$ uniquely in the form

$$a = \sum_{0 > i} a_i [t]^i + \sum_{s=1}^{n-1} \sum_{\substack{0 > j \\ (j,p)=1}} V^s(b_{s,j} [t]^j),$$

with $a_i \in W_n(K)$ and $b_{s,j} \in W_{n-s}(K)$. Now, $a \in \text{fil}_r^{\log} W_n(L)/W_n(\mathcal{O}_L)$ is equivalent to the following equality in $W_n(K((t)))/W_n(K[[t]])$

$$\begin{aligned} 0 &= [t]^r F^{n-1}(a) \\ &= \sum_{0 > i} F^{n-1}(a_i) [t]^{ip^{n-1}+r} + \sum_{s=1}^{n-1} \sum_{\substack{0 > j \\ (j,p)=1}} V^s(F^{n-1}(b_{s,j})) \cdot [t]^{jp^{n-1}-s+r}. \end{aligned}$$

This yields the statement. \square

Corollary 7.17. *Let $r = p^e r_0 \geq 1$ with $e \geq 0$ and $(r_0, p) = 1$. Let $L \in \Phi$ have local parameter $t \in \mathcal{O}_L$ and let $\sigma : K \hookrightarrow \mathcal{O}_L$ be a coefficient field. Set $\text{gr}_r^{\log} W_n(L) := \text{fil}_r^{\log} W_n(L)/\text{fil}_{r-1}^{\log} W_n(L)$, $n \geq 1$.*

(1) *Assume $e \in [0, n-1]$. There is a group isomorphism*

$$W_{e+1}(K) \xrightarrow{\cong} \text{gr}_r^{\log} W_n(L), \quad b \mapsto V^{n-1-e}(b[t]^{-r_0}) \text{ mod } \text{fil}_{r-1}^{\log} W_n(L).$$

(2) *Assume $e \geq n$. Then there is a group isomorphism*

$$W_n(K) \xrightarrow{\cong} \text{gr}_r^{\log} W_n(L), \quad b \mapsto b[t]^{-p^{e-n+1}r_0} \text{ mod } \text{fil}_{r-1}^{\log} W_n(L).$$

Proof. This follows directly from Lemma 7.16. \square

Corollary 7.18. *Let $r = p^e r_0 \geq 1$ with $e \geq 0$ and $(r_0, p) = 1$. Let $L \in \Phi$ have local parameter $t \in \mathcal{O}_L$ and let $\sigma : K \hookrightarrow \mathcal{O}_L$ be a coefficient field. Set $\text{gr}_r W_n(L) := \text{fil}_r W_n(L)/\text{fil}_{r-1} W_n(L)$, $n \geq 1$.*

(1) *Assume $e = 0$. Write $r-1 = p^{e_1} r_1$ with $e_1 \geq 0$ and $(r_1, p) = 1$. Then $\text{gr}_r W_n(L) = 0$, if $e_1 \geq n$ and, if $e_1 \in [0, n-1]$ there is a group isomorphism*

$$K \xrightarrow{\cong} \text{gr}_r W_n(L), \quad b \mapsto V^{n-1-e_1}([bt^{-r_1}]) \text{ mod } \text{fil}_{r-1} W_n(L).$$

(2) *Assume $e \in [1, n-1]$. There is a group isomorphism*

$$K \oplus W_e(K) \xrightarrow{\cong} \text{gr}_r W_n(L),$$

$$(b, c) \mapsto V^{n-1}(bt^{-(r-1)}) + V^{n-e}(c[t]^{-r_0 p}) \text{ mod } \text{fil}_{r-1} W_n(L).$$

(3) Assume $e \geq n$. Then there is a group isomorphism

$$K \oplus W_n(K) \xrightarrow{\cong} \mathrm{gr}_r W_n(L),$$

$$(b, c) \mapsto V^{n-1}(bt^{-(r-1)}) + c[t]^{-p^{e-n+1}r_0} \bmod \mathrm{fil}_{r-1} W_n(L).$$

Proof. Let $e' := \min\{e, n\}$ and recall

$$\mathrm{fil}_r W_n(L) = \mathrm{fil}_{r-1}^{\log} W_n(L) + V^{n-e'} \mathrm{fil}_r^{\log} W_{e'}(L).$$

Thus (2) and (3) follow directly from Lemma 7.16. (For the injectivity in (2) use that $V^{n-e}(c[t]^{-r_0 p}) = V^{n-e-1}(V(c)[t]^{-r_0})$.) Furthermore, it is immediate from Lemma 7.16, that there is an injective map as in (1) and that any element in the target has a representative of the form $V^{n-1-e_1}(\beta[t]^{-r_1})$ with $\beta \in W_{e_1}(K)$. Thus the statement follows if we show $V^{n-1-e_1}(V(\beta_1)[t]^{-r_1}) \in \mathrm{fil}_{r-1} W_n(L)$. But by Lemma 7.16 the element $V^{n-1-e_1}(V(\beta_1)[t]^{-r_1}) = V^{n-e_1}(\beta_1[t]^{-pr_1})$ lies in $V^{n-e_1} \mathrm{fil}_{r-1}^{\log} W_{e_1}(L) \subset \mathrm{fil}_{r-1} W_n(L)$. Hence the statement. \square

Proposition 7.19. *Let $L \in \Phi$ have residue field κ_L and local parameter $t \in \mathcal{O}_L$. Let $z_1, \dots, z_m \subset \mathcal{O}_L$ be a lift of some p -basis of κ/k . Let $\sigma_0 : K_0 \hookrightarrow \mathcal{O}_L$ be the unique coefficient field with $z_i \in K_0$, $i = 1, \dots, m$. Let x be an indeterminate and set $L_x := \mathrm{Frac}(\mathcal{O}_L[x]_{(t)}^h)$. Denote also by $\sigma_0 : K_0(x) \hookrightarrow L_x$ the canonical extension of σ_0 . Let $r \geq 1$ and $a \in \mathrm{fil}_r^F W_n(L)$. Assume one of the following:*

- (1) $(r, p) = 1$ or $r = p = 2$ or $m = 0$, and $(a, 1 - xt^{r-1})_{L_x, \sigma_0} = 0$.
- (2) $r > 2$, $p|r$, $m \geq 1$, and $(a, 1 - xt^{r-1})_{L_x, \sigma_j} = 0$, for $j = 0, 1$, where $\sigma_1 : K_1 \hookrightarrow \mathcal{O}_L$ is the unique coefficient field with $z_i/(1 + z_i^{p^e} t) \in K_1$, for all i , with $e = \mathrm{ord}_p(r)$, and we denote also by $\sigma_1 : K_1(x) \hookrightarrow \mathcal{O}_{L_x}$ the canonical extension.

Then $a \in \mathrm{fil}_{r-1}^F W_n(L)$.

Proof. Since k is perfect, a p -basis over k is the same as a separating transcendence basis over k , (e.g., [EGA IV₁, Thm 0.21.4.5]), hence there are unique coefficient fields K_0 and K_1 as in the statement (see [Bou06, IX, §3, No. 2]). By Proposition 7.5 and Corollary 4.24 we know $\mathrm{fil}_{r-1}^F W_n(L) \subset W_n(\mathcal{O}_L, \mathfrak{m}_L^{-r+1})$; furthermore for all $b \in W_n(\mathcal{O}_L, \mathfrak{m}_L^{-r+1})$ we have $(b, 1 - xt^{r-1})_{L_x, \sigma} = 0$ for all coefficient fields σ (by Corollary 4.40). Thus in the following we may replace a by $a + b$ with $b \in \mathrm{fil}_{r-1}^F W_n(L)$. We will use σ_0 to identify $\hat{L} = K_0((t))$.

Write $r = p^e r_0$ with $e \geq 0$ and $(r_0, p) = 1$. We distinguish four cases.

1st case: $e = 0$. Write $r - 1 = p^{e_1} r_1$ with $(r_1, p) = 1$ and $e_1 \geq 0$. By Corollary 7.18(1) we have $\mathrm{gr}_r W_n(L) = 0$, if $e_1 \geq n$, and there is

nothing to show; if $e_1 \in [0, n-1]$ we have

$$a \equiv \sum_{h \geq 0} F^h V^{n-1-e_1}([b_h][t]^{-r_1}) \pmod{\text{fil}_{r-1}^F W_n(L)},$$

with $b_h \in K_0$. We compute in $W_n(K_0(x))$:

$$\begin{aligned} 0 &= \left(\sum_h F^h (V^{n-1-e_1}([b_h][t]^{-r_1})), 1 - xt^{r-1} \right)_{L_x, \sigma_0}, \quad \text{by (1),} \\ &= \sum_h F^h V^{n-1-e_1} \left([b_h] \cdot ([t]^{-r_1}, 1 - xt^{r-1})_{L_x, \sigma_0} \right), \quad \text{by Lem 7.11,} \\ &= -r_1 \sum_h F^h V^{n-1-e_1}([b_h] V^{e_1}([x])), \quad \text{by Lem 7.15(1),} \\ &= -r_1 V^{n-1} \left(\sum_h b_h^{p^{e_1+h}} x^{p^h} \right). \end{aligned}$$

Hence $b_h = 0$, for all $h \geq 0$, which completes the proof of the first case.

2nd case: $r = p = 2$. By Corollary 7.18(2), (3) we have

$$a \equiv \sum_h F^h V^{n-1}(b_h t^{-1} + c_h t^{-2}) \pmod{W_n(\mathcal{O}_L)},$$

with $b_h, c_h \in K_0$. Note

$$\text{Res}_t(t^{-1} \text{dlog}(1 - xt)) = x, \quad \text{Res}_t(t^{-2} \text{dlog}(1 - xt)) = x^2.$$

Hence by (1)

$$0 = (a, 1 - xt)_{L_x, \sigma_0} = V^{n-1} \left(\sum_h b_h^{2^h} x^{2^h} + c_h^{2^h} x^{2^{h+1}} \right).$$

We obtain

$$b_0 = 0 \quad \text{and} \quad c_h = b_{h+1}^2, \quad \text{all } h \geq 0.$$

Thus reshuffling the sum defining a we obtain

$$a = \sum_h F^h V^{n-1}(b_h t^{-1} + F(b_{h+1} t^{-1})) = 2 \sum_h F^h V^{n-1}(b_h t^{-1}) = 0.$$

3rd case: $1 \leq e \leq n-1$ and $r > 2$. By Corollary 7.18(2) we have

$$a \equiv \sum_{h \geq 0} F^h \left(V^{n-1}(b_h [t]^{-(r-1)}) + V^{n-e}(c_h [t]^{-r_0 p}) \right) \pmod{\text{fil}_{r-1}^F W_n(L)},$$

where $b_h \in K_0$ and $c_h \in W_e(K_0)$. By a similar computation as in the first case, the vanishing $(a, 1 - xt^{r-1})_{L_x, \sigma_0} = 0$ together with $r-1 > r_0$ and Lemma 7.15, (1) and (2), imply $b_h = 0$, for all $h \geq 0$. Thus

$$a \equiv \sum_{h \geq 0} F^h (V^{n-e}(c_h [t]^{-r_0 p})) \pmod{\text{fil}_{r-1}^F W_n(L)}.$$

It suffices to show

$$(7.19.1) \quad c_h \in FW_e(K_0), \quad \text{all } h \geq 0.$$

Indeed, then $V^{n-e}(c_h[t]^{-r_0p}) = FV^{n-e}(c'_h[t]^{-r_0})$, for some $c'_h \in W_e(K_0)$, which lies in $F\text{fil}_{r/p}^{\log}W_n(L) \subset F\text{fil}_{r-2}^{\log}W_n(L)$ (use $r \geq 3$ for the last inclusion).

If $m = \text{trdeg}(\kappa/k) = 0$, then K_0 is perfect and (7.19.1) holds. This completes the proof of the implication: (1) $\Rightarrow a \in \text{fil}_{r-1}W_n\mathcal{O}_L$.

Now assume $m \geq 1$. We prove (7.19.1) by contradiction using $(a, 1 - xt^{r-1})_{L_x, \sigma_1} = 0$ with $\sigma_1 : K_1(x) \hookrightarrow \mathcal{O}_{L_x}$ as in (2). Thus assume not all c_h are in $FW_e(K_0)$. Let h_0 be the minimal h with $c_h \notin FW_e(K_0)$. Hence modulo $\text{fil}_{r-1}^F W_n(L)$ we can write a as $F^{h_0}(V^{n-e}(a'))$, with $a' = \sum_{h \geq h_0} F^{h-h_0}(c_h[t^{-r_0p}])$. Since $F : W_n(K_j(x)) \rightarrow W_n(K_j(x))$ and $V^{n-e} : W_e(K_j(x)) \rightarrow W_n(K_j(x))$, $j = 0, 1$, are injective, the element a' also satisfies $(a', 1 - xt^{r-1})_{L_x, \sigma_j} = 0$, $j = 0, 1$. Thus we can assume $n = e$ and $h_0 = 0$, i.e., $c_0 \notin FW_e(K_0)$ and we want to find a contradiction. Since the elements $z_1, \dots, z_m \in K_0$ from the statement form a p -basis we can write c_0 as follows:

$$c_0 = \sum_{j=0}^{e-1} V^j \left(\sum_{I \subset [0, p-1]^m} [a_{I,j}]^p [z]^I \right),$$

where $a_{I,j} \in K_0$ and $[z]^I = [z_1]^{i_1} \cdots [z_m]^{i_m}$, for $I = (i_1, \dots, i_m)$. Therefore, $c_0 \notin FW_e(K_0)$ translates into

$$(7.19.2) \quad \exists j \in [0, e-1], I \in [0, p-1]^m \setminus \{(0, \dots, 0)\} \quad \text{such that} \quad a_{I,j} \neq 0.$$

Since we want to compute the local symbol with respect to the coefficient field $\sigma_1 : K_1(x) \hookrightarrow \mathcal{O}_{L_x}$, we have to rewrite c_0 as an element in $W_n(K_1[[t]])$. Set

$$y_i := \frac{z_i}{1 + z_i^{p^e} t} \in K_1, \quad i = 1, \dots, m.$$

Then

$$c_0 = \sum_{j=0}^{e-1} V^j \left(\sum_{I \subset [0, p-1]^m} [a_{I,j}]^p [y(1 + z^{p^e} t)]^I \right),$$

where $[y(1 + z^{p^e} t)]^I := \prod_{h=1}^m [y_h(1 + z_h^{p^e} t)]^{i_h}$. Note that $a_{I,j}, z_h \in K_0 \subset K_1[[t]]$ are not constant. The composition $F^{e-1}d : W_e(-) \rightarrow \Omega^1$ is a morphism of reciprocity sheaves (see Lemma 7.7). Hence $F^{e-1}d$ commutes with the local symbol, which on Ω^1 is given by $(\alpha, f)_{L_x, \sigma_1} =$

$\text{Res}_{K_1((t))}(\alpha \wedge \text{dlog } f)$ (see Lemma 7.11). Using $F^{e-1} dF = 0$ on W_e , we obtain the following equalities in $\Omega_{K_1(x)}^1$:

$$\begin{aligned} 0 &= F^{e-1} d(a, 1 - xt^{r-1})_{L_x, \sigma_1} \\ &= (F^{e-1} d(c_0[t]^{-r_0 p}, 1 - xt^{r-1}))_{L_x, \sigma_1} \\ &= \sum_{j=0}^{e-1} \sum_I \left(F^{e-1-j} d([a_{I,j}]^p [y(1 + z^{p^e} t)]^I [t]^{-r_0 p^{j+1}}), 1 - xt^{r-1} \right)_{L_x, \sigma_1} \\ &= \sum_{j=0}^{e-1} \sum_I \text{Res}_t \left(a_{I,j}^{p^{e-j}} t^{-r} F^{e-1-j} d([y(1 + z^{p^e} t)]^I) \text{dlog}(1 - xt^{r-1}) \right). \end{aligned}$$

Write

$$a_{I,j} = \bar{a}_{I,j} + tb_{I,j}, \quad \bar{a}_{I,j} \in K_1, \quad b_{I,j} \in K_1[[t]].$$

Denote by

$$\bar{\sigma}_j : K_j \xrightarrow{\cong} \kappa_L, \quad j = 0, 1$$

the isomorphisms induced by $\sigma_j : K_j \hookrightarrow \mathcal{O}_L$. Then $\bar{\sigma}_1(\bar{a}_{I,j}) = \bar{\sigma}_0(a_{I,j})$; in particular,

$$(7.19.3) \quad a_{I,j} = 0 \iff \bar{a}_{I,j} = 0.$$

For $j \in [0, e-1]$ we have $a_{I,j}^{p^{e-j}} \equiv \bar{a}_{I,j}^{p^{e-j}} \pmod{t^2}$ and thus we obtain from the computation above

$$(7.19.4) \quad 0 = - \sum_{j=0}^{e-1} \sum_{I \subset [0, p-1]^m} \text{Res}_t \left(\bar{a}_{I,j}^{p^{e-j}} t^{-r} F^{e-1-j} d([y(1 + z^{p^e} t)]^I) d(xt^{r-1}) \right).$$

We have

$$(7.19.5) \quad \begin{aligned} &F^{e-1-j} d[y(1 + z^{p^e} t)]^I \\ &= \sum_{h=1}^m i_h(y(1 + z^{p^e} t))^{Ip^{e-1-j}} \text{dlog}(y_h(1 + z_h^{p^e} t)). \end{aligned}$$

Note

$$z_h = y_h + t\zeta_h, \quad \zeta_h \in K_1[[t]], \quad h = 1, \dots, m.$$

Thus, $z_h^{p^e} \equiv y_h^{p^e} \pmod{t^{p^e}}$. Hence the coefficient of $F^{e-1-j} d[y(1 + z^{p^e} t)]^I$ in K_1 in front of dt is equal to

$$(7.19.6) \quad f_{I,j} := q_I^{p^e} y^{Ip^{e-1-j}}, \quad \text{with } q_I = \sum_{h=1}^m i_h y_h;$$

the coefficient of $F^{e-1-j} d[y(1+z^p t)]^I$ in $\Omega_{K_1}^1$ in front of t is equal to $df_{I,j}$. (This is zero if $j \in [0, e-2]$.) Thus by (7.19.4) we have

$$0 = \sum_{j=0}^{e-1} \sum_I (\bar{a}_{I,j}^{p^{e-j}} (f_{I,j} dx + x df_{I,j})) = d \left(\sum_{j=0}^{e-1} \sum_I \bar{a}_{I,j}^{p^{e-j}} f_{I,j} \cdot x \right).$$

Hence the element in the brackets has to be a p -th power, i.e., by (7.19.6)

$$K_1^p \ni \sum_{j=0}^{e-2} \left(\sum_{I \subset [0, p-1]^m} (\bar{a}_{I,j} q_I^{p^j})^p y^I \right)^{p^{e-1-j}} \cdot x + \sum_{\substack{I \subset [0, p-1]^m \\ I \neq 0}} (\bar{a}_{I, e-1} q_I^{p^{e-1}})^p y^I x.$$

Note

$$q_I = 0 \iff I = 0.$$

Since y_1, \dots, y_m, x form a p -basis of $K_1(x)$ over k we obtain

$$\bar{a}_{I, e-1} = 0, \quad \text{for all } I \subset [0, p-1]^m \setminus \{(0, \dots, 0)\},$$

and

$$\sum_{j=0}^{e-2} \sum_{I \subset [0, p-1]^m} (\bar{a}_{I,j} q_I^{p^j})^{p^{e-1-j}} y^I p^{e-2-j} = 0.$$

Since $y_1, \dots, y_m \in K_1$ form a p -basis over k , we obtain, similar as above, $\bar{a}_{I, e-2} = 0$, for all $I \neq 0$. We may proceed in this way and obtain

$$\bar{a}_{I,j} = 0, \quad \text{for all } I \neq 0, j \geq 0.$$

By (7.19.3) this contradicts (7.19.2) and proves the statement in this case.

4th case: $e \geq n$ and $r > 2$. By Corollary 7.18(3) we have

$$a \equiv \sum_{h \geq 0} F^h \left(c_h [t]^{-p^{e-n+1} r_0} + V^{n-1} (b_h [t]^{-(r-1)}) \right) \pmod{\text{fil}_{r-1}^F W_n(L)},$$

where $c_h \in W_n(K_0)$ and $b_h \in K_0$. As before it follows from $(a, 1 - xt^{r-1})_{L_x, \sigma_0} = 0$ and Lemma 7.15 that $b_h = 0$, for all $h \geq 0$. Thus

$$a \equiv \sum_{h \geq 0} F^h (c_h [t]^{-p^{e-n+1} r_0}) \pmod{\text{fil}_{r-1}^F W_n(L)}.$$

Applying V^{e-n+1} we obtain

$$V^{e-n+1}(a) \equiv \sum_{h \geq 0} F^h V(c'_h [t]^{-r_0 p}) \pmod{\text{fil}_{r-1}^F W_{e+1}(L)},$$

where $c'_h = V^{e-n}(c_h) \in W_e(K_0)$. Since $V^{e-n+1}(a) \in \text{fil}_r W_{e+1}(L)$ we can apply the third case, in particular (7.19.1) to conclude $c_h \in FW_n(K_0)$,

and then also $a \in \text{fil}_{r-1}^F W_n(L)$. This completes the proof of the proposition. \square

Theorem 7.20. *Let $L \in \Phi$ and $r \geq 0$. Then*

$$\text{fil}_r^F W_n(L) = \widetilde{W}_n(\mathcal{O}_L, \mathfrak{m}^{-r}),$$

i.e., the Brylinski-Kato-Matsuda-Russell conductor is motivic.

Proof. We have $\text{fil}_r^F W_n(L) \subset \widetilde{W}_n(\mathcal{O}_L, \mathfrak{m}^{-r})$, by Proposition 7.5 and Theorem 4.15(4), and we know this is an equality for $r = 0$. Let $t \in \mathcal{O}_L$ be a local parameter. By Corollary 4.40 we have

$$a \in \widetilde{W}_n(\mathcal{O}_L, \mathfrak{m}^{-r}) \Rightarrow (a, 1 - xt^m)_{L_x, \sigma} = 0, \quad \text{for all } m \geq r \text{ and all } \sigma,$$

where $L_x = \text{Frac}(\mathcal{O}_L[x]_{(t)}^h)$ and σ is running through all coefficient fields $\sigma : K \hookrightarrow \mathcal{O}_L$. Furthermore, we know for any $a \in \widetilde{W}_n(\mathcal{O}_L, \mathfrak{m}^{-r})$ there exists some $m \geq r$ such that $a \in \text{fil}_m^F W_n(L)$. Hence the statement follows from Proposition 7.19. \square

8. LISSE SHEAVES OF RANK 1 AND THE ARTIN CONDUCTOR

In this section k is a perfect field of characteristic $p > 0$.

8.1. The case of finite monodromy.

8.1. Consider the constant presheaf with transfers \mathbb{Q}/\mathbb{Z} , i.e., an elementary correspondence $V \in \mathbf{Cor}(X, Y)$, with X, Y smooth and connected, acts by multiplication with $[V : X]$. By [MVW06, Lem 6.23]

$$X \mapsto H^1(X) := H_{\text{ét}}^1(X, \mathbb{Q}/\mathbb{Z}) = \text{Hom}_{\text{cont}}(\pi_1(X)^{\text{ab}}, \mathbb{Q}/\mathbb{Z})$$

is a presheaf with transfers, which we denote by H^1 in the following.

Note that $H^1 \in \mathbf{NST}$ as follows from the following Lemma.

Lemma 8.2. *Let A be an abelian group. It defines a constant étale sheaf on \mathbf{Sm} . Then the presheaf $X \mapsto H_{\text{ét}}^1(X, A)$ is a Nisnevich sheaf on \mathbf{Sm} .*

Proof. Let \mathcal{H}^i be the Nisnevich sheafification of $X \mapsto H_{\text{ét}}^i(X, A)$. Then for any $X \in \mathbf{Sm}$ we have an exact sequence

$$H_{\text{Nis}}^1(X, \mathcal{H}^0) \rightarrow H_{\text{ét}}^1(X, A) \rightarrow H_{\text{Nis}}^0(X, \mathcal{H}^1) \rightarrow H_{\text{Nis}}^2(X, \mathcal{H}^0).$$

But $\mathcal{H}^0 = A$ is constant and hence by [Voe00b, Thm 3.1.12] we have $H_{\text{Nis}}^i(X, \mathcal{H}^0) = H_{\text{Zar}}^i(X, \mathcal{H}^0) = 0$, for all $i \geq 1$. Thus the presheaf from the statement is equal to \mathcal{H}^1 . \square

Lemma 8.3. *The Artin-Schreier-Witt sequence*

$$(8.3.1) \quad 0 \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow W_n \xrightarrow{F-1} W_n \rightarrow 0$$

is an exact sequence of étale sheaves with transfers on \mathbf{Sm} , where $F : W_n \rightarrow W_n$ is the base change over $\mathrm{Spec} k$ of the Frobenius on the \mathbf{F}_p -group scheme W_n .

Proof. The exactness of the sequence (8.3.1) on $X_{\mathrm{ét}}$ is classical. The map $F - 1 : W_n \rightarrow W_n$ is a morphism of k -group schemes hence is compatible with transfers; for the inclusion $\mathbb{Z}/p^n\mathbb{Z} \hookrightarrow W_n$ this follows directly from Lemma 7.2. \square

8.4. We denote by δ_n the composition

$$\delta_n : W_n(L) \rightarrow W_n(L)/(F-1)W_n(L) \cong H_{\mathrm{ét}}^1(L, \mathbb{Z}/p^n\mathbb{Z}) := H_{p^n}^1(L),$$

which is the connecting homomorphism stemming from the Artin-Schreier-Witt sequence (8.3.1). Then we set

$$\mathrm{fil}_j H_{p^n}^1(L) := \delta_n(\mathrm{fil}_j W_n(L)) = \delta_n(\mathrm{fil}_j^F W_n(L)).$$

For $j \geq 0$, we set

$$(8.4.1) \quad \mathrm{fil}_j H^1(L) := \begin{cases} \mathrm{Im}(H^1(\mathcal{O}_L) \rightarrow H^1(L)), & \text{if } j = 0, \\ H^1(L)\{p'\} \oplus \bigcup_{n \geq 1} \mathrm{fil}_j H_{p^n}^1(L), & \text{if } j \geq 1, \end{cases}$$

with $H^1(L)\{p'\} = \bigoplus_{\ell \neq p} H_{\mathrm{ét}}^1(L, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ the prime-to- p -part of $H^1(L)$.

For $\chi \in H^1(L)$ we define

$$(8.4.2) \quad \mathrm{Art}_L(\chi) = \min\{j \geq 0 \mid \chi \in \mathrm{fil}_j H^1(L)\}.$$

Proposition 8.5. *The collection*

$$\mathrm{Art} = \{\mathrm{Art}_L : H^1(L) \rightarrow \mathbb{N}_0 \mid L \in \Phi\}$$

is a semi-continuous conductor on H^1 , as is its restriction $\mathrm{Art}^{\leq 1}$.

Proof. By Proposition 7.5 and Lemma 4.28, Art satisfies (c1)-(c6) except possibly for (c4). (For (c5) note, that $W_n(Y) \rightarrow H_{p^n}^1(Y)$ is surjective for any affine scheme over k .) It remains, to show that $\mathrm{Art}^{\leq 1}$ satisfies (c4). Let X be a smooth k -scheme and $a \in H^1(\mathbf{A}_X^1)$ with

$$(8.5.1) \quad \mathrm{Art}_{k(x)_\infty}^1(\rho_x^* a) \leq 1, \quad \text{for all closed points } x \in X_{(0)},$$

where $\rho_x : \mathrm{Spec} k(x)(t)_\infty = \mathrm{Spec} \mathrm{Frac}(\mathcal{O}_{\mathbf{P}_x^1, \infty}^h) \rightarrow \mathbf{A}_X^1$ is the natural map. We want to show : $a \in H^1(X)$. Since $H^1 = H^1\{p'\} \oplus \varinjlim_n H_{p^n}^1$ with $H^1\{p'\}$ the \mathbf{A}^1 -invariant subsheaf of prime-to- p -torsion, we can assume $a \in H_{p^n}^1(\mathbf{A}_X^1)$. Furthermore, the question is local on X , hence

we can assume $X = \text{Spec } A$ affine. We consider first the case $n = 1$. Condition (8.5.1) implies

$$(8.5.2) \quad \rho_x^* a \in \text{Im} \left(H_p^1(\mathcal{O}_{\mathbf{P}_x^1, \infty}^h) \rightarrow H_p^1(k(x)(t)_\infty) \right).$$

Denote by $a(x)$ the restriction of a to \mathbf{A}_x^1 . Since H_p^1 is a Nisnevich sheaf we conclude

$$a(x) \in H_p^1(\mathbf{P}_x^1) = H_p^1(x).$$

Thus we find a polynomial $\tilde{a} = a_0 + a_1 t + \dots + a_n t^n \in A[t]$ mapping to a such that for all closed points $x \in X$ there exist $b_x \in k(x)$ and $g_x \in k(x)[t]$ with

$$(8.5.3) \quad \tilde{a}(x) = b_x + g_x^p - g_x, \quad \text{in } k(x)[t].$$

Assume $n \geq 1$. Then, $n = p \cdot n_1$, for some $n_1 \geq 1$. We claim

$$(8.5.4) \quad a_n = c_1^p, \quad \text{some } c_1 \in A^p.$$

Indeed, write $n = p^e m$ with $e \geq 1$ and $(p, m) = 1$, and for a fixed closed point $x \in X$ write $g_x = c_0 + c_1 t + \dots + c_{p^{e-1}m} t^{p^{e-1}m}$; then (8.5.3) implies

$$a_n(x) = c_{p^{e-1}m}^p, \quad a_{p^i m}(x) = c_{p^{i-1}m}^p - c_{p^i m}, \quad i \in [1, e-1], \quad a_m(x) = -c_m.$$

Hence for all maximal ideals $\mathfrak{m} \subset A$ we have

$$a_n \equiv \sum_{j=0}^{e-1} (-a_{p^j m})^{p^{e-j}} \pmod{\mathfrak{m}}.$$

It follows that $a_n = (\sum_{j=0}^{e-1} (-a_{p^j m})^{p^{e-j-1}})^p \in A^p$, which yields (8.5.4).

Now $a^{(1)} = \tilde{a} - (c_1 t^{n_1})^p + c_1 t^{n_1}$ also has property (8.5.3) and its degree is strictly smaller than n . We can replace a by $a^{(1)}$ in the above discussion and go on in this way until we reach a polynomial $a^{(r)} \in A[t]$ whose degree is strictly smaller than p in which case (8.5.3) forces it to be constant $= c_r \in A$. We obtain

$$\tilde{a} = c_r + \sum_{i=1}^{r-1} (c_i t^{n_i})^p - c_i t^{n_i},$$

whence $a \in H_p^1(X)$.

Let $n \geq 1$. If $a \in H_{p^n}^1(\mathbf{A}_X^1)$ satisfies (8.5.1), then so does $p^{n-1}a \in H_p^1(\mathbf{A}_X^1)$. By the case $n = 1$ and the exact sequence (X is affine)

$$0 \rightarrow H_{p^{n-1}}^1(X) \rightarrow H_{p^n}^1(X) \xrightarrow{p^{n-1}} H_p^1(X) \rightarrow 0$$

we find an element $b \in H_{p^n}^1(X)$ such that $p^{n-1}(a - b) = 0$. Since $a - b$ also satisfies (8.5.1) we obtain $a - b \in H_{p^{n-1}}^1(X)$ by induction. This completes the proof. \square

Lemma 8.6. *Let K be a field of positive characteristic, x an indeterminate, and $g \in W_n(K(x))$. Assume $F(g) - g = V^{n-1}(bx)$ for some $b \in K$. Then $g \in \mathbb{Z}/p^n\mathbb{Z}$, i.e., $F(g) - g = 0$.*

Proof. If $n = 1$, then $g^p - g = bx$ forces g to be constant and hence $g^p - g = 0$, i.e., $g \in \mathbf{F}_p$. If $n \geq 2$, then $F(g) - g$ is zero when restricted to $W_{n-1}(K(x))$. Hence $g = m \cdot [1] + V^{n-1}(f)$ with $f \in K(x)$, $m \in \mathbb{Z}$. Thus $F(f) - f = bx$, and we conclude with the case $n = 1$. \square

Proposition 8.7. *Let L , $t \in \mathcal{O}_L$, $\sigma_j : K_j \hookrightarrow \mathcal{O}_L$, $j = 0, 1$, be as in Proposition 7.19. We also denote by $\sigma_j : K_j(x) \hookrightarrow \mathcal{O}_{L_x}$ the canonical extension. Let $r \geq 1$ and $a \in \text{fil}_r H_{p^n}^1(L)$. Assume one of the following:*

- (1) $(r, p) = 1$ or $r = p = 2$ or $m = 0$, and $(a, 1 - xt^{r-1})_{L_x, \sigma_0} = 0$.
- (2) $r > 2$, $p|r$, $m \geq 1$, and $(a, 1 - xt^{r-1})_{L_x, \sigma_j} = 0$, for $j = 0, 1$.

Then $a \in \text{fil}_{r-1} H_{p^n}^1(L)$.

Proof. Let $\tilde{a} \in \text{fil}_r W_n(L)$ be a lift of a . If $(a, 1 - xt^{r-1})_{L_x, \sigma_j} = 0$, for some $j \in \{0, 1\}$, then we find $g_j \in W_n(K_j(x))$ such that

$$(8.7.1) \quad (\tilde{a}, 1 - xt^{r-1})_{L_x, \sigma_j} = F(g_j) - g_j.$$

It suffices to show $\tilde{a} \in \text{fil}_{r-1}^F W_n(L)$. Write $r = p^e r_0$ with $e \geq 0$ and $(r_0, p) = 1$.

1st case: $e = 0$. Write $r - 1 = p^{e_1} r_1$ with $e_1 \geq 0$ and $(p, r_1) = 0$. If $e_1 \geq n$, then by Corollary 7.18(1) we have $\text{fil}_r H_{p^n}^1(L) = \text{fil}_{r-1} H_{p^n}^1(L)$, else we have

$$\tilde{a} \equiv V^{n-1-e_1}([b][t]^{-r_1}) \pmod{\text{fil}_{r-1} W_n(L)},$$

for some $b \in K_0$. Thus

$$\begin{aligned} F(g_0) - g_0 &= (\tilde{a}, 1 - xt^{r-1})_{L_x, \sigma_0}, && \text{by (1),} \\ &= V^{n-1-e_1} \left([b]([t]^{-r_1}, 1 - xt^{r-1})_{L_x, \sigma_0} \right), && \text{by Lem 7.11,} \\ &= -r_1 V^{n-1}(b^{p^{e_1}} x), && \text{by Lem 7.15(1).} \end{aligned}$$

Lemma 8.6 implies $F(g_0) - g_0 = 0$. Hence $\tilde{a} \in \text{fil}_{r-1}^F W_n(L)$ by Proposition 7.19(1).

2nd case: $r = p = 2$. As in the proof of Proposition 7.19 (2nd case) we have $\tilde{a} \equiv V^{n-1}(bt^{-1} + ct^{-2}) \pmod{W_n(\mathcal{O}_L)}$, with $b, c \in K_0$, and

$$g_0^2 - g_0 = (\tilde{a}, 1 - xt)_{L_x, \sigma_0} = V^{n-1}(bx + cx^2).$$

This implies $c = b^2$; hence $a \in H_{p^n}^1(\mathcal{O}_L) = \text{fil}_1 H_{p^n}^1(L)$.

3rd case: $1 \leq e \leq n - 1$ and $r > 2$. By Corollary 7.18(2) we have

$$\tilde{a} \equiv V^{n-1}(b_j[t]^{-(r-1)}) + V^{n-e}(c_j[t]^{-r_0 p}) \pmod{\text{fil}_{r-1}^F W_n(L)},$$

where $b_j \in K_j$ and $c_j \in W_e(K_j)$, $j = 0, 1$. By Lemma 7.15(1) we have

$$(\mathbb{V}^{n-1}(b_j[t]^{-(r-1)}), 1 - xt^{r-1})_{L_x, \sigma_j} = -(r-1) \mathbb{V}^{n-1}(b_j x);$$

by Lemma 7.15(2) we have

$$(\mathbb{V}^{n-e}(c_j[t]^{-ro_p}), 1 - xt^{r-1})_{L_x, \sigma_j} = 0.$$

Thus by (2)

$$F(g_j) - g_j = (\tilde{a}, 1 - xt^{r-1})_{L_x, \sigma_j} = -(r-1) \mathbb{V}^{n-1}(b_j x).$$

By Lemma 8.6 we have $F(g_j) - g_j = 0$, for $j = 0, 1$. Hence $\tilde{a} \in \text{fil}_{r-1}^F W_n(L)$ by Proposition 7.19.

4th case: $e \geq n$ and $r > 2$. By Corollary 7.18(3) we have

$$a \equiv c_j[t]^{-p^{e-n+1}ro} + \mathbb{V}^{n-1}(b_j[t]^{-(r-1)}) \pmod{\text{fil}_{r-1}^F W_n(L)},$$

where $c_j \in W_n(K_j)$ and $b_j \in K_j$, for $j = 0, 1$. As in the third case the following equality follows from Lemma 7.15 for $j = 0, 1$

$$F(g_j) - g_j = (\tilde{a}, 1 - xt^{r-1})_{L_x, \sigma_j} = -(r-1) \mathbb{V}^{n-1}(b_j x).$$

Hence $\tilde{a} \in \text{fil}_{r-1}^F W_n(L)$ as above. This completes the proof. \square

Theorem 8.8. *Let $L \in \Phi$ and $r \geq 0$. Then*

$$\text{fil}_r H^1(L) = \widetilde{H}^1(\mathcal{O}_L, \mathfrak{m}^{-r}),$$

i.e., the Artin conductor is motivic, $\text{Art} = c^{H^1}$. Furthermore, $(c^{H^1})^{\leq 1}$ is a conductor of level 1.

Proof. The last statement follows from the first and Proposition 8.5. By Corollary 4.29 it suffices to show the corresponding statement on the subsheaf of p^n -torsion, for all $n \geq 1$. Here the proof is the same as in Theorem 7.20 if we replace everywhere W_n by $H_{p^n}^1$, fil^F by fil , the reference to Proposition 7.5 by a reference to Proposition 8.5, and the reference to Proposition 7.19 by a reference to Proposition 8.7. \square

8.2. Lisse sheaves of rank 1. In this subsection we fix a prime number $\ell \neq p$, an algebraic closure $\overline{\mathbb{Q}}_\ell$ of \mathbb{Q}_ℓ , and a compatible system of primitive roots of unity $\{\zeta_n\} \subset \overline{\mathbb{Q}}_\ell^\times$.

8.9. We denote by $\text{Lisse}^1(X)$ the group of isomorphism classes of lisse $\overline{\mathbb{Q}}_\ell$ -sheaves on X of rank 1, with group structure given by \otimes . Note that

$$(8.9.1) \quad \text{Lisse}^1(X) \cong \varinjlim_{E/\mathbb{Q}_\ell} H_{\text{ét}}^1(X, \mathcal{O}_E^\times) := \varinjlim_{E/\mathbb{Q}_\ell} \varprojlim_n H_{\text{ét}}^1(X, (\mathcal{O}_E/\mathfrak{m}_E^n)^\times),$$

where E runs over sub-extensions of $\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$ which are finite over \mathbb{Q}_ℓ , and \mathcal{O}_E and \mathfrak{m}_E denote the ring of integers and the maximal ideal, respectively. Indeed, a sheaf $M \in \text{Lisse}^1(X)$ corresponds uniquely to a

continuous morphism $\pi_1^{\text{ab}}(X) \rightarrow \overline{\mathbb{Q}}_\ell^\times$, which in particular implies that it factors as a continuous morphism $\pi_1^{\text{ab}}(X) \rightarrow E^\times$, with some E as above (e.g., [Del80, 1.1]). Since any representation of a profinite group in a finite dimensional E -vector space has an \mathcal{O}_E -lattice, we see that such a morphism factors via a continuous map

$$\pi_1^{\text{ab}}(X) \rightarrow \text{Aut}_{\mathcal{O}_E}(\mathfrak{m}_E^{-j} \mathcal{O}_E) = \mathcal{O}_E^\times.$$

The isomorphism classes of such maps correspond uniquely to elements in $\varprojlim_n H_{\text{ét}}^1(X, (\mathcal{O}_E/\mathfrak{m}_E^n)^\times)$. By 8.1 and Lemma 8.2 the isomorphism (8.9.1) induces the structure of a Nisnevich sheaf with transfers on $X \mapsto \text{Lisse}^1(X)$, i.e.,

$$\text{Lisse}^1 \in \mathbf{NST}.$$

Write

$$|\mathcal{O}_E/\mathfrak{m}_E| = \ell^{r_E}, \quad \ell^{r_E} - 1 = p^{s_E} \cdot h_E, \quad \text{with } (h_E, p) = 1, s_E \geq 0.$$

Then $\mu_{\ell^{r_E-1}}(\overline{\mathbb{Q}}_\ell) \subset \mathcal{O}_E^\times$ and the roots of unity fixed at the beginning of this subsection induce a canonical isomorphism

$$\mathcal{O}_E^\times \cong \mathbb{Z}/p^{s_E} \times \mathbb{Z}/h_E \times U_E^{(1)}.$$

Since $U_E^{(1)}$ is a pro- ℓ group this yields the following decomposition

$$\text{Lisse}^1 = \text{Lisse}^{1,p'} \oplus H_{p^\infty}^1 \quad \text{in } \mathbf{NST},$$

where

$$X \mapsto \text{Lisse}^{1,p'}(X) := \varinjlim_{E/\mathbb{Q}_\ell} \varprojlim_n H_{\text{ét}}^1(X, \mathbb{Z}/h_E \times U_E^{(1)}/U_E^{(n)}),$$

$$X \mapsto H_{p^\infty}^1(X) := \varinjlim_{E/\mathbb{Q}_\ell} H_{\text{ét}}^1(X, \mathbb{Z}/p^{s_E}) = H_{p^\infty}^1(X).$$

Let $L \in \Phi$. For $j \geq 0$ we define

$$(8.9.2) \quad \text{fil}_j \text{Lisse}^1(L) := \begin{cases} \text{Im}(\text{Lisse}^1(\mathcal{O}_L) \rightarrow \text{Lisse}^1(L)), & \text{if } j = 0, \\ \text{Lisse}^{1,p'}(L) \oplus \text{fil}_j H_{p^\infty}^1(L), & \text{if } j \geq 1, \end{cases}$$

where $\text{fil}_j H_{p^\infty}^1(L) = \cup_n \text{fil}_j H_{p^n}^1(L)$ is defined in 8.4.

Corollary 8.10. *Let the notation be as in 8.9 above. Then*

- (1) $\text{Lisse}^1 \in \mathbf{RSC}_{\text{Nis}}$;
- (2) *the motivic conductor is given by*

$$c_L^{\text{Lisse}^1}(M) = \min\{j \geq 0 \mid M \in \text{fil}_j \text{Lisse}^1(L)\};$$

furthermore it restricts to a level 1 conductor.

(3) let $X \in \mathbf{Sm}$ be proper over k and $U \subset X$ dense open, then

$$h_{\mathbf{A}^1}^0(\mathrm{Lisse}^1)(U) = \mathrm{Lisse}^{1,p'}(U) \oplus H_{p^\infty}^1(X),$$

see 4.30 for notation.

Proof. Note $\mathrm{Lisse}^{1,p'} \in HI_{\mathrm{Nis}}$. Hence (1) and (2) follow directly from Theorem 8.8 together with the Corollaries 4.29 and Lemma 4.20. For (3) observe that by Theorem 8.8 and the definition of the Artin conductor, we have $H_{p^\infty}^1(\mathcal{O}_L, \mathfrak{m}_L^{-1}) = H_{p^\infty}^1(\mathcal{O}_L)$; hence the statement follows from Corollary 4.33. \square

Remark 8.11. Let $U \in \mathbf{Sm}$ and denote by $\pi_1^{\mathrm{ab},t}(U/k)$ the abelian tame fundamental group in the sense of [KS10, 7]; it is a quotient of $\pi_1^{\mathrm{ab}}(U)$. Denote by $\mathrm{Tame}^1(U)$ the subgroup of $\mathrm{Lisse}^1(U)$ consisting of those lisse sheaves of rank one whose corresponding representation factors via $\pi_1^{\mathrm{ab},t}(U/k)$. Then

$$h_{\mathbf{A}^1}^0(\mathrm{Lisse}^1)(U) = \mathrm{Tame}^1(U).$$

Indeed, we classically have $\mathrm{Tame}^1(C) = \mathrm{Lisse}^{1,p'}(C) \oplus H_{p^\infty}^1(\overline{C})$, in case $C \in \mathbf{Sm}$ is a curve over k with smooth compactification \overline{C} . Hence this \subset inclusion follows from Corollary 8.10(3) and the description of $\pi_1^{\mathrm{ab},t}(U/k)$ via curve-tameness, see [KS10]. The other inclusion follows from the \mathbf{A}^1 -invariance of Tame^1 .

9. TORSORS UNDER FINITE GROUP SCHEMES OVER A PERFECT FIELD

In this section k is a perfect field of positive characteristic p . We fix an algebraic closure \bar{k} of k . The term k -group is short for *commutative group scheme of finite type over k* .

Lemma 9.1. *Let G be a finite k -group. Then there exists an exact sequence of sheaves on $(\mathrm{Sch}/k)_{\mathrm{fppf}}$, the fppf-site on k -schemes,*

$$(9.1.1) \quad 0 \rightarrow G \rightarrow H_1 \rightarrow H_2 \rightarrow 0,$$

with H_i , $i = 1, 2$, smooth k -groups. Furthermore, if we denote by $u : (\mathrm{Sch}/k)_{\mathrm{fppf}} \rightarrow (\mathrm{Sch}/k)_{\mathrm{\acute{e}t}}$ the morphism from the fppf-site to the étale site, then the above sequence induces a canonical isomorphism

$$(9.1.2) \quad Ru_*G \cong [H_1 \rightarrow H_2]$$

in the derived category of abelian sheaves on $(\mathrm{Sch}/k)_{\mathrm{\acute{e}t}}$. In particular, for all $n \geq 0$ the presheaf on \mathbf{Sm}

$$(9.1.3) \quad \mathbf{Sm} \ni X \mapsto H^n(X_{\mathrm{fppf}}, G) \cong H^n(X_{\mathrm{\acute{e}t}}, H_1 \rightarrow H_2),$$

admits the structure of a presheaf with transfers. This transfers structure does not depend on the choice of the sequence (9.1.1) (up to isomorphism).

Proof. By a result of Raynaud (see [BBM82, 3.1.1]), there exists a closed immersion $G \hookrightarrow A$, with A an abelian variety. By [SGA 3₁, Exp VI_A, Thm 3.2], the fppf-quotient sheaf $(A/G)_{\text{fppf}}$ is representable by a k -group A/G and the quotient map $A \rightarrow A/G$ is finite and faithfully flat. Hence A/G is reduced and hence a smooth k -group. This shows the existence of a sequence (9.1.1). By [Gro68, Thm (11.7)] a smooth k -group is acyclic for the direct image functor

$$u_* : \text{Shv}((\text{Sch}/k)_{\text{fppf}}) \rightarrow \text{Shv}((\text{Sch}/k)_{\text{ét}}).$$

Hence (9.1.1) is a u_* -acyclic resolution of the fppf-sheaf G , which yields the canonical isomorphism (9.1.2). Since $H_1 \rightarrow H_2$ is a complex of étale sheaves with transfers, the presheaf (9.1.3) has transfers, by [MVW06, Lem 6.23]. Finally, we have to show that this transfer structure does not depend on the resolution (9.1.1). Assume $0 \rightarrow G \rightarrow L_1 \rightarrow L_2 \rightarrow 0$ is a second such exact sequence. We obtain a commutative diagram with exact rows in $(\text{Sch}/k)_{\text{fppf}}$

$$\begin{array}{ccccccc} 0 & \longrightarrow & G & \longrightarrow & L_1 \times H_1 & \longrightarrow & (L_1 \times H_1)/G \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & G & \longrightarrow & H_1 & \longrightarrow & H_2 \longrightarrow 0, \end{array}$$

where the vertical arrows are induced by projection and the top horizontal arrow on the left is the diagonal embedding of G ; we have also such a sequence with H replaced by L in the lower line. This yields the isomorphism $[H_1 \rightarrow H_2] \cong [L_1 \rightarrow L_2]$ in the derived category of étale sheaves with transfers, proving the final statement. \square

Notation 9.2. Let G be a finite k -group. Then we denote by $H^1(G) \in \mathbf{PST}$ the presheaf with transfers from Lemma 9.1,

$$X \mapsto H^1(G)(X) := H^1(X_{\text{fppf}}, G).$$

Lemma 9.3. Let $\text{Gal}(\bar{k}/k)$ be the absolute Galois group of k and G an étale k -group. Then the following functor defined by the Galois cohomology groups

$$(9.3.1) \quad \mathbf{Sm} \ni X \mapsto H^n(\text{Gal}(\bar{k}/k), G(X_{\bar{k}}))$$

is a proper sheaf in $\mathbf{RSC}_{\text{Nis}}$ in the sense of Definition 4.26.

Proof. The composition

$$\mathbf{Cor}_k(X, Y) \rightarrow \mathbf{Cor}_{\bar{k}}(X_{\bar{k}}, Y_{\bar{k}}) \rightarrow \text{Hom}_{\mathbf{Ab}}(G(Y_{\bar{k}}), G(X_{\bar{k}}))$$

factors through the homomorphism of Galois-modules; hence (9.3.1) \in **PST**. Since G is étale we have $G(X_{\bar{k}}) = G(\bar{k})^{\pi_0(X_{\bar{k}})}$. It follows that (9.3.1) is \mathbf{A}^1 -invariant and restrictions to dense open subsets are isomorphisms. Hence it is a Nisnevich sheaf and proper. \square

Lemma 9.4. *Let G be an étale k -group. Then the exact sequence*

$$E(X) : 0 \rightarrow H^1(\mathrm{Gal}(\bar{k}/k), G(X_{\bar{k}})) \rightarrow H^1(G)(X) \rightarrow K^1(X) \rightarrow 0,$$

with

$$K^1(X) := \mathrm{Ker}(H^1(X_{\bar{k}, \text{ét}}, G_{\bar{k}})^{\mathrm{Gal}(\bar{k}/k)} \rightarrow H^2(\mathrm{Gal}(\bar{k}/k), G(X_{\bar{k}}))),$$

coming from the E_2 -page of the Hochschild-Serre spectral sequence, defines an exact sequence $X \mapsto E(X)$ in **PST**.

Proof. First note that by Grothendieck's theorem (see Lemma 9.1) we have $H^1(G)(X) = H^1(X_{\text{ét}}, G)$, so that the sequence $E(X)$, indeed is induced by the Hochschild-Serre spectral sequence. We show that transfers act on the whole spectral sequence. By a limit argument it suffices to consider finite Galois extensions L/k and the corresponding spectral sequence. Let $G \rightarrow I^\bullet$ be an injective resolution in $\mathrm{Sh}_{\text{ét}}(\mathbf{Cor}_k)$, the category of étale sheaves with transfers. Then

$$(9.4.1) \quad H^i(X_{\text{ét}}, G) = H^i(I^\bullet(X)) = H^i(I^\bullet(X_L))^{\mathrm{Gal}(L/k)}, \quad i \geq 0,$$

for all $X \in \mathbf{Sm}$, see [MVW06, Lem 6.23]. Moreover, $H^i(X_{L, \text{ét}}, I^n) = 0 = H^i(X_{\text{ét}}, I^n)$, for $i \geq 1$ and $n \geq 0$, see *loc. cit.* Hence

$$(9.4.2) \quad H^i(\mathrm{Gal}(L/k), I^n(X_L)) = 0.$$

Let $C^\bullet(\mathrm{Gal}(L/k), M)$ be the complex of cochains computing the cohomology of the $\mathrm{Gal}(L/k)$ -module M . By (9.4.1), (9.4.2) the cohomology groups $H_{\text{ét}}^i(X, G)$ are the cohomology groups of the total complex associated to the double complex $C^\bullet(\mathrm{Gal}(L/k), I^\bullet(X_L))$. The Hochschild-Serre spectral sequence arises from a filtration of this complex. Furthermore, the canonical map $\mathbf{Cor}_k(X, Y) \times \mathrm{Gal}(L/k) \rightarrow \mathbf{Cor}_k(X_L, Y_L)$, $(\alpha, \sigma) \mapsto (\alpha \otimes_k L) \circ (\mathrm{id}_{X \times_k Y} \times \sigma)$ induces the structure of a complex of presheaves with $\mathrm{Gal}(L/k)$ -equivariant transfers on $X \mapsto I^\bullet(X_L)$. Hence $X \mapsto C^\bullet(\mathrm{Gal}(L/k), I^\bullet(X_L))$ is a double complex in **PST**. This proves the Lemma. \square

Lemma 9.5. *Let G be an étale k -group of order prime to p . Then $H^1(G) \in \mathbf{HI}_{\mathrm{Nis}}$ (see 9.2 for notation).*

Proof. In this case $G_{\bar{k}}$ is a constant finite k -group of order prime to p . By [Voe00a, Cor 5.29] the presheaf $X \mapsto K^1(X)$ from Lemma 9.4 is \mathbf{A}^1 -invariant and by Lemma 8.2 and Lemma 9.3 it is a Nisnevich sheaf. Thus the claim follows from the Lemmas 9.4, 9.3. \square

Lemma 9.6. *Let G be an étale k -group of p -primary order. Then $H^1(G) \in \mathbf{RSC}_{\text{Nis}}$ and the motivic conductor $c^{H^1(G)}$ is given by*

$$c_L^{H^1(G)} : H^1(G)(L) \rightarrow \bigoplus_i H_{\text{ét}}^1(\text{Spec } L_i, G_{\bar{k}}) \xrightarrow{\max_i \{c_{L_i}^{H^1(G_{\bar{k}})}\}} \mathbb{N}_0,$$

where $L \otimes_k \bar{k} = \prod_i L_i$ and $c^{H^1(G_{\bar{k}})}$ is computed in Theorem 8.8 (note that $G_{\bar{k}} = \bigoplus_j \mathbb{Z}/p^{n_j}$). In particular, $(c^{H^1(G)})^{\leq 1}$ is a conductor. Moreover, if X is smooth proper and $U \subset X$ is dense open, then $h_{\mathbf{A}^1}^0(H^1(G))(U) = H^1(G)(X)$ (see 4.30 for notation).

Proof. Note in this case $H^2(\text{Gal}(\bar{k}/k), G(X_{\bar{k}})) = 0$ (e.g. [SGA 4₃, Exp X, Thm 5.1]). Thus the first statement follows from Lemma 9.4, Lemma 9.3, Lemma 8.2, Lemma 4.27, Proposition 4.19, Proposition 4.21, and Theorem 8.8. For the final statement observe that

$$\widetilde{H^1(G)}(\mathcal{O}_L, \mathfrak{m}_L^{-1}) = H^1(G)(\mathcal{O}_L).$$

This follows directly from the explicit description of the motivic conductor on $H^1(G_{\bar{k}})$ in Theorem 8.8. Hence the final statement follows from Corollary 4.33. \square

Lemma 9.7. *Let G be an infinitesimal finite k -group. Then*

$$H^1(X_{\text{fppf}}, G) \cong H^1(X_{\bar{k}, \text{fppf}}, G_{\bar{k}})^{\text{Gal}(\bar{k}/k)}, \quad \text{for all } X \in \mathbf{Sm}.$$

Furthermore, this isomorphism induces an isomorphism in \mathbf{NST} (cf. Proposition 4.21 for notation)

$$H^1(G) \cong (R_{\bar{k}/k} H^1(G_{\bar{k}}))^{\text{Gal}(\bar{k}/k)}.$$

Proof. Since G is infinitesimal, we have $G(Y) = 0$ for all reduced schemes Y over k . There is also a Hochschild-Serre spectral sequence for the fppf-cohomology (e.g., [Mil80, III, Rem 2.21]); by the above remark the fppf-version of the exact sequence $E(X)$ from Lemma 9.4 yields the first isomorphism. By Lemma 9.1 this isomorphism is compatible with the transfer structure. It remains to show that $H^1(G)$ is a Nisnevich sheaf. By the remark from the beginning of this proof any sequence (9.1.1) yields an injection $H_1 \hookrightarrow H_2$ when restricted to \mathbf{Sm} . Thus the isomorphism (9.1.2) implies

$$Ru_* G \cong (H_2/H_1)_{\text{ét}}[-1]$$

in the derived category of étale sheaves on \mathbf{Sm} , where $(H_2/H_1)_{\text{ét}}$ denotes the étale sheafification of the presheaf $X \mapsto H_2(X)/H_1(X)$. Hence

$$H^1(G)(X) = H^0(X, (H_2/H_1)_{\text{ét}}).$$

It follows that $H^1(G)$ is even an étale sheaf. \square

Lemma 9.8. *Assume G is an infinitesimal finite k -group of multiplicative type. Then $H^1(G) \in \mathbf{HI}_{\text{Nis}}$.*

Proof. By Lemma 9.7 we may assume $k = \bar{k}$. In this case G is diagonalizable and we find an exact sequence (9.1.1) with $H_i = \mathbf{G}_m^{n_i}$, some $n_i \geq 1$, see [DG70a, IV, §1, 1.5 Cor]. The statement follows from the \mathbf{A}^1 -invariance of $X \mapsto H^i(X_{\text{Zar}}, \mathbf{G}_m)$, $i = 0, 1$, and Hilbert 90. \square

9.9. We denote

$$\alpha_p := \text{Ker}(F : \mathbf{G}_a \rightarrow \mathbf{G}_a),$$

where F is the absolute Frobenius on the additive group. Then α_p is a unipotent infinitesimal finite k -group. Let $L \in \Phi$ and let $t \in \mathcal{O}_L$ be a local parameter. Recall from 7.3 that $\text{fil}_j \mathbf{G}_a(L) := \text{fil}_j W_1(L)$ is given by

$$(9.9.1) \quad \text{fil}_j \mathbf{G}_a(L) = \begin{cases} \mathcal{O}_L, & \text{if } j = 0 \\ \frac{1}{t^{j-1}} \cdot \mathcal{O}_L, & \text{if } (j, p) = 1 \\ \frac{1}{t^j} \cdot \mathcal{O}_L, & \text{if } p \mid j. \end{cases}$$

We denote by

$$(9.9.2) \quad \text{fil}_j H^1(\alpha_p)(L)$$

the image of $\text{fil}_j \mathbf{G}_a(L)$ under the connecting homomorphism

$$\delta : \mathbf{G}_a(L) \rightarrow H^1(\alpha_p)(L) = H^1(\text{Spec } L_{\text{fppf}}, \alpha_p).$$

Note that $\text{fil}_j H^1(\alpha_p)(L)$ is also equal to the image of the Frobenius saturated filtration $\text{fil}_j^F W_1(L)$.

Proposition 9.10. *We have $H^1(\alpha_p) \in \mathbf{RSC}_{\text{Nis}}$ and the motivic conductor $c^{H^1(\alpha_p)}$ on $H^1(\alpha_p)$ is given by*

$$(9.10.1) \quad c_L^{H^1(\alpha_p)}(b) = \min\{j \geq 0 \mid b \in \text{fil}_j H^1(\alpha_p)(L)\}.$$

In particular, either $b \in H^1(\alpha_p)(\mathcal{O}_L)$ or $c_L^{H^1(\alpha_p)}(b) \geq 2$. Furthermore, it restricts to a level 2 conductor.

Proof. Denote the collection of maps $H^1(\alpha_p)(L) \rightarrow \mathbb{N}_0$ defined by the right hand side of (9.10.1) by c . By Proposition 7.5 and Lemma 4.28, c satisfies (c1)-(c6) except possibly for (c4). (For (c5) note, that $\mathbf{G}_a(Y) \rightarrow H^1(\alpha_p)(Y)$ is surjective for any affine scheme Y over k .) We claim that $c^{\leq 2}$ satisfies (c4). Let X be a smooth k -scheme and $b \in H^1(\alpha_p)(\mathbf{A}_X^1)$ with

$$(9.10.2) \quad c_{k(x)_\infty}(\rho_x^* b) \leq 1, \quad \text{for all } x \in X \text{ with } \text{trdeg}(k(x)/k) \leq 1,$$

where $\rho_x : \text{Spec } k(x)(t)_\infty = \text{Spec } \text{Frac}(\mathcal{O}_{\mathbf{P}_x^1, \infty}^h) \rightarrow \mathbf{A}_X^1$ is the natural map. We want to show : $b \in H^1(\alpha_p)(X)$. This is equivalent to $b = \pi^* i^* b$

in $H^1(\alpha_p)(\mathbf{A}_X^1)$; by the definition of c and Lemma 9.7, we can therefore assume k is algebraically closed. Furthermore, the question is local on X , hence we can assume $X = \text{Spec } A$ affine. Note, for a general $\beta \in H^1(\alpha_p)(L) \setminus H^1(\alpha_p)(\mathcal{O}_L)$ we have $c_L(\beta) \geq 2$, as follows directly from (9.9.1). Hence condition (9.10.2) implies

$$\rho_x^* b \in \text{Im}(H^1(\alpha_p)(\mathcal{O}_{\mathbf{P}_x^1, \infty}^h) \rightarrow H^1(\alpha_p)(k(x)(t)_\infty)).$$

Denote by $b(x)$ the restriction of b to \mathbf{A}_x^1 . Since $H^1(\alpha_p)$ is a Nisnevich sheaf we conclude

$$b(x) \in H^1(\alpha_p)(\mathbf{P}_x^1) = H^1(\alpha_p)(x).$$

Thus we find a polynomial $\tilde{b} = b_0 + b_1 t + \dots + b_n t^n \in A[t]$ mapping to b such that for all points $x \in X$ with $\text{trdeg}(k(x)/k) \leq 1$ there exist $c_x \in k(x)$ and $g_x \in k(x)[t]$ with

$$(9.10.3) \quad \tilde{b}(x) = c_x + g_x^p, \quad \text{in } k(x)[t].$$

It follows immediately that $\tilde{b} \in A[t^p]$ and it remains to show $b_i \in A^p$, for all $i \geq 1$, since then $b = b_0$ in $H^1(\alpha_p)(\mathbf{A}_X^1)$. Thus we are reduced to show the following: Let $X = \text{Spec } A \rightarrow \mathbf{A}^d = \text{Spec } k[x_1, \dots, x_d]$ be an étale map and $a \in A \setminus A^p$. Then there exists a smooth connected curve $i : C \hookrightarrow X$ such that $i^* a \in \mathcal{O}(C) \setminus \mathcal{O}(C)^p$. If $a \notin A^p$ we find a variable - say x_1 - such that $a = a_0 + a_1 x_1 + \dots + a_n x_1^n$, where $a_i \in A^p[x_2, \dots, x_d] := B$ and $a \notin B[x_1^p]$. A tuple $\lambda = (\lambda_2, \dots, \lambda_d) \in k^{d-1}$ induces a closed immersion $i_\lambda : \mathbf{A}^1 \rightarrow \mathbf{A}^d$ given by $x_1 \mapsto x_1$, $x_i \mapsto \lambda_i$, $i = 2, \dots, d$. Denote by C_λ the pullback of X along i_λ . Since k is algebraically closed we find a tuple λ such that $a|_{C_\lambda} \notin \mathcal{O}(C_\lambda)^p$. This proves the above claim; hence $c^{\leq 2}$ satisfies (c4).

Corollary 4.24 yields $c^{H^1(\alpha_p)} \leq c$. To show the other inequality it suffices by Corollary 4.40 to show the following: Let $L \in \Phi$, $t \in \mathcal{O}_L$ a local parameter, and let $\sigma : K \hookrightarrow \mathcal{O}_L$ be some coefficient field; extend it in the canonical way to $\sigma : K(x) \hookrightarrow \mathcal{O}_{L_x}$, where $L_x = \text{Frac}(\mathcal{O}_L[x]_{(t)}^h)$. Assume $b \in \text{fil}_r H^1(\alpha_p)(L)$, $r \geq 1$. Then the following implication holds

$$(9.10.4) \quad (b, 1 - xt^{r-1})_{L_x, \sigma} = 0 \quad \text{for all } \sigma \Rightarrow b \in \text{fil}_{r-1} H^1(\alpha_p)(L),$$

where the local symbol on the left hand side is the one from 4.37 for $H^1(\alpha_p)$, and σ runs through all coefficient fields of \mathcal{O}_L . By (LS6) the local symbol on $H^1(\alpha_p)$ is given by

$$(b, 1 - xt^{r-1})_{L_x, \sigma} = \delta(\text{Res}_{t, \sigma}(\tilde{b} \text{dlog}(1 - xt^{r-1}))),$$

where $\tilde{b} \in \text{fil}_r \mathbf{G}_a(L)$ is a lift of b , $\delta : \mathbf{G}_a(K(x)) \rightarrow H^1(\alpha_p)(K(x))$ is the connecting homomorphism, and we use the isomorphism $L_x = K(x)((t))$ defined by σ and t to compute the residue symbol on the

right. To prove the implication (9.10.4) it suffices to consider b modulo fil_r . Fix $\sigma : K \hookrightarrow \mathcal{O}_L$.

1st case: $(r, p) = 1 = (r - 1, p)$. In this case $\tilde{b} \equiv c/t^{r-1} \pmod{\text{fil}_{r-1}\mathbf{G}_a(L)}$, for some $c \in K$. Hence

$$\text{Res}_{t,\sigma}(\tilde{b} \text{dlog}(1 - xt^{r-1})) = -(r - 1)cx.$$

Since $\delta(-(r - 1)cx) = 0$ iff $cx \in K(x)^p$, this is only possible if $c = 0$.

2nd case: $p \mid r - 1$. In this case $\text{fil}_r H^1(\alpha_p)(L) = \text{fil}_{r-1} H^1(\alpha_p)(L)$, and there is nothing to show.

3rd case: $p \mid r$. In this case $\tilde{b} \equiv c/t^{r-1} + e/t^r \pmod{\text{fil}_{r-1}\mathbf{G}_a(L)}$, for some $c, e \in K$. By the same argument as in the 1st case we obtain the following implication

$$(b, 1 - xt^{r-1})_{Lx,\sigma} = 0 \Rightarrow (\tilde{b}, 1 - xt^{r-1})_{Lx,\sigma} = 0 \quad \text{in } \mathbf{G}_a(K(x)).$$

Since this holds for all σ , Proposition 7.19 (in the case $n = 1$) yields $\tilde{b} \in \text{fil}_{r-1}^F \mathbf{G}_a(L)$, hence $b \in \text{fil}_{r-1} H^1(\alpha_p)(L)$. This completes the proof. \square

Proposition 9.11. *Let G be a finite unipotent infinitesimal k -group.*

- (1) $H^1(G) \in \mathbf{RSC}_{\text{Nis}}$;
- (2) the motivic conductor $c^{H^1(G)}$ restricts to a level 2 conductor;
- (3) if X is a proper smooth k -scheme and $U \subset X$ is open dense, then $h_{\mathbf{A}^1}^0(H^1(G))(U) = H^1(G)(X)$ (see 4.30 for notation).

Proof. (1). We find an exact sequence in the category of k -groups

$$0 \rightarrow G \rightarrow H_1 \rightarrow H_2 \rightarrow 0$$

with H_i smooth unipotent k -groups. Indeed, by [DG70a, V, §1, 4.2, 4.7] we find a closed immersion $G \hookrightarrow W_n^N := H_1$, for some n, N , and by [DG70a, IV, §2, 2.3] the quotient $H_2 := H_1/G$ is again unipotent, and it is automatically reduced, hence is smooth. As in the proof of Lemma 9.7 we find $H^1(G) \cong (H_2/H_1)_{\text{ét}}$, where $(H_2/H_1)_{\text{ét}}$ is the étale sheaf associated to the presheaf $\mathbf{Sm} \ni X \mapsto H_2(X)/H_1(X)$. Let $v : \mathbf{Sm}_{\text{ét}} \rightarrow \mathbf{Sm}_{\text{Nis}}$ be the natural morphism of sites. Since H_1 is smooth unipotent, it is a successive extension of \mathbf{G}_a 's, hence $R^1 v_* H_1 = 0$. We obtain an isomorphism in **NST**

$$H^1(G) \cong (H_2/H_1)_{\text{Nis}},$$

where $(H_2/H_1)_{\text{Nis}}$ is the Nisnevich sheaf associated to the presheaf $X \mapsto H_2(X)/H_1(X)$. Thus $H^1(G) \in \mathbf{RSC}_{\text{Nis}}$ follows from $H_i \in \mathbf{RSC}_{\text{Nis}}$ and [Sai20, Thm 0.1], which states that Nisnevich sheafification preserves SC-reciprocity.

(2). By [DG70a, IV, 5.8] G admits a descending sequence

$$(9.11.1) \quad 0 = G_n \subset G_{n-1} \subset \dots \subset G_0 = G$$

with successive quotients $G_{r-1}/G_r \cong \alpha_p$. In particular, $H^2(X_{\text{fppf}}, G) = 0$, for all *affine* smooth k -schemes X . Note that this induces for all $r \in [1, n]$ an exact sequence in **NST**

$$(9.11.2) \quad 0 \rightarrow H^1(G_r) \rightarrow H^1(G_{r-1}) \rightarrow H^1(\alpha_p) \rightarrow 0.$$

Indeed, by Lemma 9.7 this sequence is in **NST**; hence it suffices to check its exactness on any smooth affine k -scheme X , in which case it follows from $H^0(X_{\text{fppf}}, \alpha_p) = 0 = H^2(X_{\text{fppf}}, G_r)$. By Proposition 9.10 the motivic conductor of $H^1(\alpha_p)$ restricts to a level 2 conductor and by induction we may assume that so does the motivic conductor of $H^1(G_{r-1})$. We deduce that the motivic conductor of $H^1(G_r)$ restricts to a level 2 conductor from (9.11.2) and a similar argument as at the end of the proof of Proposition 8.5.

(3). We claim

$$(9.11.3) \quad \widetilde{H^1(G)}(\mathcal{O}_L, \mathfrak{m}_L^{-1}) = H^1(G)(\mathcal{O}_L).$$

The claim is true for $G = \alpha_p$, by the explicit formula of the motivic conductor in Proposition 9.10. Consider the sequence (9.11.1) and assume the claim is proven for G_r . Let $b \in \widetilde{H^1(G_{r-1})}(\mathcal{O}_L, \mathfrak{m}_L^{-1})$. By the exact sequence (9.11.2) and the claim for α_p we find a $c \in H^1(G_{r-1})(\mathcal{O}_L)$ such that $b - c$ is in the image of $H^1(G_r)(L)$. By Proposition 4.19 we find

$$b - c \in \widetilde{H^1(G_r)}(\mathcal{O}_L, \mathfrak{m}_L^{-1}) = H^1(G_r)(\mathcal{O}_L),$$

which proves (9.11.3). Hence (3) follows from Corollary 4.33. \square

In summary:

Theorem 9.12. *Let G be a finite k -group. Then:*

- (1) $H^1(G) \in \mathbf{RSC}_{\text{Nis}}$;
- (2) *the motivic conductor of $H^1(G)$ restricts to conductor of level 2, and if G has no infinitesimal unipotent factor, to a conductor of level 1;*
- (3) *write $G = G' \times G_{\text{unip}}$ with G_{unip} unipotent and G' without any unipotent subgroup, and let X be smooth proper over k and $U \subset X$ dense open. Then*

$$h_{\mathbf{A}^1}^0(H^1(G))(U) = H^1(G')(U) \oplus H^1(G_{\text{unip}})(X).$$

Proof. By [DG70a, IV, §3, 5.9] we can decompose G uniquely into a product

$$G = G_{em} \times G_{eu} \times G_{im} \times G_{iu},$$

where G_{em} is étale multiplicative, i.e., it is an étale k -group without p -torsion, G_{eu} is étale unipotent, i.e., it is an étale k -group with p -primary torsion, G_{im} is infinitesimal and of multiplicative type, and G_{iu} is an infinitesimal unipotent k -group. Hence the statement follows from Lemma 9.5, Lemma 9.6, Lemma 9.8, and Proposition 9.11. \square

Remark 9.13. Let G be a finite unipotent k -group. Note that by Theorem 9.12(3) above, the functor $X \mapsto H^1(X_{\text{fppf}}, G)$ is a birational invariant for smooth proper k -schemes. This gives a new proof of this (probably) well-known result (it follows, e.g., also from [CR11]).

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