

# CANCELLATION THEOREMS FOR RECIPROCITY SHEAVES

ALBERTO MERICI AND SHUJI SAITO

ABSTRACT. We prove cancellation theorems for reciprocity sheaves and cube-invariant modulus sheaves with transfers of Kahn–Saito–Yamazaki, generalizing Voevodsky’s cancellation theorem for  $\mathbf{A}^1$ -invariant sheaves with transfers. As an application, we get some new formulas for internal hom’s of the sheaves  $\Omega^i$  of absolute Kähler differentials.

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## 0. INTRODUCTION

We fix once and for all a perfect field  $k$ . Let  $\mathbf{Sm}$  be the category of separated smooth schemes of finite type over  $k$ . Let  $\mathbf{Cor}$  be the category of finite correspondences:  $\mathbf{Cor}$  has the same objects as  $\mathbf{Sm}$  and morphisms in  $\mathbf{Cor}$  are finite correspondences. Let  $\mathbf{PST}$  be the category of additive presheaves of abelian groups on  $\mathbf{Cor}$ , called presheaves with transfers. Let  $\mathbf{NST} \subset \mathbf{PST}$  be the full subcategory of Nisnevich sheaves, i.e. those objects  $F \in \mathbf{PST}$  whose restrictions  $F_X$  to the small étale site  $X_{\text{ét}}$  over  $X$  are Nisnevich sheaves for all  $X \in \mathbf{Sm}$ . By a fundamental result of Voevodsky, the inclusion  $\mathbf{NST} \rightarrow \mathbf{PST}$  has an exact

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left adjoint  $a_{\mathbf{Nis}}^V$  such that for any  $F \in \mathbf{PST}$  and  $X \in \mathbf{Sm}$ ,  $(a_{\mathbf{Nis}}^V F)_X$  is the Nisnevich sheafification of  $F_X$  as a presheaf on  $X_{\mathbf{Nis}}$ . In Voevodsky's theory of motives, a fundamental role is played by  $\mathbf{A}^1$ -invariant objects  $F \in \mathbf{NST}$ , namely such  $F$  that  $F(X) \rightarrow F(X \times \mathbf{A}^1)$  induced by the projection  $X \times \mathbf{A}^1 \rightarrow X$  are isomorphisms for all  $X \in \mathbf{Sm}$ . The  $\mathbf{A}^1$ -invariant objects form a full abelian subcategory  $\mathbf{HI}_{\mathbf{Nis}} \subset \mathbf{NST}$  that carries a symmetric monoidal structure  $\otimes_{\mathbf{HI}}^{\mathbf{Nis}}$  such that

$$F \otimes_{\mathbf{HI}}^{\mathbf{Nis}} G = h_0^{\mathbf{A}^1, \mathbf{Nis}} a_{\mathbf{Nis}}^V (F \otimes_{\mathbf{PST}} G) \quad \text{for } F, G \in \mathbf{HI}_{\mathbf{Nis}},$$

where  $h_0^{\mathbf{A}^1, \mathbf{Nis}}$  is a left adjoint to the inclusion functor  $\mathbf{HI}_{\mathbf{Nis}} \rightarrow \mathbf{NST}$ , which sends an object of  $\mathbf{NST}$  to its maximal  $\mathbf{A}^1$ -invariant quotient in  $\mathbf{NST}$ . For integers  $n > 0$ , the twists of  $F \in \mathbf{HI}_{\mathbf{Nis}}$  are defined as

$$F(1) = F \otimes_{\mathbf{HI}}^{\mathbf{Nis}} \mathbf{G}_m, \quad F(n) := F(n-1) \otimes_{\mathbf{HI}}^{\mathbf{Nis}} \mathbf{G}_m.$$

where  $\mathbf{G}_m \in \mathbf{NST}$  is given by  $X \rightarrow \Gamma(X, \mathcal{O}^\times)$  for  $X \in \mathbf{Sm}$ .

Noting that  $- \otimes_{\mathbf{HI}}^{\mathbf{Nis}} \mathbf{G}_m$  is an endo-functor on  $\mathbf{HI}_{\mathbf{Nis}}$ , we get a natural map:

$$(0.1) \quad \iota_{F,G} : \text{Hom}_{\mathbf{PST}}(F, G) \rightarrow \text{Hom}_{\mathbf{PST}}(F(1), G(1)) \quad \text{for } F, G \in \mathbf{HI}_{\mathbf{Nis}}.$$

One key ingredient in Voevodsky's theory is the Cancellation theorem:

**Theorem 0.1.** ([14]) *For  $F, G \in \mathbf{HI}_{\mathbf{Nis}}$ ,  $\iota_{F,G}$  is an isomorphism.*

The purpose of this paper is to generalize Voevodsky's Cancellation theorem to reciprocity sheaves. The category  $\mathbf{RSC}_{\mathbf{Nis}}$  of reciprocity sheaves was introduced in [4] and [5] as a full subcategory of  $\mathbf{NST}$  that contains  $\mathbf{HI}_{\mathbf{Nis}}$  as well as interesting non- $\mathbf{A}^1$ -invariant objects such as the additive group scheme  $\mathbf{G}_a$ , the sheaf of absolute Kähler differentials  $\Omega^i$  and the de Rham-Witt sheaves  $W_n \Omega^i$ . In [10], a lax monoidal structure  $(-, -)_{\mathbf{RSC}_{\mathbf{Nis}}}$  on  $\mathbf{RSC}_{\mathbf{Nis}}$  is defined in such a way that

$$(F, G)_{\mathbf{RSC}_{\mathbf{Nis}}} = F \otimes_{\mathbf{HI}}^{\mathbf{Nis}} G \quad \text{for } F, G \in \mathbf{HI}_{\mathbf{Nis}}.$$

It allows us to define the twists for  $F \in \mathbf{RSC}_{\mathbf{Nis}}$  recursively as

$$F\langle 1 \rangle := (F, \mathbf{G}_m)_{\mathbf{RSC}_{\mathbf{Nis}}}, \quad F\langle n \rangle := (F\langle n-1 \rangle, \mathbf{G}_m)_{\mathbf{RSC}_{\mathbf{Nis}}}.$$

Some examples of twists were computed in [10]: If  $F \in \mathbf{HI}_{\mathbf{Nis}}$ , then  $F\langle n \rangle = F(n)$ , in particular  $\mathbb{Z}\langle n \rangle \cong \mathcal{K}_n^M$  (the Milnor  $K$ -sheaf), and  $\mathbf{G}_a\langle n \rangle \cong \Omega^n$  if  $\text{ch}(k) = 0$ .

By the fact that  $(-, \mathbf{G}_m)_{\mathbf{RSC}_{\mathbf{Nis}}}$  is an endo-functor on  $\mathbf{RSC}_{\mathbf{Nis}}$ , we get a natural map (cf. (5.6)) :

$$(0.2) \quad \iota_{F,G} : \text{Hom}_{\mathbf{PST}}(F, G) \rightarrow \text{Hom}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle) \quad \text{for } F, G \in \mathbf{RSC}_{\mathbf{Nis}},$$

which coincides with (0.1) if  $F, G \in \mathbf{HI}_{\text{Nis}}$ . We will also get a natural map in  $\mathbf{NST}$ :

$$(0.3) \quad \lambda_F : F \rightarrow \underline{\mathbf{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, F(n)) \text{ for } F \in \mathbf{RSC}_{\text{Nis}},$$

using the functoriality of  $(-, \mathbf{G}_m)_{\mathbf{RSC}_{\text{Nis}}}$ , where  $\underline{\mathbf{Hom}}_{\mathbf{PST}}$  denotes the internal hom in  $\mathbf{PST}$ .

The main result of this paper is the following:

**Theorem 0.2** (Theorems 5.3 and 5.1). *The maps  $\iota_{F,G}$  and  $\lambda_F$  are isomorphisms.*

As an application of the above theorem, we prove the following.

**Corollary 0.3.** (Theorem 6.2) *Assume  $\text{ch}(k) = 0$ . For integers  $m, n \geq 0$ , there are natural isomorphisms in  $\mathbf{NST}$ :*

$$\begin{aligned} \underline{\mathbf{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m) &\cong \Omega^{m-n} \oplus \Omega^{m-n-1} \\ \underline{\mathbf{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, \Omega^m) &\cong \Omega^{m-n}, \end{aligned}$$

where  $\Omega^i = 0$  for  $i < 0$  by convention.

Let  $\mathbf{PS}$  be the category of additive presheaves of abelian groups on  $\mathbf{Sm}$  (without transfers). Note that  $\mathbf{PST}$  is viewed as a subcategory of  $\mathbf{PS}$ . By a lemma due to Kay Rülling (see Lemma 1.1), we have a natural isomorphism in  $\mathbf{PS}$ :

$$(0.4) \quad \underline{\mathbf{Hom}}_{\mathbf{PST}}(G, \Omega^m) \cong \underline{\mathbf{Hom}}_{\mathbf{PS}}(G, \Omega^m) \text{ for any } G \in \mathbf{PST},$$

where  $\underline{\mathbf{Hom}}_{\mathbf{PS}}$  is the internal hom in  $\mathbf{PS}$ . Thanks to (0.4), the isomorphisms of Corollary 0.3 and its explicit descriptions (6.1) and (6.3) imply

$$\underline{\mathbf{Hom}}_{\mathbf{PS}}(\Omega^n, \Omega^m) = \{\omega_1 \wedge (-) + \omega_2 \wedge d(-) \mid \omega_1 \in \Omega_k^{m-n}, \omega_2 \in \Omega_k^{m-n-1}\},$$

$$\underline{\mathbf{Hom}}_{\mathbf{PS}}(\mathcal{K}_n^M, \Omega^m) = \{\omega \wedge \text{dlog}(-) \mid \omega \in \Omega_k^{m-n}\},$$

where  $\text{dlog} : \mathcal{K}_n^M \rightarrow \Omega^m$  is the map  $\{x_1, \dots, x_n\} \rightarrow \text{dlog}x_1 \wedge \dots \wedge \text{dlog}x_n$ . It would be an interesting question if there is a direct proof of these formulas which does not use the machinery of modulus sheaves with transfers explained below.

Reciprocity sheaves are closely related to *modulus sheaves with transfers* introduced in [2] and [3]: Voevodsky's category  $\mathbf{Cor}$  of finite correspondences is enlarged to a new category  $\underline{\mathbf{MCor}}$  of *modulus pairs*: Its objects are pairs  $\mathcal{X} = (X, D)$  where  $X$  is a separated scheme of finite type over  $k$  and  $D$  is an effective Cartier divisor on  $X$  such that  $\mathcal{X}^\circ := X - |D| \in \mathbf{Sm}$  ( $\mathcal{X}^\circ$  is called the interior of  $\mathcal{X}$ ). The morphisms are finite correspondences on interiors satisfying some admissibility and properness conditions. Let  $\mathbf{MCor} \subset \underline{\mathbf{MCor}}$  be the full subcategory of

such objects  $(X, D)$  that  $X$  is proper over  $k$ . We then define  $\underline{\mathbf{MPST}}$  (resp.  $\mathbf{MPST}$ ) as the category of additive presheaves of abelian groups on  $\underline{\mathbf{MCor}}$  (resp.  $\mathbf{MCor}$ ). We have a functor

$$\underline{\omega} : \underline{\mathbf{MCor}} \rightarrow \mathbf{Cor} ; (\overline{X}, X_\infty) \rightarrow \overline{X} - |X_\infty|,$$

and two adjunctions

$$\mathbf{MPST} \begin{array}{c} \xleftarrow{\tau^*} \\ \xrightarrow{\tau_!} \end{array} \underline{\mathbf{MPST}}, \quad \underline{\mathbf{MPST}} \begin{array}{c} \xleftarrow{\underline{\omega}^*} \\ \xrightarrow{\underline{\omega}_!} \end{array} \mathbf{PST},$$

where  $\underline{\omega}^*$  is induced by  $\underline{\omega}$  and  $\underline{\omega}_!$  is its left Kan extension, and  $\tau^*$  is induced by the inclusion  $\tau : \mathbf{MCor} \rightarrow \underline{\mathbf{MCor}}$  and  $\tau_!$  is its left Kan extension, which turned out to be exact and fully faithful.

For  $F \in \underline{\mathbf{MPST}}$  and  $\mathfrak{X} = (X, D) \in \underline{\mathbf{MCor}}$  write  $F_{\mathfrak{X}}$  for the presheaf on the small étale site  $X_{\text{ét}}$  over  $X$  given by  $U \rightarrow F(\mathfrak{X}_U)$  for  $U \rightarrow X$  étale, where  $\mathfrak{X}_U = (U, D \times_X U) \in \underline{\mathbf{MCor}}$ . We say  $F$  is a Nisnevich sheaf if so is  $F_{\mathfrak{X}}$  for all  $\mathfrak{X} \in \underline{\mathbf{MCor}}$ . We write  $\underline{\mathbf{MNST}} \subset \underline{\mathbf{MPST}}$  for the full subcategory of Nisnevich sheaves.

The replacement of the  $\mathbf{A}^1$ -invariance in this new framework is the  $\square$ -invariance, where  $\square := (\mathbf{P}^1, \infty) \in \mathbf{MCor}$ : Let  $\mathbf{CI} \subset \mathbf{MPST}$  be the full subcategory of those objects  $F$  that  $F(\mathcal{X}) \rightarrow F(\mathcal{X} \otimes \square)$  induced by the projection  $\mathcal{X} \otimes \square \rightarrow \mathcal{X}$  are isomorphisms for all  $\mathcal{X} \in \mathbf{MCor}$ . Let  $\mathbf{CI}^\tau \subset \underline{\mathbf{MPST}}$  be the essential image of  $\mathbf{CI}$  under  $\tau_!$  and define  $\mathbf{CI}_{\text{Nis}}^\tau = \mathbf{CI}^\tau \cap \underline{\mathbf{MNST}}$ . We further define the full subcategory  $\mathbf{CI}_{\text{Nis}}^{\tau, sp} \subset \mathbf{CI}_{\text{Nis}}^\tau$  of *semipure* objects  $F$ , namely such objects that the natural map  $F(X, D) \rightarrow F(X - D, \emptyset)$  are injective for all  $(X, D) \in \underline{\mathbf{MCor}}$ . We will define a symmetric monoidal structure  $\otimes_{\mathbf{CI}}^{\text{Nis}, sp}$  on  $\mathbf{CI}_{\text{Nis}}^{\tau, sp}$  (see §1(15)).

The relationship between reciprocity sheaves and  $\square$ -invariant modulus sheaves with transfers is encoded in

$$\mathbf{RSC}_{\text{Nis}} = \underline{\omega}_!(\mathbf{CI}_{\text{Nis}}^{\tau, sp}).$$

There is a pair of adjoint functors

$$\mathbf{CI}_{\text{Nis}}^{\tau, sp} \begin{array}{c} \xleftarrow{\underline{\omega}^{\mathbf{CI}}} \\ \xrightarrow{\underline{\omega}_!} \end{array} \mathbf{RSC}_{\text{Nis}}$$

such that  $\underline{\omega}^{\mathbf{CI}} F = \underline{\omega}^* F$  for  $F \in \mathbf{HI}_{\text{Nis}}$ . Moreover, the lax monoidal structure on  $\mathbf{RSC}_{\text{Nis}}$  is induced by the symmetric monoidal structure on  $\mathbf{CI}_{\text{Nis}}^{\tau, sp}$  via  $\underline{\omega}_!$ . The endo-functor  $- \otimes_{\mathbf{CI}}^{\text{Nis}, sp} \underline{\omega}^* \mathbf{G}_m$  on  $\mathbf{CI}_{\text{Nis}}^{\tau, sp}$  induces a natural map for  $F \in \mathbf{CI}_{\text{Nis}}^{\tau, sp}$ :

$$\iota_F : F \rightarrow \underline{\mathbf{Hom}}_{\underline{\mathbf{MPST}}}(\underline{\omega}^* \mathbf{G}_m, F \otimes_{\mathbf{CI}}^{\text{Nis}, sp} \underline{\omega}^* \mathbf{G}_m),$$

where  $\underline{\mathbf{Hom}}_{\underline{\mathbf{MPST}}}$  denotes the internal hom in  $\underline{\mathbf{MPST}}$ . Now Theorem 0.2 will be a consequence of the following result:

**Theorem 0.4** (Cor 3.5). *For  $F \in \mathbf{RSC}_{\text{Nis}}$  and  $\tilde{F} = \underline{\omega}^{\mathbf{CI}} F \in \mathbf{CI}_{\text{Nis}}^{\tau, sp}$ , the map  $\iota_{\tilde{F}}$  is an isomorphism.*

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**Conventions.** In the whole paper we fix a perfect base field  $k$ . Let  $\widetilde{\mathbf{Sm}}$  be the category of  $k$ -schemes  $X$  which are essentially smooth over  $k$ , i.e.  $X$  is a limit  $\varprojlim_{i \in I} X_i$  over a filtered set  $I$ , where  $X_i$  is smooth over  $k$  and all transition maps are étale. Note  $\text{Spec } K \in \widetilde{\mathbf{Sm}}$  for a function field  $K$  over  $k$  thanks to the assumption that  $k$  is perfect. We frequently allow  $F \in \mathbf{PST}$  to take values on objects of  $\widetilde{\mathbf{Sm}}$  by  $F(X) := \varinjlim_{i \in I} F(X_i)$  for  $X$  as above.

## 1. RECOLLECTION ON MODULUS SHEAVES WITH TRANSFERS

In this section we recall the definitions and basic properties of modulus sheaves with transfers from [2] and [7] (see also [5] for a more detailed summary).

- (1) Denote by  $\mathbf{Sch}$  the category of separated schemes of finite type over  $k$  and by  $\mathbf{Sm}$  the full subcategory of smooth schemes. For  $X, Y \in \mathbf{Sm}$ , an integral closed subscheme of  $X \times Y$  that is finite and surjective over a connected component of  $X$  is called a *prime correspondence from  $X$  to  $Y$* . The category  $\mathbf{Cor}$  of finite correspondences has the same objects as  $\mathbf{Sm}$ , and for  $X, Y \in \mathbf{Sm}$ ,  $\mathbf{Cor}(X, Y)$  is the free abelian group on the set of all prime correspondences from  $X$  to  $Y$  (see [6]). We consider  $\mathbf{Sm}$  as a subcategory of  $\mathbf{Cor}$  by regarding a morphism in  $\mathbf{Sm}$  as its graph in  $\mathbf{Cor}$ .

Let  $\mathbf{PST} = \text{Fun}(\mathbf{Cor}, \mathbf{Ab})$  be the category of additive presheaves of abelian groups on  $\mathbf{Cor}$  whose objects are called *presheaves*

with transfers. Let  $\mathbf{NST} \subseteq \mathbf{PST}$  be the category of Nisnevich sheaves with transfers and let

$$a_{\mathbf{Nis}}^V : \mathbf{PST} \rightarrow \mathbf{NST}$$

be Voevodsky's Nisnevich sheafification functor, which is an exact left adjoint to the inclusion  $\mathbf{NST} \rightarrow \mathbf{PST}$ . Let  $\mathbf{HI} \subseteq \mathbf{PST}$  be the category of  $\mathbf{A}^1$ -invariant presheaves and put  $\mathbf{HI}_{\mathbf{Nis}} = \mathbf{HI} \cap \mathbf{NST} \subseteq \mathbf{NST}$ . The product  $\times$  on  $\mathbf{Sm}$  yields a symmetric monoidal structure on  $\mathbf{Cor}$ , which induces a symmetric monoidal structure on  $\mathbf{PST}$  in the usual way.

- (2) We recall the definition of the category  $\mathbf{MCor}$  from [2, Definition 1.3.1]. A pair  $\mathcal{X} = (X, D)$  of  $X \in \mathbf{Sch}$  and an effective Cartier divisor  $D$  on  $X$  is called a *modulus pair* if  $M - |M_\infty| \in \mathbf{Sm}$ . Let  $\mathcal{X} = (X, D_X)$ ,  $\mathcal{Y} = (Y, D_Y)$  be modulus pairs and  $\Gamma \in \mathbf{Cor}(X - D_X, Y - D_Y)$  be a prime correspondence. Let  $\bar{\Gamma} \subseteq X \times Y$  be the closure of  $\Gamma$ , and let  $\bar{\Gamma}^N \rightarrow X \times Y$  be the normalization. We say  $\Gamma$  is *admissible* (resp. *left proper*) if  $(D_X)_{\bar{\Gamma}^N} \geq (D_Y)_{\bar{\Gamma}^N}$  (resp. if  $\bar{\Gamma}$  is proper over  $X$ ). Let  $\mathbf{MCor}(\mathcal{X}, \mathcal{Y})$  be the subgroup of  $\mathbf{Cor}(X - D_X, Y - D_Y)$  generated by all admissible left proper prime correspondences. The category  $\mathbf{MCor}$  has modulus pairs as objects and  $\mathbf{MCor}(\mathcal{X}, \mathcal{Y})$  as the group of morphisms from  $\mathcal{X}$  to  $\mathcal{Y}$ .
- (3) There is a canonical pair of adjoint functors  $\lambda \dashv \underline{\omega}$ :

$$\lambda : \mathbf{Cor} \rightarrow \mathbf{MCor} \quad X \mapsto (X, \emptyset),$$

$$\underline{\omega} : \mathbf{MCor} \rightarrow \mathbf{Cor} \quad (X, D) \mapsto X - |D|,$$

- (4) There is a full subcategory  $\mathbf{MCor} \subset \mathbf{MCor}$  consisting of *proper modulus pairs*, where a modulus pair  $(X, D)$  is *proper* if  $X$  is proper. Let  $\tau : \mathbf{MCor} \hookrightarrow \mathbf{MCor}$  be the inclusion functor and  $\omega = \underline{\omega}\tau$ .
- (5) For all  $n > 0$  there is an endofunctor  $(-)^{(n)}$  on  $\mathbf{MCor}$  preserving  $\mathbf{MCor}$ , such that  $(X, D)^{(n)} = (X, nD)$  where  $nD$  is the  $n$ -th thickening of  $D$ .
- (6) We have two categories of *modulus presheaves with transfers*:

$$\mathbf{MPST} = \mathit{Fun}(\mathbf{MCor}, \mathbf{Ab}) \text{ and } \underline{\mathbf{MPST}} = \mathit{Fun}(\underline{\mathbf{MCor}}, \mathbf{Ab}).$$

Let  $\mathbb{Z}_{\text{tr}}(\mathcal{X}) = \mathbf{MCor}(-, \mathcal{X}) \in \underline{\mathbf{MPST}}$  be the representable presheaf for  $\mathcal{X} \in \mathbf{MCor}$ . In this paper we frequently write  $\mathcal{X}$  for  $\mathbb{Z}_{\text{tr}}(\mathcal{X})$  for simplicity.

- (7) The adjunction  $\lambda \dashv \underline{\omega}$  induce a string of 4 adjoint functors  $(\lambda_{!} = \underline{\omega}^{\dagger}, \lambda^{*} = \underline{\omega}_{!}, \lambda_{*} = \underline{\omega}^{*}, \underline{\omega}_{*})$ :

$$\begin{array}{ccc} & \xleftarrow{\underline{\omega}^{\dagger}} & \\ & \xrightarrow{\underline{\omega}_{!}} & \mathbf{PST} \\ \mathbf{MPST} & & \\ & \xleftarrow{\underline{\omega}_{*}^{*}} & \\ & \xrightarrow{\underline{\omega}_{*}} & \end{array}$$

where  $\underline{\omega}_{!}, \underline{\omega}_{*}$  are localisations and  $\underline{\omega}^{\dagger}$  and  $\underline{\omega}^{*}$  are fully faithful.

- (8) The functor  $\omega$  yields a string of 3 adjoint functors  $(\omega_{!}, \omega^{*}, \omega_{*})$ :

$$\begin{array}{ccc} & \xrightarrow{\omega_{!}} & \\ & \xleftarrow{\omega^{*}} & \mathbf{PST} \\ \mathbf{MPST} & & \\ & \xleftarrow{\omega_{*}} & \end{array}$$

where  $\omega_{!}, \omega_{*}$  are localisations and  $\omega^{*}$  are fully faithful.

- (9) The functor  $\tau$  yields a string of 3 adjoint functors  $(\tau_{!}, \tau^{*}, \tau_{*})$ :

$$\begin{array}{ccc} & \xrightarrow{\tau_{!}} & \\ & \xleftarrow{\tau^{*}} & \mathbf{MPST} \\ \mathbf{MPST} & & \\ & \xrightarrow{\tau_{*}} & \end{array}$$

where  $\tau_{!}, \tau_{*}$  are fully faithful and  $\tau^{*}$  is a localisation;  $\tau_{!}$  has a pro-left adjoint  $\tau^{\dagger}$ , hence is exact. We will denote by  $\mathbf{MPST}^{\tau}$  the essential image of  $\tau_{!}$  in  $\mathbf{MPST}$ . Moreover,  $\omega_{!} = \underline{\omega}_{!}\tau_{!}$  and  $\omega^{*} = \tau^{*}\omega^{*}$ .

- (10) The modulus pair  $\overline{\square} := (\mathbf{P}^1, \infty)$  has an interval structure induced by the one of  $\mathbf{A}^1$  (see [5, Lem. 2.1.3]). We say  $F \in \mathbf{MPST}$  is  $\overline{\square}$ -invariant if  $p^{*} : F(\mathcal{X}) \rightarrow F(\mathcal{X} \otimes \overline{\square})$  is an isomorphism for any  $\mathcal{X} \in \mathbf{MCor}$ , where  $p : \mathcal{X} \otimes \overline{\square} \rightarrow \mathcal{X}$  is the projection. Let  $\mathbf{CI}$  be the full subcategory of  $\mathbf{MPST}$  consisting of all  $\overline{\square}$ -invariant objects and  $\mathbf{CI}^{\tau} \subset \mathbf{MPST}$  be the essential image of  $\mathbf{CI}$  under  $\tau_{!}$ .
- (11) Recall from [5, Theorem 2.1.8] that  $\mathbf{CI}$  is a Serre subcategory of  $\mathbf{MPST}$ , and that the inclusion functor  $i^{\overline{\square}} : \mathbf{CI} \rightarrow \mathbf{MPST}$  has a left adjoint  $h_0^{\overline{\square}}$  and a right adjoint  $h_0^{\overline{\square}}$  given for  $F \in \mathbf{MPST}$  and  $\mathcal{X} \in \mathbf{MCor}$  by

$$\begin{aligned} h_0^{\overline{\square}}(F)(\mathcal{X}) &= \text{Coker}(i_0^{*} - i_1^{*} : F(\mathcal{X} \otimes \overline{\square}) \rightarrow F(\mathcal{X})), \\ h_0^{\overline{\square}}(F)(\mathcal{X}) &= \text{Hom}(h_0^{\overline{\square}}(\mathcal{X}), F). \end{aligned}$$

For  $\mathcal{X} \in \mathbf{MCor}$ , we write  $h_0^{\overline{\square}}(\mathcal{X}) = h_0^{\overline{\square}}(\mathbb{Z}_{\text{tr}}(\mathcal{X})) \in \mathbf{CI}$ , and by abuse of notation, we let  $h_0^{\overline{\square}}(\mathcal{X})$  denote also for  $\tau_{!}h_0^{\overline{\square}}(\mathcal{X}) \in \mathbf{CI}^{\tau}$ .

- (12) For  $F \in \mathbf{MPST}$  and  $\mathcal{X} = (X, D) \in \mathbf{MCor}$ , write  $F_{\mathcal{X}}$  for the presheaf on the small étale site  $X_{\text{ét}}$  over  $X$  given by  $U \rightarrow F(\mathcal{X}_U)$  for  $U \rightarrow X$  étale, where  $\mathcal{X}_U = (U, D|_U) \in \mathbf{MCor}$ . We say  $F$  is

a Nisnevich sheaf if so is  $F_{\mathcal{X}}$  for all  $\mathcal{X} \in \underline{\mathbf{MCor}}$  (see [2, Section 3]). We write  $\underline{\mathbf{MNST}} \subset \underline{\mathbf{MPST}}$  for the full subcategory of Nisnevich sheaves and put

$$\underline{\mathbf{MNST}}^\tau = \underline{\mathbf{MNST}} \cap \underline{\mathbf{MPST}}^\tau, \quad \mathbf{CI}_{\text{Nis}}^\tau = \mathbf{CI}^\tau \cap \underline{\mathbf{MNST}}^\tau.$$

By [2, Prop. 3.5.3] and [3, Theorem 2], the inclusion functor  $i_{\text{Nis}} : \underline{\mathbf{MNST}} \rightarrow \underline{\mathbf{MPST}}$  has an exact left adjoint  $\underline{a}_{\text{Nis}}$  such that  $\underline{a}_{\text{Nis}}(\underline{\mathbf{MPST}}^\tau) \subset \underline{\mathbf{MNST}}^\tau$ . The functor  $\underline{a}_{\text{Nis}}$  has the following description: For  $F \in \underline{\mathbf{MPST}}$  and  $\mathcal{Y} \in \underline{\mathbf{MCor}}$ , let  $F_{\mathcal{Y}, \text{Nis}}$  be the usual Nisnevich sheafification of  $F_{\mathcal{Y}}$ . Then, for  $(X, D) \in \underline{\mathbf{MCor}}$  we have

$$\underline{a}_{\text{Nis}} F(X, D) = \varinjlim_{f: Y \rightarrow X} F_{(Y, f^*D), \text{Nis}}(Y)$$

where the colimit is taken over all proper maps  $f : Y \rightarrow X$  that induce isomorphisms  $Y - |f^*D| \xrightarrow{\sim} X - |D|$ .

- (13) The functors  $\underline{\omega}^*$  and  $\underline{\omega}_!$  respect  $\underline{\mathbf{MNST}}$  and  $\mathbf{NST}$  and induce a pair of adjoint functors (which for simplicity we write  $\underline{\omega}_!$  and  $\underline{\omega}^*$ ). Moreover, we have

$$\underline{\omega}_! \underline{a}_{\text{Nis}} = \underline{a}_{\text{Nis}}^V \underline{\omega}_!.$$

For  $F \in \mathbf{PST}$ , we have  $F \in \mathbf{HI}$  (resp  $F \in \mathbf{HI}_{\text{Nis}}$ ) if and only if  $\underline{\omega}^* F \in \mathbf{CI}^\tau$  (resp  $\underline{\omega}^* F \in \mathbf{CI}_{\text{Nis}}^\tau$ ).

- (14) We say that  $F \in \underline{\mathbf{MPST}}$  is *semi-pure* if the unit map

$$u : F \rightarrow \underline{\omega}^* \underline{\omega}_! F$$

is injective. For  $F \in \underline{\mathbf{MPST}}$  (resp.  $F \in \underline{\mathbf{MNST}}$ ), let  $F^{sp} \in \underline{\mathbf{MPST}}$  (resp.  $F^{sp} \in \underline{\mathbf{MNST}}$ ) be the image of  $F \rightarrow \underline{\omega}^* \underline{\omega}_! F$  (called the semi-purification of  $F$ ). For  $F \in \underline{\mathbf{MPST}}$  we have

$$\underline{a}_{\text{Nis}}(F^{sp}) \simeq (\underline{a}_{\text{Nis}} F)^{sp}.$$

This follows from the fact that  $\underline{a}_{\text{Nis}}$  is exact and commutes with  $\underline{\omega}^* \underline{\omega}_!$ . For  $F \in \underline{\mathbf{MPST}}^\tau$  we have  $F^{sp} \in \underline{\mathbf{MPST}}^\tau$  since  $\tau$  is exact and  $\underline{\omega}^* \underline{\omega}_! \tau_! = \tau_! \omega^* \omega_!$ .

- (15) Let  $\mathbf{CI}_{\text{Nis}}^{\tau, sp} \subset \mathbf{CI}^\tau$  be the full subcategory of semipure objects and consider the full subcategory

$$\mathbf{CI}_{\text{Nis}}^{\tau, sp} = \mathbf{CI}^{\tau, sp} \cap \underline{\mathbf{MNST}}^\tau \subset \mathbf{CI}_{\text{Nis}}^\tau.$$

By [7, Th. 0.1 and 0.4], we have  $\underline{a}_{\text{Nis}}(\mathbf{CI}_{\text{Nis}}^{\tau, sp}) \subset \mathbf{CI}_{\text{Nis}}^{\tau, sp}$ .

- (16)  $\underline{\mathbf{MCor}}$  is equipped with a symmetric monoidal structure given by

$$(X, D_X) \otimes (Y, D_Y) := (X \times Y, D_X \times Y + X \times D_Y),$$



and  $\mathbf{MCor}$  is clearly a  $\otimes$ -subcategory. Notice that the product is not a categorical product since the diagonal map is not admissible. It is admissible as a correspondence

$$(X, D_X)^{(n)} \rightarrow (X, D_X) \otimes (X, D_X) \quad \text{for } n \geq 2$$

The symmetric monoidal structure  $\otimes$  on  $\mathbf{MCor}$  (resp.  $\mathbf{MCor}$ ) induces a symmetric monoidal structure on  $\mathbf{MPST}$  (resp.  $\mathbf{MPST}$ ) in the usual way, and  $\tau_!$ ,  $\omega_!$  and  $\underline{\omega}_!$  from (9), (8) and (7) are all monoidal (see [10]).

(17) For  $F, G \in \mathbf{MPST}$  we write (cf. (9) and (11))

$$\begin{aligned} F \otimes_{\mathbf{CI}} G &= \tau_! h_0^{\square} (\tau^* F \otimes_{\mathbf{MPST}} \tau^* G) \in \mathbf{CI}^\tau, \\ F \otimes_{\mathbf{CI}}^{sp} G &= (F \otimes_{\mathbf{CI}} G)^{sp} \in \mathbf{CI}^{\tau, sp}, \\ F \otimes_{\mathbf{CI}}^{\text{Nis}} G &= \underline{a}_{\text{Nis}} (F \otimes_{\mathbf{CI}} G) \in \mathbf{CI}_{\text{Nis}}^\tau, \\ F \otimes_{\mathbf{CI}}^{\text{Nis}, sp} G &= \underline{a}_{\text{Nis}} (F \otimes_{\mathbf{CI}}^{sp} G) \in \mathbf{CI}_{\text{Nis}}^{\tau, sp}. \end{aligned}$$

The product  $\otimes_{\mathbf{CI}}$  (resp.  $\otimes_{\mathbf{CI}}^{sp}$ , resp.  $\otimes_{\mathbf{CI}}^{\text{Nis}}$ , resp.  $\otimes_{\mathbf{CI}}^{\text{Nis}, sp}$ ) defines a symmetric monoidal structure on  $\mathbf{CI}^\tau$  (resp.  $\mathbf{CI}^{\tau, sp}$ , resp.  $\mathbf{CI}_{\text{Nis}}^\tau$ , resp.  $\mathbf{CI}_{\text{Nis}}^{\tau, sp}$ ) (see Lemma 3.1).

(18) We write  $\mathbf{RSC} \subseteq \mathbf{PST}$  for the essential image of  $\mathbf{CI}$  under  $\omega_!$  (which is the same as the essential image of  $\mathbf{CI}^{\tau, sp}$  under  $\omega_!$  since  $\omega_! = \underline{\omega}_! \tau_!$  and  $\underline{\omega}_! F = \underline{\omega}_! F^{sp}$ ). Put  $\mathbf{RSC}_{\text{Nis}} = \mathbf{RSC} \cap \mathbf{NST}$ . The objects of  $\mathbf{RSC}$  (resp.  $\mathbf{RSC}_{\text{Nis}}$ ) are called reciprocity presheaves (resp. sheaves). We have  $\mathbf{HI} \subseteq \mathbf{RSC}$  and it contains also smooth commutative group schemes (which may have non-trivial unipotent part), and the sheaf  $\Omega^i$  of Kähler differentials, and the de Rham-Witt sheaves  $W\Omega^i$  (see [4] and [5]).

(19) By [5, Prop. 2.3.7] we have a pair of adjoint functors:

$$(1.1) \quad \mathbf{CI} \begin{array}{c} \xleftarrow{\omega^{\mathbf{CI}}} \\ \xrightarrow{\omega_!} \end{array} \mathbf{RSC},$$

where  $\omega^{\mathbf{CI}} = h_{\square}^0 \omega^*$  and it is fully faithful. It induces a pair of adjoint functors:

$$(1.2) \quad \mathbf{CI}^\tau \begin{array}{c} \xleftarrow{\underline{\omega}^{\mathbf{CI}}} \\ \xrightarrow{\underline{\omega}_!} \end{array} \mathbf{RSC},$$

where  $\underline{\omega}^{\mathbf{CI}} = \tau_! h_{\square}^0 \omega^*$  and it is fully faithful. Indeed, let  $F = \tau_! \hat{F}$  for  $\hat{F} \in \mathbf{CI}$  and  $G \in \mathbf{RSC}$ . In view of (11) and the exactness and full faithfulness of  $\tau_!$ , we have

$$\begin{aligned} \text{Hom}_{\mathbf{CI}^\tau}(F, \tau_! h_{\square}^0 \omega^* G) &\simeq \text{Hom}_{\mathbf{CI}}(\hat{F}, h_{\square}^0 \omega^* G) \simeq \\ \text{Hom}_{\mathbf{MPST}}(\hat{F}, \omega^* G) &\simeq \text{Hom}_{\mathbf{MPST}}(\tau_! \hat{F}, \underline{\omega}^* G) \simeq \text{Hom}_{\mathbf{RSC}}(\underline{\omega}_! F, G). \end{aligned}$$

(1.2) induce pair of adjoint functors:

$$(1.3) \quad \mathbf{CI}_{\text{Nis}}^{\tau, sp} \begin{array}{c} \xleftarrow{\omega^{\mathbf{CI}}} \\ \xrightarrow{\omega_!} \end{array} \mathbf{RSC}_{\text{Nis}},$$

If  $F \in \mathbf{CI}^\tau$ , the adjunction induces a canonical map

$$F \rightarrow \omega^{\mathbf{CI}} \omega_! F$$

which is injective if  $F \in \mathbf{CI}^{\tau, sp}$ .

We end this section with some lemmas that will be needed in the rest of the paper.

The proof of the following Lemma is due to Kay Rülling. We thank him for letting us include it in our paper.

**Lemma 1.1.** *Let  $p$  be the exponential characteristic of the base field  $k$ . Let  $F \in \mathbf{PST}$  such that*

- (1) *for all dominant étale maps  $U \rightarrow X$  in  $\mathbf{Sm}$  the pullback  $F(X) \rightarrow F(U)$  is injective,*
- (2)  *$F$  has no  $p$ -torsion.*

*Then, for any  $G \in \mathbf{PST}$ , the natural map*

$$\underline{\mathbf{Hom}}_{\mathbf{PST}}(G, F) \rightarrow \underline{\mathbf{Hom}}_{\mathbf{PS}}(G, F)$$

*is an isomorphism.*

*Proof.* (Kay Rülling) First we prove  $\mathbf{Hom}_{\mathbf{PST}}(G, F) = \mathbf{Hom}_{\mathbf{PS}}(G, F)$ , i.e. for any morphism  $\varphi : G \rightarrow F$  of presheaves on  $\mathbf{Sm}$  is also a morphism in  $\mathbf{PST}$ . We have to show  $\varphi(f^*a) = f^*\varphi(a)$  in  $F(X)$ , for  $a \in G(Y)$  and  $f \in \mathbf{Cor}(X, Y)$  a prime correspondence. By (1) we can reduce to the case  $X = \text{Spec } K$ , with  $K$  a function field over  $k$ . In this case we can write  $f^* = h_*g^*$ , where  $h : \text{Spec } L \rightarrow \text{Spec } K$  is induced by a finite field extension  $L/K$  and  $g : \text{Spec } L \rightarrow Y$  is a morphism. Since  $\varphi$  is a morphism of presheaves on  $\mathbf{Sm}$ , we are reduced to show

$$(*) \quad h_*\varphi(a) = \varphi(h_*a), \quad a \in G(L).$$

It suffices to consider the following two cases:

*1st case:  $L/K$  is finite separable.* Let  $E/K$  be a finite Galois extension containing  $L/K$  and denote by  $j : \text{Spec } E \rightarrow \text{Spec } K$  the induced morphism and by  $\sigma_i : \text{Spec } E \rightarrow \text{Spec } L$  the morphism induced by all  $K$ -embeddings of  $L$  into  $E$ . Since  $G \in \mathbf{PST}$  we obtain in  $G(E)$

$$j^*h_*a = (h^t \circ j)^*a = \sum_i \sigma_i^*(a).$$

Thus

$$j^* \varphi(h_* a) = \varphi(j^* h_* a) = \varphi\left(\sum_i \sigma_i^*(a)\right) = \sum_i \sigma_i^* \varphi(a) = j^* h_* \varphi(a).$$

Since  $j^* : F(L) \rightarrow F(E)$  is injective by (1) this shows (\*) in this case.

*2nd case:  $L/K$  is purely inseparable of degree  $p$ .* In this case we have  $h^* h_* = (h^t \circ h) : G(L) \rightarrow G(L)$  is multiplication by  $p$  as well as  $h_* h^* : G(K) \rightarrow G(K)$ . Thus

$$h^* \varphi(h_* a) = \varphi(h^* h_* a) = p\varphi(a) = h^* h_* \varphi(a);$$

applying  $h_*$  yields

$$p\varphi(h_* a) = p h_* \varphi(a);$$

thus (\*) follows from (2).

Next we prove the analogous statement for internal hom's. Indeed, note that for  $X \in \mathbf{Sm}$ ,  $\underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F) \in \mathbf{PST}$  also satisfies (1) and (2) above and that we have

$$(**) \quad \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F) = F(X \times -) = \underline{\mathrm{Hom}}_{\mathbf{PS}}(h_X, F) \quad \text{in } \mathbf{PS},$$

where  $h_X = \mathrm{Hom}_{\mathbf{Sm}}(-, X)$ . Thus for  $G \in \mathbf{PST}$

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathbf{PST}}(G, F)(X) &= \mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), \underline{\mathrm{Hom}}_{\mathbf{PST}}(G, F)) \\ &= \mathrm{Hom}_{\mathbf{PST}}(G \otimes^{\mathbf{PST}} \mathbb{Z}_{\mathrm{tr}}(X), F) \\ &= \mathrm{Hom}_{\mathbf{PST}}(G, \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F)) \\ &= \mathrm{Hom}_{\mathbf{PS}}(G, \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F)), && \text{by } (*) \\ &= \mathrm{Hom}_{\mathbf{PS}}(G, \underline{\mathrm{Hom}}_{\mathbf{PS}}(h_X, F)), && \text{by } (**) \\ &= \mathrm{Hom}_{\mathbf{PS}}(G \otimes^{\mathbf{PS}} h_X, F) \\ &= \mathrm{Hom}_{\mathbf{PS}}(h_X, \underline{\mathrm{Hom}}_{\mathbf{PS}}(G, F)) \\ &= \underline{\mathrm{Hom}}_{\mathbf{PS}}(G, F)(X). \end{aligned}$$

This completes the proof of Lemma 1.1.  $\square$

**Lemma 1.2.** *For  $F \in \mathbf{PST}$  and  $X \in \mathbf{Sm}$ , we have a natural isomorphism*

$$\underline{\omega}^* \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F) \simeq \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\mathrm{tr}}(X, \emptyset), \underline{\omega}^* F).$$

*Proof.* For  $\mathcal{Y} = (Y, E) \in \mathbf{MCor}$  with  $V = Y - |E|$ , we have natural isomorphisms

$$\begin{aligned} \underline{\omega}^* \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F)(\mathcal{Y}) &\simeq \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F)(V) \simeq \mathrm{Hom}_{\mathbf{PST}}(X \times V, F) \\ &\simeq \mathrm{Hom}_{\mathbf{MPST}}((X, \emptyset) \otimes \mathcal{Y}, \underline{\omega}^* F) \simeq \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\mathrm{tr}}(X, \emptyset), \underline{\omega}^* F)(\mathcal{Y}). \end{aligned}$$

This proves the lemma.  $\square$

**Lemma 1.3.** *For  $F \in \underline{\mathbf{MPST}}$  and  $X \in \mathbf{Sm}$ , we have a natural isomorphism*

$$\omega_! \underline{\mathbf{Hom}}_{\underline{\mathbf{MPST}}}(\mathbb{Z}_{\text{tr}}(X, \emptyset), F) \simeq \underline{\mathbf{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\text{tr}}(X), \omega_! F).$$

*Proof.* For  $Y \in \mathbf{Sm}$ , we have natural isomorphisms

$$\begin{aligned} \omega_! \underline{\mathbf{Hom}}_{\underline{\mathbf{MPST}}}(\mathbb{Z}_{\text{tr}}(X, \emptyset), F)(Y) &\simeq \mathbf{Hom}_{\mathbf{PST}}(X \times Y, \omega_! F) \\ &\simeq \underline{\mathbf{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\text{tr}}(X), \omega_! F)(Y). \end{aligned}$$

This proves the lemma.  $\square$

**Lemma 1.4.** *A complex in  $C^\bullet$  in  $\mathbf{NST}$  such that  $C^n \in \mathbf{RSC}$  for all  $n \in \mathbb{Z}$  is exact if and only if  $C^\bullet(K)$  is exact as a complex of abelian groups for any function field  $K$ .*

*Proof.* The cohomology sheaves  $H^n(C^\bullet)$  are in  $\mathbf{RSC}_{\text{Nis}}$  by [7, Th.0.1]. Hence the lemma follows from the injectivity of  $F(X) \rightarrow F(k(X))$  for  $X \in \mathbf{Sm}$  from [7, Th. 0.2].  $\square$

**Lemma 1.5.** *For  $G \in \mathbf{RSC}$  and  $F \in \mathbf{PST}$  such that  $F$  is a quotient of a finite sum of representable sheaves,  $\underline{\mathbf{Hom}}_{\mathbf{PST}}(F, G) \in \mathbf{RSC}$ .*

*Proof.* First assume  $F = \mathbb{Z}_{\text{tr}}(X)$  with  $X \in \mathbf{Sm}$ . Put  $\tilde{G} = \omega^{\mathbf{CI}} G \in \mathbf{CI}^\tau$  (cf. (19)). Note that  $\tilde{G}$  is semipure and the adjunction (1.2) implies  $\omega_! \tilde{G} \simeq G$ . Lemma 1.3 implies a natural isomorphism

$$\underline{\mathbf{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\text{tr}}(X), G) \simeq \omega_! \underline{\mathbf{Hom}}_{\underline{\mathbf{MPST}}}((X, \emptyset), \tilde{G}).$$

Thus it suffices to show

$$\underline{\mathbf{Hom}}_{\underline{\mathbf{MPST}}}((X, \emptyset), \tilde{G}) \in \mathbf{CI}^\tau.$$

The  $\bar{\square}$ -invariance follows directly from the one for  $\tilde{G}$ . The fact that it is in  $\mathbf{MPST}^\tau$  follows from [7, Lemma 1.27].

Now assume there is a surjection  $\bigoplus_{i=1}^{i=n} \mathbb{Z}_{\text{tr}}(X_i) \rightarrow F$  in  $\mathbf{PST}$ , where  $X_i \in \mathbf{Sm}$ . It induces an injection

$$\underline{\mathbf{Hom}}_{\mathbf{PST}}(F, G) \hookrightarrow \prod_{i=1}^n \underline{\mathbf{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\text{tr}}(X_i), G).$$

Since  $\underline{\mathbf{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\text{tr}}(X_i), G) \in \mathbf{RSC}$  as shown above and  $\mathbf{RSC} \subset \mathbf{PST}$  is closed under finite products and subobjects, we get  $\underline{\mathbf{Hom}}_{\mathbf{PST}}(F, G) \in \mathbf{RSC}$  as desired. This completes the proof.  $\square$

**Lemma 1.6.** *Let  $F \in \mathbf{MNST}^\tau$  be such that  $F^{sp} \in \mathbf{CI}_{\text{Nis}}^\tau$ . For any function field  $K$ , we have*

$$H^i(\mathbf{P}_K^1, F_{(\mathbf{P}_K^1, 0+\infty)}) = 0 \text{ for } i > 0.$$

*Proof.* If  $F$  is semi-pure, the assertion follows from [7, Th. 9.1]. In general we use the exact sequence in  $\underline{\mathbf{MNST}}$ :

$$0 \rightarrow C \rightarrow F \rightarrow F^{sp} \rightarrow 0$$

to reduce to the above case noting  $H^i(\mathbf{P}_K^1, C_{(\mathbf{P}_K^1, 0+\infty)}) = 0$  for  $i > 0$  since  $C_{(\mathbf{P}_K^1, 0+\infty)}$  is supported on  $\{0, \infty\}$ .  $\square$

**Lemma 1.7.** *For  $F \in \mathbf{CI}^\tau$  and a function field  $K$ , we have*

$$\underline{a}_{\text{Nis}} F(K) \xrightarrow{\simeq} \underline{a}_{\text{Nis}} F(\overline{\square} \otimes K).$$

*Proof.* We consider the exact sequence in  $\underline{\mathbf{MPST}}$ :

$$0 \rightarrow C \rightarrow F \rightarrow F^{sp} \rightarrow 0 \quad \text{with } \omega_1 C = 0.$$

From this we get an exact sequence in  $\underline{\mathbf{MNST}}$ :

$$0 \rightarrow \underline{a}_{\text{Nis}} C \rightarrow \underline{a}_{\text{Nis}} F \rightarrow \underline{a}_{\text{Nis}} F^{sp} \rightarrow 0.$$

Since  $C_{(\mathbf{P}_K^1, 0+\infty)}$  is supported on  $\{0_K, \infty_K\}$ , we have by [2, Th.1]

$$(\underline{a}_{\text{Nis}} C)_{(\mathbf{P}_K^1, 0+\infty)} = C_{(\mathbf{P}_K^1, 0+\infty)}.$$

Hence the diagram gives rise to a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C(K) & \longrightarrow & F(K) & \longrightarrow & F^{sp}(K) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C(\overline{\square} \otimes K) & \longrightarrow & \underline{a}_{\text{Nis}} F(\overline{\square} \otimes K) & \longrightarrow & \underline{a}_{\text{Nis}} F^{sp}(\overline{\square} \otimes K) \longrightarrow 0 \end{array}$$

The lower sequence is exact thanks to

$$\text{Ext}_{\underline{\mathbf{MNST}}}^1(\mathbb{Z}_{\text{tr}}(\mathbf{P}_K^1, 0+\infty), \underline{a}_{\text{Nis}} C) \simeq H_{\text{Nis}}^1(\mathbf{P}_K^1, C_{(\mathbf{P}_K^1, 0+\infty)}) = 0,$$

by [2, Th.1] and the fact that  $C_{(\mathbf{P}_K^1, 0+\infty)}$  is supported on  $\{0_K, \infty_K\}$ . The left (resp. right) vertical map is an isomorphism since  $C \in \mathbf{CI}^\tau$  (resp. thanks to [7, Th. 10.1]). This completes the proof.  $\square$

Let  $\mathbf{A}_t^1 = \text{Spec } k[t]$  be the affine line with the coordinate  $t$ . Consider the map in  $\mathbf{PST}$ :

$$\lambda_{\mathbf{G}_m} : \mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1 - \{0\}) \rightarrow \mathbf{G}_m$$

given by  $t \in \mathbf{G}_m(\mathbf{A}_t^1 - \{0\}) = k[t, t^{-1}]$ , and the map in  $\mathbf{PST}$ :

$$\lambda_{\mathbf{G}_a} : \mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1) \rightarrow \mathbf{G}_a$$

given by  $t \in \mathbf{G}_a(\mathbf{A}_t^1) = k[t]$ . Note that  $\lambda_{\mathbf{G}_m}$  and  $\lambda_{\mathbf{G}_a}$  factor through

$$\text{Coker}(\mathbb{Z} \xrightarrow{i_1} \mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1 - \{0\})) \quad \text{and} \quad \text{Coker}(\mathbb{Z} \xrightarrow{i_0} \mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1)),$$

with  $i_1$  and  $i_0$  induced by the points  $1 \in \mathbf{A}_t^1 - \{0\}$  and  $0 \in \mathbf{A}_t^1$  respectively.

**Lemma 1.8.** (1) *The composite map*

$$\omega_! \mathbb{Z}_{\text{tr}}(\mathbf{P}^1, 0 + \infty) \simeq \mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1 - \{0\}) \xrightarrow{\lambda_{\mathbf{G}_m}} \mathbf{G}_m$$

*induces an isomorphism*

$$(1.4) \quad a_{\text{Nis}}^V \omega_! h_0^{\square}(\overline{\square}_{\mathbf{G}_m}) \xrightarrow{\simeq} \mathbf{G}_m,$$

where  $\overline{\square}_{\mathbf{G}_m} = \text{Coker}(\mathbb{Z} \xrightarrow{i_1} \mathbb{Z}_{\text{tr}}(\mathbf{P}^1, 0 + \infty)) \in \mathbf{MPST}$ .

(2) *The composite map*

$$\omega_! \mathbb{Z}_{\text{tr}}(\mathbf{P}^1, 2\infty) \simeq \mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1) \xrightarrow{\lambda_{\mathbf{G}_a}} \mathbf{G}_a$$

*induces an isomorphism*

$$(1.5) \quad a_{\text{Nis}}^V \omega_! h_0^{\square}(\overline{\square}_{\mathbf{G}_a}) \xrightarrow{\simeq} \mathbf{G}_a,$$

where  $\overline{\square}_{\mathbf{G}_a} = \text{Coker}(\mathbb{Z} \xrightarrow{i_0} \mathbb{Z}_{\text{tr}}(\mathbf{P}^1, 2\infty)) \in \mathbf{MPST}$ .

*Proof.* We prove only (2). The proof of (1) is similar. By [5, Cor. 2.3.5] and [7, Th. 0.1], we have  $a_{\text{Nis}}^V \omega_! h_0^{\square}(\overline{\square}_{\mathbf{G}_a}) \in \mathbf{RSC}_{\text{Nis}}$ . Hence, by Lemma 1.4, it suffices to show that the map  $\mathbb{Z}_{\text{tr}}(\mathbf{A}^1)(K) \xrightarrow{\lambda_{\mathbf{G}_a}} \mathbf{G}_a(K) = K$  for a function field  $K$ , induces an isomorphism  $\omega_! h_0^{\square}(\overline{\square}_{\mathbf{G}_a})(K) \simeq K$ . We know that  $\mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1)(K)$  is identified with the group of 0-cycles on  $\mathbf{A}_K^1 = \mathbf{A}^1 \otimes_k K$ . Then, by [5, Th. 3.2.1], the kernel of  $\mathbb{Z}_{\text{tr}}(\mathbf{A}^1)(K) \rightarrow \omega_! h_0^{\square}(\overline{\square}_{\mathbf{G}_a})(K)$  is generated by the class of  $0 \in \mathbf{A}_K^1$  and  $\text{div}_{\mathbf{A}_K^1}(f)$  for  $f \in K(t)^\times$  such that  $f \in 1 + \mathfrak{m}_\infty^2 \mathcal{O}_{\mathbf{P}_K^1, \infty}$ , where  $\mathfrak{m}_\infty$  is the maximal ideal of the local ring  $\mathcal{O}_{\mathbf{P}_K^1, \infty}$  of  $\mathbf{P}_K^1$  at  $\infty$ . Now (2) follows by an elementary computation.  $\square$

## 2. SOME LEMMAS ON CONTRACTIONS

For an integer  $a \geq 1$  put  $\overline{\square}^{(a)} = (\mathbf{P}^1, a(0 + \infty)) \in \mathbf{MCor}$  and

$$\overline{\square}_{\text{red}}^{(a)} = \text{Ker}(\mathbb{Z}_{\text{tr}}(\overline{\square}^{(a)}) \rightarrow \mathbb{Z} = \mathbb{Z}_{\text{tr}}(\text{Spec } k, \emptyset)).$$

The inclusion  $\mathbf{A}^1 - \{0\} \hookrightarrow \mathbf{A}^1$  induces an admissible map  $\overline{\square}^{(a)} \rightarrow \overline{\square}$  for all  $a$ . Note that the composite map

$$(2.1) \quad \overline{\square}_{\text{red}}^{(1)} \hookrightarrow \overline{\square}^{(1)} \rightarrow \overline{\square}_{\mathbf{G}_m}$$

is an isomorphism, where  $\overline{\square}_{\mathbf{G}_m}$  is from (1.4).

For  $F \in \mathbf{MPST}$ , we write

$$(2.2) \quad \begin{aligned} \gamma F &= \underline{\text{Hom}}_{\mathbf{MPST}}(\overline{\square}_{\text{red}}^{(1)}, F) \text{ for } F \in \mathbf{CI}^T, \\ \gamma_{\text{Nis}} F &= \underline{a}_{\text{Nis}} \gamma F \in \mathbf{MNST}. \end{aligned}$$

Note

$$\gamma_{\text{Nis}} F = \gamma F \text{ for } F \in \mathbf{CI}_{\text{Nis}}^r.$$

We also write for an integer  $n \geq 1$

$$(2.3) \quad \gamma^n F = \underline{\text{Hom}}_{\mathbf{MPST}}((\overline{\square}_{\text{red}}^{(1)})^{\otimes \mathbf{CI}^n}, F) \text{ for } F \in \mathbf{CI}^r.$$

Note

$$(2.4) \quad \gamma^n F \cong \underline{\text{Hom}}_{\mathbf{MPST}}((\overline{\square}_{\text{red}}^{(1)})^{\otimes \mathbf{MPST}^n}, F) \cong \overbrace{\gamma \gamma \cdots \gamma}^{n \text{ times}} F.$$

The proof of the following Lemma is due to Kay Rülling. We thank him for letting us include it in our paper.

**Lemma 2.1.** *The unit map*

$$(2.5) \quad \underline{a}_{\text{Nis}} h_0^{\square}(\overline{\square}^{(1)})^{sp} \xrightarrow{\cong} \omega^* \omega_! \underline{a}_{\text{Nis}} h_0^{\square}(\overline{\square}^{(1)}) \cong \omega^*(\mathbf{G}_m \oplus \mathbb{Z})$$

is an isomorphism, where the second isomorphism in (2.5) holds by Lemma 1.8 and (2.1).

*Proof.* (Kay Rülling) The unit map is injective by semipurity. It remains to show the surjectivity. By definition of the sheafification functor, it suffices to show the surjectivity on  $(\text{Spec } R, (f))$ , where  $R$  is an integral local  $k$ -algebra and  $f \in R \setminus \{0\}$ , such that  $R_f$  is regular. Denote by

$$\psi : \mathbb{Z}_{\text{tr}}(\mathbf{P}^1, 0 + \infty)(R, f) \rightarrow R_f^\times \oplus \mathbb{Z}$$

the precomposition of (2.5) evaluated at  $(R, f)$  with the quotient map  $\mathbb{Z}_{\text{tr}}(\mathbf{P}^1, 0 + \infty)(R, f) \rightarrow \underline{a}_{\text{Nis}} h_0^{\square}(\overline{\square}^{(1)})^{sp}$ .

We show that  $\psi$  is surjective. To this end, observe that for  $a \in R_f^\times$  we find  $N \geq 0$  and  $b \in R$  such that

$$(2.6) \quad ab = f^N, \quad \text{and} \quad af^N \in R.$$

Set  $W := V(t^N - a) \subset \text{Spec } R_f[t, 1/t]$  and  $K := \text{Frac}(R)$ .

The map  $\mathbf{Cor}(K, \mathbf{A}^1 - \{0\}) \rightarrow \text{Pic}(\mathbf{P}_K^1, 0 + \infty) \cong K^\times \oplus \mathbb{Z}$  which induces the second isomorphism of (2.5) sends a prime correspondence  $V(a_0 + a_1 t + \dots a_r t^r)$  to  $((-1)^r a_0/a_r, r)$ , hence we have:

$$(2.7) \quad \psi(V(a_0 + a_1 t + \dots a_r t^r)) = ((-1)^r a_0/a_r, r)$$

provided that  $V(a_0 + a_1 t + \dots a_r t^r) \in \mathbf{MCor}((R, f), (\mathbf{P}^1, 0 + \infty))$ .

For any  $a \in R_f^\times$ , consider  $h = t^N - a$  and let  $h = \prod_i h_i$  be the decomposition into monic irreducible factors in  $K[t, 1/t]$  and denote by  $W_i \subset \text{Spec } R_f[t, 1/t]$  the closure of  $V(h_i)$ . (Note that  $W_i = W_j$  for  $i \neq j$  is allowed.)

The  $W_i$  correspond to the components of  $W$  which are dominant over  $R_f$ ; since  $W$  is finite and surjective over  $R_f$ , so are the  $W_i$ . We claim

$$(2.8) \quad W_i \in \underline{\mathbf{MCor}}((R, f), (\mathbf{P}^1, 0 + \infty))$$

Indeed, let  $I_i$  (resp.  $J_i$ ) be the ideal of the closure of  $W_i$  in  $\text{Spec } R[t]$  (resp.  $\text{Spec } R[z]$  with  $z = 1/t$ ). By (2.6)

$$bt^N - f^N \in I_i \quad \text{and} \quad f^N - f^N az^N \in J_i.$$

Hence  $(f/t)^N \in R[t]/I_i$  and  $(f/z)^N \in R[z]/J_i$ . It follows that  $f/t$  (resp.  $f/z$ ) is integral over  $R[t]/I_i$  (resp.  $R[z]/J_i$ ); thus (2.8) holds. We claim

$$\psi\left(\sum_i W_i\right) = ((-1)^{N+1}a, N).$$

Indeed, it suffices to show this after restriction to the generic point of  $R$ , in which case it follows directly from the definition of the  $W_i$  and (2.7). Since  $\psi(V(t \pm 1)) = (-\pm 1, 1)$ , this implies the surjectivity of  $\psi$  and proves the lemma.  $\square$

**Corollary 2.2.** (1) *There is a natural isomorphism*

$$\underline{a}_{\text{Nis}} h_0^{\square}(\overline{\square}_{\text{red}}^{(1)})^{sp} \simeq \underline{\omega}^* \mathbf{G}_m.$$

(2) *For  $F \in \mathbf{CI}_{\text{Nis}}^{\tau, sp}$ ,  $\gamma F \in \underline{\mathbf{MNST}}$  and we have a natural isomorphism*

$$(2.9) \quad \gamma F \simeq \underline{\mathbf{Hom}}_{\underline{\mathbf{MPST}}}(\underline{\omega}^* \mathbf{G}_m, F).$$

**Lemma 2.3.** *Consider an exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  in  $\underline{\mathbf{MNST}}$ .*

(1) *Assume  $A, B, C \in \mathbf{CI}^{\tau}$ . Then the following sequence in  $\mathbf{NST}$*

$$0 \rightarrow \underline{\omega}_1 \gamma A \rightarrow \underline{\omega}_1 \gamma B \rightarrow \underline{\omega}_1 \gamma C \rightarrow 0$$

*is exact.*

(2) *Assume  $\underline{\omega}_1 A = 0$  and  $C$  is semi-pure. Then the following sequence*

$$0 \rightarrow \gamma A(K) \rightarrow \gamma B(K) \rightarrow \gamma C(K) \rightarrow 0$$

*is exact for any function field  $K$ .*

*Proof.* First assume  $A, B, C \in \mathbf{CI}^{\tau}$ . Then all terms of the sequence are in  $\mathbf{RSC}_{\text{Nis}}$ . By Lemma 1.4, it suffices to show the exactness of

$$0 \rightarrow \gamma A(K) \rightarrow \gamma B(K) \rightarrow \gamma C(K) \rightarrow 0$$



for a function field  $K$ . By (2.2), this follows from

$$\mathrm{Ext}_{\underline{\mathbf{MNST}}}^1(\mathbb{Z}_{\mathrm{tr}}(\mathbf{P}_K^1, 0 + \infty), A) = 0.$$

By using [2, Th.1] we can compute

$$\mathrm{Ext}_{\underline{\mathbf{MNST}}}^1(\mathbb{Z}_{\mathrm{tr}}(\mathbf{P}_K^1, 0 + \infty), A) \simeq H_{\mathrm{Nis}}^1(\mathbf{P}_K^1, A_{(\mathbf{P}_K^1, 0 + \infty)}),$$

where we used the fact that any proper birational map  $X \rightarrow \mathbf{P}_K^1$  is an isomorphism. Thus the vanishing follows from Lemma 1.6.

Next we assume  $\underline{\omega}_1 A = 0$  and  $C$  is semi-pure. For a function field  $K$ , we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A(\mathbf{P}_K^1, \infty) & \longrightarrow & B(\mathbf{P}_K^1, \infty) & \longrightarrow & C(\mathbf{P}_K^1, \infty) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A(\mathbf{P}_K^1, 0 + \infty) & \longrightarrow & B(\mathbf{P}_K^1, 0 + \infty) & \longrightarrow & C(\mathbf{P}_K^1, 0 + \infty) \longrightarrow 0 \end{array}$$

where the sequences are exact since

$$\mathrm{Ext}_{\underline{\mathbf{MNST}}}^1(\mathbb{Z}_{\mathrm{tr}}(\mathbf{P}_K^1, 0 + \infty), A) \simeq H_{\mathrm{Nis}}^1(\mathbf{P}^1, A_{(\mathbf{P}_K^1, 0 + \infty)}) = 0,$$

by [2, Th.1] and the fact that  $A_{(\mathbf{P}_K^1, 0 + \infty)}$  is supported on  $\{0, \infty\}$  by the assumption. The right vertical map is injective by the semi-purity of  $C$ . This implies the desired assertion.  $\square$

**Proposition 2.4.** (1) Take  $F \in \mathbf{CI}_{\mathrm{Nis}}^\tau$  and assume  $F$  is semi-pure. For  $M \in \underline{\mathbf{MCor}}_{ls}$ , there exists a map functorial in  $M$ :

$$(2.10) \quad \gamma F(M) \rightarrow H^1(\mathbf{P}^1 \otimes M, F).$$

Moreover, if  $M$  is henselian local, it is an isomorphism.

(2) Let  $F \in \underline{\mathbf{MNST}}^\tau$  be such that  $F^{sp} \in \mathbf{CI}_{\mathrm{Nis}}^\tau$ . For  $X \in \mathbf{Sm}$ , there exists a map functorial in  $X$ :

$$(2.11) \quad \gamma F(X) \rightarrow H^1(\mathbf{P}^1 \times X, F).$$

Moreover, it is an isomorphism either if  $F \in \mathbf{CI}_{\mathrm{Nis}}^\tau$  and  $X$  is henselian local, or if  $X = K$  is a function field and the natural map  $F(K) \rightarrow F(\overline{\square} \otimes K)$  is an isomorphism.

*Proof.* Let  $L = (\mathbf{P}^1, 0)$ . We prove (1). By [7, Lem. 7.1], there exists an exact sequence of sheaves on  $(\mathbf{P}^1 \times \overline{M})_{\mathrm{Nis}}$ :

$$(2.12) \quad 0 \rightarrow F_{\mathbf{P}^1 \otimes M} \rightarrow F_{L \otimes M} \rightarrow i_* \gamma F_M \rightarrow 0,$$

where  $i : \overline{M} \rightarrow \mathbf{P}^1 \times \overline{M}$  is induced by  $0 \in \mathbf{P}^1$ . Taking cohomology, we get the map (2.10). If  $M$  is henselian local, we have

$$(2.13) \quad H^1(L \otimes M, F) \simeq H^1(M, F) = 0$$

thanks to [7, Th .9.3]. This implies that the map is an isomorphism.

Next we prove (2). Consider the exact sequence of sheaves on  $(\mathbf{P}^1 \times X)_{\text{Nis}}$ :

$$(2.14) \quad 0 \rightarrow F_{\mathbf{P}^1 \times X} \rightarrow F_{L \otimes X} \rightarrow i_* \lambda_X F \rightarrow 0,$$

where  $\lambda_X F = i^*(F_{L \otimes X}/F_{\mathbf{P}^1 \times X})$ . The injectivity of the first map follows from [7, Th.3.1] noting  $F_{\mathbf{P}^1 \times X} = F_{\mathbf{P}^1 \times X}^{sp}$ .<sup>1</sup> Taking cohomology over an étale  $U \rightarrow X$ , we get a map natural in  $U$ :

$$\lambda_X F(U) \rightarrow H^1(\mathbf{P}^1 \times U, F).$$

To define the map (2.11), it suffices to show the following.

*Claim 2.5.* There exists a natural map of sheaves on  $X_{\text{Nis}}$ :

$$\varphi_{F,X} : (\gamma_{\text{Nis}} F)_X \rightarrow \lambda_X F.$$

It is an isomorphism if  $F \in \mathbf{CI}_{\text{Nis}}^\tau$ . If  $F \in \mathbf{MNST}^\tau$  and  $F^{sp} \in \mathbf{CI}_{\text{Nis}}^\tau$ , then  $\varphi_{F,K} : (\gamma F)_K \rightarrow \lambda_K F$  is an isomorphism for a function field  $K$ .

By definition,  $\lambda_X F$  is the sheaf associated to the presheaf

$$(2.15) \quad \widetilde{\lambda_X F} : U \rightarrow \varinjlim_V F(V, 0_V)/F(V, \emptyset),$$

where  $V$  ranges over étale neighborhoods of  $0_U = i(U) \subset \mathbf{P}^1 \times U$ . On the other hand, we have

$$(\gamma F)_X(U) = F(\mathbf{P}^1 \times U, 0 + \infty)/F(\mathbf{P}^1 \times U, \infty).$$

Since the above colimit does not change when taken over étale neighborhood of  $0_U \subset \mathbf{A}^1 \times U$ , there is a natural map

$$(\gamma F)_X(U) \rightarrow F(\mathbf{A}^1 \times U, 0)/F(\mathbf{A}^1 \times U, \emptyset) \rightarrow \widetilde{\lambda_X F}(U),$$

which induces the desired map  $\varphi_{F,X}$ .

Next we show  $\varphi_{F,X}$  is an isomorphism if  $F \in \mathbf{CI}_{\text{Nis}}^\tau$ , or if  $F \in \mathbf{MNST}^\tau$  with  $F^{sp} \in \mathbf{CI}_{\text{Nis}}^\tau$  and  $X = K$  is a function field. If  $F$  is semi-pure, the assertion follows from [7, Lem. 7.1]. In general we consider the exact sequence in  $\mathbf{MNST}$ :

$$(2.16) \quad 0 \rightarrow C \rightarrow F \rightarrow F^{sp} \rightarrow 0 \quad \text{with } \omega_1 C = 0.$$

It gives rise to a commutative diagram of sheaves on  $(\mathbf{P}^1 \times X)_{\text{Nis}}$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_{\mathbf{P}^1 \times X} & \longrightarrow & F_{\mathbf{P}^1 \times X} & \longrightarrow & F_{\mathbf{P}^1 \times X}^{sp} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_{L \otimes X} & \longrightarrow & F_{L \otimes X} & \longrightarrow & F_{L \otimes X}^{sp} \end{array}$$

<sup>1</sup>The point is that  $X$  has the empty modulus.

where the upper (resp. lower) sequence is exact by the exactness of  $\omega_! : \underline{\mathbf{MNST}} \rightarrow \mathbf{NST}$  (resp. the left-exactness of  $b^* : \underline{\mathbf{MNST}} \rightarrow \underline{\mathbf{MNST}}^{\text{fin}}$ ). The right vertical map is injective by [7, Th. 3.1]. This implies the exactness of the lower sequence of the following commutative diagram in  $\underline{\mathbf{MNST}}$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\gamma C)_X & \longrightarrow & (\gamma F)_X & \longrightarrow & (\gamma F^{sp})_X \longrightarrow 0 \\ & & \downarrow \varphi_{C,X} & & \downarrow \varphi_{F,X} & & \downarrow \varphi_{F^{sp},X} \\ 0 & \longrightarrow & \lambda_X C & \longrightarrow & \lambda_X F & \longrightarrow & \lambda_X F^{sp} \end{array}$$

The upper sequence is exact by Lemma 2.3. Since we know that  $\varphi_{F^{sp},X}$  is an isomorphism, it suffices to show that  $\varphi_{C,X}$  is an isomorphism. Indeed, for an étale  $U \rightarrow X$ , we have

$$\begin{aligned} (\gamma C)_X(U) &= C(\mathbf{P}^1 \times U, 0 + \infty) / C(\mathbf{P}^1 \times U, \infty) \\ &\simeq \varinjlim_V C(V, 0_V) / C(V, \emptyset) = \widetilde{\lambda_X C}(U), \end{aligned}$$

where  $V$  are as in (2.15) and the isomorphism comes from the excision noting that  $C_{(\mathbf{P}^1 \times U, 0 + \infty)}$  (resp.  $C_{(\mathbf{P}^1 \times U, \infty)}$ ) is supported on  $\{0_U, \infty_U\}$  (resp.  $\infty_U$ ). This proves that  $\varphi_{C,X}$  is an isomorphism and completes the proof of the claim.

To show the second assertion of (2), first note that  $F(\mathbf{P}^1 \times X) \rightarrow F(L \otimes X)$  is surjective since  $F(X) \xrightarrow{\simeq} F(L \otimes X)$  by the assumption. Hence it suffices to show  $H^1(L \otimes X, F) = 0$ . If  $F$  is semi-pure, this follows from (2.13). In general it is reduced to the above case using (2.16) and noting  $H^1(L \otimes X, C) = 0$  since  $C_{L \otimes X}$  is supported on  $0 \times X$ . This completes the proof of the lemma.  $\square$

**Corollary 2.6.** *Let  $G \in \mathbf{CI}^\tau$ .*

(1) *There is a natural isomorphism*

$$\gamma_{\underline{a}_{\text{Nis}}} G(K) \simeq H^1(\mathbf{P}_K^1, \underline{a}_{\text{Nis}} G).$$

(2) *The natural map*

$$\gamma_{\underline{a}_{\text{Nis}}} G(K) \rightarrow \gamma_{\underline{a}_{\text{Nis}}} G^{sp}(K)$$

*is an isomorphism for any function field  $K$ .*

*Proof.* By Lemma 1.7,  $F = \underline{a}_{\text{Nis}} G$  satisfies the second assumption of Proposition 2.4(2). By [7, Th. 10.1]  $F^{sp} = \underline{a}_{\text{Nis}} G^{sp} \in \mathbf{CI}^\tau$ . Hence (1)

follows from Proposition 2.4(2). (2) follows from isomorphisms

$$\begin{aligned} \gamma_{\underline{a}_{\text{Nis}}}G(K) &\simeq H^1(\mathbf{P}_K^1, \underline{a}_{\text{Nis}}G) \simeq H^1(\mathbf{P}_K^1, \underline{\omega}_! \underline{a}_{\text{Nis}}G) \\ &\simeq H^1(\mathbf{P}_K^1, \underline{\omega}_! \underline{a}_{\text{Nis}}G^{sp}) \simeq H^1(\mathbf{P}_K^1, \underline{a}_{\text{Nis}}G^{sp}) \simeq \gamma_{\underline{a}_{\text{Nis}}}G^{sp}(K), \end{aligned}$$

where the last isomorphism follows also from Proposition 2.4.  $\square$

**Lemma 2.7.** *Let  $F \in \mathbf{CI}^\tau$ .*

(1) *The natural map*

$$\gamma F(K) \rightarrow \gamma_{\underline{a}_{\text{Nis}}}F(K)$$

*is an isomorphism for any function field  $K$ .*

(2) *The natural map  $\underline{a}_{\text{Nis}}\gamma F^{sp} \rightarrow \gamma_{\underline{a}_{\text{Nis}}}F^{sp}$  is injective.*

(3) *The natural map  $\underline{\omega}_! \underline{a}_{\text{Nis}}\gamma F^{sp} \rightarrow \underline{\omega}_! \gamma_{\underline{a}_{\text{Nis}}}F^{sp}$  is an isomorphism.*

*Proof.* Consider the exact sequence in  $\underline{\mathbf{MPST}}$ :

$$(2.17) \quad 0 \rightarrow C \rightarrow F \rightarrow F^{sp} \rightarrow 0 \quad \text{with } \underline{\omega}_! C = 0.$$

Note  $C, F^{sp} \in \mathbf{CI}^\tau$ . It gives rise to an exact sequence in  $\underline{\mathbf{MNST}}$ :

$$0 \rightarrow \underline{a}_{\text{Nis}}C \rightarrow \underline{a}_{\text{Nis}}F \rightarrow \underline{a}_{\text{Nis}}F^{sp} \rightarrow 0$$

and a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \gamma C(K) & \longrightarrow & \gamma F(K) & \longrightarrow & \gamma F^{sp}(K) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \gamma_{\underline{a}_{\text{Nis}}}C(K) & \longrightarrow & \gamma_{\underline{a}_{\text{Nis}}}F(K) & \longrightarrow & \gamma_{\underline{a}_{\text{Nis}}}F^{sp}(K) \longrightarrow 0 \end{array}$$

The upper sequence is exact thanks to (2.2). The lower sequence is exact by Lemma 2.3(2) noting  $\underline{\omega}_! \underline{a}_{\text{Nis}}C = 0$ . Since  $C_{(\mathbf{P}_K^1, 0+\infty)}$  is supported on  $\{0_K, \infty_K\}$ , we have

$$(\underline{a}_{\text{Nis}}C)_{(\mathbf{P}_K^1, 0+\infty)} = C_{(\mathbf{P}_K^1, 0+\infty)}.$$

Hence the left vertical map is an isomorphism. Hence we may assume that  $F$  is semi-pure. By [7, Th. 10.1], we have  $\underline{a}_{\text{Nis}}F \in \mathbf{CI}^\tau$ . By [7, Lem. 5.8], we have natural isomorphisms

$$\gamma F(K) \simeq F(\mathbf{A}_K^1, 0)/F(\mathbf{A}_K^1, \emptyset),$$

$$\gamma_{\underline{a}_{\text{Nis}}}F(K) \simeq \underline{a}_{\text{Nis}}F(\mathbf{A}_K^1, 0)/\underline{a}_{\text{Nis}}F(\mathbf{A}_K^1, \emptyset).$$

Hence (1) follows from [7, Th.4.1].

To show (2) and (3), first note that  $F^{sp} \in \mathbf{CI}^\tau$  and  $\gamma F^{sp}$  is semi-pure by the assumption. By [7, Th. 10.1],  $\underline{a}_{\text{Nis}}\gamma F^{sp}$  and  $\gamma_{\underline{a}_{\text{Nis}}}F^{sp}$  are in  $\mathbf{CI}_{\text{Nis}}^{\tau, sp}$  and hence  $\underline{\omega}_! \underline{a}_{\text{Nis}}\gamma F^{sp}$  and  $\underline{\omega}_! \gamma_{\underline{a}_{\text{Nis}}}F^{sp}$  are in  $\mathbf{RSC}_{\text{Nis}}$ . Hence (2) (resp. (3)) follows from (1) and [7, Cor. 3.3].  $\square$

□

**Lemma 2.8.** *Consider a sequence  $A \rightarrow B \rightarrow C$  in  $\mathbf{CI}^\tau$  such that*

$$\underline{\omega}_! \underline{a}_{\text{Nis}} A \rightarrow \underline{\omega}_! \underline{a}_{\text{Nis}} B \rightarrow \underline{\omega}_! \underline{a}_{\text{Nis}} C \rightarrow 0$$

*is exact in  $\mathbf{NST}$ . Then the following sequence*

$$\gamma_{\underline{a}_{\text{Nis}}} A(K) \rightarrow \gamma_{\underline{a}_{\text{Nis}}} B(K) \rightarrow \gamma_{\underline{a}_{\text{Nis}}} C(K) \rightarrow 0$$

*is exact for any function field  $K$ .*

*Proof.* The lemma follows from Corollary 2.6(1) and the right exactness of the functor

$$H^1(\mathbf{P}_K, \underline{\omega}_!(-)) : \underline{\mathbf{MNST}} \rightarrow \mathbf{Ab}.$$

□

**Proposition 2.9.** *For  $F \in \mathbf{CI}_{\text{Nis}}^{\tau, sp}$ , there is a natural isomorphism*

$$\underline{\omega}_! \gamma F \simeq \underline{\omega}_! \underline{\mathbf{Hom}}_{\underline{\mathbf{MPST}}}(\underline{\omega}^* \mathbf{G}_m, F) \simeq \underline{\mathbf{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, \underline{\omega}_! F).$$

*Proof.* The first isomorphism follows from Corollary 2.2. For  $F \in \underline{\mathbf{MPST}}$  and  $X \in \mathbf{Sm}$ , put

$$F^X = \underline{\mathbf{Hom}}_{\underline{\mathbf{MPST}}}(\mathbb{Z}_{\text{tr}}(X, \emptyset), F).$$

Note that  $F \in \mathbf{CI}_{\text{Nis}}^{\tau, sp}$  implies  $F^X \in \mathbf{CI}_{\text{Nis}}^{\tau, sp}$ . We compute

$$\begin{aligned} \underline{\omega}_! \gamma F(X) &= \underline{\mathbf{Hom}}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\text{red}}^{(1)}, F)(X, \emptyset) \\ &\simeq \underline{\mathbf{Hom}}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\text{red}}^{(1)}, F^X) = \underline{\omega}_! \gamma F^X(k), \end{aligned}$$

$$\begin{aligned} \underline{\mathbf{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, \underline{\omega}_! F)(X) &= \underline{\mathbf{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, \underline{\mathbf{Hom}}_{\mathbf{PST}}(X, \underline{\omega}_! F)) \\ &\simeq \underline{\mathbf{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, \underline{\omega}_! F^X)(k), \end{aligned}$$

where the last isomorphism comes from Lemma 1.3. Hence it suffices to show that there exists a natural isomorphism

$$\underline{\mathbf{Hom}}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\text{red}}^{(1)}, F) \simeq \underline{\mathbf{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, \underline{\omega}_! F).$$

Recall that

$$\mathbf{G}_m \simeq \text{Coker}(\iota : \mathbb{Z} \rightarrow h_0^{\mathbf{A}^1}(\mathbf{A}^1 - \{0\})),$$

where  $h_0^{\mathbf{A}^1}(\mathbf{A}^1 - \{0\}) = h_0^{\mathbf{A}^1}(\mathbb{Z}_{\text{tr}}(\mathbf{A}^1 - \{0\}))$  with  $h_0^{\mathbf{A}^1} : \mathbf{PST} \rightarrow \mathbf{HI}$  the left adjoint to the inclusion, and  $\iota$  is induced by the section  $\text{Spec } k \rightarrow \mathbf{A}^1$  given by  $1 \in \mathbf{A}^1$ . Hence the assertion follows from the lemma below. □

**Lemma 2.10.** For  $F \in \mathbf{CI}_{\text{Nis}}^{\tau, sp}$  the natural map

$$F(\mathbf{P}^1, 0 + \infty) \rightarrow F(\mathbf{A}^1 - \{0\}) = \text{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\text{tr}}(\mathbf{A}^1 - \{0\}), \underline{\omega}_! F)$$

induces an isomorphism

$$F(\mathbf{P}^1, 0 + \infty) \simeq \text{Hom}_{\mathbf{PST}}(h_0^{\mathbf{A}^1}(\mathbf{A}^1 - \{0\}), \underline{\omega}_! F).$$

*Proof.* If  $F \simeq \underline{\omega}^{\mathbf{CI}} G$  for  $G \in \mathbf{RSC}_{\text{Nis}}$ , this follows from [11, Cor.4.38].

In general, note that the natural map  $u : F \rightarrow \tilde{F} := \underline{\omega}^{\mathbf{CI}} \omega_! F$  is injective by the semipurity of  $F$  and it induces an isomorphism  $\underline{\omega}_! F \simeq \underline{\omega}_! \tilde{F}$ . Hence it suffices to show that  $u$  induces an isomorphism

$$F(\mathbf{P}^1, 0 + \infty) \simeq \tilde{F}(\mathbf{P}^1, 0 + \infty).$$

This follows from Lemma 2.8 since  $F(\mathbf{P}^1, 0 + \infty) = \gamma(F)(k) \oplus F(k)$  and Lemma 2.8 gives an isomorphism  $\gamma(F)(k) \simeq \gamma(\tilde{F})(k)$ .  $\square$

### 3. WEAK CANCELLATION THEOREM

Recall the notation from §1(17).

**Lemma 3.1.** There is natural isomorphisms for  $F, G, H \in \mathbf{CI}^\tau$

$$(3.1) \quad (F \otimes_{\mathbf{CI}}^{sp} G) \otimes_{\mathbf{CI}}^{sp} H \simeq (F \otimes_{\mathbf{CI}} G \otimes_{\mathbf{CI}} H)^{sp} \simeq F \otimes_{\mathbf{CI}}^{sp} (G \otimes_{\mathbf{CI}}^{sp} H).$$

*Proof.* Since  $\otimes_{\mathbf{CI}}$  is associative, it suffices to show a natural isomorphism

$$(F \otimes_{\mathbf{CI}} G)^{sp} \simeq (F^{sp} \otimes_{\mathbf{CI}} G)^{sp} \text{ for } F, G \in \mathbf{CI}^\tau.$$

We have an exact sequence in  $\mathbf{CI}^\tau$ :

$$0 \rightarrow C \rightarrow F \rightarrow F^{sp} \rightarrow 0 \text{ with } \underline{\omega}_! C = 0.$$

Since  $(-) \otimes_{\mathbf{CI}} G : \mathbf{CI}^\tau \rightarrow \mathbf{CI}^\tau$  is right exact, we get an exact sequence

$$C \otimes_{\mathbf{CI}} G \rightarrow F \otimes_{\mathbf{CI}} G \rightarrow F^{sp} \otimes_{\mathbf{CI}} G \rightarrow 0.$$

Since  $C \otimes_{\mathbf{CI}} G$  is a quotient of  $C \otimes_{\mathbf{MPST}} G$  and  $\underline{\omega}_! : \mathbf{MPST} \rightarrow \mathbf{PST}$  is monoidal and exact, we have  $\underline{\omega}_!(C \otimes_{\mathbf{CI}} G) = 0$  so that we get an isomorphism  $F \otimes_{\mathbf{CI}} G \simeq F^{sp} \otimes_{\mathbf{CI}} G$ . This implies the desired assertion.  $\square$

For  $F, G \in \mathbf{CI}_{\text{Nis}}^\tau$ , we write (cf. §1(17) )

$$F \otimes_{\mathbf{CI}}^{\text{Nis}, sp} G = \underline{a}_{\text{Nis}}(F \otimes_{\mathbf{CI}}^{sp} G) \in \mathbf{CI}_{\text{Nis}}^{\tau, sp}.$$

(3.1) implies

(3.2)

$$(F \otimes_{\mathbf{CI}}^{\text{Nis}, sp} G) \otimes_{\mathbf{CI}}^{\text{Nis}, sp} H \simeq \underline{a}_{\text{Nis}}(F \otimes_{\mathbf{CI}} G \otimes_{\mathbf{CI}} H)^{sp} \simeq F \otimes_{\mathbf{CI}}^{\text{Nis}, sp} (G \otimes_{\mathbf{CI}}^{\text{Nis}, sp} H).$$

since  $\underline{a}_{\text{Nis}}$  is monoidal. For  $F \in \mathbf{CI}_{\text{Nis}}^\tau$  and an integer  $d \geq 0$ , we put

$$F(d) = (\overline{\square}_{\text{red}}^{(1)})^{\otimes_{\mathbf{CI}} \text{Nis}, sp d} \otimes_{\mathbf{CI}}^{\text{Nis}, sp} F.$$

Note  $F(d) = F(m)(n)$  with  $d = m + n$  by (3.2).

For  $F \in \mathbf{CI}^\tau$  and  $f \in F(\mathcal{X})$  with  $\mathcal{X} \in \underline{\mathbf{MCor}}$ , consider the composite map

$$\overline{\square}_{\text{red}}^{(1)} \otimes_{\underline{\mathbf{MPST}}} \mathbb{Z}_{\text{tr}}(\mathcal{X}) \xrightarrow{id_{\overline{\square}_{\text{red}}^{(1)}} \otimes f} \overline{\square}_{\text{red}}^{(1)} \otimes_{\underline{\mathbf{MPST}}} F \rightarrow \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} F.$$

This gives rise to a natural map

$$(3.3) \quad \iota_F : F \rightarrow \gamma(\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} F),$$

which induces

$$(3.4) \quad \iota_F^{sp} : F^{sp} \rightarrow \gamma(\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{sp} F),$$

If  $F \in \mathbf{CI}_{\text{Nis}}^\tau$ , this induces a natural map

$$(3.5) \quad \iota_F : F^{sp} \rightarrow \gamma F(1).$$

which generalizes to a natural map for  $n \in \mathbb{Z}_{\geq 1}$  (cf. (2.3))

$$(3.6) \quad \iota_F^n : F^{sp} \rightarrow \gamma^n F(n).$$

*Question 3.2.* For  $F \in \mathbf{CI}_{\text{Nis}}^{\tau, sp}$ , is the map (3.5) an isomorphism?

We will prove the following variant.

**Theorem 3.3.** *For  $F \in \mathbf{CI}^\tau$ , the map (3.4) is an isomorphism.*

Before going into its proof, we give some consequences.

**Corollary 3.4.** *For  $F \in \mathbf{CI}^\tau$  the map (3.4) gives an isomorphism*

$$\underline{\omega}_! \iota_F : \underline{\omega}_! \underline{a}_{\text{Nis}} F \xrightarrow{\sim} \underline{\omega}_! \gamma \underline{a}_{\text{Nis}} (\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{sp} F).$$

For  $F \in \mathbf{CI}_{\text{Nis}}^\tau$ , the map (3.6) induces an isomorphism

$$\underline{\omega}_! \iota_F^n : \underline{\omega}_! F \xrightarrow{\sim} \underline{\omega}_! \gamma^n F(n).$$

*Proof.* The functors  $\underline{\omega}_!$  and  $\underline{a}_{\text{Nis}}$  are exact and  $\underline{\omega}_! \underline{a}_{\text{Nis}} G \cong \underline{\omega}_! \underline{a}_{\text{Nis}} G^{sp}$  for all  $G \in \underline{\mathbf{MPST}}$ . Hence Theorem 3.3 gives a natural isomorphism

$$\underline{\omega}_! \underline{a}_{\text{Nis}} \iota_F : \underline{\omega}_! \underline{a}_{\text{Nis}} F \xrightarrow{\sim} \underline{\omega}_! \underline{a}_{\text{Nis}} \gamma (\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{sp} F).$$

This proves the first assertion since Lemma 2.7(3) implies

$$\underline{\omega}_! \underline{a}_{\text{Nis}} \gamma (\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{sp} F) \simeq \underline{\omega}_! \gamma \underline{a}_{\text{Nis}} (\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{sp} F).$$

The second assertion for the case  $n = 1$  follows directly from the first.

For  $n > 1$ , we proceed by the induction on  $n$  to assume

$$(3.7) \quad \underline{\omega}_! \iota_F^{n-1} : \underline{\omega}_! F \xrightarrow{\sim} \underline{\omega}_! \gamma^{n-1} F(n-1).$$

Then we have isomorphisms

$$\begin{aligned} \underline{\omega}_! \gamma^n F(n) &\stackrel{(*1)}{\simeq} \underline{\omega}_! \gamma \gamma^{n-1} F(n) \stackrel{(*2)}{\simeq} \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, \underline{\omega}_! \gamma^{n-1} F(n)) = \\ &\underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, \underline{\omega}_! \gamma^{n-1} F(1)(n-1)) \stackrel{(*3)}{\simeq} \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, \underline{\omega}_! F(1)) \\ &\stackrel{(*4)}{\simeq} \underline{\omega}_! \gamma F(1) \stackrel{(*5)}{\simeq} F, \end{aligned}$$

where (\*1) (resp. (\*2), resp. (\*3), resp. (\*4), resp. (\*5)) follows from (2.4) (resp. Proposition 2.9, resp. (3.7), resp. Proposition 2.9, resp. the case  $n = 1$ ). This completes the proof.  $\square$

**Corollary 3.5.** *For  $F \in \mathbf{RSC}$  and  $\tilde{F} = \underline{\omega}^{\mathbf{CI}} F \in \mathbf{CI}_{\mathrm{Nis}}^r$  (cf. (1.3)), the map (3.6)  $\iota_{\tilde{F}}^n : \tilde{F} \rightarrow \gamma^n \tilde{F}(n)$  is an isomorphism.*

*Proof.* We have a commutative diagram

$$\begin{array}{ccc} \tilde{F} & \xrightarrow{\iota_{\tilde{F}}^n} & \gamma^n \tilde{F}(n) \\ \downarrow \cong & & \downarrow \hookrightarrow \\ \underline{\omega}^{\mathbf{CI}} \underline{\omega}_! \tilde{F} & \xrightarrow{\underline{\omega}^{\mathbf{CI}} \underline{\omega}_! \iota_{\tilde{F}}^n} & \underline{\omega}^{\mathbf{CI}} \underline{\omega}_! \gamma^n \tilde{F}(n) \end{array}$$

where the vertical arrows come from the adjunction (1.3). The left (resp. right) vertical arrow is an isomorphism (resp. injective) since  $\underline{\omega}_! \underline{\omega}^{\mathbf{CI}} \simeq id$  (resp. by the semipurity of  $\gamma^n \tilde{F}(n)$ ). Since  $\underline{\omega}^{\mathbf{CI}} \underline{\omega}_! \iota_{\tilde{F}}^n$  is an isomorphism by Corollary 3.4, this implies  $\iota_{\tilde{F}}^n$  is an isomorphism by Snake Lemma.  $\square$

**Corollary 3.6.** *For  $F \in \mathbf{CI}_{\mathrm{Nis}}^{r,sp}$ , there is a natural injective map*

$$\tilde{\rho}_F : \gamma^n F(n) \rightarrow \tilde{F} := \underline{\omega}^{\mathbf{CI}} \underline{\omega}_! F$$

*whose composite with the map (3.6)  $\iota_{\tilde{F}}^n : F \rightarrow \gamma^n F(n)$  coincides with the unit map  $u : F \rightarrow \tilde{F}$  for the adjunction (1.3). In particular (3.6) is injective.*

*Proof.* Define  $\tilde{\rho}_F$  as the composite

$$\gamma^n F(n) \xrightarrow{u} \gamma^n \tilde{F}(n) \xrightarrow{(\iota_{\tilde{F}}^n)^{-1}} \tilde{F},$$

where the second map is the inverse of the isomorphism  $\iota_{\tilde{F}}^n : \tilde{F} \cong \gamma^n \tilde{F}(n)$  from Corollary 3.5. Clearly we have  $\tilde{\rho}_F \circ \iota_{\tilde{F}}^n = u$ . We easily see that  $\tilde{\rho}_F$  coincides with the composite

$$\gamma^n F(n) \xrightarrow{u} \underline{\omega}^{\mathbf{CI}} \underline{\omega}_! \gamma^n F(n) \xrightarrow{\underline{\omega}^{\mathbf{CI}} (\underline{\omega}_! \iota_{\tilde{F}}^n)^{-1}} \underline{\omega}^{\mathbf{CI}} \underline{\omega}_! F = \tilde{F},$$



where the first map is injective by the semipurity of  $\gamma^n F(n)$  and the second map is induced by the inverse of the isomorphism  $\underline{\omega}_1 \iota_F^n : \underline{\omega}_1 F \cong \underline{\omega}_1 \gamma^n F(n)$  from Corollary 3.4. This completes the proof.  $\square$

In the rest of this section we prove the following.

**Proposition 3.7.** *For  $F \in \mathbf{CI}^r$ , the map (3.4)  $\iota_F^{sp}$  is split injective.*

For the proof of Proposition 3.7 we first recall the construction of [14]. Take  $X, Y \in \mathbf{Sm}$ . For an integer  $n > 0$  consider the rational function on  $\mathbf{A}_{x_1}^1 \times \mathbf{A}_{x_2}^1$ :

$$g_n = \frac{x_1^{n+1} - 1}{x_1^{n+1} - x_2}.$$

Let  $D_{XY}(g_n)$  be the divisor of the pullback of  $g_n$  to  $(\mathbf{A}_{x_1}^1 - 0) \times X \times (\mathbf{A}_{x_2}^1 - 0) \times Y$ . Take an elementary correspondence

$$(3.8) \quad Z \in \mathbf{Cor}((\mathbf{A}_{x_1}^1 - 0) \times X, (\mathbf{A}_{x_2}^1 - 0) \times Y).$$

Let  $\bar{Z} \subset \mathbf{P}_{x_1}^1 \times X \times \mathbf{P}_{x_2}^1 \times Y$  be the closure of  $Z$  and  $\bar{Z}^N$  be its normalization.

**Lemma 3.8.** (1) *Let  $N > 0$  be an integer such that*

$$(3.9) \quad N(0_1 + \infty_1)_{|\bar{Z}^N} \geq (0_2 + \infty_2)_{|\bar{Z}^N}.$$

*Then, for any integer  $n \geq N$ ,  $Z$  intersects transversally with  $|D_{XY}(g_n)|$  and any component of the intersection  $Z \cdot D_{XY}(g_n)$  is finite and surjective over  $X$ . Thus we get*

$$\rho_n(Z) \in \mathbf{Cor}(X, Y)$$

*as the image of  $Z \cdot D_{XY}(g_n)$  in  $X \times Y$ .*

- (2) *If  $Z = Id_{(\mathbf{A}^1 - 0)} \otimes W$  for  $W \in \mathbf{Cor}(X, Y)$ , then one can take  $N = 1$  in (1) and  $\rho_n(Z) = W$ .*
- (3) *For any  $Z$  as in (3.8) such that  $\rho_n(Z)$  is defined and for any  $f \in \mathbf{Cor}(X', Y')$  with  $X', Y' \in \mathbf{Sm}$ ,  $\rho_n(Z \otimes f)$  for*

$$Z \otimes f \in \mathbf{Cor}((\mathbf{A}_{x_1}^1 - 0) \times (X \times X'), (\mathbf{A}_{x_2}^1 - 0) \times (Y \times Y'))$$

*is defined and we have*

$$\rho_n(Z \otimes f) = \rho_n(Z) \otimes f \in \mathbf{Cor}(X \times X', Y \times Y').$$

- (4) *For an integer  $N > 0$  let*

$$\mathbf{Cor}^{(N)}((\mathbf{A}_{x_1}^1 - 0) \times X, (\mathbf{A}_{x_2}^1 - 0) \times Y)$$

be the subgroup of  $\mathbf{Cor}((\mathbf{A}_{x_1}^1 - 0) \times X, (\mathbf{A}_{x_2}^1 - 0) \times Y)$  generated by elementary correspondences satisfying the condition of Lemma 3.8(1). Then the presheaf on  $\mathbf{Sm}$  given by

$$X \rightarrow \mathbf{Cor}^{(N)}((\mathbf{A}_{x_1}^1 - 0) \times X, (\mathbf{A}_{x_2}^1 - 0) \times Y)$$

is a Nisnevich sheaf.

*Proof.* The assertions are proved in [14, Lem. 4.1 and 4.2] except that (4) follows from the fact that the condition (3.9) is Nisnevich local on  $X$ .  $\square$

For an integer  $a \geq 1$  put  $\overline{\square}^{(a)} = (\mathbf{P}^1, a(0 + \infty)) \in \mathbf{MCor}$ . Take  $\mathcal{X} = (\overline{X}, X_\infty), \mathcal{Y} = (\overline{Y}, Y_\infty) \in \mathbf{MCor}$  with  $X = \overline{X} - |X_\infty|$  and  $Y = \overline{Y} - |Y_\infty|$ . For  $a \geq 1$  take an elementary correspondence

$$Z \in \mathbf{MCor}(\overline{\square}^{(a)} \otimes \mathcal{X}, \overline{\square}^{(1)} \otimes \mathcal{Y}).$$

By definition  $Z \in \mathbf{Cor}(X, Y)$  satisfying

$$(3.10) \quad (0_2 + \infty_2)|_{\overline{Z}^N} + (Y_\infty)|_{\overline{Z}^N} \leq a(0_1 + \infty_1)|_{\overline{Z}^N} + (X_\infty)|_{\overline{Z}^N},$$

where  $\overline{Z}^N$  is the normalization of the closure  $\overline{Z}$  of  $Z$  in  $\mathbf{P}_{x_1}^1 \times X \times \mathbf{P}_{x_2}^1 \times \overline{Y}$ .

For integers  $n, m \geq N \geq a$ , we consider the rational function on  $\mathbf{A}_{x_1}^1 \times \mathbf{A}_t^1 \times \mathbf{A}_{x_2}^1$ :

$$h = tg_n + (1 - t)g_m.$$

Let  $D_{X\mathbf{A}^1Y}(h)$  be the divisor of the pullback of  $h$  to  $(\mathbf{A}_{x_1}^1 - 0) \times X \times \mathbf{A}_t^1 \times (\mathbf{A}_{x_2}^1 - 0) \times Y$ . By [14, Rem. 4.2],  $Z \times \mathbf{A}_t^1$  intersects transversally with  $|D_{X\mathbf{A}^1Y}(h)|$  and any component of the intersection  $(Z \times \mathbf{A}_t^1) \cdot D_{X\mathbf{A}^1Y}(h)$  is finite and surjective over  $X \times \mathbf{A}_t^1$ . Thus we get

$$\rho_h(Z \times \mathbf{A}_t^1) \in \mathbf{Cor}(X \times \mathbf{A}_t^1, Y).$$

It is easy to see

$$(3.11) \quad i_0^* \rho_h(Z \times \mathbf{A}_t^1) = \rho_m(Z) \quad \text{and} \quad i_1^* \rho_h(Z \times \mathbf{A}_t^1) = \rho_n(Z).$$

**Lemma 3.9.** For  $n, m \geq N \geq a$ ,  $\rho_h(Z \times \mathbf{A}_t^1) \in \mathbf{MCor}(\mathcal{X} \otimes \overline{\square}, \mathcal{Y})$ .

*Proof.* Let  $V$  be any component of  $(Z \times \mathbf{A}_t^1) \cdot D_{X\mathbf{A}^1Y}(h)$  and  $\overline{V}$  be its closure in

$$\mathbf{P}_{x_1}^1 \times \overline{X} \times \mathbf{P}_t^1 \times \mathbf{P}_{x_2}^1 \times \overline{Y}.$$

Let  $W \subset X \times \mathbf{A}_t^1 \times Y$  be the image of  $V$  and  $\overline{W}$  be its closure in  $\overline{X} \times \mathbf{P}_t^1 \times \overline{Y}$ . Then we have  $\overline{W} = \pi(\overline{V})$ , where

$$\pi : \mathbf{P}_{x_1}^1 \times \overline{X} \times \mathbf{P}_t^1 \times \mathbf{P}_{x_2}^1 \times \overline{Y} \rightarrow \overline{X} \times \mathbf{P}_t^1 \times \overline{Y}$$

is the projection. We want to show

$$(Y_\infty)|_{\overline{W}^N} \leq (\overline{X} \times \infty)|_{\overline{W}^N} + (X_\infty \times \mathbf{P}_t^1)|_{\overline{W}^N}.$$

Since  $\pi : \overline{V}^N \rightarrow \overline{W}^N$  is proper and surjective, this is reduced to showing

$$(Y_\infty)_{|\overline{V}^N} \leq (\overline{X} \times \infty)_{|\overline{V}^N} + (X_\infty \times \mathbf{P}_t^1)_{|\overline{V}^N}.$$

By (3.10) and [9, Lemma 2.1], we have

$$(Y_\infty)_{|\overline{V}^N} + (0_2 + \infty_2)_{|\overline{V}^N} \leq a(0_1 + \infty_1)_{|\overline{V}^N} + (X_\infty \times \mathbf{P}_t^1)_{|\overline{V}^N}.$$

Thus it suffices to show

$$a(0_1 + \infty_1)_{|\overline{V}^N} \leq (0_2 + \infty_2)_{|\overline{V}^N} + \infty_{|\overline{V}^N}.$$

By the containment lemma [9, Proposition 2.4], this follows from

$$(3.12) \quad a(0_1 + \infty_1)_{|T} \leq (0_2 + \infty_2)_{|T} + \infty_{|T},$$

where  $T$  is any component of the closure of the divisor of  $h$  on  $(\mathbf{A}_{x_1}^1 - 0) \times X \times \mathbf{A}_t^1 \times (\mathbf{A}_{x_2}^1 - 0)$ . By an easy computation  $T$  is contained in one of the closures  $\overline{D(H)}$ ,  $\overline{D(J_n)}$ ,  $\overline{D(J_m)}$  of the divisors of

$$H = t((x_1^{n+1} - x_1^{m+1})(1 - x_2) - x_2 x_1^{m+1}) + x_1^{n+1}(x_1^{m+1} - 1) + x_2,$$

$$J_n = x_1^{n+1} - x_2, \quad J_m = x_1^{m+1} - x_2$$

respectively. It is easy to see that  $\overline{D(H)}$ ,  $\overline{D(J_n)}$ ,  $\overline{D(J_m)}$  do not intersect with  $\infty_1 \times \mathbf{P}_t^1 \times \mathbf{P}_{x_2}^1$ . By the assumption  $n, m \geq N \geq a$ , the ideals  $(J_n, x_1^a)$ ,  $(J_m, x_1^a) \subset k[x_1, x_2]$  contains  $x_2$ , which implies (3.12) (without the last term) if  $T$  is contained in  $\overline{D(J_m)}$  or  $\overline{D(J_n)}$ .

On the other hand, the ideal  $(H, x_1^a) \subset k[x_1, x_2, t]$  contains  $x_2$ . Note that over  $\mathbf{P}_t^1 - 0 = \text{Spec } k(u)$  with  $u = t^{-1}$ ,  $\overline{D(H)}$  is the zero divisor of

$$H' = (x_1^{n+1} - x_1^{m+1})(1 - x_2) - x_2 x_1^{m+1} + u x_1^{n+1}(x_1^{m+1} - 1) + u x_2,$$

and the ideal  $(H', x_1^a) \subset k[x_1, x_2, u]$  contains  $u x_2$ . This show (3.12) if  $T \subset \overline{D(H)}$  and completes the proof of the claim.  $\square$

**Lemma 3.10.** *For  $n \geq a$  we have  $\rho_n(Z) \in \underline{\mathbf{MCor}}(\mathcal{X}, \mathcal{Y})$ .*

*Proof.* This follows from Lemma 3.9 and (3.11).  $\square$

For an integer  $N \geq a$  let

$$\mathbf{MCor}^{(N)}(\overline{\square}_{red}^{(a)} \otimes \mathcal{X}, \overline{\square}_{red}^{(1)} \otimes \mathcal{Y}) \subset \mathbf{MCor}(\overline{\square}_{red}^{(a)} \otimes \mathcal{X}, \overline{\square}_{red}^{(1)} \otimes \mathcal{Y})$$

be the subgroup generated by elementary correspondences lying

$$\mathbf{Cor}^{(N)}((\mathbf{A}^1 - 0) \times X, (\mathbf{A}^1 - 0) \times Y).$$

By Lemma 3.10, we get a map for  $n \geq N \geq a$

$$(3.13) \quad \rho_n^{(a)} : \mathbf{MCor}^{(N)}(\overline{\square}_{red}^{(a)} \otimes \mathcal{X}, \overline{\square}_{red}^{(1)} \otimes \mathcal{Y}) \rightarrow \mathbf{MCor}(\mathcal{X}, \mathcal{Y}).$$

The map (3.13) induces a map of cubical complexes

$$(3.14) \quad \rho_n^{(a)\bullet} : \mathbf{MCor}^{(N)}(\overline{\square}_{red}^{(a)} \otimes \mathcal{X} \otimes \overline{\square}^\bullet, \overline{\square}_{red}^{(1)} \otimes \mathcal{Y}) \rightarrow \mathbf{MCor}(\mathcal{X} \otimes \overline{\square}^\bullet, \mathcal{Y}).$$

By the construction the following diagram is commutative if  $n \geq N \geq b \geq a$ :

(3.15)

$$\begin{array}{ccc} \mathbf{MCor}^{(N)}(\overline{\square}_{red}^{(a)} \otimes \mathcal{X} \otimes \overline{\square}^\bullet, \overline{\square}_{red}^{(1)} \otimes \mathcal{Y}) & \xrightarrow{\rho_n^{(a)\bullet}} & \mathbf{MCor}(\mathcal{X} \otimes \overline{\square}^\bullet, \mathcal{Y}) \\ \downarrow \beta^* & \nearrow \rho_n^{(b)\bullet} & \\ \mathbf{MCor}^{(N)}(\overline{\square}_{red}^{(b)} \otimes \mathcal{X} \otimes \overline{\square}^\bullet, \overline{\square}_{red}^{(1)} \otimes \mathcal{Y}) & & \end{array}$$

where  $\beta^*$  is induced by the natural map  $\beta : \overline{\square}_{red}^{(b)} \rightarrow \overline{\square}_{red}^{(a)}$ .

**Corollary 3.11.** *For  $m, n \geq N \geq a$ ,  $\rho_{n,a}^\bullet$  and  $\rho_{a,m}^\bullet$  are homotopic.*

*Proof.* By Lemma 3.9, we get a map

(3.16)

$$s_{m,n} = \rho_h(- \times \mathbf{A}_t^1) : \mathbf{MCor}^{(N)}(\overline{\square}_{red}^{(a)} \otimes \mathcal{X}, \overline{\square}_{red}^{(1)} \otimes \mathcal{Y}) \rightarrow \mathbf{MCor}(\mathcal{X} \otimes \overline{\square}, \mathcal{Y})$$

such that  $\partial \cdot s_{m,n} = \rho_m^{(a)} - \rho_a^{(a)}$ , where

$$\partial = i_0^* - i_1^* : \mathbf{MCor}(\mathcal{X} \otimes \overline{\square}, \mathcal{Y}) \rightarrow \mathbf{MCor}(\mathcal{X}, \mathcal{Y}).$$

Let

$$s_{m,n}^i : \mathbf{MCor}^{(N)}(\overline{\square}_{red}^{(a)} \otimes \mathcal{X} \otimes \overline{\square}^i, \overline{\square}_{red}^{(1)} \otimes \mathcal{Y}) \rightarrow \mathbf{MCor}(\mathcal{X} \otimes \overline{\square}^{i+1}, \mathcal{Y})$$

be the map (3.16) defined replacing  $\mathcal{X}$  by  $\mathcal{X} \otimes \overline{\square}^i$ . Then it is easy to check that these give the desired homotopy.  $\square$

We now consider

$$\begin{aligned} L_a(\mathcal{Y})^{(N)} &= \underline{\mathbf{Hom}}_{\mathbf{MPST}}^{(N)}(\overline{\square}_{red}^{(a)}, \overline{\square}_{red}^{(1)} \otimes \mathbb{Z}_{\text{tr}}(\mathcal{Y})) \\ &= \mathbf{MCor}^{(N)}(\overline{\square}_{red}^{(a)} \otimes (-), \overline{\square}_{red}^{(1)} \otimes \mathcal{Y}). \end{aligned}$$

It is a subobject of

$$L_a(\mathcal{Y}) = \underline{\mathbf{Hom}}_{\mathbf{MPST}}(\overline{\square}_{red}^{(a)}, \overline{\square}_{red}^{(1)} \otimes \mathbb{Z}_{\text{tr}}(\mathcal{Y})) \in \mathbf{MPST}.$$

The above construction gives a map of complexes in  $\mathbf{MPST}$ :

$$\rho_N^{(a)\bullet} : C_\bullet L_a(\mathcal{Y})^{(N)} \rightarrow C_\bullet(\mathcal{Y}),$$

where  $C^\bullet(-)$  is the Suslin complex. Let

$$\rho_N^{(a)} : H_i(C_\bullet L_a(\mathcal{Y})^{(N)}) \rightarrow H_i(C_\bullet(\mathcal{Y}))$$

be the map in **MPST** induced on cohomology presheaves. Thanks to Corollary 3.11, the diagram

$$\begin{array}{ccc} H_i(C_\bullet L_a(\mathcal{Y})^{(N)}) & \xrightarrow{\rho_N^{(a)}} & h_i^{\square}(\mathcal{Y}) \\ \downarrow & \nearrow & \\ H_i(C_\bullet L_a(\mathcal{Y})^{(N')}) & \xrightarrow{\rho_{N'}^{(a)}} & \end{array}$$

commutes for integers  $N' \geq N$ . Hence we get maps

$$\rho^{(a)} : H_i(C_\bullet L_a(\mathcal{Y})) \rightarrow h_i^{\square}(\mathcal{Y}).$$

Putting  $\Phi = \overline{\square}_{red}^{(1)} \otimes \mathcal{Y}$ , we have

$$C_\bullet(L_a(\mathcal{Y})) = \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\overline{\square}_{red}^{(a)}, \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\overline{\square}^\bullet, \Phi)).$$

Recall that for  $F \in \mathbf{MPST}$  and  $\mathcal{X} \in \mathbf{MCor}$ , we have by the Hom-tensor adjunction an isomorphism:

$$h_0^{\square} \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}), F) \cong \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}), h_0^{\square}(F)).$$

Hence, we get an isomorphism

$$H_0(C_\bullet L_a(\mathcal{Y})) \simeq \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\overline{\square}_{red}^{(a)}, h_0^{\square}(\Phi)),$$

where  $h_i^{\square}(\Phi) = H_i(C_\bullet(\Phi))$  and we have an isomorphism

$$h_0^{\square}(\Phi) \simeq h_0^{\square}(\overline{\square}_{red}^{(1)} \otimes \mathcal{Y}) = \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y}.$$

Hence we get a natural map

$$(3.17) \quad \rho_{\mathcal{Y}}^{(a)} : \gamma_a(\overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y}) \rightarrow h_0^{\square}(\mathcal{Y}).$$

where

$$\gamma_a(F) := \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\overline{\square}_{red}^{(a)}, F) \text{ for } F \in \mathbf{MPST}.$$

In view of (3.15), the following diagram is commutative:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\overline{\square}_{red}^{(a)}, h_0^{\square}(\Phi)) & \xrightarrow{\rho_{\mathcal{Y}}^{(a)}} & h_0^{\square}(\mathcal{Y}) \\ \downarrow \beta^* & \nearrow & \\ \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\overline{\square}_{red}^{(b)}, h_0^{\square}(\Phi)) & \xrightarrow{\rho_{\mathcal{Y}}^{(b)}} & \end{array}$$

Now take any  $F \in \mathbf{CI}^r$  and consider a resolution in **MPST**:

$$A \rightarrow B \rightarrow F \rightarrow 0,$$

where  $A, B$  are the direct sum of  $h_0^{\overline{\square}}(\mathcal{Y})$  for varying  $\mathcal{Y} \in \mathbf{MCor}$ . We then get a commutative diagram

$$\begin{array}{ccccccc} \gamma_a(\overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} A) & \rightarrow & \gamma_a(\overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} B) & \rightarrow & \gamma_a(\overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} F) & \rightarrow & 0 \\ \downarrow \rho_A^{(a)} & & \downarrow \rho_B^{(a)} & & & & \\ A & \longrightarrow & B & \longrightarrow & F & \longrightarrow & 0, \end{array}$$

where the vertical maps are induced by (3.17). The upper sequence is exact by the right-exactness of  $\otimes_{\mathbf{CI}}$  and the fact that  $\overline{\square}_{red}^{(a)}$  is a projective object of  $\underline{\mathbf{MPST}}$ . Thus we get the induced map in  $\underline{\mathbf{MPST}}$ :

$$(3.18) \quad \rho_F^{(a)} : \gamma_a(\overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} F) \rightarrow F.$$

Write  $\rho_F = \rho_F^{(1)}$ .

*Claim 3.12.* The map  $\rho_F$  splits  $\iota_F$ .

*Proof.* By the construction of  $\rho_F$ , this is reduced to the case  $F = h_0^{\overline{\square}}(\mathcal{Y})$  for  $\mathcal{Y} \in \mathbf{MCor}$ , which follows from Lemma 3.8(2).  $\square$

Finally Proposition 3.7 follows from the following:

**Lemma 3.13.** For  $F \in \mathbf{CI}^T$ ,  $\rho_F$  from (3.18) factors through

$$\rho_F^{sp} : \gamma(\overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}}^{sp} F) \rightarrow F^{sp}.$$

Moreover it splits the map  $\iota_F^{sp}$  from (3.4).

*Proof.* Take  $\mathcal{X} \in \mathbf{MCor}$  and let  $\varphi$  be in the kernel of

$$\mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X}, \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} F) \rightarrow \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X}, \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}}^{sp} F).$$

Note that the map is surjective since  $\overline{\square}_{red}^{(a)} \otimes \mathcal{X}$  is a projective object of  $\underline{\mathbf{MPST}}$  by Yoneda's lemma. By the definition of semi-purification there exists an integer  $m > 0$  such that

$$\beta_m^* \varphi = 0 \text{ in } \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(m)} \otimes \mathcal{X}^{(m)}, \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} F),$$

where  $\beta_m : \overline{\square}_{red}^{(m)} \otimes \mathcal{X}^{(m)} \rightarrow \overline{\square}_{red}^{(1)} \otimes \mathcal{X}$ . Then the maps from (3.18) induce a commutative diagram

$$\begin{array}{ccccc}
 & & \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X}, \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} F) & \xrightarrow{\rho_F} & F(\mathcal{X}) \\
 & & \downarrow & & \downarrow \theta_m^* \\
 \beta_m^* \curvearrowright & & \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X}^{(m)}, \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} F) & \xrightarrow{\rho_F} & F(\mathcal{X}^{(m)}) \\
 & & \downarrow & \nearrow \rho_F^{(m)} & \\
 & & \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(m)} \otimes \mathcal{X}^{(m)}, \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} F) & & 
 \end{array}$$

where  $\theta_m^*$  is induced by  $\theta_m : \mathcal{X}^{(m)} \rightarrow \mathcal{X}$ . We have

$$\theta_m^* \rho_F(\varphi) = \rho_F^{(m)} \beta_m^*(\varphi) = 0.$$

Hence  $\rho_F(\varphi)$  lies in the kernel of  $\theta_m^*$ , which is contained in the kernel of the map

$$sp_{\mathcal{X}} : F(\mathcal{X}) \rightarrow F^{sp}(\mathcal{X})$$

by the definition of semi-purification. Hence the composite map

$$sp_{\mathcal{X}} \circ \rho_F : \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X}, \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} F) \rightarrow F^{sp}(\mathcal{X})$$

factors through  $\text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X}, \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}}^{sp} F)$  inducing the desired map  $\rho_F^{sp}$ . Finally, to show the last assertion, consider the commutative diagram

$$\begin{array}{ccccc}
 F & \xrightarrow{\iota_F} & \gamma(\overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} F) & \xrightarrow{\rho_F} & F \\
 \downarrow & & \downarrow & & \downarrow \\
 F^{sp} & \xrightarrow{\iota_F^{sp}} & \gamma(\overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}}^{sp} F) & \xrightarrow{\rho_F^{sp}} & F^{sp}
 \end{array}$$

where  $\rho_F \iota_F = id_F$  by Claim 3.12. This implies  $\rho_F^{sp} \iota_F^{sp} = id_{F^{sp}}$  since  $F \rightarrow F^{sp}$  is surjective. This completes the proof of Lemma 3.13.  $\square$

#### 4. COMPLETION OF THE PROOF OF THE MAIN THEOREM

Take  $\mathcal{Y} \in \mathbf{MCor}$  and put

$$\Psi = \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y} \quad \text{and} \quad \Psi^{sp} = \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}}^{sp} \mathcal{Y}.$$

In this section we prove the following result:

**Proposition 4.1.** *For every  $\varphi \in \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X}, \Psi)$ , there exists  $f \in \mathbf{MCor}(\mathcal{X}, \mathcal{Y})$  such that  $\varphi$  and  $id_{\overline{\square}_{red}^{(1)}} \otimes f$  have the same image in  $\text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X}, \Psi^{sp})$ .*

First we deduce Theorem 3.3 follows from Proposition 4.1. By Proposition 3.7 it suffices to show the surjectivity of the map (3.4)  $\iota_F^{sp}$ . Proposition 4.1 implies that the following composition

$$h_0^{\square}(\mathcal{Y}) \rightarrow \gamma(\overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y}) \rightarrow \gamma(\overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}}^{sp} \mathcal{Y})$$

is surjective. Since  $\gamma(h_0^{\square}(Y) \otimes_{\mathbf{CI}}^{sp} \overline{\square}_{red}^{(1)})$  is semi-pure, it factors through  $h_0^{\square}(\mathcal{Y})^{sp}$ , proving the desired surjectivity for  $F = h_0^{\square}(\mathcal{Y})$ .

For a general  $F \in \mathbf{CI}^T$  consider a surjection

$$q : \bigoplus_{\mathcal{Y} \rightarrow F} h_0^{\square}(\mathcal{Y}) \rightarrow F$$

which gives a commutative diagram

$$\begin{array}{ccc} \bigoplus h_0^{\square}(\mathcal{Y})^{sp} & \xrightarrow{\oplus \iota_{\mathcal{Y}}^{sp}} & \bigoplus \gamma(\overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}}^{sp} \mathcal{Y}) \\ \downarrow q^{sp} & & \downarrow \\ F^{sp} & \xrightarrow{\iota_F^{sp}} & \gamma(\overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}}^{sp} F) \end{array}$$

where the top arrow is surjective and the vertical arrows are surjective since representable presheaves are projective objects of  $\mathbf{MPST}$  by Yoneda's lemma and the functors  $(-)^{sp}$  and  $\overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}}$  commute with direct sums and preserves surjective maps. This proves the desired surjectivity of  $\iota_F$ .

The proof of Proposition 4.1 requires a construction analogous to the one in [15]. Write

$$\overline{\square}_T^{(1)} = (\mathbf{P}_T^1, 0 + \infty) \text{ for a variable } T \text{ over } k,$$

where  $\mathbf{P}_T^1$  is the compactification of  $\mathbf{A}_T^1 = \text{Spec } k[T]$ . We also put

$$\overline{\square}_{T,red}^{(1)} = (1 - e)\overline{\square}_T^{(1)} \in \mathbf{MPST}.$$

For  $X \in \mathbf{Sm}$  and  $a \in \Gamma(X, \mathcal{O}^\times)$ , let  $[a] \in \mathbf{Cor}(X, \mathbf{A}_z^1 - \{0\})$  be the map given by  $z \rightarrow a$ .

**Lemma 4.2.** *The correspondences*

$$[T], [U], [TU], [1] \in \mathbf{Cor}((\mathbf{A}_T^1 - \{0\}) \times (\mathbf{A}_U^1 - \{0\}), (\mathbf{A}^1 - \{0\}))$$

lie in  $\mathbf{MCor}(\overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)}, \overline{\square}^{(1)})$ . Moreover we have

$$[T] + [U] - [TU] - [1] = 0 \in \text{Hom}_{\mathbf{MPST}}(\overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)}, h_0^{\square}(\overline{\square}^{(1)})).$$



*Proof.* The first assertion follows from the fact

$$[T] = \mu(id \otimes [1]), \quad [U] = \mu(id \otimes [1]), \quad [TU] = \mu$$

where  $\mu : (\mathbf{A}_T^1 - \{0\}) \times (\mathbf{A}_U^1 - \{0\}) \rightarrow (\mathbf{A}_W^1 - \{0\})$  is the multiplication, which is admissible by [7, Claim 1.21].

To show the second assertion, consider as in [16, p.142] the finite correspondence  $Z$  given by the following algebraic subset:

$$(4.1) \quad \{V^2 - (W(T+U) + (1-W)(TU+1))V + TU = 0\} \\ \in \mathbf{Cor}((\mathbf{A}_T^1 - \{0\}) \times (\mathbf{A}_U^1 - \{0\}) \times \mathbf{A}_W^1, \mathbf{A}_V^1 - \{0\})$$

Let

$$i_0, i_1 : (\mathbf{A}_T^1 - 0) \times (\mathbf{A}_U^1 - 0) \times (\mathbf{A}_V^1 - 0) \rightarrow (\mathbf{A}_T^1 - 0) \times (\mathbf{A}_U^1 - 0) \times \mathbf{A}_W^1 \times (\mathbf{A}_V^1 - 0)$$

be the maps induced by the inclusion of  $0_W$  and  $1_W$  in  $\mathbf{A}_W^1$ . It is clear that  $(i_0^* - i_1^*)(Z) = ([T] + [U]) - ([TU] + [1])$  since

$$V^2 - (TU+1)V + TU = (V - TU)(V - 1), \\ V^2 - (T+U)V + TU = (V - T)(V - U)$$

We have to check that the correspondence is admissible. Consider the compactification  $(\mathbf{P}^1)^{\times 4}$  and put coordinates with the usual convention  $[0 : 1] = \infty$  and  $[1 : 0] = 0$ :

$$([T_0 : T_\infty], [U_0 : U_\infty], [W_0 : W_\infty], [V_0 : V_\infty]).$$

Then the closure of  $Z$  is the hypersurface given by the following polyhomogeneous polynomial:

$$V_\infty^2 W_0 T_0 U_0 - (W_\infty(T_0 U_\infty + T_\infty U_0) + (W_0 - W_\infty)(T_\infty U_\infty + T_0 U_0)) V_\infty V_0 \\ + T_\infty U_\infty W_0 V_0^2.$$

We have to check that it satisfies the modulus condition: letting

$$\varphi : \bar{Z} \rightarrow (\mathbf{P}^1)^{\times 4}$$

be the inclusion and letting

$$D_1 = (\{0\} + \{\infty\}) \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 + \mathbf{P}^1 \times (\{0\} + \{\infty\}) \times \mathbf{P}^1 \times \mathbf{P}^1 + \mathbf{P}^1 \times \mathbf{P}^1 \times \{\infty\} \times \mathbf{P}^1, \\ D_2 = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times (\{0\} + \{\infty\}),$$

we have to check the following inequality:

$$(4.2) \quad \varphi^*(D_1) \geq \varphi^*(D_2).$$

Consider the Zariski cover of  $(\mathbf{P}^1)^{\times 4}$  given by:

$$\left\{ \mathcal{U}_{\alpha, \beta, \gamma, \delta} = (\mathbf{P}^1 - \alpha)(\mathbf{P}^1 - \beta)(\mathbf{P}^1 - \gamma)(\mathbf{P}^1 - \delta), \alpha, \beta, \gamma, \delta \in \{0, \infty\} \right\}.$$

Define  $t_\alpha = T_\infty/T_0$  if  $\alpha = \infty$  and  $t_\alpha = T_0/T_\infty$  if  $\alpha = 0$  and  $u_\beta, w_\gamma, v_\delta$  similarly. Then

$$\mathcal{U}_{\alpha,\beta,\gamma,\delta} = \text{Spec}(k[t_\alpha, u_\beta, w_\gamma, v_\delta]).$$

On this cover, the Cartier divisors  $D_1$  and  $D_2$  are given by the following system of local equations:

$$D_1 = \left\{ (\mathcal{U}_{\alpha,\beta,0,\delta}, t_\alpha u_\beta w_0), (\mathcal{U}_{\alpha,\beta,\infty,\delta}, t_\alpha u_\beta) \right\} \quad D_2 = \left\{ (\mathcal{U}_{\alpha,\beta,\gamma,\delta}, v_\delta) \right\}$$

A straightforward computation on all the charts shows (4.2).  $\square$

*Remark 4.3.* The same proof works for all  $aT$  and  $bU$  and  $[abTU] + [1]$  are  $\square$ -homotopic for  $a, b \in k$ . In particular,  $[T] + [-U]$  and  $[-TU] + [1]$  are.

**Corollary 4.4.**  $[TU] = 0 \in \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{T,\text{red}}^{(1)} \otimes \overline{\square}_{U,\text{red}}^{(1)}, h_0^\square(\overline{\square}^{(1)}))$ .

*Proof.* This follows from Lemma 4.2 since

$$\begin{aligned} [TU]((1-e) \otimes (1-e)) &= [TU] - [TU](1 \otimes e) - [TU](e \otimes 1) + [TU](e \otimes e) \\ &= [TU] - [T] - [U] + [1] \text{ in } \text{Hom}_{\mathbf{MPST}}(\overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)}, \overline{\square}^{(1)}). \end{aligned}$$

$\square$

For  $X \in \mathbf{Sm}$  and  $a, b \in \Gamma(X, \mathcal{O}^\times)$ , let

$$[a, b] \in \mathbf{Cor}(X, (\mathbf{A}_z^1 - \{0\}) \otimes (\mathbf{A}_w^1 - \{0\}))$$

be the map given by  $z \rightarrow a, w \rightarrow b$ .

**Corollary 4.5.** *We have*

$$\begin{aligned} [T, V] + [U, V] - [TU, V] - [1, V] &= 0 \\ \text{in } \mathbf{MCor}(\overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)} \otimes \overline{\square}_V^{(1)}, h_0^\square(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)})). \end{aligned}$$

*Proof.* This follows from Lemma 4.2 noting the end functor  ${}_-\otimes \overline{\square}^{(1)}$  on  $\mathbf{MPST}$  is additive and  $h_0^\square(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)})$  is a quotient of  $h_0^\square(\overline{\square}^{(1)}) \otimes \overline{\square}^{(1)}$ .

Write

$$\overline{\square}_T^{(2)} = (\mathbf{P}_T^1, 2(0 + 2\infty)), \quad \overline{\square}_{T,\text{red}}^{(2)} = (1 - e)\overline{\square}_T^{(2)} \in \mathbf{MPST}.$$

**Proposition 4.6.** *The correspondences*

$$[U, T], [T^{-1}, U] \in \mathbf{Cor}((\mathbf{A}_T^1 - \{0\}) \times (\mathbf{A}_U^1 - \{0\}), (\mathbf{A}^1 - \{0\}) \times (\mathbf{A}^1 - \{0\}))$$

*lie in*  $\mathbf{MCor}(\overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)}, \overline{\square}^{(1)} \otimes \overline{\square}^{(1)})$ . *Moreover the class of correspondence*

$$[U, T] - [T^{-1}, U] \in \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{T,\text{red}}^{(1)} \otimes \overline{\square}_{U,\text{red}}^{(1)}, h_0^\square(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)}))$$

lies in the kernel of the map

$$h_0^{\square}(\square^{(1)} \otimes \square^{(1)})(\square_{T,red}^{(1)} \otimes \square_{U,red}^{(1)}) \rightarrow h_0^{\square}(\square^{(1)} \otimes \square^{(1)})(\square_{T,red}^{(2)} \otimes \square_{U,red}^{(2)})$$

*Proof.* (see [15, Corollary 9]) The first assertion is easily checked. To show the second, consider the map in **MCor**:

$$\square_S^{(2)} \rightarrow \square_T^{(1)} \otimes \square_U^{(1)} ; T \rightarrow S, U \rightarrow S^{-1}.$$

Composing this with the correspondences of 4.2, we get

$$[S] + [S^{-1}] = 0 \in \text{Hom}_{\mathbf{MPST}}(\square_{S,red}^{(2)}, h_0^{\square}(\square^{(1)})),$$

where we used the fact that  $[1] \circ (1 - e) = 0$ . This implies

$$(4.3) \quad [S, V] + [S^{-1}, V] = 0 \in \text{Hom}_{\mathbf{MPST}}(\square_{S,red}^{(2)} \otimes \square_V^{(1)}, h_0^{\square}(\square^{(1)} \otimes \square^{(1)})).$$

again noting the end functor  $-\otimes \square_V^{(1)}$  on **MCor** is additive and  $h_0^{\square}(\square^{(1)} \otimes \square^{(1)})$  is a quotient of  $h_0^{\square}(\square^{(1)}) \otimes \square^{(1)}$ .

On the other hand, by tensoring the correspondence of 4.4 with another copy of itself we get

$$(4.4) \quad [TU, VW] = 0$$

$$\text{in } \text{Hom}_{\mathbf{MPST}}((\square_{T,red}^{(1)} \otimes \square_{U,red}^{(1)} \otimes \square_{V,red}^{(1)} \otimes \square_{W,red}^{(1)}, h_0^{\square}(\square^{(1)} \otimes \square^{(1)})).$$

There is a map in **MCor**:

$$\begin{aligned} \square_{S_1}^{(2)} \otimes \square_{S_2}^{(2)} &\rightarrow \square_T^{(1)} \otimes \square_U^{(1)} \otimes \square_V^{(1)} \otimes \square_W^{(1)} ; \\ &T \rightarrow S_1, U \rightarrow S_2, V \rightarrow -S_1, W \rightarrow S_2, \end{aligned}$$

which induces an element of

$$\text{Hom}_{\mathbf{MPST}}(\square_{S_1,red}^{(2)} \otimes \square_{S_2,red}^{(2)}, \square_{T,red}^{(1)} \otimes \square_{U,red}^{(1)} \otimes \square_{V,red}^{(1)} \otimes \square_{W,red}^{(1)}).$$

Composing this with (4.4) and changing variables  $(S_1, S_2)$  to  $(T, U)$ , we get

$$(4.5) \quad [TU, -TU] = 0 \in \text{Hom}_{\mathbf{MPST}}(\square_{T,red}^{(2)} \otimes \square_{U,red}^{(2)}, h_0^{\square}(\square^{(1)} \otimes \square^{(1)})).$$

We claim the following equalities in  $\text{Hom}_{\mathbf{MPST}}(\square_{T,red}^{(1)} \otimes \square_{U,red}^{(1)}, h_0^{\square}(\square^{(1)} \otimes \square^{(1)}))$ :

$$\begin{aligned} [TU, -TU] &= [T, -TU] + [U, -TU], \\ [T, -TU] &= [T, -T] + [T, U], \quad [U, -TU] = [U, T] + [U, -U], \\ [T, -T] &= [U, -U] = 0. \end{aligned}$$

Indeed, composing the correspondence of 4.5 with the map in **MCor**:

$$\square_T^{(1)} \otimes \square_U^{(1)} \rightarrow \square_T^{(1)} \otimes \square_U^{(1)} \otimes \square_V^{(1)}$$

given by  $V \rightarrow -TU$  which is admissible by [7, Claim 1.21], we get

$$\begin{aligned} [TU, -TU] + [1, -TU] - [T, -TU] - [U, -TU] &= 0 \\ &\text{in } \text{Hom}_{\text{MPST}}(\overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)}, h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)})). \end{aligned}$$

The first equality follows from this since

$$[1, -TU] = 0 \text{ in } \text{Hom}_{\text{MPST}}(\overline{\square}_{T,red}^{(1)} \otimes \overline{\square}_{U,red}^{(1)}, \overline{\square}^{(1)} \otimes \overline{\square}^{(1)}).$$

The second and third equalities follow from 4.5 by the similar argument. The last equality holds since

$$\begin{aligned} [T, -T] \circ ((1-e) \otimes (1-e)) &= [T, -T] - [T, -T] - [1, -1] + [1, -1] = 0 \\ &\text{in } \text{Hom}_{\text{MPST}}(\overline{\square}_{T,red}^{(1)} \otimes \overline{\square}_{U,red}^{(1)}, \overline{\square}^{(1)} \otimes \overline{\square}^{(1)}). \end{aligned}$$

By the above claim, (4.5) implies

$$(4.6) \quad [T, U] + [U, T] = 0 \text{ in } \text{Hom}_{\text{MPST}}(\overline{\square}_{T,red}^{(2)} \otimes \overline{\square}_{U,red}^{(2)}, h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)})).$$

Putting (4.3) and (4.6) together we conclude that

$$[T, U] - [U^{-1}, T] = 0 \text{ in } \text{Hom}_{\text{MPST}}(\overline{\square}_{T,red}^{(2)} \otimes \overline{\square}_{U,red}^{(2)}, h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)})).$$

This completes the proof of Proposition 4.6.  $\square$

Take  $\mathcal{Y} \in \mathbf{MCor}$  and  $\mathcal{X} \in \mathbf{MCor}$  and

$$\varphi \in \text{Hom}_{\text{MPST}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X}, \overline{\square}_{red}^{(1)} \otimes \mathcal{Y})$$

It induces

$$\varphi_{\overline{\square}} \in \text{Hom}_{\text{MPST}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X}, \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y}).$$

Let

$$\varphi^* \in \text{Hom}_{\text{MPST}}(\mathcal{X} \otimes \overline{\square}_{red}^{(1)}, \mathcal{Y} \otimes \overline{\square}_{red}^{(1)})$$

be induced from  $\varphi$ . It induces

$$\varphi_{\overline{\square}}^* \in \text{Hom}_{\text{MPST}}(\mathcal{X} \otimes \overline{\square}_{red}^{(1)}, \mathcal{Y} \otimes_{\mathbf{CI}} \overline{\square}_{red}^{(1)}).$$

We then put

$$\varphi \otimes Id_{\overline{\square}_{red}^{(1)}} \in \text{Hom}_{\text{MPST}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{red}^{(1)}, \overline{\square}_{red}^{(1)} \otimes \mathcal{Y} \otimes \overline{\square}_{red}^{(1)}),$$

$$Id_{\overline{\square}_{red}^{(1)}} \otimes \varphi^* \in \text{Hom}_{\text{MPST}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{red}^{(1)}, \overline{\square}_{red}^{(1)} \otimes \mathcal{Y} \otimes \overline{\square}_{red}^{(1)}),$$

which induce

$$\varphi_{\overline{\square}} \otimes Id_{\overline{\square}_{red}^{(1)}} \in \text{Hom}_{\text{MPST}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{red}^{(1)}, \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y} \otimes_{\mathbf{CI}} \overline{\square}_{red}^{(1)}),$$

$$Id_{\overline{\square}_{red}^{(1)}} \otimes \varphi_{\overline{\square}}^* \in \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{red}^{(1)}, \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y} \otimes_{\mathbf{CI}} \overline{\square}_{red}^{(1)}).$$

We have

$$\varphi \otimes Id_{\overline{\square}_{red}^{(1)}} = (\sigma \otimes Id_{\mathcal{Y}}) \circ (Id_{\overline{\square}_{red}^{(1)}} \otimes \varphi^*) \circ (\sigma \otimes Id_{\mathcal{X}}),$$

where

$$\sigma : \overline{\square}_{red}^{(1)} \otimes \overline{\square}_{red}^{(1)} \rightarrow \overline{\square}_{red}^{(1)} \otimes \overline{\square}_{red}^{(1)}$$

is the permutation of the two copies of  $\overline{\square}_{red}^{(1)}$ . Let

$$\iota : \overline{\square}_{red}^{(1)} \rightarrow \overline{\square}_{red}^{(1)}$$

be the map given by  $T \rightarrow T^{-1}$  for a coordinate  $T$  and put

$$\sigma' = \sigma - Id_{\overline{\square}_{red}^{(1)}} \otimes \iota.$$

We can write

$$\varphi \otimes id_{\overline{\square}_{red}^{(1)}} = Id_{\overline{\square}_{red}^{(1)}} \otimes \varphi^* + (\sigma' \otimes Id_{\mathcal{Y}}) \circ p + q \circ (\sigma' \otimes Id_{\mathcal{X}}),$$

for some

$$p, q \in \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{red}^{(1)}, \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y} \otimes_{\mathbf{CI}} \overline{\square}_{red}^{(1)}).$$

Put

$$\Gamma_{\mathcal{X}} = \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} \mathcal{X} \otimes_{\mathbf{CI}} \overline{\square}_{red}^{(1)} \quad \Gamma_{\mathcal{Y}} = \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y} \otimes_{\mathbf{CI}} \overline{\square}_{red}^{(1)}.$$

Hence we can write

$$(4.7) \quad \varphi_{\overline{\square}} \otimes id_{\overline{\square}_{red}^{(1)}} = Id_{\overline{\square}_{red}^{(1)}} \otimes \varphi_{\overline{\square}}^* + \sigma'_{\overline{\square}, \mathcal{Y}} \circ p + q_{\overline{\square}} \circ \sigma'_{\overline{\square}, \mathcal{X}},$$

where

$$\sigma'_{\overline{\square}, \mathcal{Y}} : \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y} \otimes_{\mathbf{CI}} \overline{\square}_{red}^{(1)} \rightarrow \Gamma_{\mathcal{Y}}$$

$$\sigma'_{\overline{\square}, \mathcal{X}} : \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} \mathcal{X} \otimes_{\mathbf{CI}} \overline{\square}_{red}^{(1)} \rightarrow \Gamma_{\mathcal{X}}$$

$$q_{\overline{\square}} : \Gamma_{\mathcal{X}} \rightarrow \Gamma_{\mathcal{Y}}$$

are induced by  $\sigma' \otimes Id_{\mathcal{Y}}$ ,  $\sigma' \otimes Id_{\mathcal{X}}$  and  $q$  respectively. For an integer  $n > 0$  let  $\mathcal{X}^{(n)} := (X, nD)$  if  $\mathcal{X} = (X, D)$ . Then we consider the map

$$\text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{red}^{(1)}, \Gamma_{\mathcal{Y}}) \xrightarrow{\beta_n^*} \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(n)} \otimes \mathcal{X}^{(n)} \otimes \overline{\square}_{red}^{(n)}, \Gamma_{\mathcal{Y}})$$

induced by the natural map  $\beta_n : \overline{\square}_{red}^{(n)} \otimes \mathcal{X}^{(n)} \otimes \overline{\square}_{red}^{(n)} \rightarrow \overline{\square}_{red}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{red}^{(1)}$ .

*Claim 4.7.* The maps  $\sigma'_{\overline{\square}, \mathcal{Y}} \circ p$  and  $q_{\overline{\square}} \circ \sigma'_{\overline{\square}, \mathcal{X}}$  lie in the kernel of

$$\text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{red}^{(1)}, \Gamma_{\mathcal{Y}}) \xrightarrow{\beta_2^*} \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(2)} \otimes \mathcal{X}^{(2)} \otimes \overline{\square}_{red}^{(2)}, \Gamma_{\mathcal{Y}})$$

*Proof.* By Proposition 4.6, the composite map

$$\overline{\square}_{red}^{(2)} \otimes \overline{\square}_{red}^{(2)} \xrightarrow{\beta_2} \overline{\square}_{red}^{(1)} \otimes \overline{\square}_{red}^{(1)} \xrightarrow{\sigma'} \overline{\square}_{red}^{(1)} \otimes \overline{\square}_{red}^{(1)} \rightarrow h_0^{\overline{\square}}(\overline{\square}_{red}^{(1)}) \otimes_{\mathbf{CI}} h_0^{\overline{\square}}(\overline{\square}_{red}^{(1)})$$

is zero. This immediately implies the claim for  $q_{\overline{\square}} \circ \sigma'_{\overline{\square}, \mathcal{X}}$ . We now show the claim for  $\sigma'_{\overline{\square}, \mathcal{Y}} \circ p$ . For  $M \in \underline{\mathbf{MCor}}$  and  $N \in \mathbf{MCor}$ , write

$$\Lambda_{M,N} = \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes M \otimes \overline{\square}_{red}^{(1)}, \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} N \otimes_{\mathbf{CI}} \overline{\square}_{red}^{(1)}),$$

$$\Lambda_{M,N}^{(n)} = \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(n)} \otimes M^{(n)} \otimes \overline{\square}_{red}^{(n)}, \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} N \otimes_{\mathbf{CI}} \overline{\square}_{red}^{(1)}).$$

For  $p \in \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{red}^{(1)}, \overline{\square}_{red}^{(1)} \otimes \mathcal{Y} \otimes \overline{\square}_{red}^{(1)})$ , there is a commutative diagram

$$(4.8) \quad \begin{array}{ccc} \Lambda_{\mathcal{Y}, \mathcal{Y}} & \xrightarrow{p^*} & \Lambda_{\mathcal{X}, \mathcal{Y}} \\ \downarrow \beta_2^* & & \downarrow \beta_2^* \\ \Lambda_{\mathcal{Y}, \mathcal{Y}}^{(2)} & \xrightarrow{(p^{(2)})^*} & \Lambda_{\mathcal{X}, \mathcal{Y}}^{(2)}, \end{array}$$

where  $p^{(2)} \in \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(2)} \otimes \mathcal{X}^{(2)} \otimes \overline{\square}_{red}^{(2)}, \overline{\square}_{red}^{(2)} \otimes \mathcal{Y} \otimes \overline{\square}_{red}^{(2)})$  is induced by  $p$ . The claim for  $\sigma'_{\overline{\square}, \mathcal{Y}} \circ p$  follows from this.  $\square$

We now complete the proof of Proposition 4.1. Let

$$(4.9) \quad \Phi = \overline{\square}_{red}^{(1)} \otimes \mathcal{Y} \quad \text{and} \quad \Psi = \overline{\square}_{red}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y} = h_0^{\overline{\square}}(\Phi).$$

We consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{red}^{(1)}, \Phi) & \xrightarrow{\rho_1} & \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\mathcal{X} \otimes \overline{\square}_{red}^{(1)}, \Psi) \\ \downarrow \beta_n^* & & \downarrow \beta_n^* \\ \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(n)} \otimes \mathcal{X}^{(n)} \otimes \overline{\square}_{red}^{(n)}, \Phi) & \xrightarrow{\rho_n} & \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\mathcal{X}^{(n)} \otimes \overline{\square}_{red}^{(n)}, \Psi) \end{array}$$

where the horizontal maps come from (3.17) replacing  $\mathcal{Y}$  with  $\mathcal{Y} \otimes \overline{\square}_{red}^{(1)}$ . By Lemma 3.8(2) and (3) we have

$$\rho_1(\varphi_{\overline{\square}} \otimes id_{\overline{\square}_{red}^{(1)}}) = \rho(\varphi_{\overline{\square}}) \otimes Id_{\overline{\square}_{red}^{(1)}} \quad \text{and} \quad \rho_1(Id_{\overline{\square}_{red}^{(1)}} \otimes \varphi_{\overline{\square}}^*) = \varphi_{\overline{\square}}^*,$$

where  $\rho(\varphi_{\overline{\square}})$  is the image of  $\varphi_{\overline{\square}}$  under the map from (3.17):

$$(4.10) \quad \rho_{\mathcal{X}} : \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X}, \Psi) \rightarrow \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\mathcal{X}, h_0^{\overline{\square}}(\mathcal{Y})),$$

By (4.7) and Claim 4.7 we get  $\beta_n^*(\varphi_{\overline{\square}}^* - \rho(\varphi_{\overline{\square}}) \otimes Id_{\overline{\square}_{red}^{(1)}}) = 0$  so that

$$(4.11) \quad \beta_n^*(\varphi_{\overline{\square}} - Id_{\overline{\square}_{red}^{(1)}} \otimes \rho(\varphi_{\overline{\square}})) = 0 \in \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(n)} \otimes \mathcal{X}^{(n)}, \Psi).$$

Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X}, \Psi) & \longrightarrow & \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(1)} \otimes \mathcal{X}, \Psi^{sp}) \\ \downarrow \beta_n^* & & \downarrow \beta_n^* \\ \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(n)} \otimes \mathcal{X}^{(n)}, \Psi) & \longrightarrow & \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{red}^{(n)} \otimes \mathcal{X}^{(n)}, \Psi^{sp}) \end{array}$$

The two horizontal maps are surjective since representable presheaves are projective objects of  $\underline{\mathbf{MPST}}$  by the Yoneda lemma and  $\Psi \rightarrow \Psi^{sp}$  is surjective. The map  $\beta_n^*$  on the right hand side is injective since  $\Psi^{sp}$  is semi-pure. Hence Proposition 4.1 follows from (4.11).

## 5. IMPLICATIONS ON RECIPROCITY SHEAVES

Let  $\mathbf{RSC}_{\mathrm{Nis}}$  be the category of reciprocity sheaves (see §1 (18)). Recall that for simplicity, we denote for all  $F \in \mathbf{RSC}_{\mathrm{Nis}}$  (cf. §1 (19))

$$\tilde{F} := \underline{\omega}^{\mathbf{CI}} F \in \mathbf{CI}_{\mathrm{Nis}}^{\tau, sp}.$$

By [10] there is a *lax* monoidal structure on  $\mathbf{RSC}_{\mathrm{Nis}}$  given by

$$(F, G)_{\mathbf{RSC}_{\mathrm{Nis}}} := \underline{\omega}_1(\tilde{F} \otimes_{\mathbf{CI}}^{\mathrm{Nis}} \tilde{G}) = \underline{\omega}_1(\tilde{F} \otimes_{\mathbf{CI}}^{\mathrm{Nis}, sp} \tilde{G}).$$

Following [10, 5.21], we define

$$F\langle 0 \rangle := F, \quad F\langle n \rangle := (F\langle n-1 \rangle, \mathbf{G}_m)_{\mathbf{RSC}_{\mathrm{Nis}}} \text{ for } n \geq 1.$$

By Corollary 2.2(1) and the fact that  $\underline{\omega}_1 = \underline{\omega}_1(-)^{sp}$ , we have

$$F\langle n \rangle \cong \underline{\omega}_1(\widetilde{F\langle n-1 \rangle(1)}).$$

By recursiveness of the definition we have

$$(5.1) \quad (F\langle n \rangle)\langle m \rangle \cong F\langle n+m \rangle.$$

There exist a natural map  $F\langle n \rangle \rightarrow \underline{\omega}_1(\tilde{F} \otimes_{\mathbf{CI}} (\underline{\omega}^* \mathbf{G}_m)^{\otimes_{\mathbf{CI}} n})$  but it is not known whether this is an isomorphism. By [10, Prop. 5.6 and Cor. 5.22], we have isomorphisms

$$(5.2) \quad \mathbb{Z}\langle n \rangle \cong \mathcal{K}_n^M, \quad \mathbf{G}_a\langle n \rangle \cong \Omega^n \text{ if } ch(k) = 0.$$

By [10, 5.21 (4)], there is a natural surjection for  $F \in \mathbf{RSC}_{\mathrm{Nis}}$

$$(5.3) \quad F \otimes_{\mathbf{NST}} \mathcal{K}_n^M \rightarrow F\langle n \rangle.$$

For an affine  $X = \mathrm{Spec} A \in \mathbf{Sm}$ , the composite map

$$(5.4) \quad \mathbf{G}_a(A) \otimes_{\mathbb{Z}} \mathbf{G}_m(A)^{\otimes_{\mathbb{Z}} n} \rightarrow (\mathbf{G}_a \otimes_{\mathbf{NST}} \mathbf{G}_m^{\otimes_{\mathbf{NST}} n})(A) \xrightarrow{(5.3)} \mathbf{G}_a\langle n \rangle(A) \xrightarrow{(5.2)} \Omega_A^n$$

sends  $a \otimes f_1 \otimes \cdots \otimes f_n$  with  $a \in A$  and  $f_i \in A^\times$  to  $ad\log f_1 \wedge \cdots \wedge d\log f_n$ .

We have a map natural in  $X \in \mathbf{Sm}$ :

$$\begin{aligned} F(X) = \mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F) &\rightarrow \mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X) \otimes_{\mathbf{NST}} \mathcal{K}_n^M, F \otimes_{\mathbf{NST}} \mathcal{K}_n^M) \\ &\rightarrow \mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X) \otimes_{\mathbf{NST}} \mathcal{K}_n^M, F\langle n \rangle), \end{aligned}$$

where the last map is induced by (5.3). Thus we get a map

$$(5.5) \quad F \rightarrow \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, F\langle n \rangle).$$

**Theorem 5.1.** *For  $F \in \mathbf{RSC}_{\mathrm{Nis}}$ , the map (5.5) is an isomorphism.*

The proof will be given later. First we prove the following.

**Proposition 5.2.** *The map (5.5) is an isomorphism for  $n = 1$ .*

*Proof.* By Proposition 2.9 we have an isomorphism

$$\underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, F\langle 1 \rangle) \cong \omega_! \gamma(\tilde{F}\langle 1 \rangle).$$

Hence the proposition follows from Corollary 3.4  $\square$

For  $F, G \in \mathbf{RSC}_{\mathrm{Nis}}$  let

$$(5.6) \quad \iota_{F,G} : \mathrm{Hom}_{\mathbf{PST}}(F, G) \rightarrow \mathrm{Hom}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle)$$

be the composite map

$$\begin{aligned} \mathrm{Hom}_{\mathbf{PST}}(F, G) &\xrightarrow{\omega^{\mathbf{CI}}} \mathrm{Hom}_{\mathbf{MPST}}(\tilde{F}, \tilde{G}) \xrightarrow{-\otimes_{\mathbf{CI}}^{\mathrm{Nis}} \omega^* \mathbf{G}_m} \\ &\mathrm{Hom}_{\mathbf{MPST}}(\tilde{F} \otimes_{\mathbf{CI}}^{\mathrm{Nis}} \omega^* \mathbf{G}_m, \tilde{G} \otimes_{\mathbf{CI}}^{\mathrm{Nis}} \omega^* \mathbf{G}_m) \xrightarrow{\omega_!} \mathrm{Hom}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle). \end{aligned}$$

**Theorem 5.3.** *For  $F, G \in \mathbf{RSC}_{\mathrm{Nis}}$ ,  $\iota_{F,G}$  is an isomorphism.*

*Proof.* We have isomorphisms (cf. §1 (19))

$$\begin{aligned} (5.7) \quad \mathrm{Hom}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle) &= \mathrm{Hom}_{\mathbf{PST}}(\omega_!(\tilde{F} \otimes_{\mathbf{CI}}^{\mathrm{Nis}, \mathrm{sp}} \overline{\square}_{\mathrm{red}}^{(1)}), \omega_!(\tilde{G} \otimes_{\mathbf{CI}}^{\mathrm{Nis}, \mathrm{sp}} \overline{\square}_{\mathrm{red}}^{(1)})) \\ &\cong \mathrm{Hom}_{\mathbf{MPST}}(\tilde{F} \otimes_{\mathbf{CI}}^{\mathrm{Nis}, \mathrm{sp}} \overline{\square}_{\mathrm{red}}^{(1)}, \omega^{\mathbf{CI}} \omega_!(\tilde{G} \otimes_{\mathbf{CI}}^{\mathrm{Nis}, \mathrm{sp}} \overline{\square}_{\mathrm{red}}^{(1)})) \\ &\cong \mathrm{Hom}_{\mathbf{MPST}}(\tilde{F} \otimes_{\mathbf{CI}}^{\mathrm{Nis}, \mathrm{sp}} \omega^* \mathbf{G}_m, \omega^{\mathbf{CI}} \omega_!(\tilde{G} \otimes_{\mathbf{CI}}^{\mathrm{Nis}, \mathrm{sp}} \overline{\square}_{\mathrm{red}}^{(1)})) \\ &\cong \mathrm{Hom}_{\mathbf{MPST}}(\tilde{F} \otimes_{\mathbf{MPST}} \omega^* \mathbf{G}_m, \omega^{\mathbf{CI}} \omega_!(\tilde{G} \otimes_{\mathbf{CI}}^{\mathrm{Nis}, \mathrm{sp}} \overline{\square}_{\mathrm{red}}^{(1)})) \\ &\cong \mathrm{Hom}_{\mathbf{MPST}}(\tilde{F}, \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\omega^* \mathbf{G}_m, \omega^{\mathbf{CI}} \omega_!(\tilde{G} \otimes_{\mathbf{CI}}^{\mathrm{Nis}, \mathrm{sp}} \overline{\square}_{\mathrm{red}}^{(1)}))), \end{aligned}$$

where the first (resp. second, resp. third) isomorphism follows from (1.2) (resp. Corollary 2.2, resp. the fact  $\omega^{\mathbf{CI}} \omega_! \tau_!(\tilde{G} \otimes_{\mathbf{CI}}^{\mathrm{Nis}, \mathrm{sp}} \overline{\square}_{\mathrm{red}}^{(1)}) \in \mathbf{CI}_{\mathrm{Nis}}^{\tau, \mathrm{sp}}$ ). Note that for  $H \in \mathbf{CI}^{\tau, \mathrm{sp}}$ , the natural map  $H \rightarrow \omega^{\mathbf{CI}} \omega_! H$  is injective.



Hence we get injective maps

$$\begin{aligned}
 (5.8) \quad & \text{Hom}_{\underline{\text{MPST}}}(\tilde{F}, \underline{\text{Hom}}_{\underline{\text{MPST}}}(\underline{\omega}^* \mathbf{G}_m, \tilde{G} \otimes_{\underline{\text{CI}}}^{\text{Nis}, sp} \overline{\square}_{red}^{(1)})) \\
 & \hookrightarrow \text{Hom}_{\underline{\text{MPST}}}(\tilde{F}, \underline{\text{Hom}}_{\underline{\text{MPST}}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\underline{\text{CI}}} \underline{\omega}_! (\tilde{G} \otimes_{\underline{\text{CI}}}^{\text{Nis}, sp} \overline{\square}_{red}^{(1)}))) \\
 & \hookrightarrow \text{Hom}_{\underline{\text{MPST}}}(\tilde{F}, \underline{\omega}^{\underline{\text{CI}}} \underline{\omega}_! \underline{\text{Hom}}_{\underline{\text{MPST}}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\underline{\text{CI}}} \underline{\omega}_! (\tilde{G} \otimes_{\underline{\text{CI}}}^{\text{Nis}, sp} \overline{\square}_{red}^{(1)}))) \\
 & \stackrel{(*1)}{\simeq} \text{Hom}_{\underline{\text{MPST}}}(\tilde{F}, \underline{\omega}^{\underline{\text{CI}}} \underline{\text{Hom}}_{\underline{\text{PST}}}(\mathbf{G}_m, \underline{\omega}_! (\tilde{G} \otimes_{\underline{\text{CI}}}^{\text{Nis}, sp} \overline{\square}_{red}^{(1)}))) \\
 & \stackrel{(*2)}{\simeq} \text{Hom}_{\underline{\text{MPST}}}(\tilde{F}, \underline{\omega}^{\underline{\text{CI}}} \underline{\text{Hom}}_{\underline{\text{PST}}}(\mathbf{G}_m, G\langle 1 \rangle)),
 \end{aligned}$$

where the isomorphism (\*1) comes from Proposition 2.9 and  $\underline{\omega}_! \underline{\omega}^{\underline{\text{CI}}} \simeq id$  (cf. §1 (19)) and (\*2) follows from Corollary 2.2. These maps fit into a commutative diagram

$$\begin{array}{ccc}
 & & \text{Hom}_{\underline{\text{MPST}}}(\tilde{F}, \tilde{G}) \\
 & \swarrow \alpha & \uparrow \simeq \underline{\omega}^{\underline{\text{CI}}} \\
 & \text{Hom}_{\underline{\text{MPST}}}(\tilde{F}, \underline{\text{Hom}}_{\underline{\text{MPST}}}(\underline{\omega}^* \mathbf{G}_m, \tilde{G} \otimes_{\underline{\text{CI}}}^{\text{Nis}, sp} \overline{\square}_{red}^{(1)})) & \text{Hom}_{\underline{\text{PST}}}(F, G) \\
 & \downarrow \hookrightarrow & \downarrow \iota_{F,G} \\
 \text{Hom}_{\underline{\text{MPST}}}(\tilde{F}, \underline{\text{Hom}}_{\underline{\text{MPST}}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\underline{\text{CI}}} \underline{\omega}_! (\tilde{G} \otimes_{\underline{\text{CI}}}^{\text{Nis}, sp} \overline{\square}_{red}^{(1)}))) & \xleftarrow[\simeq]{(5.7)} & \text{Hom}_{\underline{\text{PST}}}(F\langle 1 \rangle, G\langle 1 \rangle) \simeq \underline{\omega}^{\underline{\text{CI}}} \\
 & \downarrow \hookrightarrow & \searrow \simeq \underline{\omega}^{\underline{\text{CI}}} \\
 \text{Hom}_{\underline{\text{MPST}}}(\tilde{F}, \underline{\omega}^{\underline{\text{CI}}} \underline{\text{Hom}}_{\underline{\text{PST}}}(\mathbf{G}_m, G\langle 1 \rangle)) & \xleftarrow[\simeq]{\beta} & \text{Hom}_{\underline{\text{MPST}}}(\tilde{F}, \tilde{G})
 \end{array}$$

The two right vertical isomorphisms follow from the full faithfulness of  $\underline{\omega}^{\underline{\text{CI}}}$ . The isomorphism  $\alpha$  (resp.  $\beta$ ) follows from Corollaries 3.5 and 2.2 (resp. Proposition 5.2) and the squares are commutative by construction, since the maps  $\alpha$  and  $\beta$  are both induced by the natural map  $\tilde{G} \rightarrow \underline{\text{Hom}}_{\underline{\text{MPST}}}(\underline{\omega}^* \mathbf{G}_m, \tilde{G} \otimes_{\underline{\text{CI}}}^{\text{Nis}} \underline{\omega}^* \mathbf{G}_m)$  and the left vertical maps are viewed as inclusions under the identifications

$$\begin{aligned}
 \underline{\omega}_! \underline{\text{Hom}}_{\underline{\text{MPST}}}(\underline{\omega}^* \mathbf{G}_m, \tilde{G} \otimes_{\underline{\text{CI}}}^{\text{Nis}, sp} \overline{\square}_{red}^{(1)}) & \simeq \underline{\text{Hom}}_{\underline{\text{PST}}}(\mathbf{G}_m, G\langle 1 \rangle) \\
 & \simeq \underline{\omega}_! \underline{\text{Hom}}_{\underline{\text{MPST}}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\underline{\text{CI}}} \underline{\omega}_! (\tilde{G} \otimes_{\underline{\text{CI}}}^{\text{Nis}, sp} \overline{\square}_{red}^{(1)}))
 \end{aligned}$$

coming from Lemma 1.3 and Proposition 2.9. This proves that the map  $\iota_{F,G}$  is an isomorphism as desired.  $\square$

**Corollary 5.4.** *For  $F, G \in \mathbf{RSC}_{\text{Nis}}$ , there exists a natural injective map in  $\mathbf{NST}$  for internal hom:*

$$(5.9) \quad \underline{\text{Hom}}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle) \hookrightarrow \underline{\text{Hom}}_{\mathbf{PST}}(F, G),$$

which coincides with the inverse of (5.6) on the  $k$ -valued points.

*Proof.* The surjective map  $F \otimes_{\mathbf{NST}} \mathbf{G}_m \rightarrow F\langle 1 \rangle$  in  $\mathbf{NST}$  from (5.3) induces an injective map

$$\begin{aligned} \underline{\text{Hom}}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle) &\hookrightarrow \underline{\text{Hom}}_{\mathbf{PST}}(F \otimes_{\mathbf{NST}} \mathbf{G}_m, G\langle 1 \rangle) \\ &\simeq \underline{\text{Hom}}_{\mathbf{PST}}(F, \underline{\text{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, G\langle 1 \rangle)) \end{aligned}$$

and the latter is isomorphic to  $\underline{\text{Hom}}_{\mathbf{PST}}(F, G)$  by Proposition 5.2. This completes the proof.  $\square$

We now prove Theorem 5.1. Consider the map induced by (5.3):

$$q : \underline{\text{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, F \otimes_{\mathbf{NST}} \mathcal{K}_n^M) \rightarrow \underline{\text{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, F\langle n \rangle).$$

The map (5.5) is then the composition of  $q$  and the map

$$(5.10) \quad F \rightarrow \underline{\text{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, F \otimes_{\mathbf{NST}} \mathcal{K}_n^M); \quad s \mapsto s \otimes id_{\mathcal{K}_n^M}.$$

On the other hand, we have isomorphisms  $\mathcal{K}_{i-1}^M\langle 1 \rangle \cong \mathcal{K}_i^M$  for all  $i \geq 1$  by (5.2). Hence the map (5.9) for  $F = \mathcal{K}_{i-1}^M$  gives an injective map

$$(5.11) \quad \underline{\text{Hom}}_{\mathbf{PST}}(\mathcal{K}_i^M, F\langle i \rangle) \rightarrow \underline{\text{Hom}}_{\mathbf{PST}}(\mathcal{K}_{i-1}^M, F\langle i-1 \rangle).$$

Composing (5.11) for all  $i \leq n$ , we get an injective map

$$(5.12) \quad \underline{\text{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, F\langle n \rangle) \hookrightarrow F$$

which by definition sends  $q(s \otimes id_{\mathcal{K}_n^M})$  to  $s$  for a section  $s$  of  $F$ . Hence the composition

$$F \xrightarrow{(5.5)} \underline{\text{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, F\langle n \rangle) \xrightarrow{(5.12)} F$$

is the identity, so (5.5) is an isomorphism, which completes the proof of Theorem 5.1.

Let  $G \in \mathbf{RSC}_{\text{Nis}}$  and  $X \in \mathbf{Sm}$ . By Lemma 1.3 we have a natural isomorphism

$$\omega_! \underline{\text{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G}) \simeq \underline{\text{Hom}}_{\mathbf{PST}}(X, G).$$

Hence, the unit map  $id \rightarrow \omega^{\mathbf{CI}} \omega_!$  from (1.3) induces a natural map

$$(5.13) \quad \underline{\text{Hom}}_{\mathbf{MPST}}((X, \emptyset), \omega^{\mathbf{CI}} G) \rightarrow \omega^{\mathbf{CI}} \underline{\text{Hom}}_{\mathbf{PST}}(X, G).$$

It is injective by the semipurity of  $\underline{\mathrm{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\mathrm{tr}}(X, \emptyset), \underline{\omega}^{\mathbf{CI}}F)$ , and becomes an isomorphism after taking  $\underline{\omega}_!$ . Moreover the following diagram is commutative:

$$(5.14) \quad \begin{array}{ccc} \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \underline{\omega}^{\mathbf{CI}}G) & \xrightarrow{(5.13)} & \underline{\omega}^{\mathbf{CI}} \underline{\mathrm{Hom}}_{\mathbf{PST}}(X, G) \\ \downarrow \hookrightarrow & & \downarrow \hookrightarrow \\ \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \underline{\omega}^*G) & \xrightarrow{\cong} & \underline{\omega}^* \underline{\mathrm{Hom}}_{\mathbf{PST}}(X, G) \end{array}$$

where the isomorphism comes from Lemma 1.2.

For  $G \in \mathbf{RSC}_{\mathrm{Nis}}$  and  $X \in \mathbf{Sm}$ , we define the following condition:

(♣) $_X$  The maps (5.13) is an isomorphism.

**Theorem 5.5.** *Let  $F, G \in \mathbf{RSC}_{\mathrm{Nis}}$ . Assume one of the following:*

- (a)  $G$  satisfies (♣) $_X$  for any  $X \in \mathbf{Sm}$ .
- (b)  $G$  satisfies (♣) $_{\mathrm{Spec}(K)}$  for any function field  $K$  over  $k$  and  $F$  is the quotient of a direct sum of representable objects.

Then (5.9) is an isomorphism.

*Proof.* Assume the condition (a). Letting  $\tilde{G} = \underline{\omega}^{\mathbf{CI}}G$ , we have isomorphisms for  $X \in \mathbf{Sm}$

$$(5.15) \quad \begin{aligned} \underline{\mathrm{Hom}}_{\mathbf{PST}}(F, G)(X) &= \mathrm{Hom}_{\mathbf{PST}}(F, \underline{\mathrm{Hom}}_{\mathbf{PST}}(X, G)) \\ &\stackrel{(*)1}{\cong} \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\tilde{F}, \underline{\omega}^{\mathbf{CI}} \underline{\mathrm{Hom}}_{\mathbf{PST}}(X, G)) \stackrel{(*)2}{\cong} \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\tilde{F}, \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G})), \end{aligned}$$

where the isomorphism (\*1) (resp. (\*2)) comes from the full faithfulness of  $\underline{\omega}^{\mathbf{CI}}$  (resp. (♣) $_X$ ). Moreover, we have isomorphisms

$$(5.16) \quad \begin{aligned} \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G}) &\stackrel{(*)3}{\cong} \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \tilde{G}(1))) \\ &\cong \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G}(1))), \end{aligned}$$

where the isomorphism (\*3) comes from Corollaries 3.5 and 2.2. We also have isomorphisms

$$(5.17) \quad \begin{aligned} \underline{\mathrm{Hom}}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle)(X) &= \mathrm{Hom}_{\mathbf{PST}}(F\langle 1 \rangle, \underline{\mathrm{Hom}}_{\mathbf{PST}}(X, G\langle 1 \rangle)) \\ &\stackrel{(*)4}{\cong} \mathrm{Hom}_{\mathbf{PST}}(\underline{\omega}_!(\tilde{F} \otimes_{\mathbf{CI}}^{\mathrm{Nis}} \underline{\omega}^* \mathbf{G}_m), \underline{\omega}_! \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G}(1))) \\ &\cong \mathrm{Hom}_{\mathbf{MPST}}(\tilde{F} \otimes_{\mathbf{MPST}} \underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \underline{\omega}_! \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G}(1))) \\ &\cong \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\tilde{F}, \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \underline{\omega}_! \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G}(1))), \end{aligned}$$

where (\*4) comes from Lemma 1.3. These maps fit into a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\underline{\mathbf{MPST}}}(\tilde{F}, \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}((X, \emptyset), \tilde{G})) & & \\
\downarrow (5.16) \simeq & \swarrow (5.15) & \\
\mathrm{Hom}_{\underline{\mathbf{MPST}}}(\tilde{F}, \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\omega^* \mathbf{G}_m, \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}((X, \emptyset), \tilde{G}(1)))) & & \underline{\mathrm{Hom}}_{\underline{\mathbf{PST}}}(F, G)(X) \\
\downarrow (\dagger) \hookrightarrow & & \uparrow (5.9) \\
\mathrm{Hom}_{\underline{\mathbf{MPST}}}(\tilde{F}, \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\omega^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \omega_! \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}((X, \emptyset), \tilde{G}(1)))) & \xleftarrow[(5.17)]{\simeq} & \underline{\mathrm{Hom}}_{\underline{\mathbf{PST}}}(F\langle 1 \rangle, G\langle 1 \rangle)(X)
\end{array}$$

where the injective map  $(\dagger)$  comes from the counit map  $id \rightarrow \underline{\omega}^{\mathbf{CI}} \omega_!$  from the adjunction (1.3). The diagram commutes since the map (5.16) is induced by the map

$$\begin{aligned}
\underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}((X, \emptyset), \tilde{G}) &\rightarrow \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\omega^* \mathbf{G}_m, \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}((X, \emptyset), \tilde{G}(1))) \\
&\simeq \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}((X, \emptyset) \otimes \omega^* \mathbf{G}_m, \tilde{G} \otimes_{\mathbf{CI}}^{\mathrm{Nis}, sp} \omega^* \mathbf{G}_m)
\end{aligned}$$

given by  $f \mapsto f \otimes id_{\omega^* \mathbf{G}_m}$ , and the map (5.9) is induced by the surjection  $F \otimes_{\mathbf{NST}} \mathbf{G}_m \rightarrow F\langle 1 \rangle$  from (5.3) and the isomorphism inverse of (5.5):

$$\underline{\mathrm{Hom}}_{\underline{\mathbf{PST}}}(F \otimes \mathbf{G}_m, G\langle 1 \rangle) \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\underline{\mathbf{PST}}}(F, G)$$

given by  $f \otimes id_{\mathbf{G}_m} \mapsto f$ , and the maps (5.15) and  $(\dagger)$  are inclusions under the identifications

$$\begin{aligned}
\omega_! \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\omega^* \mathbf{G}_m, \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(X, \emptyset), \tilde{G}(1)) &\simeq \underline{\mathrm{Hom}}_{\underline{\mathbf{PST}}}(\mathbf{G}_m \otimes X, G\langle 1 \rangle) \\
&\simeq \omega_! \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\omega^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \omega_! \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}((X, \emptyset), \tilde{G} \otimes_{\mathbf{CI}}^{\mathrm{Nis}, sp} \overline{\square}_{red}^{(1)}))
\end{aligned}$$

coming from Lemma 1.3 and Proposition 2.9. This proves that (5.9) is an isomorphism.

Next assume the condition (b). In view of Lemma 1.5, we have  $\underline{\mathrm{Hom}}_{\underline{\mathbf{PST}}}(F, G)$  and  $\underline{\mathrm{Hom}}_{\underline{\mathbf{PST}}}(F\langle 1 \rangle, G\langle 1 \rangle)$  are in  $\mathbf{RSC}_{\mathrm{Nis}}$ . Hence, by Lemma 1.4, it is enough to prove that (5.9) induces an isomorphism

$$\underline{\mathrm{Hom}}_{\underline{\mathbf{PST}}}(F\langle 1 \rangle, G\langle 1 \rangle)(K) \cong \underline{\mathrm{Hom}}_{\underline{\mathbf{PST}}}(F, G)(K)$$

for any function field  $K$ . This follows from the same computations as above.  $\square$

**Lemma 5.6.**  $F \in \mathbf{HI}_{\mathrm{Nis}}$  satisfies  $(\clubsuit)_X$  for all  $X \in \mathbf{Sm}$ .

*Proof.* We have

$$\begin{aligned} \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}((X, \emptyset), \underline{\omega}^{\mathbf{CI}} F) &= \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}((X, \emptyset), \underline{\omega}^* F) \underset{(*1)}{\cong} \underline{\omega}^* \underline{\mathrm{Hom}}_{\mathbf{PST}}(X, F) \\ &\underset{(*2)}{\cong} \underline{\omega}^{\mathbf{CI}} \underline{\mathrm{Hom}}_{\mathbf{PST}}(X, F), \end{aligned}$$

where the isomorphism (\*1) (resp. (\*2)) follows from Lemma 1.2 (resp. the fact that  $\underline{\mathrm{Hom}}_{\mathbf{PST}}(X, F) \in \mathbf{HI}$ ). This completes the proof.  $\square$

**Lemma 5.7.** *If  $\mathrm{ch}(k) = 0$ ,  $\Omega^i$  satisfies  $(\clubsuit)_X$  for all  $X \in \mathbf{Sm}$ .*

*Proof.* Put  $\Gamma = \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), \Omega^i)$  and

$$G = \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\mathbb{Z}_{\mathrm{tr}}(X, \emptyset), \underline{\omega}^{\mathbf{CI}} \Omega^i), \quad G^* = \underline{\omega}^{\mathbf{CI}} \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), \Omega^i).$$

Note that  $\Gamma \in \mathbf{RSC}_{\mathrm{Nis}}$  by Lemma 1.5. By [11, Cor. 6.8], for  $\mathcal{Y} = (Y, D) \in \underline{\mathbf{MCor}}$  where  $Y \in \mathbf{Sm}$  and  $D_{\mathrm{red}}$  is a simple normal crossing divisor, we have

$$(5.18) \quad G(\mathcal{Y}) = \Gamma(Y \times X, \Omega^i(\log D_{\mathrm{red}} \times X)((D - D_{\mathrm{red}}) \times X)).$$

Hence the conductor  $c^G$  associated to  $G$  in the sense of [11, Def. 4.14] is given as follows (note that Lemma 1.3 implies  $G \in \mathbf{CI}(\Gamma)$  under the notation of loc. cite.): Let  $\Phi$  be as [11, Def. 4.1]. For

$$a \in G(L) = H^0(X \otimes_k L, \Omega^i) \quad \text{with } L \in \Phi,$$

put  $c_L^G(a) = 0$  if  $a \in H^0(X \otimes_k \mathcal{O}_L, \Omega^i)$ . Otherwise, put

$$c_L^G(a) = \min \left\{ n \geq 1 \mid a \in H^0(X \otimes_k \mathcal{O}_L, \frac{1}{t^{n-1}} \cdot \Omega_{X \otimes_k \mathcal{O}_L}^i(\log)) \right\},$$

where  $t$  is a local paramter of  $\mathcal{O}_L$  and  $\Omega_{X \otimes_k \mathcal{O}_L}^\bullet(\log)$  is the differential graded subalgebra of  $\Omega_{X \otimes_k L}^\bullet$  generated by  $\Omega_{X \otimes_k \mathcal{O}_L}^\bullet$  and  $d \log t$  (cf. [11, §6.1 6.3]). Moreover, one easily sees that for  $\mathcal{Y} = (Y, D) \in \underline{\mathbf{MCor}}$  as (5.18),

$$G(\mathcal{Y}) = \{a \in G(Y - D) \mid c_L^G(a) \leq v_L(D) \text{ for any } L \in \Phi\}$$

(see [11, Notation 4.2] for  $v_L(D)$ ). Hence, by [11, Th. 4.15(4)], it suffices to show  $c^{G^*} = c^G$ . We know  $c^{G^*} \leq c^G$  by loc. cite so that it suffices to show the following: Let  $L \in \Phi$  and  $a \in G(L)$ . For  $r \in \mathbb{Z}_{\geq 0}$ , we have

$$c_L^{G^*}(a) \leq r \Rightarrow c_L^G(a) \leq r.$$

We prove it by the descending induction on  $r$ . By [11, Cor. 4.44] this is reduced to showing the following: Choose a ring homomorphism  $K \hookrightarrow \mathcal{O}_L$  such that  $K \rightarrow \mathcal{O}_L \rightarrow \mathcal{O}_L/(t)$  is an identity and extend it in the canonical way to  $\sigma : K(x) \hookrightarrow \mathcal{O}_{L_x}$ , where  $x$  is a variable

and  $L_x = \text{Frac}(\mathcal{O}_L[x]_{(t)}^h)$ . Assume  $c_L^G(a) \leq r + 1$ . Then the following implication holds

$$(5.19) \quad (a, 1 - xt^r)_{L_x, \sigma} = 0 \in G(K(x)) \Rightarrow c_L^G(a) \leq r,$$

where  $(-, -)_{L_x, \sigma}$  is the local symbol for  $\Gamma = \underline{\text{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\text{tr}}(X), \Omega^i)$  from [11, §4.3 4.41]. Since the local symbol is uniquely determined by the properties (LS1) - (LS4) from [11, §4.3 4.38], we see that it is given by

$$(a, 1 - xt^r)_{L_x, \sigma} = \text{Res}_t(a \, \text{dlog}(1 - xt^r)),$$

where

$$\text{Res}_t : \Gamma(L_x) = H^0(X \otimes_k L_x, \Omega^{i+1}) \rightarrow \Gamma(K(x)) = H^0(X \otimes_k K(x), \Omega^i)$$

is induced by the residue map  $\Omega_{L_x}^{i+1} \rightarrow \Omega_{K(x)}^i$ , which is defined using the isomorphism  $L_x \simeq K(x)((t))$  induced by  $\sigma : K(x) \hookrightarrow \mathcal{O}_{L_x}$ . To prove the implication (5.19), we may assume after replacing  $a$  by  $a - b$  for some  $b \in \Gamma(L)$  with  $c_L^G(b) \leq r$ ,

$$a = \frac{1}{t^r} \alpha + \beta \frac{dt}{t^{r+1}} \text{ for } \alpha \in H^0(X \otimes_k K, \Omega^i), \beta \in H^0(X \otimes_k K, \Omega^{i-1}).$$

Then we compute in  $H^0(X \otimes_k K(x), \Omega^i)$

$$\text{Res}_t(a \, \text{dlog}(1 - xt^r)) = -r\alpha + \beta dx.$$

This shows (5.19) and completes the proof.  $\square$

## 6. INTERNAL HOM'S FOR $\Omega^n$

In this section, we assume  $\text{ch}(k) = 0$ . Note that a section of  $\underline{\text{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)$  over  $X \in \mathbf{Sm}$  is given by a collection of maps

$$\varphi_Y : H^0(Y, \Omega^n) \rightarrow H^0(X \times Y, \Omega^m) \text{ for } Y \in \mathbf{Sm},$$

which are natural in  $Y \in \mathbf{Cor}$ . For

$$(\alpha, \beta) \in H^0(X, \Omega^{m-n}) \oplus H^0(X, \Omega^{m-n-1}),$$

we define

$$\varphi_{Y, \alpha, \beta}^{n, m} : H^0(Y, \Omega^n) \rightarrow H^0(X \times Y, \Omega^m); \omega \rightarrow p_X^* \alpha \wedge p_Y^* \omega + p_X^* \beta \wedge p_Y^* d\omega,$$

where  $p_X : X \times Y \rightarrow X$  and  $p_Y : X \times Y \rightarrow Y$  are the projections. The naturalness of  $\varphi_{Y, \alpha, \beta}^{n, m}$  in  $Y \in \mathbf{Cor}$  follows from [1]. Thus we get a natural map in  $\mathbf{NST}$ :

$$(6.1) \quad \Omega^{m-n} \oplus \Omega^{m-n-1} \rightarrow \underline{\text{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m); (\alpha, \beta) \rightarrow \{\varphi_{Y, \alpha, \beta}^{n, m}\}_{Y \in \mathbf{Sm}},$$

where  $\Omega^i = 0$  for  $i < 0$  by convention. Taking the sections over  $\text{Spec } k$ , we get a natural map

$$(6.2) \quad \Phi^{n, m} : \Omega_k^{m-n} \oplus \Omega_k^{m-n-1} \rightarrow \text{Hom}_{\mathbf{PST}}(\Omega^n, \Omega^m).$$

We also consider the composite map in **NST**:

$$(6.3) \quad \Omega^{m-n} \xrightarrow{(6.1)} \underline{\mathrm{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m) \xrightarrow{\mathrm{dlog}^*} \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, \Omega^m),$$

where the second map is induced by the map  $\mathrm{dlog} : \mathcal{K}_n^M \rightarrow \Omega^n$ . Taking the sections over  $\mathrm{Spec} k$ , we get a natural map

$$(6.4) \quad \Psi^{n,m} : \Omega_k^{m-n} \rightarrow \mathrm{Hom}_{\mathbf{PST}}(\mathcal{K}_n^M, \Omega^m).$$

The main result of this subsection is the following.

**Theorem 6.1.** *The maps (6.1) and (6.3) are isomorphisms.*

First we prove the following.

**Proposition 6.2.** *The maps (6.2) and (6.4) are isomorphisms.*

This follows from Lemmas 6.3, 6.4 and 6.5 below. For  $i \geq 0$ , let us fix the isomorphisms

$$(6.5) \quad \sigma^i : \Omega^{i-1}\langle 1 \rangle \xrightarrow{\simeq} \Omega^i, \quad \varsigma^i : \mathcal{K}_{i-1}^M\langle 1 \rangle \xrightarrow{\simeq} \mathcal{K}_i^M$$

coming from (5.1) and (5.2)

**Lemma 6.3.** (1) *The following diagram is commutative:*

$$\begin{array}{ccc} \Omega_k^{m-n} \oplus \Omega_k^{m-n-1} & \xrightarrow{\Phi^{n,m}} & \mathrm{Hom}_{\mathbf{PST}}(\Omega^n, \Omega^m) \\ \downarrow \Phi^{n-1,m-1} & & \uparrow \\ \mathrm{Hom}_{\mathbf{PST}}(\Omega^{n-1}, \Omega^{m-1}) & \xrightarrow{(5.6)} & \mathrm{Hom}_{\mathbf{PST}}(\Omega^{n-1}\langle 1 \rangle, \Omega^{m-1}\langle 1 \rangle) \end{array}$$

where the right vertical map is induced by  $\sigma^m$  and  $(\sigma^n)^{-1}$  of (6.5).

(2) *The following diagram is commutative:*

$$\begin{array}{ccc} \Omega_k^{m-n} & \xrightarrow{\Psi^{n,m}} & \mathrm{Hom}_{\mathbf{PST}}(\mathcal{K}_n^M, \Omega^m) \\ \downarrow \Psi^{n-1,m-1} & & \uparrow \\ \mathrm{Hom}_{\mathbf{PST}}(\mathcal{K}_{n-1}^M, \Omega^{m-1}) & \xrightarrow{(5.6)} & \mathrm{Hom}_{\mathbf{PST}}(\mathcal{K}_{n-1}^M\langle 1 \rangle, \Omega^{m-1}\langle 1 \rangle) \end{array}$$

where the right vertical map is induced by  $\sigma^m$  and  $(\varsigma^n)^{-1}$  of (6.5).

*Proof.* By [10, Cor. 5.22], for an affine  $X = \mathrm{Spec} A \in \mathbf{Sm}$  and  $i \geq 0$ , the composite map

$$\theta^i : \Omega_A^{i-1} \otimes_{\mathbb{Z}} A^\times \rightarrow (\Omega^{i-1} \otimes_{\mathbf{NST}} \mathbf{G}_m)(A) \xrightarrow{(5.3)} \Omega^{i-1}\langle 1 \rangle(A) \xrightarrow{\sigma^i} \Omega_A^i$$

sends  $\omega \otimes f$  with  $\omega \in \Omega_A^{i-1}$  and  $f \in A^\times$  to  $\omega \wedge \mathrm{dlog} f$ . Moreover, for  $\varphi \in \mathrm{Hom}_{\mathbf{PST}}(\Omega^{n-1}, \Omega^{m-1})$  and  $\varphi' = \sigma^m \circ \varphi \langle 1 \rangle \circ (\sigma^n)^{-1}$ , the diagram

$$\begin{array}{ccc} \Omega_A^{n-1} \otimes_{\mathbb{Z}} A^\times & \xrightarrow{\theta^n} & \Omega_A^n \\ \downarrow \varphi \otimes \mathrm{id}_{A^\times} & & \downarrow \varphi' \\ \Omega_A^{m-1} \otimes_{\mathbb{Z}} A^\times & \xrightarrow{\theta^m} & \Omega_A^m \end{array}$$

is commutative. Hence (1) follows from the equation

$$\alpha \wedge (\omega \wedge \mathrm{dlog} f) + \beta \wedge d(\omega \wedge \mathrm{dlog} f) = (\alpha \wedge \omega + \beta \wedge d\omega) \wedge \mathrm{dlog} f,$$

where  $\alpha \in \Omega_k^{m-n}$  and  $\beta \in \Omega_k^{m-n-1}$ .

(2) follows from (1) and the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{K}_{n-1}^M \langle 1 \rangle & \xrightarrow{\mathrm{dlog} \langle 1 \rangle} & \Omega^{n-1} \langle 1 \rangle \\ \downarrow \zeta^n & & \downarrow \sigma^n \\ \mathcal{K}_n^M & \xrightarrow{\mathrm{dlog}} & \Omega^n \end{array}$$

which can be verified using (5.4). □

**Lemma 6.4.** *For an integer  $n \geq 1$ , we have*

$$(6.6) \quad \mathrm{Hom}_{\mathbf{PST}}(\Omega^n, \mathbf{G}_a) = \mathrm{Hom}_{\mathbf{PST}}(\mathcal{K}_n^M, \mathbf{G}_a) = 0.$$

*Proof.* We have isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbf{PST}}(\Omega^n, \mathbf{G}_a) &\simeq \mathrm{Hom}_{\mathbf{PST}}(\omega_1(\widetilde{\Omega}^{n-1} \otimes_{\mathbf{CI}} \underline{\omega}^* \mathbf{G}_m), \mathbf{G}_a) \\ &\simeq \mathrm{Hom}_{\mathbf{MPST}}(\widetilde{\Omega}^{n-1} \otimes_{\mathbf{CI}} \underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \mathbf{G}_a) \\ &\simeq \mathrm{Hom}_{\mathbf{MPST}}(\widetilde{\Omega}^{n-1} \otimes_{\mathbf{MPST}} \underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \mathbf{G}_a) \\ &\simeq \mathrm{Hom}_{\mathbf{MPST}}(\widetilde{\Omega}^{n-1}, \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \mathbf{G}_a)). \end{aligned}$$

where the first isomorphism is induced by  $(\sigma^n)^{-1}$ , inverse of (6.5), and the second from (1.2). Similarly we have an isomorphism using  $(\zeta^n)^{-1}$  instead of  $(\sigma^n)^{-1}$

$$\mathrm{Hom}_{\mathbf{PST}}(\mathcal{K}_n^M, \mathbf{G}_a) \simeq \mathrm{Hom}_{\mathbf{MPST}}(\mathcal{K}_{n-1}^M, \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \mathbf{G}_a)).$$



We compute

$$\begin{aligned} \underline{\mathrm{Hom}}_{\underline{\mathrm{MPST}}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathrm{CI}} \mathbf{G}_a) &\simeq \mathrm{Hom}_{\underline{\mathrm{MPST}}}(\overline{\square}_{red}^{(1)}, \underline{\omega}^{\mathrm{CI}} \mathbf{G}_a) \\ &\simeq \mathrm{Coker}(\underline{\omega}^{\mathrm{CI}} \mathbf{G}_a(k) \rightarrow \underline{\omega}^{\mathrm{CI}} \mathbf{G}_a(\mathbf{P}^1, 0 + \infty)) \\ &\simeq \mathrm{Coker}(k \rightarrow H^0(\mathbf{P}^1, \mathcal{O})) = 0 \end{aligned}$$

where the first (resp. last) isomorphism follows from Corollary 2.2(1) (resp. [11, Cor. 6.8]). This completes the proof of Lemma 6.4.  $\square$

**Lemma 6.5.** *The maps (6.2) and (6.4) are isomorphisms for  $n = 0$ .*

*Proof.* The assertion for (6.4) is obvious since  $\mathcal{K}_n^M = \mathbb{Z}$  for  $n = 0$ . We prove it for (6.2). We have isomorphisms

$$\begin{aligned} (6.7) \quad \mathrm{Hom}_{\mathrm{PST}}(\mathbf{G}_a, \Omega^i) &\simeq \mathrm{Hom}_{\mathrm{PST}}(a_{\mathrm{Nis}}^V \omega_! h_0^{\overline{\square}}(\overline{\square}_{\mathbf{G}_a}), \Omega^i) \\ &\simeq \mathrm{Hom}_{\underline{\mathrm{MPST}}}(h_0^{\overline{\square}}(\overline{\square}_{\mathbf{G}_a}), \omega^{\mathrm{CI}} \Omega^i) \\ &\simeq \mathrm{Hom}_{\underline{\mathrm{MPST}}}(\overline{\square}_{\mathbf{G}_a}, \omega^{\mathrm{CI}} \Omega^i) \\ &\simeq \mathrm{Ker}(H^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1}^i(\log \infty)(\infty)) \xrightarrow{i_0^*} \Omega_k^i), \end{aligned}$$

where the first (resp. last) isomorphism follows from (1.5) (resp. [11, Cor. 6.8]). The standard exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^1} \otimes_k \Omega_k^1 \rightarrow \Omega_{\mathbf{P}^1}^1 \rightarrow \Omega_{\mathbf{P}^1/k}^1 \rightarrow 0$$

induces an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^1} \otimes_k \Omega_k^i \rightarrow \Omega_{\mathbf{P}^1}^i \rightarrow \Omega_{\mathbf{P}^1/k}^1 \otimes_k \Omega_k^{i-1} \rightarrow 0$$

noting  $\Omega_{\mathbf{P}^1/k}^i = 0$  for  $i > 1$ . Here  $\Omega_k^{i-1} = 0$  if  $i = 0$  by convention. It induces an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^1}(\infty) \otimes_k \Omega_k^i \rightarrow \Omega_{\mathbf{P}^1}^i(\log \infty)(\infty) \rightarrow \Omega_{\mathbf{P}^1/k}^1(2\infty) \otimes_k \Omega_k^{i-1} \rightarrow 0,$$

since  $\mathcal{O}_{\mathbf{P}^1}(\log \infty) = \mathcal{O}_{\mathbf{P}^1}$  and  $\Omega_{\mathbf{P}^1/k}^1(\log \infty) = \Omega_{\mathbf{P}^1/k}^1(\infty)$ . Letting  $t$  be the standard coordinate of  $\mathbf{P}^1$ , we have

$$H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(\infty)) = k \cdot 1 \oplus k \cdot t, \quad H^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1/k}^1(2\infty)) = k \cdot dt,$$

and  $dt$  lifts canonically to a section  $dt \in H^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1}^1(\log \infty)(\infty))$ . Hence we get an isomorphism

$$(6.8) \quad H^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1}^i(\log \infty)(\infty)) \simeq (k \cdot 1 \oplus k \cdot t) \otimes_k \Omega_k^i \oplus (k \cdot dt) \otimes_k \Omega_k^{i-1}.$$

Thus the last group of (6.7) is isomorphic to

$$k \cdot t \otimes_k \Omega_k^i \oplus k \cdot dt \otimes_k \Omega_k^{i-1} \simeq \Omega_k^i \oplus \Omega_k^{i-1}.$$

Hence, from (6.7), we get a natural isomorphism

$$(6.9) \quad \Omega_k^{i-1} \oplus \Omega_k^i \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^i).$$

Next we claim that the map (6.9) coincides with (6.2) for  $n = 0$ . By Lemma 1.8(2), we have a commutative diagram

$$(6.10) \quad \begin{array}{ccc} \mathbb{Z}_{\mathrm{tr}}(\mathbf{A}_t^1) & \xrightarrow{\lambda_{\mathbf{G}_a}} & \mathbf{G}_a \\ \downarrow \simeq & & \uparrow (1.5) \\ \omega_! \mathbb{Z}_{\mathrm{tr}}(\mathbf{P}^1, 2\infty) & \longrightarrow & \omega_! h_0^{\square}(\square_{\mathbf{G}_a}) \end{array}$$

where  $\lambda_{\mathbf{G}_a}$  is given by  $t \in \mathbf{G}_a(\mathbf{A}_t^1) = k[t]$ . The standard isomorphism

$$\Omega^i(\mathbf{A}_t^1) \simeq (\Omega_k^i \otimes_k k[t]) \oplus (\Omega_k^{i-1} \otimes_k k[t] dt)$$

induces a natural isomorphism

$$(6.11) \quad \mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(\mathbf{A}_t^1), \Omega^i) = \Omega^i(\mathbf{A}_t^1) \simeq \Omega_k^i[t] \oplus \Omega_k^{i-1}[t] \wedge dt,$$

where

$$\Omega_k^i[t] = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \Omega_k^i \cdot t^m, \quad \Omega_k^{i-1}[t] \wedge dt = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \Omega_k^{i-1} \wedge t^m dt.$$

The map  $\lambda_{\mathbf{G}_a}$  induces the inclusion

$$\lambda_{\mathbf{G}_a}^* : \mathrm{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^i) \hookrightarrow \mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(\mathbf{A}_t^1), \Omega^i) = \Omega^i(\mathbf{A}_t^1)$$

such that

$$(6.12) \quad \lambda_{\mathbf{G}_a}^*(\varphi) = \varphi_{\Omega_{\mathbf{A}_t^1}^i}(t) \text{ for } \varphi \in \mathrm{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^i),$$

where  $\varphi_{\Omega_{\mathbf{A}_t^1}^i} : \mathbf{G}_a(\mathbf{A}_t^1) = k[t] \rightarrow \Omega^i(\mathbf{A}_t^1)$  is induced by  $\varphi$ . The following claim follows from (6.7), (6.8) and (6.10).

*Claim 6.6.* The image of  $\lambda_{\mathbf{G}_a}^*$  is identified under (6.11) with

$$\Omega_k^i \cdot t \oplus \Omega_k^{i-1} \wedge dt \subset \Omega_k^i[t] \oplus \Omega_k^{i-1}[t] \wedge dt,$$

and the composite map

$$\Omega_k^i \oplus \Omega_k^{i-1} \xrightarrow{(6.9)} \mathrm{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^i) \xrightarrow{\lambda_{\mathbf{G}_a}^*} \Omega_k^i \cdot t \oplus \Omega_k^{i-1} \wedge dt$$

is given by the obvious identifications  $\Omega_k^i = \Omega_k^i \cdot t$  and  $\Omega_k^{i-1} = \Omega_k^{i-1} \wedge dt$ .

Let

$$(6.13) \quad \mathrm{Hom}_{\mathbf{G}_a}(\mathbf{G}_a, \Omega^m) \subset \mathrm{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^m)$$

be the subgroup of  $\mathbf{G}_a$ -linear morphisms. There is a natural isomorphism

$$\xi : \Omega_k^m \cong \mathrm{Hom}_{\mathbf{G}_a}(\mathbf{G}_a, \Omega^m) ; \quad \omega \mapsto \{\lambda \mapsto \lambda\omega\} \quad (\lambda \in \mathbf{G}_a).$$

(6.13) is a direct summand since we have a splitting given by

$$\mathrm{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^m) \rightarrow \mathrm{Hom}_{\mathbf{G}_a}(\mathbf{G}_a, \Omega^m) ; \quad \varphi \mapsto \{\lambda \mapsto \lambda\varphi(1)\}.$$

The other summand is

$$\mathrm{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^m)^0 := \{\varphi \mid \varphi(1) = 0\}.$$

There is a natural map

$$\xi' : \Omega_k^{m-1} \rightarrow \mathrm{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^m)^0 ; \quad \omega \mapsto \{\alpha \mapsto \omega \wedge d\alpha\}.$$

By (6.12), under the identification (6.11), we have

$$\lambda_{\mathbf{G}_a}^*(\xi(\omega)) = \omega \cdot t, \quad \lambda_{\mathbf{G}_a}^*(\xi'(\eta)) = \eta \wedge dt \quad (\omega \in \Omega^i, \eta \in \Omega^{i-1}).$$

Hence the composite map

$$\Omega_k^i \oplus \Omega_k^{i-1} \xrightarrow{\xi \oplus \xi'} \mathrm{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^i) \xrightarrow{\lambda_{\mathbf{G}_a}^*} \Omega_k^i \cdot t \oplus \Omega_k^{i-1} \wedge dt$$

is given by the obvious identifications  $\Omega_k^i = \Omega_k^i \cdot t$  and  $\Omega_k^{i-1} = \Omega_k^{i-1} \wedge dt$ . By Claim 6.6 this proves the desired claim and completes the proof of Lemma 6.5.  $\square$

To deduce Theorem 6.1 from Proposition 6.2, we need some preliminaries.

Let  $K$  be the function field of  $S \in \mathbf{Sm}$  and define  $\mathbf{Cor}_K$ ,  $\mathbf{PST}_K$ ,  $\underline{\mathbf{MCor}}_K$ ,  $\underline{\mathbf{MPST}}_K$ , etc. defined as  $\mathbf{Cor}$ ,  $\mathbf{PST}$ ,  $\underline{\mathbf{MCor}}$ ,  $\underline{\mathbf{MPST}}$ , etc. where the base field  $k$  is replaced by  $K$ . We have then a map

$$(6.14) \quad r_K : \mathrm{Hom}_{\mathbf{PST}_K}(\Omega^n, \Omega^m) \rightarrow \underline{\mathrm{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)(K) ; \quad \varphi \rightarrow \{\psi_Y\}_{Y \in \mathbf{Sm}},$$

where  $\psi_Y$  for  $Y \in \mathbf{Sm}$  is the composite map

$$H^0(Y, \Omega^n) \rightarrow H^0(Y \times_k K, \Omega^n) \rightarrow H^0(Y \times_k K, \Omega^m),$$

where the second map is  $\varphi_{Y \times_k K}$  (note  $Y \times_k K \in \mathbf{Sm}_K$ ) and the first is the pullback by the projection  $p_Y : Y \times_k K \rightarrow Y$ . Similarly we can define a map

$$(6.15) \quad r_K : \mathrm{Hom}_{\mathbf{PST}_K}(\mathcal{K}_n^M, \Omega^m) \rightarrow \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, \Omega^m)(K).$$

By definitions, the following diagrams are commutative.

$$\begin{array}{ccc} \Omega_K^{m-n} \oplus \Omega_K^{m-n-1} & \xrightarrow{(6.2)} & \mathrm{Hom}_{\mathbf{PST}_K}(\Omega^n, \Omega^m) \\ & \searrow (6.1) & \downarrow r_K \\ & & \underline{\mathrm{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)(K) \end{array}$$

$$\begin{array}{ccc} \Omega_K^{m-n} & \xrightarrow{(6.4)} & \mathrm{Hom}_{\mathbf{PST}_K}(\mathcal{K}_n^M, \Omega^m) \\ & \searrow (6.3) & \downarrow r_K \\ & & \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, \Omega^m)(K) \end{array}$$

In view of Lemma 1.4, Theorem 6.1 follows from Proposition 6.2 and the following.

**Lemma 6.7.** *The maps (6.14) and (6.15) are isomorphisms.*

For the proof we need the following.

**Lemma 6.8.** *For  $\mathcal{X} = (X, D) \in \mathbf{MCor}$  and  $\mathcal{X}_K = (X_K, D_K)$  with  $X_K = X \times_k K$  and  $D_K = D \times_k K$ , we have a natural isomorphism*

$$\mathrm{Hom}_{\underline{\mathbf{MPST}}_K}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}_K), \underline{\omega}^{\mathbf{CI}_K} \Omega^n) \cong \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\mathbb{Z}_{\mathrm{tr}}(\mathcal{X}), \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(K, \underline{\omega}^{\mathbf{CI}} \Omega^n)).$$

*Proof.* By [2, Pr. 1.9.2 c)] we may assume  $X \in \mathbf{Sm}$  and  $D_{\mathrm{red}}$  is a simple normal crossing divisor. From the explicit computation of  $\underline{\omega}^{\mathbf{CI}} \Omega^m$  in [11, Cor. 6.8],

$$\begin{aligned} (\underline{\omega}^{\mathbf{CI}_K} \Omega^m)(X_K, D_K) &= H^0(X_K, \Omega_{X_K}^m(\log(D_K))(D_K - D_{K,\mathrm{red}})) \\ &= (\underline{\omega}^{\mathbf{CI}} \Omega^m)(X_K, D_K) := \varinjlim_{U \subset S} (\underline{\omega}^{\mathbf{CI}} \Omega^m)(X \times_k U, D \times_k U), \end{aligned}$$

where  $U$  ranges over the open subsets of  $S$ . This proves the lemma.  $\square$

We now prove Lemma 6.7. We only prove the assertion for (6.14). The proof for (6.15) is similar. Put

$$\overline{\square}_{\Omega^n} = \overline{\square}_{\mathbf{G}_a} \otimes_{\mathbf{MPST}} \overline{\square}_{\mathbf{G}_m}^{\otimes n},$$

where  $\overline{\square}_{\mathbf{G}_a}$  and  $\overline{\square}_{\mathbf{G}_m}$  are from Lemma 1.8. By (1.4) and (1.5) and (5.2), we have an isomorphism in  $\mathbf{PST}$ :

$$(6.16) \quad a_{\mathrm{Nis}}^V \omega_! h_0^{\overline{\square}}(\overline{\square}_{\Omega^n}) \xrightarrow{\cong} \Omega^n.$$

Let  $\bar{\square}_K = (\mathbf{P}_K^1, \infty) \in \mathbf{MCor}_K$  and  $\bar{\square}_{\Omega^n, K} \in \mathbf{MPST}_K$  be defined as  $\bar{\square}_{\Omega^n}$ . We have isomorphisms

$$(6.17) \quad \begin{aligned} \mathrm{Hom}_{\mathbf{PST}_K}(\Omega^n, \Omega^m) &\simeq \mathrm{Hom}_{\mathbf{PST}_K}(\omega_! h_0^{\bar{\square}_K}(\bar{\square}_{\Omega^n, K}), \Omega^m) \simeq \\ &\mathrm{Hom}_{\mathbf{MPST}_K}(\bar{\square}_{\Omega^n, K}, \underline{\omega}^{\mathbf{CI}_K} \Omega^m) \simeq \mathrm{Hom}_{\mathbf{MPST}}(\bar{\square}_{\Omega^n}, \underline{\mathrm{Hom}}_{\mathbf{MPST}}(K, \underline{\omega}^{\mathbf{CI}} \Omega^m)), \end{aligned}$$

where the last isomorphism comes from Lemma 6.8. On the other hand, we have isomorphisms

$$(6.18) \quad \begin{aligned} \underline{\mathrm{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)(K) &= \mathrm{Hom}_{\mathbf{PST}}(\Omega^n, \underline{\mathrm{Hom}}_{\mathbf{PST}}(K, \Omega^m)) \simeq \\ &\mathrm{Hom}_{\mathbf{PST}}(\omega_! h_0^{\bar{\square}}(\bar{\square}_{\Omega^n}), \underline{\mathrm{Hom}}_{\mathbf{PST}}(K, \Omega^m)) \simeq \mathrm{Hom}_{\mathbf{MPST}}(\bar{\square}_{\Omega^n}, \underline{\omega}^{\mathbf{CI}} \underline{\mathrm{Hom}}_{\mathbf{PST}}(K, \Omega^m)). \end{aligned}$$

Hence Lemma 6.7 follows from Lemma 5.7 and the following.

*Claim 6.9.* The following diagram is commutative.

$$(6.19) \quad \begin{array}{ccc} \mathrm{Hom}_{\mathbf{PST}_K}(\Omega^n, \Omega^m) & \xrightarrow{(6.17)} & \mathrm{Hom}_{\mathbf{MPST}}(\bar{\square}_{\Omega^n}, \underline{\mathrm{Hom}}_{\mathbf{MPST}}(K, \underline{\omega}^{\mathbf{CI}} \Omega^m)) \\ \downarrow r_K & & \downarrow \\ \underline{\mathrm{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)(K) & \xrightarrow{(6.18)} & \mathrm{Hom}_{\mathbf{MPST}}(\bar{\square}_{\Omega^n}, \underline{\omega}^{\mathbf{CI}} \underline{\mathrm{Hom}}_{\mathbf{PST}}(K, \Omega^m)) \end{array}$$

where the right vertical map is induced by the map (5.13).

To show the above claim, write  $\mathbf{A}_{\Omega^n} = \mathbf{A}^1 \times (\mathbf{A}^1 - \{0\})^n$  and  $\mathbf{A}_{\Omega^n, K} = \mathbf{A}_{\Omega^n} \otimes_k K$ . Take the standard coordinates  $y$  on  $\mathbf{A}^1$  and  $(x_1, \dots, x_n)$  on  $(\mathbf{A}^1 - \{0\})^n$  so that

$$\mathbf{A}_{\Omega^n} = \mathrm{Spec} k[y, x_1, \dots, x_n][x_1^{-1}, \dots, x_n^{-1}].$$

By the definition of  $\bar{\square}_{\Omega^n}$ , we have natural maps in  $\mathbf{MPST}$

$$(6.20) \quad \mathbb{Z}_{\mathrm{tr}}(\mathbf{A}_{\Omega^n}, \emptyset) \rightarrow (\mathbf{P}^1, 2\infty) \otimes (\mathbf{P}^1, 0 + \infty)^{\otimes n} \rightarrow \bar{\square}_{\Omega^n},$$

which induces a map in  $\mathbf{PST}$ :

$$(6.21) \quad \lambda_{\Omega^n} : \mathbb{Z}_{\mathrm{tr}}(\mathbf{A}_{\Omega^n}) \rightarrow \omega_! \bar{\square}_{\Omega^n} \rightarrow \Omega^n,$$

where the last map is induced by (6.16). Let

$$(6.22) \quad \lambda_{\Omega^n, K} : \mathbb{Z}_{\mathrm{tr}}(\mathbf{A}_{\Omega^n, K}) \rightarrow \Omega^n$$

be defined as (6.21) replacing  $k$  by  $K$ . By the definition of  $\lambda_{\mathbf{G}_m}$  and  $\lambda_{\mathbf{G}_a}$  (cf. Lemma 1.8) and (5.4),  $\lambda_{\Omega^n}$  corresponds to

$$(6.23) \quad \omega_0 := y \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \in \Omega^n(\mathbf{A}_{\Omega^n}).$$

The map (6.20) induces an injective maps

$$(6.24) \quad \mathrm{Hom}_{\mathbf{MPST}}(\bar{\square}_{\Omega^n}, \underline{\mathrm{Hom}}_{\mathbf{MPST}}(K, \underline{\omega}^{\mathbf{CI}} \Omega^m)) \hookrightarrow H^0(\mathbf{A}_{\Omega^n, K}, \Omega^m),$$

$$(6.25) \quad \mathrm{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(\overline{\square}_{\Omega^n}, \underline{\omega}^{\mathrm{CI}} \underline{\mathrm{Hom}}_{\mathbf{PST}}(K, \Omega^m)) \hookrightarrow H^0(\mathbf{A}_{\Omega^n, K}, \Omega^m),$$

which are compatible with the right vertical map in (6.19) since applying  $\underline{\omega}_1$ , the map (5.13) is identified with the identity on  $\underline{\mathrm{Hom}}_{\mathbf{PST}}(K, \Omega^m)$  via the isomorphism in Lemma 1.3. Hence it suffices to show the commutativity of the diagram

$$(6.26) \quad \begin{array}{ccc} \mathrm{Hom}_{\mathbf{PST}_K}(\Omega^n, \Omega^m) & \xrightarrow{\alpha} & H^0(\mathbf{A}_{\Omega^n, K}, \Omega^m) \\ \downarrow r_K & \nearrow \beta & \\ \underline{\mathrm{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)(K) & & \end{array}$$

where  $\alpha$  (resp.  $\beta$ ) is the composite of (6.17) and (6.24) (resp. (6.18) and (6.25)). By the definition,  $\alpha$  is induced by the map  $\lambda_{\Omega^n, K}$  from (6.22). As  $\lambda_{\Omega^n, K}$  is given by the image  $\omega_{0, K}$  of  $\omega_0$  from (6.23) under the pullback map  $p^* : \Omega^n(\mathbf{A}_{\Omega^n}) \rightarrow \Omega^n(\mathbf{A}_{\Omega^n, K})$ , we have

$$\alpha(\varphi) = \varphi_{\mathbf{A}_{\Omega^n, K}}(\omega_{0, K}) \quad \text{for } \varphi \in \mathrm{Hom}_{\mathbf{PST}_K}(\Omega^n, \Omega^m),$$

where  $\varphi_{\mathbf{A}_{\Omega^n, K}} : \Omega^n(\mathbf{A}_{\Omega^n, K}) \rightarrow \Omega^m(\mathbf{A}_{\Omega^n, K})$  is induced by  $\varphi$ . On the other hand, by the definition of  $\beta$ , we have a commutative diagram

$$\begin{array}{ccc} H^0(\mathbf{A}_{\Omega^n, K}, \Omega^m) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbf{PST}}(\mathbf{A}_{\Omega^n}, \underline{\mathrm{Hom}}_{\mathbf{PST}}(K, \Omega^m)) \\ \beta \uparrow & & \uparrow \lambda_{\Omega^n}^* \\ \underline{\mathrm{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)(K) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbf{PST}}(\Omega^n, \underline{\mathrm{Hom}}_{\mathbf{PST}}(K, \Omega^m)) \end{array}$$

where  $\lambda_{\Omega^n}^*$  is induced by  $\lambda_{\Omega^n}$  from (6.21). Hence we have

$$\beta(\psi) = \psi_{\mathbf{A}_{\Omega^n}}(\omega_0) \quad \text{for } \psi \in \underline{\mathrm{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)(K),$$

where  $\psi_{\mathbf{A}_{\Omega^n}} : \Omega^n(\mathbf{A}_{\Omega^n}) \rightarrow \underline{\mathrm{Hom}}_{\mathbf{PST}}(K, \Omega^m)(\mathbf{A}_{\Omega^n}) = \Omega^m(\mathbf{A}_{\Omega^n, K})$  is induced by  $\psi$ . Then, for  $\varphi \in \mathrm{Hom}_{\mathbf{PST}_K}(\Omega^n, \Omega^m)$ , we get

$$\beta(r_K(\varphi)) = r_K(\varphi)_{\mathbf{A}_{\Omega^n}}(\omega_0) = \varphi_{\mathbf{A}_{\Omega^n, K}}(p^*\omega_0) = \varphi_{\mathbf{A}_{\Omega^n, K}}(\omega_{0, K}) = \alpha(\varphi),$$

which proves the commutativity of (6.26).

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INSTITUT FÜR MATHEMATIK, UNIVERSITÄT ZÜRICH, WINTERTHURERSTRASSE  
190, CH-8057 ZÜRICH, SWITZERLAND

*Email address:* [alberto.merici@math.uzh.ch](mailto:alberto.merici@math.uzh.ch)

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO,  
3-8-1 KOMABA, TOKYO 153-8941, JAPAN

*Email address:* [sshuji@msb.biglobe.ne.jp](mailto:sshuji@msb.biglobe.ne.jp)