Noether-Lefschetz locus for Beilinson-Hodge cycles on open complete intersections

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The infinitesimal method in Hodge theory is fruitful in various aspects of algebraic geometry. The idea originates from Griffiths work [Gri] where the Poincaré residue representation of cohomology of a hypersurface played a crucial role in proving the infinitesimal Torelli theorem for hypersurfaces. Since then many important applications of the idea have been made in different geometric problems such as the generic Torelli problem and the Noether-Lefschetz theorem for Hodge cycles and the study of algebraic cycles (see [G1], Lectures 7 and 8). In this lecture we explain that the method can be applied to study an analog of the Noether-Lefschetz theorem in the context of Beilinson's Hodge conjecture.

Let $X \subset \mathbb{P}^n$ be a smooth projective variety over $\mathbb{C}$. Recall that the Hodge conjecture for X predicts that the space of Hodge cycles in codimension q on X is generated by cohomology classes of algebraic subvarieties on X, namely the cycle class map from the the Chow group to the singular cohomology of X:

$$CH^q(X) \otimes \mathbb{Q} \to H^{2q}(X, \mathbb{Q}(q)) \cap F^q H^{2q}(X, \mathbb{C})$$

is surjective, where $\mathbb{Q}(q) = (2\pi\sqrt{-1})^q \mathbb{Q} \subset \mathbb{C}$ and F^* denotes the Hodge filtration. Let

$$H^{2q}(X, \mathbb{Q}(q))_{triv} \subset H^{2q}(X, \mathbb{Q}(q)) \cap F^q H^{2q}(X, \mathbb{C}),$$

be the (one-dimensional) $\mathbb{Q}$-subspace generated by the class of the section on X of a linear subspace of codimension q in $\mathbb{P}^n$. It is called the space of trivial cycles.

Now let S be a non-singular quasi-projective variety over $\mathbb{C}$ and assume that we are given $\mathcal{X} \hookrightarrow \mathbb{P}^n_S$, an algebraic family over S of smooth projective varieties. Let X_t be the fiber of $\mathcal{X}$ over $t \in S$. Then the Noether-Lefschetz locus for Hodge cycles in codimension q on $\mathcal{X}/S$ is defined to be

$$S^q_{NL} = \{t \in S |\ F^q H^{2q}(X_t, \mathbb{C}) \cap H^{2q}(X_t, \mathbb{Q}(q)) \neq H^{2q}(X_t, \mathbb{Q}(q))_{triv}\}.$$

It is the locus of such $t \in S$ that there exist non-trivial Hodge cycles in codimension q on X_t and hence that the Hodge conjecture is non-trivial for X_t. One can prove S^q_{NL} is the union of countable number of (not necessarily proper) closed algebraic subsets of S.

Now we take S to be the moduli space of smooth hypersurfaces of degree d in $\mathbb{P}^3$ and $\mathcal{X}/S$ to be the universal family of hypersurfaces. Let S_{NL} denote S^q_{NL} for $q = 1$. It is the locus of those surfaces that possess curves which are not complete intersections of the given surface with another surface. The celebrated theorem of Noether-Lefschetz affirms that every component of S_{NL} has positive codimension in S when $d \geq 4$. M. Green and C. Voisin have refined it to show the following striking theorem:

Theorem 0.1 *([G2], [G3], [V]) For every irreducible component T of S_{NL},* $\mathrm{codim}(T) \geq d-3$*. If $d \geq 5$, the only irreducible component of S_{NL} having codimension $d-3$ is the family of surfaces of degree d containing a line.*

Let U be a (non-complete) smooth variety over $\mathbb{C}$. Beilinson's Hodge conjecture for U predicts the surjectivity of the regulator maps:

$$reg_U^q \ : \ CH^q(U,q) \otimes \mathbb{Q} \to H^q(U, \mathbb{Q}(q)) \cap F^q H^q(U, \mathbb{C}),$$

where $CH^q(U,q)$ denotes Bloch's higher Chow group and $H^q(U,\mathbb{C})$ is endowed with the mixed Hodge structure defined by Deligne (see [J] for the definitions). Taking a smooth compactification $U \subset X$ with $Z = X \setminus U$, a simple normal crossing divisor on X, we have the following formula for the values of reg_U^q on decomposable elements in $CH^q(U,q)$;

$$reg_U^q(\{g_1, \ldots, g_q\}) = \frac{dg_1}{g_1} \wedge \cdots \wedge \frac{dg_q}{g_q} \in H^0(X, \Omega_X^q(\log Z)) = F^q H^q(U, \mathbb{C}),$$

where $\{g_1, \ldots, g_q\} \in CH^q(U,q)$ is the products of $g_j \in CH^1(U,1) = \Gamma(U, \mathcal{O}_{Zar}^*)$. Such elements of $CH^q(U,q)$ is called *decomposable* and we let

$$CH^q(U,q)_{dec} \subset CH^q(U,q) \otimes \mathbb{Q}$$

denote the subspace generated by decomposable elements. The description of reg_U^q on non-decomposable elements are rather complicate. We put

$$H^q(U, \mathbb{Q}(q))_{triv} := reg_U^q(CH^q(U,q)_{dec}) \subset H^q(U, \mathbb{Q}(q)) \cap F^q H^q(U, \mathbb{C}).$$

Now let S be a non-singular quasi-projective variety over $\mathbb{C}$ and assume given $\mathcal{U} \to S$, an algebraic family over S of non-complete smooth varieties. Let U_t be the fiber of $\mathcal{U}$ over $t \in S$. Then the Noether-Lefschetz locus for Beilinson-Hodge cycles on $\mathcal{U}/S$ is defined as

$$S_{NL}^q = \{t \in S |\ H^q(U_t, \mathbb{Q}(q)) \cap F^q H^q(U_t, \mathbb{C}) \neq H^q(U_t, \mathbb{Q}(q))_{triv}\}.$$

In this lecture we explain some results on Noether-Lefschetz locus for Beilinson-Hodge cycles that are analogous to 0.1 proved in [AS1] and [AS2].

References

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