CHOW GROUP OF 0-CYCLES WITH MODULUS AND HIGHER DIMENSIONAL CLASS FIELD THEORY

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Abstract. One of the main results of this paper is a proof of the rank one case of an existence conjecture on lisse $\overline{\mathbb{Q}}_\ell$-sheaves on a smooth variety U over a finite field due to Deligne and Drinfeld. The problem is translated into the language of higher dimensional class field theory over finite fields, which describes the abelian fundamental group of U by Chow groups of zero cycles with moduli. A key ingredient is the construction of a cycle theoretic avatar of refined Artin conductor in ramification theory originally studied by Kazuya Kato.

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Introduction

One of the main results of this paper is a proof of the rank one case of an existence conjecture on lisse $\overline{\mathbb{Q}}_\ell$-sheaves on a smooth variety U over a finite field suggested by Deligne [EK], which was motivated by work of Drinfeld [Dr], see also [D]. The conjecture says that a compatible system of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on the integral closed curves on U, satisfying a certain boundedness condition for ramification at infinity, should arise from a lisse sheaf on U. It thus reduces the understanding of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on U to that of lisse $\overline{\mathbb{Q}}_\ell$-sheaves on curves on U. A precise formulation is as follows.

Skeleton sheaves. Let U be a smooth variety over a finite field k. Choose a compactification $U \subset X$ with X normal and proper over k such that $X \setminus U$ is the support of an effective Cartier divisor on X. Consider a family $(V_Z)_Z$, where Z runs through all closed integral curves on U and where V_Z is a semi-simple lisse $\overline{\mathbb{Q}}_\ell$-sheaf on the normalization $\tilde{Z}$ of Z. We say that that the family $(V_Z)_Z$ is compatible if for two different curves Z_1, Z_2 the sheaves V_{Z_1} and V_{Z_2} become isomorphic up to semi-simplification after pullback to the finite scheme $(\tilde{Z}_1 \times_X \tilde{Z}_2)_{\mathrm{red}}$. Such compatible families are also called *skeleton sheaves* and have been studied by Deligne and Drinfeld, see [EK].

Let Z^N be the normalization of the closure of Z in X, which is by definition the smooth compactification of $\tilde{Z}$. Let $\psi_Z : Z^N \to X$ be the natural map and Z_∞ be the set of points $y \in Z^N$ such that $\psi_Z(y) \notin U$. We say that a skeleton sheaf $(V_Z)_Z$ has bounded ramification if there exists an effective Cartier divisor D on X with $|D| \subset X \setminus U$ and such that for all integral curves Z on U, the following inequality of Cartier divisors on Z^N holds:

$$\sum_{y \in Z_\infty} \mathrm{ar}_y(V_Z)[y] \;\leq\; \psi_Z^* D.$$

Here $\psi_Z^* D$ is the pullback of D by ψ_Z and $\mathrm{ar}_y(V_Z)$ is the local Artin conductor of V_Z at the point y, see [Se1].

Question I (Deligne). *For any skeleton sheaf $(V_Z)_Z$ with bounded ramification, does there exist a lisse $\overline{\mathbb{Q}}_\ell$-sheaf V on U such that for all curves Z on U, the restrictions of V to $\tilde{Z}$ become isomorphic to V_Z after semi-simplification?*

In this paper we prove the following:

Theorem II. *Question I has a positive answer in rank one and for* $\mathrm{ch}(k) \neq 2$.

Class field theory. Using classical class field theory for curves over finite fields, Theorem II is translated into the language of higher dimensional class field theory over finite fields, and follows from Theorem III explained below. Instead of the class group involving higher K-theory which was used in earlier work, see [KS1] for example, we use a relative Chow group of zero cycles with modulus.

The principal idea is to describe the abelian fundamental group $\pi_1^{\mathrm{ab}}(U)$ of U in terms of the Chow groups $\mathrm{C}(X,D)$ with modulus D, where D is an effective Cartier divisor with support $|D|$ in $X \setminus U$. We define

$$\mathrm{C}(X,D) = \mathrm{Coker}\Big(\bigoplus_{Z \subset U} k(Z)_D^\times \xrightarrow{\mathrm{div}_Z} Z_0(U) \Big),$$

where $Z_0(U)$ is the group of zero-cycles on U and Z ranges over the integral closed curves on U. Here $\mathrm{div}_Z : k(Z)^\times \to Z_0(U)$ is the divisor map on the function field $k(Z)$. The group $k(Z)_D^\times$ is the congruence subgroup of elements of $k(Z)^\times$ which are congruent to 1 modulo the ideal $I_D = \mathcal{O}_X(-D)$ at all infinite places of $k(Z)$.

We define

$$k(Z)_D^\times = \bigcap_{y \in Z_\infty} \mathrm{Ker}\big(\mathcal{O}_{Z^N,y}^\times \to (\mathcal{O}_{Z^N,y}/I_D\mathcal{O}_{Z^N,y})^\times \big) \;\subset k(Z)^\times,$$

where $\mathcal{O}_{Z^N,y}$ is the local ring of Z^N at y. Thus $\mathrm{C}(X,D)$ is an extension of the Chow group of zero-cycles of U which has been used repeatedly, see [ESV] [LW] [Ru]. It is also an extension of Suslin's singular homology $H_0^{sing}(U,\mathbb{Z})$, see [SV] and Remark 1.5 below. In case $\dim(X) = 1$ it is the class group with modulus D used in classical class field theory.

We then introduce our class group of U as

$$\mathrm{C}(U) := \varprojlim_D \mathrm{C}(X,D) \;,$$

where D runs through all effective Cartier divisors on X with $|D| \subset X \setminus U$ and endow it with the inverse limit topology where $\mathrm{C}(X,D)$ is endowed with the discrete topology. We show that the topological group $\mathrm{C}(U)$ is independent of the compactification X of U, and construct a continuous map of topological groups, called the reciprocity map,

$$\rho_U : \mathrm{C}(U) \to \pi_1^{\mathrm{ab}}(U),$$

such that its composite with the natural map $Z_0(U) \to \mathrm{C}(U)$ is induced by the Frobenius maps $Frob_x : \mathbb{Z} \to \pi_1^{\mathrm{ab}}(U)$ for closed points x of U. The reciprocity map induces a continuous map

$$\rho_U^0 : \mathrm{C}(U)^0 \to \pi_1^{\mathrm{ab}}(U)^0. \tag{0.1}$$

Here $\pi_1^{\mathrm{ab}}(U)^0 = \mathrm{Ker}\big(\pi_1^{\mathrm{ab}}(U) \to \pi_1^{\mathrm{ab}}(\mathrm{Spec}(k))\big)$ and $\mathrm{C}(U)^0 = \mathrm{Ker}(\mathrm{C}(U) \xrightarrow{\deg} \mathbb{Z})$, where deg is induced by the degree map $Z_0(U) \to \mathbb{Z}$. Now our main result, see also Theorem 3.3, is the following.

Theorem III (Existence Theorem)**.** *Assuming* $\mathrm{ch}(k) \neq 2$, ρ_U^0 *is an isomorphism of topological groups.*

In case $\dim(X) = 1$ the theorem is one of the main results in classical class field theory. In higher dimension a special case of Theorem III, describing only the tame quotient of $\pi_1^{\mathrm{ab}}(U)$, is shown in [SS] (see also [Wi] and [KeSc]).

In [KS1] an analog of Theorem III is shown with $\mathrm{C}(U)$ replaced by a different class group $\mathrm{C}^{KS}(U)$ explained below, which can be described in terms of higher local fields associated to chains of subvarieties on a compactification X of U. Recall $\mathrm{C}(U)$ is defined only in terms of points and curves on U. There is a factorization of the reciprocity map

$$\mathrm{C}(U) \to \mathrm{C}^{KS}(U) \to \pi_1^{\mathrm{ab}}(U) \tag{0.2}$$

and the main result of [KS1] over a finite field, see also [Ra, Thm. 6.2], is a direct consequence of our Theorem III if $\mathrm{ch}(k) \neq 2$.

Using ramification theory in local class field theory, Theorem III implies.

Corollary IV. *Assume* $\mathrm{ch}(k) \neq 2$. *For an effective divisor D on X with $|D| \subset X \setminus U$, ρ_U induces an isomorphism of finite groups*

$$\mathrm{C}(X, D)^0 \xrightarrow{\sim} \pi_1^{\mathrm{ab}}(X, D)^0,$$

where $\pi_1^{\mathrm{ab}}(X, D)$ is the quotient of $\pi_1^{\mathrm{ab}}(X)$ which classifies abelian étale coverings of U with ramification over $X \setminus U$ bounded by the divisor D.

The finiteness of $\mathrm{C}(X, D)^0$ is equivalent to the rank one case of Deligne's finiteness theorem (see [EK, Thm. 8.1]). Our arguments yield an alternative proof of this finiteness result.

Ramification theory. The Pontryagin dual $\mathrm{fil}_D H^1(U)$ of $\pi_1^{\mathrm{ab}}(X, D)$ is the group of continuous characters $\chi : \pi_1^{\mathrm{ab}}(U) \to \mathbb{Q}/\mathbb{Z}$ such that for any integral curve $Z \subset U$, its restriction $\chi|_Z : \pi_1^{\mathrm{ab}}(\tilde{Z}) \to \mathbb{Q}/\mathbb{Z}$ to the normalization $\tilde{Z}$ of Z satisfies the following inequality of Cartier divisors on Z^N, the smooth compactification of $\tilde{Z}$:

$$\sum_{y \in Z_\infty} \mathrm{ar}_y(\chi|_Z)[y] \;\leq\; \psi_Z^* D,$$

where $\mathrm{ar}_y(\chi|_Z) \in \mathbb{Z}_{\geq 0}$ is the Artin conductor of $\chi|_Z$ at $y \in Z_\infty$ and $\psi_Z^* D$ is the pullback of D by the natural map $\psi_Z : Z^N \to X$ (see Definition 2.9).

Our proof of Theorem III depends in an essential way on ramification theory due to Kato [Ka1] and its variant by Matsuda [Ma]. Let K_λ be the henselization of $K = k(U)$ at the generic point λ of an irreducible component C_λ of $X \setminus U$ and let $H^1(K_\lambda)$ be the group of continuous characters $G_{K_\lambda} \to \mathbb{Q}/\mathbb{Z}$, where G_{K_λ} is the absolute Galois group of K_λ. They introduced a ramification filtration $\mathrm{fil}_m H^1(K_\lambda)$

($m \in \mathbb{Z}_{\geq 0}$) on $H^1(K_\lambda)$ which generalizes the ramification filtration for local fields with perfect residue fields (see [Se1]), and defined a natural injective map

$$\mathrm{rar}_{K_\lambda} : \mathrm{fil}_m H^1(K_\lambda)/\mathrm{fil}_{m-1} H^1(K_\lambda) \hookrightarrow \Omega^1_X(mC_\lambda) \otimes_{\mathcal{O}_X} k(C_\lambda) \quad (m > 1) \tag{0.3}$$

which we call refined Artin conductor (indeed what Kato originally defined is refined Swan conductor and we use a variant for Artin conductor introduced by Matsuda), where $k(C_\lambda)$ is the function field of C_λ. In case $C = X \setminus U$ is a simple normal crossing divisor on smooth X, results from the ramification theory imply

$$\mathrm{fil}_D H^1(U) = \mathrm{Ker}\big(H^1(U) \to \bigoplus_{\lambda \in I} H^1(K_\lambda)/\mathrm{fil}_{m_\lambda} H^1(K_\lambda)\big).$$

Here $H^1(U)$ denotes the group of continuous characters $\chi : \pi_1^{\mathrm{ab}}(U) \to \mathbb{Q}/\mathbb{Z}$, I is the set of the generic points λ of C and m_λ is the multiplicity of C_λ in D.

Now the basic strategy of the proof of Theorem III is as follows (see §3 for the details). By an argument due to Wiesend we are allowed to replace X by an alteration $f : X' \to X$ and U by a smooth open $U' \subset f^{-1}(U)$. Then a Lefschetz theorem for $\pi_1^{\mathrm{ab}}(X, D)$ (cf. [KeS]) reduces the proof to the case where X is a smooth projective surface and $C = X \setminus U$ is a simple normal crossing divisor. The proof then proceeds by induction on the multiplicity of D reducing to the tame case $D = C$. A key point is the construction of a natural map, which we call the *cycle conductor*, defined for Cartier divisors D such that $D \geq 2C$:

$$\mathrm{cc}_{X,D} : \mathrm{C}(X, D)^\vee := \mathrm{Hom}(\mathrm{C}(X, D), \mathbb{Q}/\mathbb{Z}) \to H^0(C, \Omega^1_X(D + \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C), \tag{0.4}$$

where $\Xi \subset X$ is an auxiliary effective Cartier divisor independent of D such that it contains none of the irreducible components of C. It satisfies

$$\mathrm{Ker}(\mathrm{cc}_{X,D}) = \mathrm{C}(X, D - C)^\vee \subset \mathrm{C}(X, D)^\vee, \tag{0.5}$$

and the following diagram

$$\begin{array}{ccccc}
\mathrm{fil}_D H^1(U) & \longrightarrow & \mathrm{fil}_{m_\lambda} H^1(K_\lambda) & \xrightarrow{\mathrm{rar}_{K_\lambda}} & \Omega^1_X(D) \otimes_{\mathcal{O}_X} k(C_\lambda) \\
\downarrow{\scriptstyle \Psi_{X,D}} & & & & \uparrow \\
\mathrm{C}(X, D)^\vee & & \xrightarrow{\qquad \mathrm{cc}_{X,D} \qquad} & & H^0(C, \Omega^1_X(D + \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C)
\end{array}$$

commutes. Here $\Psi_{X,D}$ is the dual of the reciprocity map $\mathrm{C}(X, D) \to \pi_1^{\mathrm{ab}}(X, D)$ induced by ρ_U. Therefore we consider the cycle conductor $\mathrm{cc}_{X,D}$ as a cycle theoretic avatar of the refined Artin conductor of Kato and Matsuda.

By duality, the definition of cycle conductors is reduced to the construction of a natural map

$$\phi_{X,D} : H^1(C, \Omega^1_X(-D + C - \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \to \mathrm{C}(X, D) \tag{0.6}$$

such that the following sequence is exact

$$H^1(C, \Omega^1_X(-D + C - \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \xrightarrow{\phi_{X,D}} \mathrm{C}(X, D) \to \mathrm{C}(X, D - C) \to 0.$$

General base fields. Let now k be an arbitrary perfect field of characteristic $p > 0$ and let U be a smooth variety of dimension d over k as above. We pose the following question.

Question V. *Is the natural map*

$$\tau_U : \mathrm{C}(U) \to \mathrm{C}^{KS}(U)$$

to Kato–Saito class group over a general perfect field k an isomorphism?

Recall that the Kato–Saito class group is defined in terms of Nisnevich cohomology groups

$$\mathrm{C}^{KS}(U) = \varprojlim_{D} H^d(X_{\mathrm{Nis}}, \mathcal{K}_d^M(X,D)). \tag{0.7}$$

Here $\mathcal{K}_d^M(X,D)$ is the relative Milnor K-sheaf of [KS2] and D runs through all effective Cartier divisors on X with $|D| \subset X \setminus U$.

In case k is finite, the main result of [KS2] with (0.2) implies that the reciprocity map ρ_U^0 (cf. (0.1)) is an isomorphism if and only if τ_U is an isomorphism. One should think of Question V and our main theorem as a particular case of a Nisnevich descent and a motivic duality statement in a conjectural world of mixed motives with modulus which still has to be developed (see [KSY]). In view of this lack of a conceptional framework to approach the problem, our construction of the cycle conductor (0.4) has to be technical and ad hoc.

We hope that our technique may be used to approach Question I in the higher rank case by constructing a non-abelian version of the cycle conductor (0.4), where $\mathrm{C}(X,D)^\vee$ is replaced by the set of skeleton sheaves on X of a higher rank with ramification bounded by D and refined Artin conductor is replaced by its non-abelian version constructed in [TSa].

We give an overview of the content of the paper.

In §1 we introduce a class group $W(U)$ studied by Wiesend [Wi]. We define some filtrations on $W(U)$ and explain their basic properties. Our class group $\mathrm{C}(X,D)$ can be defined as a quotient of $W(U)$ by a certain filtration. We also introduce a tool to produce relations in $W(U)$.

In §2 we review some results on ramification theory. The first subsection treats local ramification theory for henselian discrete valuation fields whose residue fields are not necessarily perfect, originally due to Kato [Ka1], [Ka2] and [Ka3]. The refined Artin conductor, see (0.3), is introduced, which plays a key role in this paper. In the second subsection, some implications on global ramification theory are given.

In §3 the reciprocity map ρ_U is defined and the Existence Theorem is stated. The basic strategy of the proof of the Existence Theorem is explained. We explain an argument due to Wiesend, which allows us to replace X by an alteration. We reduce the proof to the case $\dim(X) = 2$ by using the Lefschetz theorem for abelian fundamental groups allowing ramification along some divisor, which is proved in [KeS].

In §4 we introduce the map $\phi_{X,D}$, see (0.6). It is the dual of the cycle conductor which is a key ingredient of the proof of the Existence Theorem. Two key theorems are stated (Theorem 4.1 and Theorem 4.4). The first theorem gives a characterization of $\phi_{X,D}$ by its local components which are defined for pairs (x, Z) where x is a regular closed point of C and $Z \subset X$ is an integral curve which intersects transversally with C at x. We also state a lemma (see Lemma 4.3) which implies the key property (0.5). The second theorem states a compatibility of $\phi_{X,D}$ with the refined Artin conductor. The proof of the Existence Theorem is completed in §5 using these key theorems.

In §6 we introduce a local component of $\phi_{X,D}$ which only depends on a regular closed point of C, but not on a curve as above. It is used in the proof of the first key Theorem 4.1 given in §7 as well as that of the second key Theorem 4.4 given in §8.

The proof of Theorem 4.1 depends on three technical key lemmas (Lemma 4.3, Lemma 6.5 and Lemma 7.12), which are restated in §9 over a general perfect field.

The proof of these lemma occupies the later sections §10 through §14. The tool to produce relations in $W(U)$ introduced in §1 will play a basic role in the proof.

There is work related to our main results by H. Russell involving a geometric method based on his joint work with K. Kato on Albanese varieties with modulus.

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1. Wiesend class group and filtrations

In the whole paper we fix a perfect field k with $\mathrm{ch}(k) = p > 0$. At many places we have to assume $p \neq 2$. Let X be a proper normal scheme over k and C be the support of an effective Cartier divisor on X and put $U = X \setminus C$. Note that C is a reduced closed subscheme of pure codimension one.

Definition 1.1. Let $Z_1(X)^+$ be the monoid of effective 1-cycles on X and

$$Z_1(X,C)^+ \subset Z_1(X)^+$$

be the submonoid of the cycles Z such that none of the prime components of Z is contained in C. Take

$$Z = \sum_{1 \le i \le r} n_i Z_i \ \in Z_1(X)^+,$$

where $Z_1, \ldots, Z_r$ are the prime components of Z and $n_i \in \mathbb{Z}_{\ge 0}$. We write

$$k(Z)^\times = k(Z_1)^\times \oplus \cdots \oplus k(Z_r)^\times,$$

$$|Z| = \bigcup_{1 \le i \le r} Z_i \ \subset X, \quad I_Z = \prod_{1 \le i \le r} (I_{Z_i})^{n_i} \ \subset \mathcal{O}_X,$$

where $I_{Z_i} \subset \mathcal{O}_X$ is the ideal of Z_i. We say Z is reduced if $n_i = 1$ for all $1 \le i \le r$ and integral if it is reduced and $r = 1$. We say Z intersects C transversally at $x \in X$ (denoted by $Z \Cap C$ at x) if $|Z|$ and C are regular at x and the intersection multiplicity

$$(Z,C)_x := \mathrm{length}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/I_Z + I_C) = 1.$$

We say Z intersects C transversally (denoted by $Z \Cap C$) if $Z \Cap C$ at all $x \in Z \cap C$.

Definition 1.2. For $Z \in Z_1(X,C)^+$, let $\psi_Z : Z^N \to |Z|$ be the normalization and put

$$Z_\infty = \{ y \in Z^N \mid \psi_Z(x) \in |Z| \cap C \}.$$

For $y \in Z_\infty$, let $k(Z)_y$ be the henselization of $k(Z)$ at y and put

$$k(Z)_\infty = \prod_{y \in Z_\infty} k(Z)_y \quad \text{and} \quad k(Z)_\infty^\times = \prod_{y \in Z_\infty} k(Z)_y^\times.$$

The Wiesend class group of U is defined as

$$W(U) = \mathrm{Coker}\Big(\bigoplus_{Z \subset X} k(Z)^\times \xrightarrow{\delta} \bigoplus_{Z \subset X} k(Z)_\infty^\times \ \oplus \ Z_0(U) \Big), \tag{1.1}$$

where Z ranges over the integral elements of $Z_1(X,C)^+$ [Wi], [KeSc]. Here we map $k(Z)^\times$ diagonally in $k(Z)^\times_\infty$ and $k(Z)^\times \to Z_0(U)$ is the composite map

$$\kappa(Z)^\times \xrightarrow{\mathrm{div}_{Z\cap U}} Z_0(Z\cap U) \hookrightarrow Z_0(U).$$

Obviously $W(U)$ depends only on U, i.e. is independent of (X,C) such that $U = X \setminus C$.

For a morphism $f : U' \to U$ of smooth varieties there is a canonical induced morphism

$$f_* : W(U') \to W(U), \tag{1.2}$$

see [Wi] and [KeSc, Sec. 7]. If f is finite we also speak of the norm map and write N_f for f_*.

Definition 1.3. Let $Z \in Z_1(X,C)^+$.

(1) Assume Z integral. For $x \in Z \cap C$, we have the natural map

$$\{\ \}_{Z,x} : \bigoplus_{y\in\psi_Z^{-1}(x)} k(Z)^\times_y \to W(U)$$

where $\psi_Z : Z^N \to Z$ is the normalization. Taking the sum of these maps for $x \in Z\cap C$, we get

$$\{\ \}_Z : k(Z)^\times_\infty \to W(U).$$

(2) In general we write $Z = \sum_{i\in I} e_i Z_i$ where $\{Z_i\}_{i\in I}$ are the prime components of Z and $e_i \in \mathbb{Z}_{\geq 0}$, and define

$$\{\ \}_{Z,x} = \sum_{i\in I} e_i\{\ \}_{Z_i,x}\ ;\ \bigoplus_{y\in\psi_Z^{-1}(x)} k(Z)^\times_y \to W(U),$$

$$\{\ \}_Z = \sum_{i\in I} e_i\{\ \}_{Z_i}\ ;\ k(Z)^\times_\infty \to W(U).$$

where $\psi_Z : Z^N \to |Z|$ is the normalization.

Let the notation be as in Definition 1.3 and $Z \in Z_1(X,C)^+$. Write $\mathcal{O}_Z = \mathcal{O}_X/I_Z$ and $\mathcal{O}^h_{Z,x}$ for the henselization of $\mathcal{O}_{Z,x}$ for $x \in |Z|$. We also write

$$\mathcal{O}^h_{Z,C\cap Z} = \prod_{x\in Z\cap C} \mathcal{O}^h_{Z,x}, \quad \mathcal{O}^h_{Z^N,C\cap Z} = \mathcal{O}^h_{Z,C\cap Z}\otimes_{\mathcal{O}_Z}\mathcal{O}_{Z^N} = \prod_{y\in\psi_Z^{-1}(Z\cap C)} \mathcal{O}^h_{Z^N,y}.$$

We have the natural maps

$$\mathcal{O}^h_{Z,C\cap Z} \to \mathcal{O}^h_{Z^N,C\cap Z} \hookrightarrow k(Z)_\infty$$

Definition 1.4. Let D be an effective Cartier divisor such that $|D| = C$ and let $I_D = \mathcal{O}_X(-D)$ be the ideal sheaf of D.

(1) We define $F^{(D)}W(X,C) \subset W(U)$ as the subgroup generated by

$$\{1 + I_D\mathcal{O}^h_{Z,C\cap Z}\}_Z$$

for all $Z \in Z_1(X,C)^+$.

(2) We define $\widehat{F}^{(D)}W(X,C) \subset W(U)$ as the subgroup generated by

$$\{1 + I_D\mathcal{O}^h_{Z^N,C\cap Z}\}_Z$$

for all $Z \in Z_1(X,C)^+$. Note $F^{(D)}W(X,C) \subset \widehat{F}^{(D)}W(X,C)$.

(3) We define $\widehat{F}^{(1)}W(U) \subset W(U)$ as the subgroup generated by
$$\{1+\mathfrak{m}\mathcal{O}^h_{Z^N,C\cap Z}\}_Z$$
for all $Z \in Z_1(X,C)^+$, where $\mathfrak{m}$ is the Jacobson radical of $\mathcal{O}^h_{Z^N,C\cap Z}$. Note that $\widehat{F}^{(1)}W(U)$ depends only on U and $\widehat{F}^{(D)}W(X,C) \subset \widehat{F}^{(1)}W(U)$ if $|D| = C$ and $\widehat{F}^{(1)}W(U)/F^{(D)}W(X,C)$ is p-primary torsion.

(4) For a dense open subset $V \subset X$ containing the generic points of C, we define
$$F^{(D)}_{\Cap V}W(X,C) = \sum_G \{1+\mathcal{O}^h_{G,G\cap C}(-D)\}_G \subset F^{(D)}W(X,C),$$
where G ranges over $Z_1(X,C)^+$ such that $G \Cap C$ and $G \cap C \subset V$. In case $V = X$ we simply denote $F^{(D)}_{\Cap V}W(X,C) = F^{(D)}_{\Cap}W(X,C)$.

Remark 1.5. It is shown in [Sc, Thm 3.1] that there is a natural isomorphism
$$W(U)/\widehat{F}^{(1)}W(U) \simeq H_0^{sing}(U,\mathbb{Z}),$$
where the right hand side is Suslin's singular homology.

Definition 1.6. Under the notation of Definition 1.4, we put
$$\mathrm{C}(X,D) = W(U)/\widehat{F}^{(D)}W(X,C).$$
By the weak approximation theorem, we have an isomorphism
$$\mathrm{C}(X,D) \simeq \mathrm{Coker}\Big(\bigoplus_{Z\subset X} k(Z)^\times_D \to Z_0(U)\Big),$$
where Z ranges over the integral elements of $Z_1(X,C)^+$, and
$$\begin{aligned} k(Z)^\times \supset k(Z)^\times_D &= \mathrm{Ker}\big(k(Z)^\times \to \prod_{y\in Z_\infty} k(Z)^\times_y/1+I_D\mathcal{O}_{Z^N,y}\big) \\ &= \bigcap_{y\in Z_\infty} \mathrm{Ker}\big(\mathcal{O}^\times_{Z^N,y} \to (\mathcal{O}_{Z^N,y}/I_D\mathcal{O}_{Z^N,y})^\times\big) . \end{aligned}$$
Thus $\mathrm{C}(X,D)$ is an extension of the Chow group of zero-cycles of U.

Lemma 1.7. *Let $f : X' \to X$ be a morphism with $f(X') \cap U \neq \varnothing$ and let D be an effective Cartier divisor on X with $|D| \subset C$. Set $U' = f^{-1}(U)$ and $D' = f^*D$. Then the pushforward* (1.2) *satisfies $f_*(\widehat{F}^{(D')}W(X',C')) \subset \widehat{F}^{(D)}W(X,C)$.*

Proof. This is a direct consequence of the definition and standard properties of the norm map for local fields. □

In what follows we assume $\dim(X) = 2$.

Definition 1.8. Let X be a projective smooth surface over k. Let $\mathrm{Div}(X)^+$ be the monoid of effective Cartier divisors on X and $\mathrm{Div}(X,C)^+ \subset \mathrm{Div}(X)^+$ be the submonoid of such Cartier divisors D that none of the prime components of D is contained in C. Then $Z_1(X)^+$ coincides with $\mathrm{Div}(X)^+$ and $Z_1(X,C)^+$ coincides with $\mathrm{Div}(X,C)^+$.

Definition 1.9. Let $\mathcal{C}$ be the category of triples (X,C), where

- X is a projective smooth surface over k,
- C is a reduced Cartier divisor on X.

A morphism $f : (X',C') \to (X,C)$ in $\mathcal{C}$ is a surjective map $f : X' \to X$ of schemes such that $C' = f^{-1}(C)_{\mathrm{red}}$. For f as above and for $D \in \mathrm{Div}(X)^+$, we let $f^*D \in \mathrm{Div}(X)^+$ be the pullback of D as a Cartier divisor. Let $\mathcal{C}_X \subset \mathcal{C}$ be the category of the objects and the morphisms in $\mathcal{C}$ over X.

Definition 1.10. Let $X = (X, C)$ be in $\mathcal{C}$.

(1) Let $\widehat{\mathcal{B}}_X \subset \mathcal{C}_X$ be the subcategory of the object $(\tilde{X}, \tilde{C})$, where $g : \tilde{X} \to X$ is the composite of successive blowups at closed points in the preimages of C.

(2) Let $\mathcal{B}_X \subset \widehat{\mathcal{B}}_X$ be the subcategory of the object $(\tilde{X}, \tilde{C})$, where $g : \tilde{X} \to X$ is the composite of successive blowups at closed points of regular loci of preimages of C.

Lemma 1.11. *Let $X = (X, C)$ be in $\mathcal{C}$ and $D \in \mathrm{Div}(X)^+$ such that $|D| = C$. For $g : (\tilde{X}, \tilde{C}) \to (X, C)$ in $\widehat{\mathcal{B}}_X$, we have*

$$F^{(D)}W(X, C) \subset F^{(g^*D)}W(\tilde{X}, \tilde{C}) \subset \widehat{F}^{(D)}W(X, C).$$

We have

$$\widehat{F}^{(D)}W(X, C) = \varinjlim_{g:\tilde{X}\to X} F^{(g^*D)}W(\tilde{X}, \tilde{C}),$$

where $g : (\tilde{X}, \tilde{C}) \to (X, C)$ ranges over $\widehat{\mathcal{B}}_X$.

Proof. Take integral $Z \in \mathrm{Div}(X, C)^+$ and let $Z' \in \mathrm{Div}(\tilde{X}, \tilde{C})^+$ be its proper transform. Then Z' is finite over Z and we have

$$\mathcal{O}^h_{Z,C\cap Z} \subset \mathcal{O}^h_{Z',\tilde{C}\cap Z'} \subset \mathcal{O}^h_{Z^N,C\cap Z}.$$

The first assertion follows from these facts. The second assertion follows from the fact that for any integral $Z \in \mathrm{Div}(X, C)^+$, there is $g : \tilde{X} \to X$ in $\widehat{\mathcal{B}}_X$ such that the proper transform of Z in $\tilde{X}$ is regular (see [SaSa, Appendix Th.A.1]). □

Definition 1.12. For $X = (X, C)$ and D as in Lemma 1.11, we put

$$F^{(D)}_{\mathcal{B}}W(X, C) = \varinjlim_{g:\tilde{X}\to X} F^{(g^*D)}W(\tilde{X}, \tilde{C}),$$

where $g : (\tilde{X}, \tilde{C}) \to (X, C)$ ranges over $\mathcal{B}_X$. Lemma 1.11 implies

$$F^{(D)}W(X, C) \subset F^{(D)}_{\mathcal{B}}W(X, C) \subset \widehat{F}^{(D)}W(X, C).$$

For $g : (\tilde{X}, \tilde{C}) \to (X, C)$ in $\mathcal{B}_X$, we have

$$F^{(g^*D)}_{\mathcal{B}}W(\tilde{X}, \tilde{C}) = F^{(D)}_{\mathcal{B}}W(X, C). \tag{1.3}$$

Remark 1.13. For $Z \in \mathrm{Div}(X, C)^+$, we have an isomorphism

$$\mathcal{O}_{Z,C\cap Z} \otimes_{\mathcal{O}_X} \mathcal{O}_C \simeq \prod_{x\in Z\cap C} \mathcal{O}^h_{Z,x} \otimes_{\mathcal{O}_X} \mathcal{O}_C.$$

For $D, D' \in \mathrm{Div}(X)^+$ with $D' \geq D$ and $|D| = |D'| = C$, we have isomorphisms

$$\frac{1 + I_D\mathcal{O}_{Z,x}}{1 + I_{D'}\mathcal{O}_{Z,x}} \simeq \frac{1 + I_D\mathcal{O}^h_{Z,x}}{1 + I_{D'}\mathcal{O}^h_{Z,x}}, \quad \frac{1 + I_D\mathcal{O}_{Z,Z\cap C}}{1 + I_{D'}\mathcal{O}_{Z,Z\cap C}} \simeq \bigoplus_{x\in Z\cap C} \frac{1 + I_D\mathcal{O}^h_{Z,x}}{1 + I_{D'}\mathcal{O}^h_{Z,x}},$$

and we have

$$\{1 + I_D\mathcal{O}^h_{Z,x}\}_{Z,x} \subset \{1 + I_D\mathcal{O}_{Z,x}\}_{Z,x} + F^{(D')}W(X, C),$$

$$\sum_{x\in Z\cap C} \{1 + I_D\mathcal{O}^h_{Z,x}\}_{Z,x} \subset \{1 + I_D\mathcal{O}_{Z,C\cap Z}\}_Z + F^{(D')}W(X, C).$$

Let $X = (X, C)$ be in $\mathcal{C}$. We introduce a tool to produce relations in $W(U)$ by using symbols in the Milnor K-group $K^M_2(k(X))$ of the function field $k(X)$ of X.

Lemma 1.14. *Let $a, b \in k(X)^\times$ and assume we can write as divisors*

$$\mathrm{div}_X(a) = Z_a^+ - Z_a^- + W_a, \quad \mathrm{div}_X(b) = Z_b^+ - Z_b^- + W_b,$$

where $Z_a^+, Z_a^-, Z_b^+, Z_b^- \in \mathrm{Div}(X, C)^+$ such that any common component of $Z_a^+ \cup Z_a^-$ and $Z_b^+ \cup Z_b^-$ does not intersect with C, and W_a, W_b have support in C. Then

$$\partial\{a, b\} := \{a\}_{Z_b^+} - \{a\}_{Z_b^-} - \{b\}_{Z_a^+} + \{b\}_{Z_a^-},$$

vanishes in $W(U)$. Here the first term $\{a\}_{Z_b^+}$ denotes $\{a_{|Z_b^+}\}_{Z_b^+}$ where $a_{|Z_b^+}$ is the image of a in $k(Z_b^+)^\times_\infty$, which is well-defined. The other terms are defined similarly.

Proof. We have a decomposition in $\mathrm{Div}(X, C)^+$:

$$Z_a^+ = \Phi_a^+ + E_a^+, \;\; Z_a^- = \Phi_a^- + E_a^-, \;\; Z_b^+ = \Phi_b^+ + E_b^+, \;\; Z_b^- = \Phi_b^- + E_b^-,$$

where $E_a^\pm$ and $E_b^\pm$ do not intersect C, and every irreducible component of $\Phi_a^\pm$ and $\Phi_b^\pm$ intersects C. The assumption implies that a (resp. b) is invertible at any generic point of $\Phi_b^\pm$ (resp. $\Phi_a^\pm$). Since $k(Z_a^\pm)^\times_\infty = k(\Phi_a^\pm)^\times_\infty$ and $k(Z_b^\pm)^\times_\infty = k(\Phi_b^\pm)^\times_\infty$, this implies that

$$a_{|Z_b^\pm} \in k(Z_b^\pm)^\times_\infty \quad \text{and} \quad b_{|Z_a^\pm} \in k(Z_a^\pm)^\times_\infty$$

are well-defined.

For $Z \in \mathrm{Div}(X, C)^+$ and $a \in k(X)^\times$ which is invertible at any generic point of Z, we write

$$a_{|Z} = \big((a_{|Z_i})^{e_i}\big)_{1 \le i \le r} \in k(Z)^\times = \prod_{1 \le i \le r} k(Z_i)^\times,$$

where $Z_1, \dots, Z_r$ are the irreducible components of Z and e_i is the multiplicity of Z_i in Z (see Definition 1.1). Put

$$\alpha^+ = a_{|\Phi_b^+} \in k(\Phi_b^+)^\times, \;\; \alpha^- = a_{|\Phi_b^-} \in k(\Phi_b^-)^\times,$$
$$\beta^+ = b_{|\Phi_a^+} \in k(\Phi_a^+)^\times, \;\; \beta^- = b_{|\Phi_a^-} \in k(\Phi_a^-)^\times.$$

For an integral curve $E \subset X$ such that $E \cap C = \varnothing$, let $\gamma_E \in k(E)^\times$ be the image $\{a, b\}$ of the tame symbol

$$\partial_E : K_2(k(X)) \to k(E)^\times.$$

Obviously, the elements

$$\delta(\alpha^+), \delta(\alpha^-), \delta(\beta^+), \delta(\beta^-), \delta(\gamma_E)$$

map to zero in $W(U)$, with δ as in (1.1). In order to finish the proof of the lemma it suffices to show the equality:

$$\partial\{a, b\} = \delta(\alpha^+) - \delta(\alpha^-) - \delta(\beta^+) + \delta(\beta^-) + \sum_E \delta(\gamma_E) \in \bigoplus_{Z \subset X} k(Z)^\times_\infty \, \oplus \, Z_0(U),$$

where the last sum ranges over the irreducible components E of $E_a^\pm \cup E_b^\pm$. This follows from the fact that the contributions of the right hand side at any closed point $x \in U$ cancel out as a consequence of the Gersten complex for K-theory

$$K_2(k(X)) \xrightarrow{\partial_y} \bigoplus_{y \in \mathrm{Spec}(\mathcal{O}_{U,x})^{(1)}} K_1(y) \to K_0(x) = \mathbb{Z}.$$

□

Lemma 1.15. *Fix $D \in \mathrm{Div}(X)^+$ with $|D| = C$, and $F \in \mathrm{Div}(X, C)^+$ and $a \in H^0(X, \mathcal{O}_X(-D + F))$ with $a \neq 0$. Define $Z \in \mathrm{Div}(X)^+$ by*

$$Z = \mathrm{div}_X(1 + a) + F.$$

(1) *We have $Z \cap C = F \cap C$.*

(2) *Let $b \in k(X)^\times$ and assume we can write as divisors*

$$\mathrm{div}_X(b) = F_1 - F_2 + W,$$

where W has support in C, and $F_1, F_2 \in \mathrm{Div}(X,C)^+$ such that $Z \cup F$ and $F_1 \cup F_2$ have no common irreducible component which passes through $F \cap C$. Then we have

$$\{1+a\}_{F_1} - \{1+a\}_{F_2} - \{b\}_Z + \{b\}_F = 0 \ \in W(U).$$

(3) *In (2) assume further $F_i = Z_i + G_i$ for $i = 1, 2$, where $Z_i, G_i \in \mathrm{Div}(X,C)^+$ such that $G_i \cap F \cap C = \varnothing$. Then we have*

$$\{1+a\}_{Z_1} - \{1+a\}_{Z_2} - \{b\}_Z + \{b\}_F \ \in F^{(D)}W(X,C).$$

(4) *Let $b \in k(X)^\times$ be such that $b = u\pi^n$, where $u \in \mathcal{O}^\times_{X,F\cap C}$ and $\pi \in \mathcal{O}_{X,F\cap C}$ is a local equation of C around $F \cap C$ and $n \in \mathbb{Z}$. Then we have*

$$\{b\}_Z - \{b\}_F \ \in F^{(D)}W(X,C).$$

Proof. For $x \in C\backslash F$, $1+a$ is regular in a neighborhood of x and its restriction to C is 1. Hence $x \notin Z$. For $x \in F \cap C$, we can write $a = u\pi/f$, where $u \in \mathcal{O}_{X,x}$ and f (resp. π) is a local equation of F (resp. D) at x. Then $f(1+a) = f + u\pi$ is a local equation of Z which vanishes at x since f and π do. So $x \in Z$, which proves (1).

By (1), the assumption of (2) implies that any common component of $Z \cup F$ and $F_1 \cup F_2$ does not intersect C. Hence (2) follows from Lemma 1.14.

As for (3) note $a|_{G_i} \in \mathcal{O}_{G_i, G_i\cap C}(-D)$ since $a \in H^0(X, \mathcal{O}_X(-D+F))$ and $G_i \cap F \cap C = \varnothing$. This implies $\{1+a\}_{G_i} \in F^{(D)}W(X,C)$ and (3) follows from (2).

The assumption of (4) implies $\mathrm{div}_X(b) = G_1 - G_2 + W$, where W has support in C, and $G_1, G_2 \in \mathrm{Div}(X,C)^+$ such that $(G_1 \cup G_2) \cap F \cap C = \varnothing$. Thus (4) follows from (3). □

2. Review of ramification theory

2.1. **Local ramification theory.** In this subsection K denotes a henselian discrete valuation field of $\mathrm{ch}(K) = p > 0$ with ring of integers $\mathcal{O}_K$ and residue field E. Let π be a prime element of $\mathcal{O}_K$ and $\mathfrak{m}_K = (\pi) \subset \mathcal{O}_K$ be the maximal ideal. By the Artin–Schreier–Witt theory, we have a natural isomorphism for $s \in \mathbb{Z}_{\geq 1}$,

$$\delta_s : W_s(K)/(1-F)W_s(K) \xrightarrow{\cong} H^1(K, \mathbb{Z}/p^s\mathbb{Z}), \tag{2.1}$$

where $W_s(K)$ is the ring of Witt vectors of length s and F is the Frobenius. We have the Brylinski–Kato filtration

$$\mathrm{fil}^{\log}_m W_s(K) = \{(a_{s-1}, \ldots, a_1, a_0) \in W_s(K) \mid p^i v_K(a_i) \geq -m\},$$

where v_K is the normalized valuation of K. In this paper we use its non-log version introduced by Matsuda [Ma]:

$$\mathrm{fil}_m W_s(K) = \mathrm{fil}^{\log}_{m-1} W_s(K) + V^{s-s'}\mathrm{fil}^{\log}_m W_{s'}(K),$$

where $s' = \min\{s, \mathrm{ord}_p(m)\}$ and $V : W_{s-1}(K) \to W_s(K)$ is the Verschiebung. We define ramification filtrations on $H^1(K) := H^1(K, \mathbb{Q}/\mathbb{Z})$ as

$$\mathrm{fil}^{\log}_m H^1(K) = H^1(K)\{p'\} \oplus \bigcup_{s\geq 1} \delta_s(\mathrm{fil}^{\log}_m W_s(K)) \qquad (m \geq 0),$$

$$\mathrm{fil}_m H^1(K) = H^1(K)\{p'\} \oplus \bigcup_{s\geq 1} \delta_s(\mathrm{fil}_m W_s(K)) \qquad (m \geq 1),$$

where $H^1(K)\{p'\}$ is the prime-to-p part of $H^1(K)$. We note that $\mathrm{fil}_m H^1(K)$ is shifted by one from Matsuda's filtration [Ma, Def.3.1.1]. We also let $\mathrm{fil}_0 H^1(K)$ be the subgroup of all unramified Galois characters.

Definition 2.1. For $\chi \in H^1(K)$ we denote the minimal m with $\chi \in \mathrm{fil}_m H^1(K)$ by $\mathrm{ar}_K(\chi)$ and call it the Artin conductor of χ.

In case the field E is perfect this definition coincides with the classical definition, see [Ka2, Prop. 6.8].

We have the following fact (cf. [Ka2] and [Ma]).

Lemma 2.2. *(1) $\mathrm{fil}_1 H^1(K)$ is the subgroup of tamely ramified characters.*

(2) $\mathrm{fil}_m H^1(K) \subset \mathrm{fil}_m^{\log} H^1(K) \subset \mathrm{fil}_{m+1} H^1(K)$.

(3) $\mathrm{fil}_m H^1(K) = \mathrm{fil}_{m-1}^{\log} H^1(K)$ if $(m,p)=1$.

The structure of graded quotients:

$$\mathrm{gr}_m H^1(K) = \mathrm{fil}_m H^1(K)/\mathrm{fil}_{m-1} H^1(K) \quad (m > 1)$$

are described as follows. Let Ω^1_K be the absolute Kähler differential module and put

$$\mathrm{fil}_m \Omega^1_K = \mathfrak{m}_K^{-m} \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K}.$$

We have an isomorphism

$$\mathrm{gr}_m \Omega^1_K = \mathrm{fil}_m \Omega^1_K / \mathrm{fil}_{m-1} \Omega^1_K \simeq \mathfrak{m}_K^{-m} \Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} E. \tag{2.2}$$

We have the maps

$$F^s d : W_s(K) \to \Omega^1_K \ ; \ (a_{s-1}, \dots, a_1, a_0) \to \sum_{i=0}^{s-1} a_i^{p^i - 1} da_i. \tag{2.3}$$

and one can check $F^s d(\mathrm{fil}_m W_s(K)) \subset \mathrm{fil}_m \Omega^1_K$.

Theorem 2.3. *([Ma, 3.2.3]) Assume $p \neq 2$ and $m > 1$.*

(1) The maps $F^s d$ induces an injective map

$$\mathrm{rar}_K : \mathrm{gr}_m H^1(K) \hookrightarrow \mathrm{gr}_m \Omega^1_K. \tag{2.4}$$

(2) If the residue field of K is perfect the map (2.4) is surjective.

The map rar_K is called the refined Artin conductor for K.

Definition 2.4. Let K be as before and $K_N^M(K)$ be the N-th Milnor K-group of K. For an integer $m \geq 1$, we define $V^m K_N^M(K) \subset K_N^M(K)$ as a subgroup generated by the elements of the form

$$\{1 + a, b_1, \dots, b_{N-1}\} \quad \text{and} \quad \{1 + a\pi, b_1, \dots, b_{N-2}, \pi\},$$

where $a \in \mathfrak{m}_K^m$ and $b_1, \dots, b_N \in \mathcal{O}_K^\times$.

The following lemma is proved by a similar argument as the proof of [BK, Lem.(4.2)].

Lemma 2.5. *Assume $\mathrm{ch}(E) \neq 2$. There is a canonical surjective map*

$$\rho_K^m : \mathfrak{m}_K^{m-1} \Omega^{N-1}_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} E \to V^{m-1} K_N^M(K)/V^m K_N^M(K),$$

such that

$$\rho_K^m(a db_1 \wedge \cdots \wedge db_{N-1}) = \{1 + ab_1 \cdots b_{N-1}, b_1, \dots, b_{N-1}\}.$$

where $a \in \mathfrak{m}_K^{m-1}$ and $b_1, \dots, b_N \in \mathcal{O}_K$.

Let K be an N-dimensional local field, namely there is a sequence of fields $k_0, \dots, k_N$ such that k_0 is finite, $k_N = K$, and for $1 \leq i \leq N$, k_i is a henselian discrete valuation field with residue field k_{i-1}. In [Ka1] Kato defined the so-called reciprocity map for K:

$$\Psi_K : H^1(K) \to \mathrm{Hom}(K_N^M(K), \mathbb{Q}/\mathbb{Z}). \tag{2.5}$$

Lemma 2.6. *Assume $\mathrm{ch}(K) = p > 0$ with $p \neq 2$.*

(*i*) *For* $m \in \mathbb{Z}_{\geq 1}$ *and* $\chi \in H^1(K)$*, we have an equivalence of conditions:*

$$\chi \in \mathrm{fil}_m H^1(K) \iff \Psi_K(\chi)(V^m K_N^M(K)) = 0.$$

(*ii*) *The following diagram is commutative*

$$\begin{array}{ccc} \mathrm{fil}_m H^1(K) & \xrightarrow{\Psi_K} & \mathrm{Hom}(K_N^M(K)/V^m K_N^M(K), \mathbb{Q}/\mathbb{Z}) \\ {\scriptstyle -\mathrm{rar}_K}\downarrow & & \downarrow{\scriptstyle (\rho_K^m)^\vee} \\ \mathfrak{m}_K^{-m}\Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} E & \xrightarrow{\sigma} & \mathrm{Hom}(\mathfrak{m}_K^{m-1}\Omega^{N-1}_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} E, \mathbb{Q}/\mathbb{Z}) \end{array} \tag{2.6}$$

where the right vertical map is induced by ρ_K^m *and* σ *is induced by the pairing*

$$\langle\ ,\ \rangle_\Omega : \mathfrak{m}_K^{-m}\Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} E \times \mathfrak{m}_K^{m-1}\Omega^{N-1}_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} E \to \mathfrak{m}_K^{-1}\Omega^N_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} E \to \mathbb{F}_p \simeq \mathbb{Z}/p\mathbb{Z},$$

where the last map is induced by $Res^\Omega_{K/\mathbb{F}_p} : \Omega^N_K \to \mathbb{F}_p$*, which is the composite of the residue map* $Res^\Omega_{K/k_0} : \Omega^N_K \to k_0$ *from* [Ka1, §2 Prop.3] *and the trace map* $k_0 \to \mathbb{F}_p$*.*

A variant of (*i*) and (*ii*) for $\mathrm{fil}^{\log}_m H^1(K)$ is stated in [Ka3, §3.5]. We will sketch a proof of the lemma in the appendix §15.

2.2. **Global ramification theory.** Let X be a normal variety over a perfect field k. Let $U \subset X$ be an open subscheme which is smooth over k and whose reduced complement $C \subset X$ is the support of an effective Cartier divisor. Our aim in this section is to introduce the abelian fundamental group $\pi_1^{\mathrm{ab}}(X, D)$ classifying abelian étale coverings of U with ramification bounded by D. Here $D \in \mathrm{Div}(X)^+$ is an effective divisor with support in C.

Let I be the set of generic points of C and $C_\lambda = \overline{\{\lambda\}}$ for $\lambda \in I$. For $\lambda \in I$ let K_λ be the henselization of $K = k(X)$ at λ. Note that K_λ is a henselian discrete valuation field with residue field $k(C_\lambda)$. We write $H^1(U)$ for the étale cohomology group $H^1(U, \mathbb{Q}/\mathbb{Z})$.

Proposition 2.7.

(1) *Assume* C *is regular at a closed point* x *and* $x \in C_\lambda$ *for* $\lambda \in I$*. Let* $F \in Z_1(X, C)^+$ *be such that* $F \pitchfork C$ *at* x *and let* $k(F)_x$ *be the henselization of* $k(F)$ *at* x*. Take* $\chi \in H^1(U)$ *and let* $\chi|_{K_\lambda} \in H^1(K_\lambda)$ *and* $\chi|_{F,x} \in H^1(k(F)_x)$ *be its restrictions. For an integer* $m \geq 0$*, we have an implication:*

$$\chi|_{K_\lambda} \in \mathrm{fil}_m H^1(K_\lambda) \implies \chi|_{F,x} \in \mathrm{fil}_m H^1(k(F)_x).$$

(2) *Assume* $C = C_\lambda$ *is regular and irreducible. Let* T_X *be the tangent sheaf of* X*. There is a dense open subset* $V_\chi \subset \mathbb{P}(T_X|_{C_\lambda})$ *(depending on* χ*) such that for any integral* $F \in Z_1(X, C)^+$ *and for any* x *with* $F \pitchfork C$ *at* x *the implication*

$$T_F(x) \in V_\chi \implies \mathrm{ar}_{K_\lambda}(\chi|_{K_\lambda}) = \mathrm{ar}_{k(F)_x}(\chi|_{F,x})$$

holds.

(3) *Assume* C *is a simple normal crossing divisor in a neighborhood of a closed point* $x \in C$*. Let* $g : X' = Bl_x(X) \to X$ *be the blowup at* x *and* $E \subset X'$ *be the exceptional divisor and* K_E *be the henselization of* K *at its generic point. For a Cartier divisor* D *supported on* C *we put*

$$m_E = \sum_{\lambda \in I_x} m_\lambda(D),$$

where I_x be the set of irreducible components of C containing x and $m_\lambda(D)$ is the multiplicity of D at λ. Then, for

$$\chi \in \mathrm{Ker}\big(H^1(U) \to \bigoplus_{\lambda \in I_x} H^1(K_\lambda)/\mathrm{fil}_{m_\lambda(D)} H^1(K_\lambda)\big),$$

we have $\chi|_{K_E} \in \mathrm{fil}_{m_E} H^1(K_E)$.

Proof. (1) and (2) follow from [Ma, (7.2.1)]. (3) is proved by the same argument as [Ka2, Th.(8.1)] using [Ma, Cor.4.2.2] instead of [Ka2, Th.(7.1)]. □

Corollary 2.8. *Assume C is a simple normal crossing divisor. For $\chi \in H^1(U)$ and a Cartier divisor D supported on C, the following are equivalent*

(1) *for all generic points λ of C we have $\chi|_{K_\lambda} \in \mathrm{fil}_{m_\lambda(D)} H^1(K_\lambda)$,*
(2) *for all integral $Z \in Z_1(X,C)^+$ and $x \in Z_\infty$, we have (see Definition 1.2)*

$$\chi|_{Z,x} \in \mathrm{fil}_{m_x(\psi_Z^* D)} H^1(k(Z)_x) \ .$$

Here $\chi|_{Z,x} \in H^1(k(Z)_x)$ is the restriction of χ and m_x is the multiplicity at x.

Proof. The implication (1)⇒(2) follows from Proposition 2.7(1) and (3) by observing that for integral $Z \in Z_1(X,C)^+$ there is a chain of blowups in closed points such that the strict transform of Z becomes smooth and such that its intersection with the total transform of C is transversal. The implication (2)⇒(1) follows from Proposition 2.7(2). □

For general X and C, not necessarily of normal crossing, we make the following definition.

Definition 2.9. For $D \in \mathrm{Div}(X)^+$ with support in C we define $\mathrm{fil}_D H^1(U)$ to be the subgroup of $\chi \in H^1(U)$ satisfying property (2) in Corollary 2.8. Define

$$\pi_1^{\mathrm{ab}}(X,D) = \mathrm{Hom}(\mathrm{fil}_D H^1(U), \mathbb{Q}/\mathbb{Z}), \tag{2.7}$$

endowed with the usual pro-finite topology of the dual.

One should think of $\pi_1^{\mathrm{ab}}(X,D)$ as the quotient of $\pi_1^{ab}(U)$ classifying abelian étale coverings of U with ramification bounded by D.

Proposition 2.10. *The filtration $\mathrm{fil}_D H^1(U)$ is exhaustive, i.e.*

$$\bigcup_D \mathrm{fil}_D H^1(U) = H^1(U),$$

where $D \in \mathrm{Div}(X)^+$ runs through all divisors with support in C.

A proof can be found in [EK, Sec. 3.3].

3. Existence Theorem

In this section k is assume to be finite. Let U be a smooth variety over k. Choose a compactification $U \subset X$ with X normal and proper over k such that the reduced subscheme $C = X \setminus U$ of X is the support of an effective Cartier divisor on X. Put $K = k(X)$. In §1 we defined the relative Chow group of zero cycles $\mathrm{C}(X,D)$, where $D \in \mathrm{Div}(X)^+$ is a Cartier divisor with support in C. We endow this relative Chow group with the discrete topology. We endow the group

$$\mathrm{C}(U) = \varprojlim_D \mathrm{C}(X,D)$$

with the inverse limit topology. Here D runs through all effective Cartier divisors on X with support in C.

Lemma 3.1. *The topological group* $\mathrm{C}(U)$ *does not depend on the choice of the compactification* X *of* U.

Proof. Let us write $\mathrm{C}(U \subset X)$ for the class group relative to the compactification X in the following. Assume $U \subset X_1$ and $U \subset X_2$ are two compactifications. Considering the normalization of the Zariski closure of the diagonal $U \to X_1 \times_k X_2$, we may assume that there is a morphism $f : X_2 \to X_1$ which is the identity on U. It is then sufficient to show that the pushforward map (1.2)

$$f_* \ : \ \mathrm{C}(U \subset X_2) \to \mathrm{C}(U \subset X_1) \tag{3.1}$$

is an isomorphism. For an effective Cartier divisor D on X_1 with support in $X_1 \setminus U$, one easily see that $f_* \ : \ \mathrm{C}(X_2, f^*D) \to \mathrm{C}(X_1, D)$ is an isomorphism (see Definition 1.4(2)). As the divisors f^*D are cofinal in the system of all divisors on X_2 with support in $X_2 \setminus U$, the isomorphy of (3.1) follows. □

In fact it is also clear from the proof that $U \mapsto \mathrm{C}(U)$ is a covariant functor from the category of smooth varieties over k to the category of topological abelian groups.

Proposition 3.2. *There is a unique continuous reciprocity homomorphism* ρ_U *making the diagram*

$$\begin{array}{ccc} Z_0(U) & \longrightarrow & \mathrm{C}(U) \\ & \searrow & \downarrow \rho_U \\ & & \pi_1^{\mathrm{ab}}(U) \end{array}$$

commutative. Here the diagonal arrow is induced by the Frobenius homomorphisms $Frob_x : \mathbb{Z} \to \pi_1^{\mathrm{ab}}(U)$ *for closed points* $x \in U$. *Moreover,* ρ_U *induces a homomorphism*

$$\rho_{X,D} : \mathrm{C}(X, D) \to \pi_1^{\mathrm{ab}}(X, D).$$

Recall that the pro-finite fundamental group $\pi_1^{\mathrm{ab}}(X, D)$ classifies abelian étale coverings of U with ramification over C bounded by the divisor D, see Definition 2.9. In what follows, for a topological abelian group M, we write

$$M^\vee = \mathrm{Hom}_{\mathrm{cont}}(M, \mathbb{Q}/\mathbb{Z}),$$

where we endow $\mathbb{Q}/\mathbb{Z}$ with the discrete topology.

Proof of Proposition 3.2. In [Wi], [KeSc] a continuous reciprocity homomorphism $r_U \ : \ \mathrm{W}(U) \to \pi_1^{\mathrm{ab}}(U)$ is constructed. In order to accomplish the proof of the proposition we need some ramification theory. It is sufficient to show that for any character

$$\chi \in (\pi_1^{\mathrm{ab}}(U))^\vee \cong H^1(U)$$

there is a divisor $D \in \mathrm{Div}(X)^+$ with support in C such that $r_U^* \chi \in \mathrm{Hom}(W(U) \to \mathbb{Q}/\mathbb{Z})$ factors through $\mathrm{C}(X, D)$. In view of Definition 2.9, ramification properties of classical local class field theory (see [Se1, Sec. XV.2]) imply that the map r_U induces a map

$$\Psi_{X,D} : \mathrm{fil}_D H^1(U) \longrightarrow \mathrm{C}(X, D)^\vee.$$

Finally, the proposition follows from Proposition 2.10. □

Define topological groups $\mathrm{C}(U)^0$ and $\pi_1^{\mathrm{ab}}(U)^0$ as kernels in the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{C}(U)^0 & \longrightarrow & \mathrm{C}(U) & \xrightarrow{f_*} & \mathrm{C}(\mathrm{Spec} k) \\ & \downarrow & & \downarrow \rho_U & & \downarrow \rho_k \\ 0 \longrightarrow & \pi_1^{\mathrm{ab}}(U)^0 & \longrightarrow & \pi_1^{\mathrm{ab}}(U) & \xrightarrow{f_*} & \pi_1^{\mathrm{ab}}(\mathrm{Spec} k) \end{array} \tag{3.2}$$

where $f : U \to \mathrm{Spec} k$ is the natural morphism. Note that $\mathrm{C}(\mathrm{Spec} k) = \mathbb{Z}$ and ρ_k maps $1 \in \mathbb{Z}$ to the Frobenius over k. Let

$$\rho_U^0 \ : \ \mathrm{C}(U)^0 \to \pi_1^{\mathrm{ab}}(U)^0$$

be the induced map.

Our main theorem says:

Theorem 3.3 (Existence Theorem)**.** *Over a finite field k with* $\mathrm{ch}(k) \neq 2$*, ρ_U^0 is an isomorphism of topological groups.*

Corollary 3.4. *Assume* $\mathrm{ch}(k) \neq 2$*. For an effective divisor $D \in \mathrm{Div}(X)^+$ with support in C, ρ_U^0 induces an isomorphism of finite groups*

$$\rho_{X,D} \ : \ \mathrm{C}(X,D)^0 \xrightarrow{\sim} \pi_1^{\mathrm{ab}}(X,D)^0.$$

Proof. In view of Definition 2.9, the corollary follows from the Theorem 3.3 by using standard ramification properties in local class field theory as is explained in [Se1, Sec. XV.2]. □

The proof of the Existence Theorem is begun in this section and completed in §5 assuming some technical lemmas that will be shown in later sections.

Now we start the proof of the Existence Theorem 3.3. Consider the following property

$\mathbf{I}_U$: ρ_U induces a surjection

$$\Psi_U : H^1(U, \mathbb{Q}/\mathbb{Z}) \to \mathrm{C}(U)^\vee = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{C}(U), \mathbb{Q}/\mathbb{Z}). \tag{3.3}$$

Note that we already know that the map (3.3) is injective by Chebotarev density theorem [Se2].

We now give an overview of the steps in the proof of the Existence Theorem 3.3.

- In Lemma 3.5 we show that property $\mathbf{I}_U$ implies the Existence Theorem for the triple (X, C, U).
- In Lemma 3.6 combined with de Jong's alteration theorem we show how to reduce the proof of $\mathbf{I}_U$ to the situation where C is a simple normal crossing divisor.
- We use a Lefschetz hyperplane theorem [KeS] (C simple normal crossing) which allows us to reduce the proof of $\mathbf{I}_U$ to the case $\dim(X) = 2$.
- In §4 we study (for $\dim(X) = 2$) ramification filtrations on the Galois side and the class group side and compare graded pieces to complete the proof of $\mathbf{I}_U$ in §5. The understanding of the filtration on the class group side is our key new ingredient.

Lemma 3.5. *Property $\mathbf{I}_U$ implies that the map ρ_U in Theorem 3.3 is an isomorphism of topological groups.*

Proof. As $\pi_1^{\mathrm{ab}}(X,D)^0$ is finite by [KeS] for any effective divisor D with $|D| \subset C$, it is enough to show that ρ_U induces an isomorphism $\mathrm{C}(X,D)^0 \to \pi_1^{\mathrm{ab}}(X,D)^0$ of abstract groups. It is sufficient to show that dually

$$\Psi_{X,D} : \mathrm{fil}_D H^1(U) \to \mathrm{C}(X,D)^\vee$$

is an isomorphism. The latter is a direct consequence of $\mathbf{I}_U$ and classical ramification theory for local fields. □

We next introduce certain reduction techniques for property $\mathbf{I}_U$, based on methods of Wiesend. In the following lemma we denote by $f : X' \to X$ an alteration with X' normal. By $U' \subset f^{-1}(U)$ we denote an open smooth subscheme of X', which is the complement of the support of an effective Cartier divisor. We use the notation

$$f : U' \to U, \quad C' = X' \setminus U'.$$

Lemma 3.6 (Wiesend trick)**.**

(i) *For $f : X' \to X$ and $U' \subset f^{-1}(U)$ as above, the implication $\mathbf{I}_{U'} \Rightarrow \mathbf{I}_U$ holds.*
(ii) *Assume that for any character $\chi \in \mathrm{C}(U)^\vee$ we can find $f : X' \to X$ and $U' \subset X'$ as above such that $f^*(\chi) = 0$. Then property $\mathbf{I}_U$ holds.*

Proof. We first explain the proof of (i). Consider the commutative diagram of abstract groups

$$\begin{array}{ccc} H^1(U) & \xrightarrow{\Psi_U} & \mathrm{C}(U)^\vee \\ \downarrow f^* & & \downarrow f^* \\ H^1(U') & \xrightarrow[\Psi_{U'}]{} & \mathrm{C}(U')^\vee \end{array} \tag{3.4}$$

It is sufficient to see that a character $\chi \in \mathrm{C}(U)^\vee$ such that $f^*(\chi)$ is of the form $\Psi_{U'}(\sigma)$ with $\sigma \in H^1(U')$ is in the image of Ψ_U. We can choose another alteration $f' : X'' \to X'$ with the property that $f'^{-1}(\sigma) = 0$. This means that without loss of generality we can assume that $f^*(\chi) = 0 \in \mathrm{C}(U')^\vee$.

Shrinking U' we can also assume that $U' \to f(U') \subset X$ is the composition of a finite surjective radicial map $U' \to U_{\text{ét}}$ and a finite étale map $U_{\text{ét}} \to f(U')$. Then the maps

$$H^1(U_{\text{ét}}) \to H^1(U') \quad \text{and} \quad \mathrm{C}(U_{\text{ét}})^\vee \to \mathrm{C}(U')^\vee$$

are isomorphisms: For the first map this is clear. As for the second, in view of the definition of the norm map for the Wiesend class groups (cf. [KeSc, Lem.7.3] and its proof), it follows from the facts that the pushforward map $Z_0(U') \to Z_0(U_{\text{ét}})$ is an isomorphism, and that for a finite surjective radicial covering $Z' \to Z$ of integral normal curves, the norm map $k(Z')^\times \to k(Z)^\times$ is an isomorphism as well as the norm map $k(Z')^\times_y \to k(Z)^\times_x$ for the henselizations at closed points $x \in Z$ and $y \in Z'$ lying over x. Therefore we can without loss of generality assume that $X' \to X$ is generically étale. In this situation we finally conclude that χ is in the image of Ψ_U by using Wiesend's method, see [KeSc, Prop. 3.7].

The proof of (ii) is a variant of the proof of (i). □

Lemma 3.7. *Assume that property $\mathbf{I}_U$ holds for all smooth varieties U with* $\dim(U) = 2$*. Then it holds for arbitrary smooth U.*

Proof. By Lemma 3.5 we obtain Corollary 3.4 for two-dimensional X. In the general case we reduce the proof of property $\mathbf{I}_U$ to the case C is simple normal crossing and X is projective by Lemma 3.6 and de Jong's alteration theorem [dJ]. This means that for such (X, C, U) we have to show that the map

$$\mathrm{C}(X, D)^0 \to \pi_1^{\mathrm{ab}}(X, D)^0$$

is an isomorphism for all D.

Let $\mathcal{L}$ be an ample line bundle on X. Let $i : Y \hookrightarrow X$ be a smooth hypersurface section, which is the zero locus of some section of $\mathcal{L}^{\otimes n}$ $(n \gg 0)$, such that $Y \times_X C$ is a reduced simple normal crossing divisor on Y and let $E = Y \times_X D$. Consider the commutative diagram

$$\begin{array}{ccccc} & & \mathrm{C}(Y,E)^0 & \xrightarrow[\rho_{Y,E}]{\sim} & \pi_1^{\mathrm{ab}}(Y,E)^0 \\ & & {\scriptstyle i_*}\downarrow & & \downarrow\wr \\ Z_0(U)^0 & \twoheadrightarrow & \mathrm{C}(X,D)^0 & \xrightarrow[\rho_{X,D}]{} & \pi_1^{\mathrm{ab}}(X,D)^0 \end{array}$$

The map $\rho_{Y,E}$ is an isomorphism by induction on dimension. The right vertical map is an isomorphism for n sufficiently large [KeS]. The map $\rho_{X,D}$ is surjective because of Chebotarev density [Se2] and the finiteness of $\pi_1^{\mathrm{ab}}(X,D)^0$, see [KeS]. So we have to show injectivity of $\rho_{X,D}$.

For an $\alpha \in \mathrm{C}(X,D)^0$ with $\rho_{X,D}(\alpha) = 0$ use a Bertini argument to choose Y as above which contains the support of a lift of α to $Z_0(U)$. Then α is in the image of i_*. A diagram chase shows that $\alpha = 0$.

□

4. Cycle conductor

Let the notation be as in §1. Let $X = (X,C)$ be in $\mathcal{C}$ and recall $\dim(X) = 2$ (cf. Definition 1.9). Let $\{C_\lambda\}_{\lambda\in I}$ be the set of prime components of C. Fix a Cartier divisor

$$D = \sum_{\lambda\in I} m_\lambda C_\lambda \quad \text{with } m_\lambda \geq 2. \tag{4.1}$$

For a Cartier divisor F on X and $Z \in \mathrm{Div}(X,C)^+$, we write

$$\mathcal{O}_Z(F) = \mathcal{O}_Z \otimes_{\mathcal{O}_X} \mathcal{O}_X(F).$$

In this section we assume k is finite and introduce the key homomorphism called the *cycle conductor* for (X,D):

$$\mathrm{cc}_{X,D} : \mathrm{C}(X,D)^\vee \to H^0(C, \Omega^1_X(D+\Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C), \tag{4.2}$$

and state its basic properties. Here $\Xi \in \mathrm{Div}(X,C)^+$ is some sufficiently big Cartier divisor introduced below. First we note the canonical duality isomorphism

$$H^0(C, \Omega^1_X(D+\Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \simeq H^1(C, \Omega^1_X(-D+C-\Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C)^\vee. \tag{4.3}$$

Indeed, letting ω_C be the dualizing sheaf of C, we have

$$\omega_C \simeq \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_C, \Omega^2_X) \simeq \Omega^2_X(C) \otimes_{\mathcal{O}_X} \mathcal{O}_C,$$

where the second isomorphism follows from the long exact sequence for $\mathcal{E}xt$ induced by the exact sequence $0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$. Thus the Serre duality implies that the pairing

$$\Omega^1_X(D+\Xi) \otimes_{\mathcal{O}_X} \Omega^1_X(-D+C-\Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C \to \Omega^2_X(C) \otimes_{\mathcal{O}_X} \mathcal{O}_C \simeq \omega_C$$

induces a perfect pairing of abelian groups

(4.4)

$$H^0(C, \Omega^1_X(D+\Xi)\otimes_{\mathcal{O}_X}\mathcal{O}_C)\times H^1(C, \Omega^1_X(-D+C-\Xi)\otimes_{\mathcal{O}_X}\mathcal{O}_C) \to H^1(C,\omega_C) \xrightarrow{Tr_{C/\mathbb{F}_p}} \mathbb{Z}/p\mathbb{Z},$$

where $Tr_{C/\mathbb{F}_p}$ is the composite $H^1(C,\omega_C) \xrightarrow{Tr_{C/k}} k \xrightarrow{Tr_{k/\mathbb{F}_p}} \mathbb{Z}/p\mathbb{Z}$. This induces (4.3). Hence, by rewriting D by $D+C$, the construction of (4.2) is reduced to that of its dual map:

$$H^1(C, \Omega^1_X(-D+C-\Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \to \mathrm{C}(X,D).$$

or equivalently that of a map (see Theorem 4.1 below):

$$\phi_{X,D} : H^1(C, \Omega^1_X(-D-\Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \to W(U)/\widehat{F}^{(D+C)}W(X,C) \tag{4.5}$$

for a Cartier divisor

$$D = \sum_{\lambda \in I} m_\lambda C_\lambda \quad \text{with } m_\lambda \geq 1. \tag{4.6}$$

Let x be a regular closed point of C. Let $Z \in \mathrm{Div}(X, C)^+$ be such that $Z \pitchfork C$ at x. Locally on a neighborhood of x, we have an exact sequence

$$0 \to \Omega^1_X(-D) \to \Omega^1_X(\log Z)(-D) \to \mathcal{O}_Z(-D) \to 0. \tag{4.7}$$

Tensored with $\mathcal{O}_C$, it induces a boundary map

$$\partial_{Z,x} : \mathcal{O}_{Z,x}(-D) \otimes \kappa(x) \to H^1_x(C, \Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_C). \tag{4.8}$$

We will let $\partial_{Z,x}$ denote also the composite of $\partial_{Z,x}$ and the natural map

$$H^1_x(C, \Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \to H^1(C, \Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_C).$$

On the other hand we define a map

$$\mu_{Z,x} : \mathcal{O}_{Z,x}(-D) \otimes \kappa(x) \to W(U)/F^{(D+C)}W(X, C) \tag{4.9}$$

as the composite of the natural injection

$$\mathcal{O}_{Z,x}(-D) \otimes \kappa(x) \to k(Z)^\times_x/(1 + I_{D+C}\mathcal{O}^h_{Z,x}) \ ; \ a \to 1 + a$$

and the map

$$k(Z)^\times_x/(1 + I_{D+C}\mathcal{O}^h_{Z,x}) \to W(U)/F^{(D+C)}W(X, C)$$

induced by $\{\ \}_{Z,x}$ (cf. Definition 1.3 and the notation below it).

We now state the first key theorem for the proof of Theorem 3.3. Its proof will be given in §7. Recall that k is assumed to be finite in this section.

Theorem 4.1. *Assume $p = \mathrm{ch}(k) \neq 2$. There exists $\Xi \in \mathrm{Div}(X, C)^+$ (cf. Definition 1.8) and a natural map*

$$\phi_{X,D} : H^1(C, \Omega^1_X(-D - \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \to W(U)/\widehat{F}^{(D+C)}W(X, C)$$

such that $C_{sing} \subset \Xi$ and Ξ is independent of D as in (4.6), and that the following conditions hold:

(*i*) *For any closed point x of $C \backslash \Xi$ and $Z \in \mathrm{Div}(X, C)^+$ such that $Z \pitchfork C$ at x, the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{O}_{Z,x}(-D) \otimes \kappa(x) & \xrightarrow{\partial_{Z,x}} & H^1(C, \Omega^1_X(-D - \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \\ & \searrow^{\mu_{Z,x}} & \downarrow{\scriptstyle \phi_{X,D}} \\ & & W(U)/\widehat{F}^{(D+C)}W(X, C). \end{array} \tag{4.10}$$

(*ii*) $\mathrm{Image}(\phi_{X,D}) = \mathrm{Image}(\widehat{F}^{(D)}W(X, C))$.

Remark 4.2. The images of $\partial_{Z,x}$ for closed points x of $C \backslash \Xi$ and $Z \in \mathrm{Div}(X, C)^+$ with $Z \pitchfork C$ at x generate $H^1(C, \Omega^1_X(-D - \Xi) \otimes_{\mathcal{O}_X} \mathcal{O}_C)$. Thus the condition (*i*) uniquely characterizes $\phi_{X,D}$.

Theorem 4.1(*ii*) follows from (*i*) and Lemma 4.3 below, whose proof will be given in §14 (see Lemma 9.3 and the leitfaden in §9). It concerns moving elements of $W(U)$ to symbols on curves transversal to C. Take any dense open subset $V \subset X$ containing the generic points of C and recall Definition 1.4.

Lemma 4.3 (moving)**.** *Assume $p \neq 2$. For any integer $N > 0$, we have*

$$\widehat{F}^{(D)}W(X, C) \subset F^{(D)}_{\pitchfork V}W(X, C) \ + \ \widehat{F}^{(D+N\cdot C)}W(X, C).$$

For D as (4.1) let

$$\mathrm{cc}_{X,D} : \mathrm{C}(X,D)^\vee \to H^0(C, \Omega^1_X(D+\Xi)\otimes_{\mathcal{O}_X}\mathcal{O}_C),$$

be the map induced by

$$\phi_{X,D-C} : H^1(C,\Omega^1_X(-D+C-\Xi)\otimes_{\mathcal{O}_X}\mathcal{O}_C) \to \mathrm{C}(X,D) = W(U)/\widehat{F}^{(D)}W(X,C)$$

using the duality (4.3). By Theorem 4.1(ii) we have an exact sequence

$$0 \to \mathrm{C}(X,D-C)^\vee \to \mathrm{C}(X,D)^\vee \xrightarrow{\mathrm{cc}_{X,D}} H^0(C,\Omega^1_X(D+\Xi)\otimes_{\mathcal{O}_X}\mathcal{O}_C). \tag{4.11}$$

The second key theorem concerns compatibility of the cycle conductor with ramification theory reviewed in §2. Its proof will be given in §8.

Theorem 4.4. *For $\lambda \in I$, let m_λ be as in* (4.1) *and*

$$\mathrm{rar}_{K_\lambda} : \mathrm{gr}_{m_\lambda} H^1(K_\lambda) \to \mathrm{fil}_{m_\lambda}\Omega^1_{K_\lambda}$$

be the refined Artin conductor for K_λ in Theorem 2.3. We note

$$\Omega^1_X(D)\otimes_{\mathcal{O}_X} k(C_\lambda) \simeq \mathrm{gr}_{m_\lambda}\Omega^1_{K_\lambda}.$$

Then the following diagram commutes

$$\begin{array}{ccccc}
\mathrm{fil}_D H^1(U) & \longrightarrow & \mathrm{fil}_{m_\lambda} H^1(K_\lambda) & \xrightarrow{-\mathrm{rar}_{K_\lambda}} & \Omega^1_X(D)\otimes_{\mathcal{O}_X} k(C_\lambda) \\
\downarrow{\scriptstyle \Psi_{X,D}} & & & & \uparrow{\scriptstyle \iota_\lambda} \\
\mathrm{C}(X,D)^\vee & & \xrightarrow{\qquad \mathrm{cc}_{X,D} \qquad} & & H^0(C,\Omega^1_X(D+\Xi)\otimes_{\mathcal{O}_X}\mathcal{O}_C)
\end{array}.$$

5. Proof of Existence theorem

Let the notation be as in §3. In this section we always assume $\mathrm{ch}(k)\neq 2$ and $\dim(X)=2$, but we do not generally assume that C is simple normal crossing or that X is smooth. We prove property $\mathbf{I}_U$ in this case. Using Lemma 3.7 we are going to reduce the proof of the Existence Theorem 3.3 to Theorems 4.1 and 4.4.

We start with the tame case, which is essentially due to Wiesend. Note that the second statement of the following proposition is motivated by the fact that an abelian étale covering $U'\to U$ is tame along C if and only if its pullback to integral $F\in Z_1(X,C)^+$ such that $F \Cap C$ is tame along $C\cap F$, see Proposition 2.7.

Proposition 5.1. *Assume* $\dim(X)=2$ *and that X/k is smooth. The reciprocity map ρ_U induces an isomorphism of finite groups*

$$\mathrm{C}(X,C)^0 \xrightarrow{\sim} \pi_1^{\mathrm{ab}}(X,C)^0.$$

Moreover the closure of the image of $F^{(C)}_{\Cap}W(X,C)$ in $\mathrm{C}(U)$ is equal to $\widehat{F}^{(C)}W(X,C)$.

Here C is considered as a reduced effective Cartier divisor.

Proof. It is shown in [KeSc, Thm. 8.3] (see also Remark 1.5) that ρ_U induces an isomorphism

$$W(U)^0/\widehat{F}^{(1)}W(U) \xrightarrow{\sim} \pi_1^{\mathrm{ab}}(X,C)^0,$$

where $W(U)^0 = \mathrm{Ker}\big(W(U)\xrightarrow{f_*} W(\mathrm{Spec}k)\big)$ for the natural map $f: U\to \mathrm{Spec}k$ (note $W(\mathrm{Spec}k)=\mathbb{Z}$). The verbatim same argument shows the proposition. □

Corollary 5.2. *Assume* $\dim(X)=2$. *For any effective Cartier divisor D on X with $|D|=C$, $\mathrm{C}(X,D)^0$ is torsion of finite exponent and $\widehat{F}^{(1)}W(U)/\widehat{F}^{(D)}W(X,C)$ is of finite exponent of p-power.*

Proof. Using de Jong's alteration result [dJ] and a norm trick we can assume without loss of generality that X/k is smooth. Then by Proposition 5.1, we have

$$W(U)^0/(F^{(C)}_{\Cap}W(X,C)+\widehat{F}^{(D)}W(X,C))\cong W(U)^0/\widehat{F}^{(C)}W(X,C),$$

and it is isomorphic to $\pi_1^{\mathrm{ab}}(X,C)^0$, which is finite by [KeSc, Th.2.7]. On the other hand $p^mF^{(C)}_{\Cap}W(X,C)\subset F^{(D)}W(X,C)$ if $p^mC\geq D$ since for $F\in \mathrm{Div}(X,C)^+$ such that $F\Cap C$ at $x\in F\cap C$, $(1+\mathcal{O}_{F,x}(-C))^{p^m}\subset 1+\mathcal{O}_{F,x}(-p^mC)$. □

Now we turn to the proof of property $\mathbf{I}_U$ in the wild case. By Wiesend's trick (Lemma 3.6) and a standard fibration technique [SGA4, XI, Prop. 3.3], we can assume that there is a proper smooth curve S over k and morphisms

$$f:X\to S \quad\text{and}\quad \sigma:S\to X$$

where f is a proper surjective morphism with smooth generic fiber and σ is a section of f. Let J be the set of generic points λ of C which lie over the generic point η of S and let $C_\lambda=\overline{\{\lambda\}}$ be the closure of $\lambda\in J$ in X. We can assume:

- $f(\sigma(S)\cap C)$ does not contain η.
- C is a Cartier divisor on X.
- The induced morphism $C_\lambda\to S$ is an isomorphism for each $\lambda\in J$.
- $f|_U:U\to S$ is smooth.

Let us fix an algebraic closure $\overline{k(S)}$ of $k(S)$. Write $\bar\eta=\mathrm{Spec}\,\overline{k(S)}$. Let us consider pairs $\Sigma=(T,\theta)$ where

- T is the normalization of S in a finite subextension of $k(S)$ in the field extension $k(S)\subset\overline{k(S)}$
- θ is an effective divisor on T.

Clearly for such $\Sigma=(T,\theta)$ there is a canonical map $T\to S$. We define a directed partial ordering on the set of all Σ by setting

$$\Sigma_1=(T_1,\theta_1)\leq\Sigma_2=(T_2,\theta_2),$$

if $k(T_1)\subset k(T_2)$, which means that the map $T_2\to S$ factors canonically through

$$T_2\xrightarrow{g_{\Sigma_2,\Sigma_1}}T_1\to S\quad\text{and if}\quad g^*_{\Sigma_2,\Sigma_1}(\theta_1)\leq\theta_2.$$

Let X_Σ be the normalization of $X\times_S T$ and write $C_\Sigma\in\mathrm{Div}(X_\Sigma)^+$ for the pullback of $\cup_{\lambda\in J}C_\lambda$ to X_Σ. We also write $\theta_\Sigma\in\mathrm{Div}(X_\Sigma)^+$ for the pullback of θ to X_Σ. By U_Σ we denote the preimage of U in $X_\Sigma\setminus\mathrm{supp}(\theta_\Sigma)$. Using the compatibility of étale cohomology with directed inverse limits of schemes we get an isomorphism

$$\varinjlim_{\Sigma}H^1(U_\Sigma)\xrightarrow{\sim}H^1(U_{\bar\eta}). \tag{5.1}$$

Thinking of $U_{\bar\eta}$ as a smooth curve over $\bar\eta$ with compactification $X_{\bar\eta}$ we endow the cohomology group $H^1(U_{\bar\eta})$ with the ramification filtration

$$\mathrm{fil}_mH^1(U_{\bar\eta})=\mathrm{Ker}\big(H^1(U_{\bar\eta})\to\bigoplus_{\lambda\in J}H^1(K_\lambda^-)/\mathrm{fil}_mH^1(K_\lambda^-)\big)$$

where K_λ^- is the quotient field of the henselization of $X_{\bar\eta}$ at the preimage of λ. Note that

$$\mathrm{fil}_mH^1(U_{\bar\eta})=\varinjlim_{\Sigma}\mathrm{fil}_mH^1(U_{\Sigma,\eta}), \tag{5.2}$$

where $U_{\Sigma,\eta}=U_\Sigma\times_S\eta$ and

$$\mathrm{fil}_mH^1(U_{\Sigma,\eta})=\mathrm{Ker}\big(H^1(U_{\Sigma,\eta})\to\bigoplus_{\lambda\in J}H^1(K_{\Sigma,\lambda})/\mathrm{fil}_mH^1(K_{\Sigma,\lambda})\big),$$

and $K_{\Sigma,\lambda}$ is the quotient field of the henselization of X_Σ at the preimage of λ. If we fix T and take $\mathrm{supp}(\theta)$ large enough, then $X_\Sigma \backslash \mathrm{supp}(\theta_\Sigma)$ is smooth and $C_\Sigma \backslash \mathrm{supp}(\theta_\Sigma)$ is a regular divisor on it. Hence Corollary 2.8 implies

$$\mathrm{fil}_m H^1(U_{\Sigma,\eta}) = \varinjlim_{\theta} \mathrm{fil}_{mC_\Sigma+\theta_\Sigma} H^1(U_\Sigma), \tag{5.3}$$

where θ ranges over the effective divisors on T and $\mathrm{fil}_{mC_\Sigma+\theta_\Sigma} H^1(U_\Sigma)$ is defined as Definition 2.9 for the effective Cartier divisor $mC_\Sigma + \theta_\Sigma$ on X_Σ. Combining (5.1), (5.2) and (5.3), we get an isomorphism

$$\mathrm{fil}_m H^1(U_{\bar\eta}) \xrightarrow{\sim} \varinjlim_{\Sigma} \mathrm{fil}_{mC_\Sigma+\theta_\Sigma} H^1(U_\Sigma). \tag{5.4}$$

Composing (5.4) with the dual reciprocity map (see Proposition 3.2), we get a homomorphism

$$\Psi_{\bar\eta}^{(m)} : \mathrm{fil}_m H^1(U_{\bar\eta}) \to \varinjlim_{\Sigma} \mathrm{C}(X_\Sigma, mC_\Sigma + \theta_\Sigma)^\vee. \tag{5.5}$$

By Wiesend's trick (Lemma 3.6), the surjectivity of $\Psi_{\bar\eta}^{(m)}$ implies $\mathbf{I}_U$.

The following result is essentially due to Wiesend.

Lemma 5.3 (Wiesend)**.** *The map $\Psi_{\bar\eta}^{(1)}$ is surjective.*

Proof. Consider $\chi \in \mathrm{C}(X_\Sigma, C_\Sigma + \theta)^\vee$ for some $\Sigma = (T, \theta)$. By Wiesend's trick Lemma 3.6 it is enough to construct a quasi-finite map $U' \to U_\Sigma$ with dense image such that the pullback of χ to U' vanishes. This is proved by the same argument as the proof of [KeSc, Prop.3.6]. For convenience of the readers we recall it (see the first part of [KeSc, §4]).

By Corollary 5.2, $\mathrm{C}(X_\Sigma, C_\Sigma + \theta_\Sigma)^0$ is of finite exponent so that the map

$$\varinjlim_{n} \mathrm{Hom}_{\mathrm{cont}}(\mathrm{C}(X_\Sigma, C_\Sigma + \theta_\Sigma), \mathbb{Z}/n) \xrightarrow{\sim} \mathrm{C}(X_\Sigma, C_\Sigma + \theta_\Sigma)^\vee$$

is an isomorphism, where n ranges over all positive integers. Thus we can find $n > 0$ such that

$$\chi \in \mathrm{Hom}_{\mathrm{cont}}(\mathrm{C}(X_\Sigma, C_\Sigma + \theta_\Sigma), \mathbb{Z}/n\mathbb{Z}) \subset \mathrm{C}(X_\Sigma, C_\Sigma + \theta_\Sigma)^\vee.$$

If we pull back χ along the section $\sigma : T \to X_\Sigma$ we get a character in $\mathrm{C}(T, \theta')^\vee$ for some effective divisor θ' on T. By one-dimensional global class field theory it comes from a cohomology element of $H^1(T \setminus |\theta'|)$ via the dual reciprocity map. By making a base change in the base T, we can assume that this cohomology element vanishes.

The maximal abelian pro-finite étale covering $U' \to (U_\Sigma)_\eta$ which splits over the image of σ, and whose Galois group is n-torsion and which is tame along $(C_\Sigma)_\eta$ is finite. Let $\chi' : W(U') \to \mathbb{Z}/n\mathbb{Z}$ be the pullback of χ via the composite map

$$W(U') \to W(U_\Sigma) \twoheadrightarrow \mathrm{C}(X_\Sigma, C_\Sigma + \theta_\Sigma),$$

where the first map is the map (1.2) induced by $U' \to U_\Sigma$. By a specialization argument for tame fundamental groups and one-dimensional class field theory for the fibers of $U_\Sigma \to T$, we can deduce that the pullback of χ' to the class group of the fibers of $U' \to T$ are trivial (see the last paragraph of page 2579 of [KeSc]). By the Chebotarev density theorem (see [Se2, Th.7]) this implies χ' is trivial as desired. □

We now prove $\Psi_{\bar\eta}^{(m)}$ is surjective for $m \geq 2$ by induction on m. Consider the exact localization sequence

$$0 \to H^1(X_{\bar\eta}) \to H^1(U_{\bar\eta}) \to \bigoplus_{\lambda \in J} H^1(K_\lambda^-) \xrightarrow{\iota} H^2(X_{\bar\eta}, \mathbb{Q}/\mathbb{Z}).$$

It induces the exact sequences in the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & H^1(X_{\bar\eta}) & \to & \mathrm{fil}_{m-1}H^1(U_{\bar\eta}) & \to & \bigoplus_{\lambda\in J} \mathrm{fil}_{m-1}H^1(K_\lambda^-) & \xrightarrow{\iota} & \iota(\mathrm{fil}_{m-1}) & \to 0 \\ & \| & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & H^1(X_{\bar\eta}) & \to & \mathrm{fil}_m H^1(U_{\bar\eta}) & \to & \bigoplus_{\lambda\in J} \mathrm{fil}_m H^1(K_\lambda^-) & \xrightarrow{\iota} & \iota(\mathrm{fil}_m) & \to 0 \end{array}$$

Here the vertical maps are the canonical inclusions. Taking cokernels of the vertical maps we get an exact sequence

$$0 \to \mathrm{gr}_m H^1(U_{\bar\eta}) \to \bigoplus_{\lambda\in J} \mathrm{gr}_m H^1(K_\lambda^-) \xrightarrow{\iota} \iota(\mathrm{fil}_m)/\iota(\mathrm{fil}_{m-1}) \to 0. \tag{5.6}$$

The map ι in (5.6) vanishes, because $\iota(\mathrm{fil}_m)/\iota(\mathrm{fil}_{m-1})$ is a subquotient of the cohomology group $H^2(X_{\bar\eta}, \mathbb{Q}/\mathbb{Z})$, which has no p-torsion by [SGA4, X, Thm. 5.1], and $\mathrm{gr}_m H^1(K_\lambda^-)$ is a p-primary torsion group. This implies the exactness of the left column of the following diagram:

$$\begin{array}{ccc} \mathrm{fil}_{m-1}H^1(U_{\bar\eta}) & \xrightarrow{\Psi_{\bar\eta}^{(m-1)}} & \varinjlim_\Sigma \mathrm{C}(X_\Sigma, (m-1)C_\Sigma + \theta_\Sigma)^\vee \\ \downarrow & & \downarrow \\ \mathrm{fil}_m H^1(U_{\bar\eta}) & \xrightarrow{\Psi_{\bar\eta}^{(m)}} & \varinjlim_\Sigma \mathrm{C}(X_\Sigma, mC_\Sigma + \theta_\Sigma)^\vee \\ \downarrow & & \downarrow \mathrm{cc} \\ \bigoplus_{\lambda\in J} \mathrm{gr}_m H^1(K_\lambda^-) & \xrightarrow{-\oplus_\lambda \mathrm{rar}_\lambda} & \bigoplus_{\lambda\in J} \Omega^1_{X_{\bar\eta}}(mC) \otimes \overline{k(C_\lambda)} \\ \downarrow & & \\ 0 & & \end{array} \tag{5.7}$$

The map cc is induced by the cycle conductor defined in §4 as follows. For $\Sigma = (T, \theta)$ take the minimal desingularization $\widetilde{X}_\Sigma \to X_\Sigma$ and let $\widetilde{C}_\Sigma$ be the proper transform of C_Σ in $\widetilde{X}_\Sigma$. Note that the projection $\widetilde{C}_\Sigma \to C_\Sigma$ is an isomorphism. Let $\widetilde{\theta}_\Sigma \in \mathrm{Div}(\widetilde{X}_\Sigma)^+$ be the pullback of θ to $\widetilde{X}_\Sigma$. If $\mathrm{supp}(\theta) \subset T$ is sufficiently large, we have $\widetilde{X}_\Sigma \setminus \mathrm{supp}(\widetilde{\theta}_\Sigma) \simeq X_\Sigma \setminus \mathrm{supp}(\theta_\Sigma)$ so that the natural map

$$\varprojlim_\theta \mathrm{C}(\widetilde{X}_\Sigma, m\widetilde{C}_\Sigma + \widetilde{\theta}_\Sigma) \to \varprojlim_\theta \mathrm{C}(X_\Sigma, mC_\Sigma + \theta_\Sigma)$$

is an isomorphism, where the limit is taken over the effective divisors on a fixed T. Hence Theorem 4.1 and (4.11) applied to $(\widetilde{X}_\Sigma, m\widetilde{C}_\Sigma + \widetilde{\theta}_\Sigma)$ implies an exact sequence

$$\varinjlim_\theta \mathrm{C}(X_\Sigma, (m-1)C_\Sigma + \theta_\Sigma)^\vee \to \varinjlim_\theta \mathrm{C}(X_\Sigma, mC_\Sigma + \theta_\Sigma)^\vee \xrightarrow{cc} \bigoplus_{\lambda\in J} \Omega^1_{X_\Sigma}(mC_\Sigma) \otimes k(C_\lambda).$$

The right vertical sequence in (5.7) is deduced from this so it is exact. The map rar_λ in (5.7) is the refined Artin conductor for K_λ^- recalled in Theorem 2.3 and it is surjective. The lower square of (5.7) commutes by Theorem 4.4. The commutativity of the upper square is obvious. A diagram chase shows that the surjectivity of $\Psi_{\bar\eta}^{(m-1)}$ implies the surjectivity of $\Psi_{\bar\eta}^{(m)}$. This finishes the induction and therefore the proof of Theorem 3.3.

6. Reciprocity at a closed point

Let the notation be as in §1. Let $X = (X, C)$ be in $\mathcal{C}$ (cf. Definition 1.9). The purpose of this section is to reduce Proposition 6.1 and Corollary 6.2 to Lemma 6.5

below. It will play a crucial role in the proof of Theorem 4.1 in §7. We note that the arguments in this section work over any perfect field k (not necessarily finite).

For a regular closed point x of C, there is a natural isomorphism

$$H^1_x(C, \Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \simeq \frac{\Omega^1_X(-D) \otimes_{\mathcal{O}_X} k(C_\lambda)}{\Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}}$$

where C_λ is the irreducible component of C containing x. Thus we have a natural map

$$\iota_x : \frac{\Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}(x)}{\Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}} \to H^1_x(C, \Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \tag{6.1}$$

We will let ι_x denote also the composite of ι_x and the natural map

$$H^1_x(C, \Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \to H^1(C, \Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_C).$$

Let $Z \in \mathrm{Div}(X, C)^+$ be such that $Z \pitchfork C$ at x. Locally on a neighborhood of x, we have a commutative diagram of exact sequences (cf. (4.7))

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^1_X(-D) & \longrightarrow & \Omega^1_X(\log Z)(-D) & \longrightarrow & \mathcal{O}_Z(-D) & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1_X(-D) & \longrightarrow & \Omega^1_X(Z-D) & \longrightarrow & \Omega^1_X(Z-D) \otimes_{\mathcal{O}_X} \mathcal{O}_Z & \longrightarrow & 0\,.
\end{array}$$

Tensored with $\mathcal{O}_C$, it induces a commutative diagram of boundary maps

$$\begin{array}{ccc}
\mathcal{O}_{Z,x}(-D) \otimes \kappa(x) & \xrightarrow{\partial_{Z,x}} & H^1_x(C, \Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_C) \\
\downarrow{d_{Z,x}} & & \downarrow{=} \\
\frac{\Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}(x)}{\Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}} & \xrightarrow{\iota_x} & H^1_x(C, \Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_C)
\end{array}$$

where $d_{Z,x}$ has the following explicit description: Let $f \in \mathcal{O}_{X,x}$ be a local equation of Z at x. Then, for $a \in \mathcal{O}_{X,x}(-D)$, we have

$$d_{Z,x}(a_{|Z}) = a_{|C} \cdot \frac{df}{f} \in \frac{\Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}(x)}{\Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}}. \tag{6.2}$$

Proposition 6.1. *Let x be a regular closed point of C. Let $F, Z_1, Z_2 \in \mathrm{Div}(X, C)^+$ be such that $F \pitchfork C$ at x, and $F \pitchfork Z_i$ and $Z_i \pitchfork C$ at x for $i = 1, 2$. For $a \in \mathcal{O}_{F,x}(-D) \otimes \kappa(x)$ and $b_i \in \mathcal{O}_{Z_i,x}(-D) \otimes \kappa(x)$ for $i = 1, 2$ satisfying*

$$d_{F,x}(a) = d_{Z_1,x}(b_1) + d_{Z_2,x}(b_2) \in \frac{\Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}(x)}{\Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}},$$

we have

$$\mu_{F,x}(a) = \mu_{Z_1,x}(b_1) + \mu_{Z_2,x}(b_2) \in W(U)/F^{(D+C)}W(X, C).$$

Before going to the proof, we give its corollary.

Corollary 6.2. *For a regular closed point $x \in C$, there exists a unique homomorphism*

$$\mu_x : \frac{\Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}(x)}{\Omega^1_X(-D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}} \to W(U)/F^{(D+C)}W(X, C)$$

such that for every $Z \in \mathrm{Div}(X,C)^+$ with $Z \Cap C$ at x, the following diagram is commutative

$$
\begin{array}{ccc}
\mathcal{O}_{Z,x}(-D)\otimes\kappa(x) & \xrightarrow{d_{Z,x}} & \dfrac{\Omega^1_X(-D)\otimes_{\mathcal{O}_X}\mathcal{O}_{C,x}(x)}{\Omega^1_X(-D)\otimes_{\mathcal{O}_X}\mathcal{O}_{C,x}} \\
& \searrow^{\mu_{Z,x}} & \downarrow{\scriptstyle \mu_x} \\
& & W(U)/F^{(D+C)}W(X,C).
\end{array}
\tag{6.3}
$$

Remark 6.3. By Proposition 6.1, Theorem 4.1(i) is equivalent to the commutativity of the following diagram for any closed point x of $C \backslash \Xi$:

$$
\begin{array}{ccc}
& & H^1(C, \Omega^1_X(-D-\Xi)\otimes_{\mathcal{O}_X}\mathcal{O}_C) \\
& \nearrow^{\iota_x} & \\
\dfrac{\Omega^1_X(-D)\otimes_{\mathcal{O}_X}\mathcal{O}_{C,x}(x)}{\Omega^1_X(-D)\otimes_{\mathcal{O}_X}\mathcal{O}_{C,x}} & & \downarrow{\scriptstyle \phi_{X,D}} \\
& \searrow_{\mu_x} & \\
& & W(U)/\widehat{F}^{(D+C)}W(X,C).
\end{array}
\tag{6.4}
$$

Now we deduce Corollary 6.2 from Proposition 6.1. Put

$$
\Lambda_x = \frac{\Omega^1_X(-D)\otimes_{\mathcal{O}_X}\mathcal{O}_{C,x}(x)}{\Omega^1_X(-D)\otimes_{\mathcal{O}_X}\mathcal{O}_{C,x}}.
\tag{6.5}
$$

We have

$$
\Omega^1_{X,x} = \mathcal{O}_{X,x}\cdot d\pi \oplus \mathcal{O}_{X,x}\cdot df.
$$

Hence any element $\xi \in \Lambda_x$ is written in a unique way as

$$
\xi = \frac{1}{f}(\alpha d\pi + \beta df) \quad \text{with } \alpha,\beta \in \mathcal{O}_{X,x}(-D) \bmod \mathcal{O}_{X,x}(-D-C).
\tag{6.6}
$$

For $Z \in \mathrm{Div}(X,C)^+$ such that $Z \Cap C$ at x, put

$$
A(Z) = \mathcal{O}_{Z,x}\otimes\kappa(x).
$$

Let $F, Z \in \mathrm{Div}(X,C)^+$ be such that $F \Cap C$, $F \Cap Z$, $Z \Cap C$ at x. From (6.2) one easily see that the map

$$
d_{F,Z} : A(F)\oplus A(Z) \to \Lambda_x \ ; \ (a,b) \to d_{F,x}(a) + d_{Z,x}(b)
$$

is an isomorphism. Define the composite map

$$
\mu_{F,Z} : \Lambda_x \overset{(d_{F,Z})^{-1}}{\longrightarrow} A(F)\oplus A(Z) \overset{\mu_{F,x}\oplus\mu_{Z,x}}{\longrightarrow} W(U)/F^{(D+C)}W(X,C).
$$

We claim $\mu_{F,Z}$ independent of the choice of F, Z as above. Indeed, assume given two choices F, Z_1 and F, Z_2 as above and $a, a' \in A(F)$ and $b_i \in A(Z_i)$ with $i = 1,2$. Then, by Proposition 6.1 an equality

$$
d_{F,x}(a) + d_{Z_1,x}(b_1) = d_{F,x}(a') + d_{Z_2,x}(b_2)
$$

implies an equality

$$
\mu_{F,x}(a) + \mu_{Z_1,x}(b_1) = \mu_{F,x}(a') + \mu_{Z_2,x}(b_2).
$$

Hence we get $\mu_{F,Z_1} = \mu_{F,Z_2}$. By the same argument we also get $\mu_{F_1,Z} = \mu_{F_2,Z}$ for two choices F_1, Z and F_2, Z as above. This implies the desired claim. The commutativity of (6.3) is obvious from the construction and the proof of Corollary 6.2 is complete.

Remark 6.4. The above argument gives the following explicit description of μ_x. For $g \in \mathcal{O}_{X,x}$, let $\mathrm{div}_{X,x}(g)$ denote the effective Cartier divisor on X obtained from $\mathrm{div}_X(g)$ by removing its components which do not contain x. Note $\mathrm{div}_{X,x}(g) = \mathrm{div}_{X,x}(ug)$ for $u \in \mathcal{O}_{X,x}^\times$. Take a system of regular parameter (π, f) of $\mathcal{O}_{X,x}$ such that π is a local parameter of C at x. Then we have (cf. (6.6))

$$\mu_x\big(\frac{1}{f}(\alpha d\pi + \beta df)\big) = \{1 + (\beta - \alpha)\}_{F,x} + \{1 + \alpha\}_{F_\pi,x}, \tag{6.7}$$

where $F = \mathrm{div}_{X,x}(f),\ F_\pi = \mathrm{div}_{X,x}(f + \pi) \in \mathrm{Div}(X, C)^+$.

We deduce Proposition 6.1 from the following lemma whose proof will be given in §10 (see Lemma 9.1 in §9).

Lemma 6.5. *Let x be a regular closed point of C. Let $F, Z_1, Z_2 \in \mathrm{Div}(X, C)^+$ be such that $F \Cap C$ at x, and $F \Cap Z_i$ and $Z_i \Cap C$ at x for $i = 1, 2$. Let (π, f) be a system of regular parameters such that*

$$F = \mathrm{div}_{X,x}(f),\ Z_1 = \mathrm{div}_{X,x}(u_1 f + \pi),\ Z_2 = \mathrm{div}_{X,x}(u_2 f + \pi),\ C = \mathrm{div}_{X,x}(\pi), \tag{6.8}$$

where $u_1, u_2 \in \mathcal{O}_{X,x}^\times$. For $\alpha \in \mathcal{O}_{X,x}(-D)$, we have

$$\{1 - (u_1 - u_2)\alpha\}_{F,x} + \{1 - u_1\alpha\}_{Z_1,x} - \{1 - u_2\alpha\}_{Z_2,x} \ \in F^{(D+C)}W(X, C).$$

By the assumption of Proposition 6.1, F, Z_1, Z_2 can be described as (6.8). For $a, b_1, b_2 \in \mathcal{O}_{X,x}(-D)$, assume an equality

$$d_{F,x}(a_{|F}) = d_{Z_1,x}((b_1)_{|Z_1}) + d_{Z_2,x}((b_2)_{|Z_2}) \ \in \ \Lambda_x$$

holds. Using (6.2) and (6.8), one can compute

$$d_{Z_i,x}((b_i)_{|Z_i}) = \frac{df + u_i^{-1} d\pi}{f} \quad \text{for } i = 1, 2. \tag{6.9}$$

Thus (6.9) is equivalent to equalities

$$\overline{a} = \overline{b_1} + \overline{b_1},\ \overline{u_2 b_1} + \overline{u_1 b_2} = 0 \ \in \ \mathcal{O}_{X,x}(-D) \otimes \kappa(x),$$

where $\overline{z} \in \mathcal{O}_{X,x}(-D) \otimes \kappa(x)$ is the residue class of $z \in \mathcal{O}_{X,x}(-D)$. This implies that there exists $\alpha \in \mathcal{O}_{X,x}(-D)$ such that

$$\overline{b_1} = \overline{u_1\alpha}, \quad \overline{b_2} = -\overline{u_2\alpha},\ \ \overline{a} = \overline{(u_1 - u_2)\alpha}.$$

Hence the desired equality of Proposition 6.1 follows from Lemma 6.5.

7. Reciprocity along the boundary

The purpose of this section is to reduce Theorem 4.1(i) to Lemma 6.5 and Lemma 7.12 below. In this section we always assume k is finite. A key point is Proposition 7.4 concerning reciprocity property of the map μ_x from Corollary 6.2 when x moves along C. First we prove a preliminary lemma.

Lemma 7.1. *Let X and C be as in Theorem 4.1. Let k'/k be a finite Galois extension with $G = \mathrm{Gal}(k'/k)$ and put $X' = X \otimes_k k'$ and $C' = C \otimes_k k'$. Assume that there exists $\Xi' \in \mathrm{Div}(X', C')^+$ such that $C'_{sing} \subset \Xi'$ and Ξ' is independent of D as in* (4.6) *and a map defined for every $D' = D \otimes_k k'$ with D as in* (4.6)*:*

$$\phi_{X',D'} : H^1(C', \Omega^1_{X'}(-D' - \Xi') \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{C'}) \to W(U_{k'})/\widehat{F}^{(D'+C')}W(X_{k'}, C_{k'}),$$

such that the following diagram commutes (cf. Remark 6.3):

$$\begin{array}{ccc} \bigoplus_{y|x} \frac{\Omega^1_{X'}(-D')\otimes_{\mathcal{O}_{X'}}\mathcal{O}_{C',y}(y)}{\Omega^1_{X'}(-D')\otimes_{\mathcal{O}_{X'}}\mathcal{O}_{C',y}} & \xrightarrow{\sum_{y|x}\iota_y} & H^1(C',\Omega^1_{X'}(-D'-\Xi')\otimes_{\mathcal{O}_{X'}}\mathcal{O}_{C'}) \\ & \searrow^{\sum_{y|x}\mu_y} & \downarrow{\scriptstyle \phi_{X',D'}} \\ & & W(U_{k'})/\widehat{F}^{(D'+C')}W(X_{k'},C_{k'}), \end{array} \tag{7.1}$$

where x is any fixed regular closed point of C not lying in the image of Ξ' and y ranges over the points of C' lying over x. Then there exists $\Xi \in \mathrm{Div}(X,C)^+$ such that $C_{sing} \subset \Xi$ and Ξ is independent of D as in (4.6) *and a map defined for every D as in* (4.6)*:*

$$\phi_{X,D} : H^1(C,\Omega^1_X(-D-\Xi)\otimes_{\mathcal{O}_X}\mathcal{O}_C) \to W(U)/\widehat{F}^{(D+C)}W(X,C),$$

which satisfies the condition (*i*) *of Theorem 4.1.*

Proof. We may replace Ξ' by the sum of its Galois conjugates to assume that there exists $\Xi \in \mathrm{Div}(X,C)^+$ such that $\Xi' = \Xi\otimes_k k'$. Note that (7.1) implies that $\phi_{X',D'}$ is G-equivariant since so are $\sum_{y|x}\iota_y$ and $\sum_{y|x}\mu_y$. The trace map induces an isomorphism

$$Tr_{k'/k} : H^1(C',\Omega^1_{X'}(-D'-\Xi')\otimes_{\mathcal{O}_{X'}}\mathcal{O}_{C'})_G \xrightarrow{\cong} H^1(C,\Omega^1_X(-D-\Xi)\otimes_{\mathcal{O}_X}\mathcal{O}_C),$$

where M_G denotes the coinvariants of a G-module M. We then define $\phi_{X,D}$ as the composite

$$\begin{aligned} H^1(C,\Omega^1_X(-D-\Xi)\otimes_{\mathcal{O}_X}\mathcal{O}_C) &\overset{(Tr_{k'/k})^{-1}}{\longrightarrow} H^1(C',\Omega^1_{X'}(-D'-\Xi')\otimes_{\mathcal{O}_{X'}}\mathcal{O}_{C'})_G \\ &\overset{\phi_{X',D'}}{\longrightarrow} \Big(W(U_{k'})/\widehat{F}^{(D'+C')}W(X_{k'},C_{k'})\Big)_G \overset{N_{k'/k}}{\longrightarrow} W(U)/\widehat{F}^{(D+C)}W(X,C), \end{aligned}$$

where the last map is induced by the norm map $N_{k'/k} : W(U_{k'}) \to W(U)$. The condition of Remark 6.3 for $\phi_{X,D}$ follows from (7.1) thanks to the commutativity of the following diagrams

$$\begin{array}{ccc} \bigoplus_{y|x} \frac{\Omega^1_{X'}(-D')\otimes_{\mathcal{O}_{X'}}\mathcal{O}_{C',y}(y)}{\Omega^1_{X'}(-D')\otimes_{\mathcal{O}_{X'}}\mathcal{O}_{C',y}} & \xrightarrow{\sum_{y|x}\mu_y} & W(U_L)/\widehat{F}^{(D'+C')}W(X_{k'},C_{k'}) \\ \downarrow{\scriptstyle Tr_{k'/k}} & & \downarrow{\scriptstyle N_{k'/k}} \\ \frac{\Omega^1_X(-D)\otimes_{\mathcal{O}_X}\mathcal{O}_{C,x}(x)}{\Omega^1_X(-D)\otimes_{\mathcal{O}_X}\mathcal{O}_{C,x}} & \xrightarrow{\mu_x} & W(U)/\widehat{F}^{(D+C)}W(X,C), \end{array}$$

$$\begin{array}{ccc} \bigoplus_{y|x} \frac{\Omega^1_{X'}(-D')\otimes_{\mathcal{O}_{X'}}\mathcal{O}_{C',y}(y)}{\Omega^1_{X'}(-D')\otimes_{\mathcal{O}_{X'}}\mathcal{O}_{C',y}} & \xrightarrow{\sum_{y|x}\iota_y} & H^1(C',\Omega^1_{X'}(-D'-\Xi')\otimes_{\mathcal{O}_{X'}}\mathcal{O}_{C'}) \\ \downarrow{\scriptstyle Tr_{k'/k}} & & \downarrow{\scriptstyle Tr_{k'/k}} \\ \frac{\Omega^1_X(-D)\otimes_{\mathcal{O}_X}\mathcal{O}_{C,x}(x)}{\Omega^1_X(-D)\otimes_{\mathcal{O}_X}\mathcal{O}_{C,x}} & \xrightarrow{\iota_x} & H^1(C,\Omega^1_X(-D-\Xi)\otimes_{\mathcal{O}_X}\mathcal{O}_C). \end{array}$$

□

Definition 7.2. Let (X,C) be in $\mathcal{C}$ (see Definition 1.9).

(1) Let $H \subset X$ be a hyperplane section. For an integer $d > 0$ let $\mathcal{L}(d) = |dH|$ be the linear system on X of hypersurface sections of degree d. For $t \in \mathcal{L}(d)$ let $F_t \subset X$ be the corresponding section. We write $Gr(1,\mathcal{L}(d))$ for the Grassmannian variety of lines in $\mathcal{L}(d)$.

(2) A pencil $\{F_t\}_{t\in L}$ of hypersurface sections parametrized by $L \in Gr(1, \mathcal{L}(d))$, is admissible for (X, C) if $\Delta_L \cap C = \varnothing$ for the axis Δ_L of L and $F_t \pitchfork C$ for almost all $t \in L$.

By [SGA7II, XVIII 6.6.1] and Lemma 7.1, we may assume by replacing k by its finite extension that for a sufficiently large d, there exists $L \in Gr(1, \mathcal{L}(d))$ admissible for (X, C). In what follows we fix such $L \in Gr(1, \mathcal{L}(d))$ and $\pi \in k(X)$ satisfying:

$$\mathrm{div}_X(\pi) = C + G_0 - G_\infty \quad \text{with } G_0, G_\infty \in \mathrm{Div}(X, C)^+. \tag{7.2}$$

We also fix a finite set $T_L \subset L$ such that

$$F_t \pitchfork C \quad \text{and} \quad F_t \cap C \cap (G_0 \cup G_\infty) = \varnothing \quad \text{for } t \in L - T_L. \tag{7.3}$$

We have the rational map

$$h_L : X \cdots \to L \ ; \ x \to t \text{ such that } x \in F_t.$$

By Definition 7.2(2) h_L is defined at any point of C and it gives rise to

$$\mathcal{O}_{L,t} \hookrightarrow \mathcal{O}_{X,x} \quad \text{for } t \in L \text{ and } x \in F_t \cap C.$$

Lemma 7.3. *For each $t \in L$, choose a prime element $f_t \in \mathcal{O}_{L,t}$.*

(1) *For $t \in L - T_L$ and $x \in F_t \cap C$, we have*

$$\Omega^1_{X,x} = \mathcal{O}_{X,x} \cdot d\pi \oplus \mathcal{O}_{X,x} \cdot df_t.$$

(2) *There exists an effective divisor θ on L independent of the choice of f_t such that $|\theta| = T_L$ and that for any $t \in T_L$ and $x \in F_t \cap C$ and for any*

$$\omega = \frac{1}{f_t}(\xi_1 d\pi + \xi_2 df_t) \in \Omega^1_X \otimes_{\mathcal{O}_X} k(C) \quad \textit{with } \xi_i \in k(C) = \prod_{\lambda\in I} k(C_\lambda),$$

we have the implication

$$\omega \in \Omega^1_X(-F_\theta) \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x} \Rightarrow \xi_i \in \mathcal{O}_{C,x}(-F_t),$$

where $F_\theta = \sum_{t\in T_L} e_t F_t$ for $\theta = \sum_{t\in T_L} e_t t$ with $e_t \in \mathbb{Z}_{\geq 1}$.

Proof. (1) follows from the fact that (π, f_t) is a system of regular parameters of $\mathcal{O}_{X,x}$ if $t \in L - T_L$ and $x \in F_t \cap C$. To show (2), note

$$\Omega^1_X \otimes_{\mathcal{O}_X} k(C) = k(C) \cdot \frac{d\pi}{f_t} \oplus k(C) \cdot \frac{df_t}{f_t}$$

and put

$$\Theta_x = \mathcal{O}_{C,x} \cdot \frac{d\pi}{f_t} \oplus \mathcal{O}_{C,x} \cdot \frac{df_t}{f_t} \ \subset \ \Omega^1_X \otimes_{\mathcal{O}_X} k(C).$$

We see that Θ_x is independent of the choice f_t, namely for $f'_t = uf_t$ with $u \in \mathcal{O}^\times_{L,t}$,

$$\Theta_x = \mathcal{O}_{C,x} \cdot \frac{d\pi}{f'_t} \oplus \mathcal{O}_{C,x} \cdot \frac{df'_t}{f'_t}.$$

Thus (2) follows from the fact that there exists an effective divisor θ on L such that $|\theta| = T_L$ and that for any $t \in T_L$ and $x \in F_t \cap C$, we have

$$\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_{C,x}(-F_\theta) \subset \mathcal{O}_{C,x}(-F_t) \cdot \Theta_x.$$

□

We fix θ as in Lemma 7.3 and put

$$\Xi = F_\theta + G_\infty \in \mathrm{Div}(X,C)^+. \tag{7.4}$$

Note that Ξ is independent of D as in (4.6). Let B be a finite subset of C such that

$$B \cap \Xi = \varnothing \quad \text{and} \quad h_L(B) \cap T_L = \varnothing. \tag{7.5}$$

Note that this implies that B consists of regular points of C by (7.3). For D as in (4.6) consider the maps

$$\begin{array}{ccc} \bigoplus_{x\in B} \Lambda_x & \xrightarrow{\psi_{L,B}} & H^1(C,\Omega^1_X(-D-\Xi)\otimes\mathcal{O}_C) \\ \downarrow{\scriptstyle \sum \mu_x} & & \\ W(U)/F^{(D+C)}W(X,C) & & \end{array}$$

where Λ_x is defined as (6.5) and $\psi_{L,B}$ is induced by (6.1). Taking B large enough, we assume that $\psi_{L,B}$ is surjective. We have

$$\mathrm{Ker}(\psi_{L,B}) = \mathrm{Image}\big(H^0(C,\Omega^1_X(-D-\Xi)\otimes\mathcal{O}_C(B))\big).$$

Proposition 7.4. *Assume $p \neq 2$. Take*

$$\omega \in H^0(C,\Omega^1_X(-D-\Xi)\otimes\mathcal{O}_C(B))$$

and let $\omega_x \in \Lambda_x$ be the image of ω for $x \in B$. Then we have

$$\sum_{x\in B} \mu_x(\omega_x) \ \in\ \widehat{F}^{(D+C)}W(X,C). \tag{7.6}$$

Remark 7.5. It suffices to show the proposition after enlarging B.

The proposition implies the existence of a map

$$\phi_{L,B} : H^1(C,\Omega^1_X(-D-\Xi)\otimes\mathcal{O}_C) \to W(U)/\widehat{F}^{(D+C)}W(X,C)$$

such that the following diagram commutes

$$\begin{array}{ccc} \bigoplus_{x\in B} \Lambda_x & \xrightarrow{\psi_{L,B}} & H^1(C,\Omega^1_X(-D-\Xi)\otimes\mathcal{O}_C) \\ \downarrow{\scriptstyle \sum \mu_x} & \swarrow{\scriptstyle \phi_{L,B}} & \\ W(U)/\widehat{F}^{(D+C)}W(X,C) & & \end{array} \tag{7.7}$$

This implies that $\phi_L = \phi_{L,B}$ is independent of B (depending only on L and Ξ). Take any $x \in C\backslash\Xi$. Note $t = h_L(x) \notin T_L$ (cf. (7.4) and Lemma 7.3(2)) so that we can choose such B that $x \in B$. Then (7.7) implies that $\phi_L = \phi_{L,B}$ satisfies (6.4) for x, which completes the proof of Theorem 4.1(i) by Remark 6.3.

Let $D = \sum_{\lambda\in I} m_\lambda C_\lambda$ be as (4.6). For $\lambda \in I$ consider the map

$$h_\lambda : H^0(C,\Omega^1_X(-D-\Xi)\otimes\mathcal{O}_C(B)) \to \Omega^1_X(-D)\otimes k(C_\lambda) \to \Omega^1_{k(C_\lambda)}\otimes\mathcal{O}_X(-D).$$

Claim 7.6. *After enlarging B, we may assume $h_\lambda(\omega) \neq 0$ for any $\lambda \in I$.*

Proof. We claim that after enlarging B, we can find

$$\xi_\lambda \in H^0(C,\Omega^1_X(-D-\Xi)\otimes\mathcal{O}_C(B)) \quad \text{for } \lambda \in I$$

such that $h_\lambda(\xi_\mu) = 0$ for $\mu \in I - \{\lambda\}$ and $h_\lambda(\xi_\lambda) \neq 0$. Admit the claim for the moment. We may assume further that $h_\lambda(\xi_\lambda) \neq h_\lambda(\omega)$ for any $\lambda \in I$ after replacing ξ_λ by $c\cdot\xi_\lambda$ with $c \in k - \{0,1\}$ if necessary. Then, putting

$$\omega_1 = \sum_{\lambda\in I} \xi_\lambda, \quad \omega_2 = \omega - \omega_1,$$

we have $h_\lambda(\omega_i) \neq 0$ for all $\lambda \in I$ and $i = 1, 2$. Noting $\omega = \omega_1 + \omega_2$, (7.6) for ω follows from that for ω_1 and ω_2, which proves Claim 7.6.

It remains to show the claim. By Bertini's theorem, we can take $F_1, F_2 \in \mathcal{L}(d_1)$ for a sufficiently large $d_1 > 0$ such that $F_i \pitchfork C$ for $i = 1, 2$. Take $f \in k(X)^\times$ such that $\mathrm{div}_X(f) = F_1 - F_2$ and put

$$df \in H^0(X, \Omega^1_X(2F_2)).$$

For a sufficiently large $d_2 >> d_1$, we can take $F \in \mathcal{L}(d_2)$ such that

$$F \pitchfork C,\ F \supset B,\ F \cap \Xi \cap C = \varnothing,\ h_L(F \cap C) \cap T_L = \varnothing,$$

and $F_\lambda \in \mathcal{L}(d_2)$ for $\lambda \in I$ such that

$$F_\lambda = D + \sum_{\mu \in I - \{\lambda\}} C_\mu + 2F_2 + \Xi + G_\lambda \quad \text{with } G_\lambda \in \mathrm{Div}(X, C)^+.$$

Then $B_F = F \cap C$ satisfies the condition (7.5) and $B \subset B_F$. Taking $g_\lambda \in k(X)^\times$ such that $\mathrm{div}_X(g_\lambda) = F_\lambda - F$ for $\lambda \in I$, we have

$$g_\lambda df \in H^0(X, \Omega^1_X(-D - \sum_{\mu \in I - \{\lambda\}} C_\mu - \Xi + F)).$$

Let ξ_λ be the image of $g_\lambda df$ under the composite map

$$H^0(X, \Omega^1_X(-D - \sum_{\mu \in I - \{\lambda\}} C_\mu - \Xi + F)) \hookrightarrow H^0(X, \Omega^1_X(-D - \Xi + F))$$
$$\to H^0(C, \Omega^1_X(-D - \Xi) \otimes \mathcal{O}_C(B_F)).$$

Then $(\xi_\lambda)_{\lambda \in I}$ satisfies the claimed condition for B_F instead of B. □

We now start the proof of Proposition 7.4. We choose an isomorphism over k:

$$\iota : L \simeq \mathbb{P}^1_k = \mathrm{Proj}(k[T_0, T_1]).$$

For a finite extension $\mathbb{F}_q$ of k, let $L(\mathbb{F}_q)$ denote the set of points $x \in L$ such that there exists an embedding $k(x) \to \mathbb{F}_q$ (note that the notation is not the standard one that means the set of k-morphisms $\mathrm{Spec}\mathbb{F}_q \to L$). Put

$$L(\mathbb{F}_q)^o = \{t \in L(\mathbb{F}_q) |\ \iota(t) \neq 0, \infty\}, \quad \text{where } 0 = (1:0), \infty = (0:1) \in \mathbb{P}^1_k.$$

By Lemma 7.1 we may assume that k is large enough so that after a coordinate transformation of $\mathbb{P}^1_k$, we have

(∗1) $h_L(B) \cup T_L \subset L(\mathbb{F}_q)^o$ for a finite extension $\mathbb{F}_q$ of k.

By Claim 7.6 we may assume

(∗2) $h_\lambda(\omega) \neq 0$ for any $\lambda \in I$.

For $t \in L - T_L$ and $x \in F_t \cap C$, let $\omega_{C,x}$ be the image of ω under the map

$$H^0(C, \Omega^1_X(-D - \Xi) \otimes \mathcal{O}_C(B)) \to \Omega^1_X(-D) \otimes \mathcal{O}_{C,x}(B) \to \Omega^1_{C,x}(-D) \otimes \mathcal{O}_C(B).$$

By (∗2), $\omega_{C,x} \neq 0$. Choose an isomorphism $s : \Omega^1_{C,x}(-D) \otimes \mathcal{O}_C(B) \simeq \mathcal{O}_{C,x}$ as $\mathcal{O}_{C,x}$-modules. Then the order of $s(\omega_{C,x}) \in \mathcal{O}_{C,x}$ is independent of the choice and is denoted by $\mathrm{ord}_x(\omega_{C,x})$. The set

$$\{x \in \bigcup_{t \in L - T_L} F_t \cap C \mid \mathrm{ord}_x(\omega_{C,x}) \neq 0\}$$

is finite for ω fixed. Therefore we can choose $\iota : L \simeq \mathbb{P}^1_k$ (possibly after replacing k by a finite extension) in such a way that the following condition holds.

(∗3) $\mathrm{ord}_x(\omega_{C,x}) = 0$ for any $x \in F_\infty \cap C$.

For $t \in L - T_L$ and $x \in F_t \cap C$, if $x \notin B$, then $\omega_x = 0 \in \Lambda_x$ so that $\mu_x(\omega_x) = 0$. Therefore Proposition 7.4 follows from the following claim.

Claim 7.7. *Under the conditions* (∗1), (∗2) *and* (∗3), *we have*

$$\sum_{t\in L(\mathbb{F}_q)^o\setminus T_L}\ \sum_{x\in F_t\cap C} \mu_x(\omega_x) \ \in\ \widehat{F}^{(D+C)}W(X,C).$$

Remark 7.8. It suffices to prove the claim after replacing $\mathbb{F}_q$ by its finite extension.

Proof. We let $0,\infty$ denote the closed points of L which correspond to $0,\infty\in\mathbb{P}^1_{\mathbb{F}_p}$ by $\iota : L\simeq\mathbb{P}^1_k=\mathrm{Proj}(k[T_0,T_1])$. Put

$$\rho = T_0/T_1\in\mathbb{F}_p(\mathbb{P}^1)\quad\text{and}\quad b=1-\rho^{q-1}, \tag{7.8}$$

considered as elements of $k(L)\subset k(X)$ via $\iota : L\simeq\mathbb{P}^1_k$ and h_L. Put

$$W_b=\mathrm{div}_X(b+\pi)+G_\infty+(q-1)F_0.\quad (\text{cf. }(7.2)) \tag{7.9}$$

Note

$$\mathrm{div}_X(b)=\sum_{t\in L(\mathbb{F}_q)^o}F_t\ -(q-1)\cdot F_0, \tag{7.10}$$

Claim 7.9. (1) $W_b\in\mathrm{Div}(X,C)^+$ *and* $W_b\cap C\subset\bigcup_{t\in L(\mathbb{F}_q)^o}F_t\cap C$.

(2) *For* $t\in L(\mathbb{F}_q)^o\setminus T_L$ *and* $x\in F_t\cap C$, $W_b\Cap C$ *at* x *and* $\mathrm{div}_{X,x}(b+\pi)$ *(cf. Remark 6.4) is the irreducible component of* W_b *containing* x.

(3) *For* $t\in L(\mathbb{F}_q)^o$ *and* $x\in F_t\cap C\cap W_b$, *we have*

$$\mathcal{O}_{W_b,x}(-F_t-G_\infty)\subset\mathcal{O}_{W_b,x}(-C).$$

Proof. (1) and (2) follow immediately from (7.2) and (7.3) and (7.10) except that $W_b\cap F_0\cap C=\varnothing$, which holds since $C=\mathrm{div}_X(\pi)$ and $W_b=\mathrm{div}_X(\sigma^{q-1}-1+\pi\sigma^{q-1})$ locally at $F_0\cap C$, where $\sigma=\rho^{-1}$ is a local parameter of F_0 at $F_0\cap C$. The last fact is checked by using (7.3) and (∗1) and $b+\pi=\dfrac{\sigma^{q-1}-1+\pi\sigma^{q-1}}{\sigma^{q-1}}$. To show (3), let π_∞ be a local parameter of G_∞ at x if $x\in G_\infty$ and $\pi_\infty=1$ otherwise. Putting $\pi'=\pi\pi_\infty$, $\pi'\mathcal{O}_{W_b,x}\subset\mathcal{O}_{W_b,x}(-C)$ by (7.2). Note $x\notin F_0$ since $F_t\cap F_0\cap C=\varnothing$ for $t\in L(\mathbb{F}_q)^o$. Hence (7.9) implies $W_b=\mathrm{div}_X(b\pi_\infty+\pi')$ locally at x. Thus we get

$$\mathcal{O}_{W_b,x}(-F_t-G_\infty)=b\pi_\infty\mathcal{O}_{W_b,x}=\pi'\mathcal{O}_{W_b,x}\subset\mathcal{O}_{W_b,x}(-C).$$

This completes the proof of Claim 7.9. □

For ω from Proposition 7.4 we write

$$\omega=\frac{1}{b}(\alpha d\pi+\beta db)\ \text{ in }\ \Omega^1_X(-D)\otimes_{\mathcal{O}_X}k(C)\quad\text{with }\alpha,\beta\in\mathcal{O}_X(-D)\otimes_{\mathcal{O}_X}k(C).$$

Recall $\omega\in H^0(C,\Omega_X(-D+\Xi)\otimes\mathcal{O}_C(B))$ and $B\subset\bigcup_{t\in L(\mathbb{F}_q)^o}F_t\cap C$. Noting (7.10) and

$$\frac{db}{b}=\frac{\rho^{q-2}d\rho}{1-\rho^{q-1}}=\frac{d\sigma}{\sigma(1-\sigma^{q-1})}\ \ (\sigma=\rho^{-1}), \tag{7.11}$$

Lemma 7.3 implies

$$\begin{aligned}&\alpha\in H^0(C,\mathcal{O}_C(-D+(q-1)F_0-\sum_{t\in T_L}F_t-G_\infty)),\\ &\beta\in H^0(C,\mathcal{O}_C(-D+(q-2)F_\infty-\sum_{t\in T_L\cup\{0\}}F_t-G_\infty)),\end{aligned} \tag{7.12}$$

where for a divisor Γ on X, we write $\mathcal{O}_C(\Gamma)=\mathcal{O}_X(\Gamma)\otimes_{\mathcal{O}_X}\mathcal{O}_C$. Since F_0 and F_∞ are ample divisors on X, the restriction maps

$$H^0(X,\mathcal{O}_X(-D+(q-1)F_0-\sum_{t\in T_L}F_t-G_\infty))\to H^0(C,\mathcal{O}_C(-D+(q-1)F_0-\sum_{t\in T_L}F_t-G_\infty))),$$

$$H^0(X,\mathcal{O}_X(-D+(q-2)F_\infty-\sum_{t\in T_L\cup\{0\}}F_t-G_\infty))\to H^0(C,\mathcal{O}_C(-D+(q-2)F_\infty-\sum_{t\in T_L\cup\{0\}}F_t-G_\infty))$$

are surjective for q sufficiently large (cf. Remark 7.8). Thus we can take

$$\begin{aligned}&\tilde{\alpha}\in H^0(X,\mathcal{O}_X(-D+(q-1)F_0-\sum_{t\in T_L}F_t-G_\infty)),\\ &\tilde{\beta}\in H^0(X,\mathcal{O}_X(-D+(q-2)F_\infty-\sum_{t\in T_L\cup\{0\}}F_t-G_\infty)),\end{aligned}\tag{7.13}$$

such that $\omega=\tilde{\omega}\otimes k(C)$ with

$$\tilde{\omega}=\frac{1}{b}(\tilde{\alpha}d\pi+\tilde{\beta}db)\in\Omega^1_X(-D)\otimes_{\mathcal{O}_X}\mathcal{O}_{X,C},$$

where $\mathcal{O}_{X,C}$ is the semi-local ring of X at the generic points of C. By Claim 7.9(2) and (6.7), for $t\in L(\mathbb{F}_q)^o\backslash T_L$ and $x\in F_t\cap C$, we have

$$\mu_x(\omega_x)=\{1+\tilde{\beta}\}_{F_t,x}-\{1+\tilde{\alpha}\}_{F_t,x}+\{1+\tilde{\alpha}\}_{W_b,x}.$$

Hence Claim 7.7 follows from the following.

Claim 7.10. *Under the assumption of* $(*3)$ *we have*

$$\sum_{t\in L(\mathbb{F}_q)^o\backslash T_L}\{1+\tilde{\beta}\}_{F_t}\ \in\ \widehat{F}^{(D+C)}W(X,C),\tag{7.14}$$

$$\sum_{t\in L(\mathbb{F}_q)^o\backslash T_L}\big(\{1+\tilde{\alpha}\}_{F_t}-\sum_{x\in F_t\cap C}\{1+\tilde{\alpha}\}_{W_b,x}\big)\ \in\ F^{(D+C)}W(X,C).\tag{7.15}$$

For the proof of the claim we need Lemma 7.12 below.

Definition 7.11. Let $F\in\mathrm{Div}(X,C)^+$ be a reduced effective Cartier divisor such that $F\Cap C$. For an integer $e>0$, let $\mathcal{P}_{D,e}(F)_{(X,C)}=\mathcal{P}_{D,e}(F)$ denote the set of $a\in H^0(X,\mathcal{O}_X(-D+eF))$ satisfying the condition:

($\circledast$) The image $a_{F\cap C}\in\mathcal{O}_{C,F\cap C}(-D+eF)$ of a is a basis as an $\mathcal{O}_{C,F\cap C}$-module.

Here we note that $\mathcal{O}_{C,F\cap C}$ is a semi-local ring. For $a\in\mathcal{P}_{D,e}(F)$, put

$$Z_a=\mathrm{div}_X(1+a)+eF.$$

By Lemma 1.15(1), $Z_a\in\mathrm{Div}(X,C)^+$ such that $Z_a\cap C=F\cap C$.

Fix $\pi,\pi_D,f\in\mathcal{O}_{X,F\cap C}$ such that locally at $F\cap C$

$$C=\mathrm{div}_X(\pi),\ D=\mathrm{div}_X(\pi_D),\ F=\mathrm{div}_X(f).$$

By the assumption (π,f) is a system of regular parameters in $\mathcal{O}_{X,F\cap C}$. The condition ($\circledast$) is equivalent to the condition that locally at $x\in F\cap C$,

$$Z_a=\mathrm{div}_X(f^e+\pi_D\cdot u)\quad\text{with } u\in\mathcal{O}^\times_{X,F\cap C}.\tag{7.16}$$

The proof of the following lemma will be given later (see Lemma 9.2 in §9).

Lemma 7.12 (increasing order)**.** *Let* Z_a *with* $a\in\mathcal{P}_{D,e}(F)$ *be as above and take* $x\in Z_a\cap C=F\cap C$. *Assume*

$$H^1(X,\mathcal{O}_X(-2D-C+(e-1)F))=H^1(C,\mathcal{O}_C(-2D+F))=0,\tag{$\star$}$$

Assume further $p\neq 2$. *There exists a constant* $c>0$ *depending only on* X *and* D *such that for* $e\geq c$, *we have*

$$\{1+f^{e+1}\mathcal{O}_{Z_a,F\cap C}\}_{Z_a,x}\subset\widehat{F}^{(D+C)}W(X,C).$$

Note that (7.16) implies $f^{e+1}|_{Z_a}=-uf\pi_D$ so that $\{1+f^{e+1}\mathcal{O}_{Z_a,F\cap C}\}_{Z_a,x}$ lies in $F^{(D)}W(X,C)$ without any assumption.

Remark 7.13. By Serre's vanishing theorem the condition $(\star)$ of Lemma 7.12 is satisfied if $F \in \mathcal{L}(d)$ for $d > 0$ sufficiently large (see Definition 7.2).

Proof of Claim 7.10. We first claim $\tilde{\beta} \in \mathcal{P}_{D,q-2}(F_\infty)$. Indeed the condition $(*3)$ implies that the image of β under the restriction map (cf. (7.12))

$$H^0(C, \mathcal{O}_C(-D + (q-2)F_\infty)) \to \mathcal{O}_{C,F_\infty \cap C}(-D + (q-2)F_\infty)$$

is a basis as an $\mathcal{O}_{C,F_\infty \cap C}$-module. Note (cf. Definition 7.11 and (7.13))

$$\begin{aligned} &\mathrm{div}_X(1+\tilde{\beta}) = Z_{\tilde{\beta}} - (q-2)F_\infty, \quad Z_{\tilde{\beta}} \cap C = F_\infty \cap C, \\ &(1+\tilde{\beta})|_{F_t} = 1 \quad \text{for } t \in T_L \cup \{0\} \quad \text{and} \quad b|_{F_\infty} = 1. \end{aligned}$$

In view of (7.10) $\mathrm{div}_X(b)$ and $\mathrm{div}_X(1+\tilde{\beta})$ have no common component which intersects C. Hence we may apply Lemma 1.15(2) to $1+\tilde{\beta}$ and b to get

$$\sum_{t \in L(\mathbb{F}_q)^o \setminus T_L} \{1+\tilde{\beta}\}_{F_t} \ + \ \{1-\rho^{q-1}\}_{Z_{\tilde{\beta}}} = 0 \ \in W(U).$$

Taking q sufficiently large (cf. Remark 7.8), Lemma 7.12 (where we take $e = q-2$ and $F = F_\infty$) implies $\{1-\rho^{q-1}\}_{Z_{\tilde{\beta}}} \in \widehat{F}^{(D+C)}W(X,C)$, which proves (7.14).

To show (7.15), put

$$Z' = \mathrm{div}_X(1+\tilde{\alpha}) + (q-1)F_0 \in \mathrm{Div}(X,C)^+. \tag{7.17}$$

By Lemma 1.15(1) we have $Z' \cap C = F_0 \cap C$. Locally at $F_0 \cap C$,

$$Z' = \mathrm{div}_X(\sigma^{q-1} + \pi_D \gamma) \ \ (\gamma \in \mathcal{O}_{X,F_0 \cap C}), \tag{7.18}$$

where $\sigma = \rho^{-1}$ and π_D is a local parameter of D at $F_0 \cap C$. By (7.9) and (7.10),

$$\mathrm{div}_X(\frac{b+\pi}{b}) = W_b - G_\infty - \sum_{t \in L(\mathbb{F}_q)^o} F_t. \tag{7.19}$$

Since $G_\infty \cap F_0 \cap C = W_b \cap F_0 \cap C = \varnothing$ by Claim 7.9(1), $\mathrm{div}_X(1+\tilde{\alpha})$ and $\mathrm{div}_X(\frac{b+\pi}{b})$ have no common component which intersects C. Hence we may apply Lemma 1.15(2) to $1+\tilde{\alpha}$ and $\frac{b+\pi}{b}$ to get

$$\begin{aligned} \{1+\tilde{\alpha}\}_{W_b} - &\sum_{t \in L(\mathbb{F}_q)^o \setminus T_L} \{1+\tilde{\alpha}\}_{F_t} \\ &- \{\frac{b+\pi}{b}\}_{Z'} + (q-1)\{\frac{b+\pi}{b}\}_{F_0} = 0 \in W(U), \end{aligned} \tag{7.20}$$

where we used the fact $(1+\tilde{\alpha})_{|G_\infty} = 1$ and $(1+\tilde{\alpha})_{|F_t} = 1$ for $t \in T_L$ in view of (7.13). We claim

$$\{1+\tilde{\alpha}\}_{W_b} - \sum_{t \in L(\mathbb{F}_q)^o \setminus T_L} \sum_{x \in F_t \cap C} \{1+\tilde{\alpha}\}_{W_b,x} \ \in F^{(D+C)}W(X,C). \tag{7.21}$$

Indeed, by Claim 7.9(1),

$$\{1+\tilde{\alpha}\}_{W_b} = \sum_{t \in L(\mathbb{F}_q)^o} \sum_{x \in F_t \cap C} \{1+\tilde{\alpha}\}_{W_b,x}.$$

For $t \in T_L$ and $x \in F_t \cap C$, we have $x \notin F_0$ since $F_0 \cap F_t \cap C = \varnothing$. In view of (7.13) this implies $\tilde{\alpha}_{|W_b} \in \mathcal{O}_{W_b,x}(-D-F_t-G_\infty)$. Since $\mathcal{O}_{W_b,x}(-D-F_t-G_\infty) \subset \mathcal{O}_{W_b,x}(-D-C)$ by Claim 7.9(3), we get $\{1+\tilde{\alpha}\}_{W_b,x} \in F^{(D+C)}W(X,C)$.

By (7.21) we are reduced to showing that the last two terms of (7.20) belong to $F^{(D+C)}W(X,C)$. The assertion follows from the fact

$$\frac{b+\pi}{b} = 1 - \frac{\pi\sigma^{q-1}}{1-\sigma^{q-1}} \quad (\sigma = \rho^{-1})$$

and that we have in view of (7.18),

$$(\frac{b+\pi}{b})_{|Z'} = 1 + \frac{\gamma\pi\pi_D}{1-\sigma^{q-1}} \in 1 + \mathcal{O}_{Z',Z'\cap C}(-D-C).$$

This reduces the proof of (7.15) and that of Theorem 4.1 to Lemma 7.12.

□

8. Compatibility with ramification theory

In this section we prove Theorem 4.4. We need some preliminaries.

8.1. **Review of class class field theory for two-dimensional local rings.** Let $(A,\mathfrak{m}_A)$ be an excellent regular henselian two-dimensional local domain with the quotient field K. Assume $F = A/\mathfrak{m}_A$ is finite. Let P be the set of prime ideals of height one in A. For $\mathfrak{p} \in P$ let $A_\mathfrak{p}$ be the henselization of A at $\mathfrak{p}$ and $K_\mathfrak{p}$ (resp. $k(\mathfrak{p})$) be the quotient (resp. residue) field of $A_\mathfrak{p}$. Let C be a reduced effective Cartier divisor on $\mathrm{Spec}(A)$ and put $U = \mathrm{Spec}(A) - C$. Let $P_C \subset P$ be the subset of $\mathfrak{p}$ lying on C. For $\lambda \in P_C$, let K_λ be the quotient field of the henselization of A at λ.

For an effective Cartier divisor D on $\mathrm{Spec}(A)$ with $|D| = C$ ($|D|$ denotes the support of D), we consider the subgroup of $H^1(U) = H^1(U,\mathbb{Q}/\mathbb{Z})$:

$$\mathrm{fil}_D H^1(U) = \mathrm{Ker}\big(H^1(U) \to \bigoplus_{\lambda\in P_C} H^1(K_\lambda)/\mathrm{fil}_{m_\lambda} H^1(K_\lambda)\big).$$

where $m_\lambda \in \mathbb{Z}_{>0}$ for $\lambda \in P_C$ is the multiplicity of λ in D (see 2.1 for the notation). We introduce an idele class group which controls $\pi_1^{\mathrm{ab}}(U)$:

$$W^{KS}(U) := \mathrm{Coker}\big(K_2(K) \xrightarrow{\partial=(\partial_\mathfrak{p},\partial_\lambda)} \bigoplus_{\mathfrak{p}\in P-P_C} k(\mathfrak{p})^\times \oplus \bigoplus_{\lambda\in P_C} K_2(K_\lambda)\big), \tag{8.1}$$

where $\partial_\mathfrak{p}$ for $\mathfrak{p} \notin P_C$ is the tame symbol and ∂_λ for $\lambda \in P_C$ is the map induced by $K \to K_\lambda$. We put

$$\mathrm{C}^{KS}(A,D) = W^{KS}(U)/F^{(D)}W^{KS}(A,C), \tag{8.2}$$

where $F^{(D)}W^{KS}(A,C) \subset W^{KS}(U)$ is the subgroup generated by the images of $V^{m_\lambda}K_2(K_\lambda) \subset K_2(K_\lambda)$ for $\lambda \in P_C$. By the reciprocity law for A we have a canonical map (cf. [Sa1, 1.9], [Sa2, (2.9)], [Sa3, Ch.I])

$$\Psi_U^{KS} : \mathrm{fil}_D H^1(U) \to \mathrm{Hom}(\mathrm{C}^{KS}(A,D),\mathbb{Q}/\mathbb{Z})$$

such that the following diagrams are commutative for $\mathfrak{p} \notin P_C$ and $\lambda \in P_C$:

$$\begin{array}{ccc} \mathrm{fil}_D H^1(U) & \xrightarrow{\Psi_U^{KS}} & \mathrm{Hom}(\mathrm{C}^{KS}(A,D),\mathbb{Q}/\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^1(k(\mathfrak{p})) & \xrightarrow{\Psi_{k(\mathfrak{p})}} & \mathrm{Hom}(k(\mathfrak{p})^\times,\mathbb{Q}/\mathbb{Z}), \end{array} \tag{8.3}$$

$$\begin{array}{ccc} \mathrm{fil}_D H^1(U) & \xrightarrow{\Psi_U^{KS}} & \mathrm{Hom}(\mathrm{C}^{KS}(A,D),\mathbb{Q}/\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{fil}_{m_\lambda} H^1(K_\lambda) & \xrightarrow{\Psi_{K_\lambda}} & \mathrm{Hom}(K_2(K_\lambda)/V^{m_\lambda}K_2(K_\lambda),\mathbb{Q}/\mathbb{Z}) \end{array}$$

where $\Psi_{k(\mathfrak{p})}$ (resp. Ψ_{K_λ}) is the map (2.5) for the 1-dimensional (resp. 2-dimensional) local field $k(\mathfrak{p})$ (resp. K_λ) (cf. (2.5)).

Remark 8.1. Let $I_D \subset A$ be the ideal defining D. For $\alpha \in I_D$ and $\mathfrak{p} \in P - P_C$, the image in $W^{KS}(U)$ of $1+\alpha \bmod \mathfrak{p} \in k(\mathfrak{p})^\times$ lies in $F^{(D)}W^{KS}(A,C)$. Indeed, let $f \in A$ be such that $\mathfrak{p} = (f)$ and put $\xi = \{1+\alpha, f\} \in K_2(K)$. Then one easily sees

$$\partial_\lambda(\xi) \in V^{m_\lambda}K_2(K_\lambda) \quad \text{for } \lambda \in P_C,$$
$$\partial_\mathfrak{q}(\xi) = \begin{cases} 1+\alpha \bmod \mathfrak{p} & \text{for } \mathfrak{q} = \mathfrak{p} \\ 0 & \text{for } \mathfrak{q} \in P - P_C - \{\mathfrak{p}\} \end{cases}$$

Now we assume $D = \mathrm{Spec}(A/(\pi^m))$ $(m \in \mathbb{Z}_{\geq 1})$, where $\pi \in A$ is such that $\lambda = (\pi) \in P$ and that $B_\lambda = A/(\pi)$ is regular. Let $\mathfrak{m}_\lambda = \mathfrak{m}_A B_\lambda$ be the maximal ideal of B_λ. Define

$$\nu_A : \pi^{m-1}\Omega^1_A \otimes_A \mathfrak{m}_\lambda^{-1} \to \mathrm{C}^{KS}(A,D)$$

as the composite

$$\pi^{m-1}\Omega^1_A \otimes_A \mathfrak{m}_\lambda^{-1} \hookrightarrow \pi^{m-1}\Omega^1_{A_\lambda} \otimes_{A_\lambda} k(\lambda) \xrightarrow{\rho^m_{K_\lambda}} K_2(K_\lambda)/V^m K_2(K_\lambda) \to \mathrm{C}^{KS}(A,D),$$

where $\rho^m_{K_\lambda}$ is the map from Lemma 2.5.

Lemma 8.2. *The above map induces a map*

$$\nu_A : \pi^{m-1}\Omega^1_A \otimes_A \mathfrak{m}_\lambda^{-1} \otimes_{B_\lambda} F \to \mathrm{C}^{KS}(A,D).$$

Proof. Choosing $f \in A$ such that $f \bmod (\pi) \in B_\lambda$ is a generator of $\mathfrak{m}_\lambda$, an element $\omega \in \pi^{m-1}\Omega^1_A \otimes_A \mathfrak{m}_\lambda^{-1}$ is written as

$$\omega = \frac{\pi^{m-1}}{f}(a d\pi + b df) \quad (a, b \in A).$$

The description of $\rho^m_{K_\lambda}$ in Lemma 2.5 shows that $\rho^m_{K_\lambda}(\nu_A(f\omega))$ is the image of

$$\gamma := \{1+\pi^m a, \pi\} + \{1+\pi^{m-1}fb, f\} \ \in K_2(K)$$

under ∂_λ in (8.1). One easily check that the images of γ under $\partial_\mathfrak{p}$ for $\mathfrak{p} \in P - \{\lambda\}$ in (8.1) vanish. This proves the desired assertion. □

Lemma 8.3. *Assume $p \neq 2$. Let (π, f) be a system of regular parameters of A.*

(1) *The image of the composite*

$$\mathrm{rar}_A : \mathrm{fil}_m H^1(U) \to \mathrm{fil}_m H^1(K_\lambda) \xrightarrow{\mathrm{rar}_{K_\lambda}} \frac{1}{\pi^m}\Omega^1_{A_\lambda} \otimes_{A_\lambda} k(\lambda)$$

is contained in $\frac{1}{\pi^m}\Omega^1_A \otimes_A B_\lambda$. The diagram

$$\begin{array}{ccccc} \mathrm{fil}_m H^1(U) & \xrightarrow{-\mathrm{rar}_A} & \frac{1}{\pi^m}\Omega^1_A \otimes_A B_\lambda & \longrightarrow & \frac{1}{\pi^m}\Omega^1_A \otimes_A F \\ \downarrow{\scriptstyle \Psi^{KS}_U} & & & \swarrow{\scriptstyle \tau_A} & \\ \left(\mathrm{C}^{KS}(A,D)\right)^\vee & \xrightarrow{(\nu_A)^\vee} & \left(\pi^{m-1}\Omega^1_A \otimes_A \mathfrak{m}_\lambda^{-1} \otimes_{B_\lambda} F\right)^\vee, & & \end{array}$$

$$\tau_A : \frac{1}{\pi^m}\Omega^1_A \otimes_A F \to \left(\pi^{m-1}\Omega^1_A \otimes_A \mathfrak{m}_\lambda^{-1} \otimes_{B_\lambda} F\right)^\vee$$

is commutative, where τ_A is induced by the pairing

$$\frac{1}{\pi^m}\Omega^1_A \otimes_A B_\lambda \times \pi^{m-1}\Omega^1_A \otimes_A \mathfrak{m}_\lambda^{-1} \to \pi^{-1}\Omega^2_A \otimes_A \mathfrak{m}_\lambda^{-1} \xrightarrow{Res_\lambda} \mathfrak{m}_\lambda^{-1}\Omega^1_{B_\lambda}$$
$$\xrightarrow{Res_{\mathfrak{m}_\lambda}} F = B_\lambda/\mathfrak{m}_\lambda \xrightarrow{Tr_{F/\mathbb{F}_p}} \mathbb{F}_p \simeq \mathbb{Z}/p\mathbb{Z}.$$

Moreover τ_A is an isomorphism.

(2) *Putting* $\mathfrak{p}_1 = (f)$, $\mathfrak{p}_2 = (\pi + f) \in P$, *we have*

$$\xi = \nu_A(\frac{1}{f}(\alpha d\pi + \beta df)) \quad \text{for } \alpha, \beta \in (\pi^{m-1})$$

is the image in $\mathrm{C}^{KS}(A, D)$ *of*

$$\eta = \{1+\beta\}_{k(\mathfrak{p}_1)} - \{1+\alpha\}_{k(\mathfrak{p}_1)} + \{1+\alpha\}_{k(\mathfrak{p}_2)} \in \bigoplus_{\mathfrak{p}\in P-P_C} k(\mathfrak{p})^\times.$$

Proof. The first (resp. second) assertion of (1) follows from [Ma, Prop.4.2.1] (resp. (2.6)). Noting that f generates $\mathfrak{m}_\lambda$, τ_A is induced by the pairing

$$\langle\ ,\ \rangle : \frac{1}{\pi^m}\Omega^1_A \otimes_A F \ \times\ \frac{\pi^{m-1}}{f}\Omega^1_A \otimes_A F \to \mathbb{F}_p$$

such that for $\alpha, \beta, \gamma, \delta \in F$

$$\langle \alpha\frac{d\pi}{\pi^m} + \beta\frac{df}{\pi^m}\ , \gamma\frac{\pi^{m-1}d\pi}{f} + \delta\frac{\pi^{m-1}df}{f}\ \rangle = \pm Tr_{F/\mathbb{F}_p}(\alpha\delta - \beta\gamma).$$

This shows that the pairing is non-degenerate so that τ_A is an isomorphism.

To show (2), note

$$\frac{1}{f}(\alpha d\pi + \beta df) = \frac{\alpha}{\pi+f}d(\pi+f) + \frac{\beta-\alpha}{f}df \ \in\ \pi^{m-1}\Omega^1_A \otimes_A \mathfrak{m}_\lambda^{-1}.$$

Hence its image under the map

$$\pi^{m-1}\Omega^1_A \otimes_A \mathfrak{m}_\lambda^{-1} \hookrightarrow \pi^{m-1}\Omega^1_{A_\lambda} \otimes_{A_\lambda} k(\lambda) \xrightarrow{\rho^m_{K_\lambda}} K_2(K_\lambda)/V^m K_2(K_\lambda)$$

is

$$\{1+\alpha, \pi+f\} + \{1+(\beta-\alpha), f\} \equiv \{1+\alpha, \pi+f\} + \{1+\beta, f\} - \{1+\alpha, f\}.$$

This implies the desired assertion. □

8.2. **Review of class field theory of Kato-Saito.** Now we come back to the global setting of §4 where k is assumed finite. We also assume $|D| = C$. For a closed point $x \in C$, let A_x be the henselization of $\mathcal{O}_{X,x}$, and U_x (resp. C_x, resp. D_x) are the base change of $U = X - C$ (resp. C, resp. D) via $\mathrm{Spec}(A_x) \to X$. Let K_x be the fraction field of A_x and P_x be the set of prime ideals of height one in A_x. For $\lambda \in I$, let K_λ be the fraction field of the henselization of $\mathcal{O}_{X,\lambda}$. We introduce the idele class group for U:

$$W^{KS}(U) = \mathrm{Coker}\big(\bigoplus_{Z\subset X} k(Z)^\times \oplus \bigoplus_{\lambda\in I} K_2(K_\lambda) \ \xrightarrow{\partial}\ Z_0(U) \oplus \bigoplus_{x\in C} W^{KS}(U_x)\big),$$

where $W^{KS}(U_x)$ is defined as in (8.1) for U_x, and Z ranges over integral curves on X not contained in C, and ∂ is induced by the maps:

$$\partial_1 : k(Z)^\times \to Z_0(U), \quad \partial_2 : k(Z)^\times \to \bigoplus_{x\in C} W^{KS}(U_x),$$

$$\partial_3 : K_2(K_\lambda) \to Z_0(U), \quad \partial_4 : K_2(K_\lambda) \to \bigoplus_{x\in C} W^{KS}(U_x),$$

where ∂_1 is the divisor map, and ∂_2 is induced by the identification (cf. Definition 1.2):

$$k(Z)^\times_\infty = \bigoplus_{x\in Z\cap C} \bigoplus_{\mathfrak{p}\in P_{x,Z}} k(\mathfrak{p})^\times, \tag{8.4}$$

where $P_{x,Z}$ denotes the set of $\mathfrak{p} \in P_x$ lying over Z, and ∂_3 is the zero map, and ∂_4 is induced by the natural map

$$K_\lambda \to \prod_{x\in C} \prod_{\xi\in P_{x,\lambda}} K_\xi,$$

where $P_{x,\lambda}$ is the set of $\xi \in P_x$ which lies over C_λ (cf. (4.1)) and K_ξ is the fraction field of the henselization of A_x at ξ. We then put

$$\mathrm{C}^{KS}(X,D) = W^{KS}(U)/F^{(D)}W^{KS}(X,C),$$

where $F^{(D)}W^{KS}(X,C) \subset W^{KS}(U)$ is the subgroup generated by the image of $F^{(D)}W^{KS}(A_x, C_x)$ for $x \in C$. We remark that the argument of [KS2, (1.6)] shows an isomorphism (cf. (0.7))

$$\mathrm{C}^{KS}(X,D) \xrightarrow{\cong} H^d(X_{\mathrm{Nis}}, \mathcal{K}_d^M(X,D)),$$

where $\mathcal{K}_d^M(X,D)$ is the relative Milnor K-sheaf introduced in [KS2, (1.3)]. By the class field theory developed in [KS1, Ch.II §3 Th.1] and [KS2, Th.9.1], we have a canonical isomorphism

$$\Psi_U^{KS} : \mathrm{fil}_D H^1(U) \xrightarrow{\cong} \mathrm{Hom}(\mathrm{C}^{KS}(X,D), \mathbb{Q}/\mathbb{Z}) \tag{8.5}$$

which fits into the commutative diagram for any closed point $x \in C$,

$$\begin{array}{ccc} \mathrm{fil}_D H^1(U) & \xrightarrow{\Psi_U^{KS}} & \mathrm{C}^{KS}(X,D)^\vee \\ \downarrow & & \downarrow \\ \mathrm{fil}_{D_x} H^1(U_x) & \xrightarrow{\Psi_{U_x}^{KS}} & \mathrm{C}_x^{KS}(X,D)^\vee \end{array} \tag{8.6}$$

In view of (8.4), there is a natural map

$$\epsilon_U : W(U) = \mathrm{Coker}\Big(\bigoplus_{Z\subset X} k(Z)^\times \to \bigoplus_{Z\subset X} k(Z)_\infty^\times \oplus Z_0(U)\Big) \to W^{KS}(U)\ . \tag{8.7}$$

By Remark 8.1 it induces a canonical map

$$\epsilon_{X,D} : W(U)/F^{(D)}W(X,C) \to \mathrm{C}^{KS}(X,D).$$

The commutativity of the diagrams (8.3) and (8.6) implies that the diagram

$$\begin{array}{ccc} \mathrm{fil}_D H^1(U) & \xrightarrow{\Psi_U^{KS}} & \mathrm{C}^{KS}(X,D)^\vee \\ \downarrow{\scriptstyle \Psi_{X,D}} & & \downarrow{\scriptstyle (\epsilon_{X,D})^\vee} \\ \mathrm{C}(X,D)^\vee & \longrightarrow & \big(W(U)/F^{(D)}W(X,C)\big)^\vee \end{array} \tag{8.8}$$

commutes, where we recall $\mathrm{C}(X,D) = W(U)/\widehat{F}^{(D)}W(X,C)$ (cf. Definition 1.6).

8.3. **Proof of Theorem 4.4.** By Lemma 8.3(1) the image of composite map

$$\mathrm{fil}_D H^1(U) \to \mathrm{fil}_{m_\lambda} H^1(K_\lambda) \xrightarrow{\mathrm{rar}_{K_\lambda}} \Omega_X^1(D) \otimes_{\mathcal{O}_X} k(C_\lambda)$$

is contained in $H^0(C_\lambda^o, \Omega_X^1(D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C_\lambda})$, where C_λ^o denotes a dense open subset of C_λ where C is smooth. Since the natural map

$$H^0(C_\lambda^o, \Omega_X^1(D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C_\lambda}) \to \prod_{x\in C_\lambda^o} \Omega_X^1(D) \otimes_{\mathcal{O}_X} \kappa(x)$$

is injective, we are reduced to showing the commutativity of the diagram

$$\begin{array}{ccccc}
\mathrm{fil}_D H^1(U) & \longrightarrow & H^0(C_\lambda^o, \Omega^1_X(D)\otimes_{\mathcal{O}_X}\mathcal{O}_{C_\lambda}) & \longrightarrow & \Omega^1_X(D)\otimes_{\mathcal{O}_X}\kappa(x) \\
\downarrow & & & \nearrow & \\
\mathrm{C}(X,D)^\vee & \xrightarrow{\mathrm{cc}_X} & H^0(C, \Omega^1_X(D+\Xi)\otimes_{\mathcal{O}_X}\mathcal{O}_C) & &
\end{array}$$

for each $x \in C_\lambda^o$. This follows from the commutative diagram (8.8) and the following commutative diagram by noting that τ_{A_x} is an isomorphism by Lemma 8.3(1):

$$\begin{array}{ccccc}
\mathrm{fil}_D H^1(U) & \longrightarrow & \mathrm{fil}_{D_x} H^1(U_x) & \xrightarrow{-\mathrm{rar}_{U_x}} & \Omega^1_X(D)\otimes_{\mathcal{O}_X}\mathcal{O}_{C,x} \\
\downarrow{\scriptstyle \Psi_U^{KS}} & \boxed{I} & \downarrow{\scriptstyle \Psi_{U_x}^{KS}} & \boxed{III} & \downarrow \\
\mathrm{C}^{KS}(X,D)^\vee & \longrightarrow & \mathrm{C}_x^{KS}(X,D)^\vee & & \Omega^1_X(D)\otimes_{\mathcal{O}_X}\kappa(x) \\
\downarrow{\scriptstyle (\epsilon_{X,D})^\vee} & \boxed{II} & \downarrow{\scriptstyle (\nu_{A_x})^\vee} & \swarrow{\scriptstyle \tau_{A_x}} & \uparrow \\
\big(W(U)/F^{(D)}W(X,C)\big)^\vee & \xrightarrow[(\mu_x)^\vee]{} & \big(\Lambda_x\big)^\vee & & \\
\uparrow & & \boxed{IV} & & \\
\mathrm{C}(X,D)^\vee & & \xrightarrow[\mathrm{cc}_X]{} & & H^0(C, \Omega^1_X(D+\Xi)\otimes_{\mathcal{O}_X}\mathcal{O}_C)
\end{array}$$

Here $\Lambda_x = \Omega^1_X(-D+C)\otimes_{\mathcal{O}_X}\mathcal{O}_{C,x}(x)\otimes_{\mathcal{O}_{C,x}}\kappa(x)$ and rar_{U_x} arises from Lemma 8.3(1). The commutativity of $\boxed{I}$ comes from (8.6), that of $\boxed{II}$ from Lemma 8.3(2) and (6.7), that of $\boxed{III}$ from Lemma 8.3(1), and that of $\boxed{IV}$ from the definition of cc_X and the commutativity of

$$\begin{array}{ccc}
H^0(C, \Omega^1_X(D+\Xi)\otimes_{\mathcal{O}_X}\mathcal{O}_C) & \longrightarrow & \Omega^1_X(D)\otimes_{\mathcal{O}_X}\kappa(x) \\
\downarrow & & \downarrow{\scriptstyle \tau_{A_x}} \\
H^1(C, \Omega^1_X(-D+C-\Xi)\otimes_{\mathcal{O}_X}\mathcal{O}_C)^\vee & \xrightarrow{(\iota_x)^\vee} & \big(\Lambda_x\big)^\vee
\end{array}$$

where the left vertical map is induced by (4.3) and ι_x comes from (6.1). This completes the proof of Theorem 4.4.

Remark 8.4. In case C is a simple normal crossing divisor on X, one can show the map (8.7) induces a map

$$\epsilon_{X,D} : \mathrm{C}(X,D) = W(U)/\widehat{F}^{(D)}W(X,C) \to \mathrm{C}^{KS}(X,D).$$

In view of (8.5), the main result Corollary 3.4 is equivalent to that the map is an isomorphism.

9. Key Lemmas

In order to finish the proof of the main results of the paper, it remains to prove three key lemmas (Lemma 4.3, Lemma 6.5 and Lemma 7.12). In this section we restate them over a general perfect field and give a leitfaden for the proofs, which occupies the remaining sections §10 through §14. In the rest of the paper, k is assumed only perfect (not necessarily finite).

Let the notation be as in §1. Let (X,C) be in $\mathcal{C}$ (see Definition 1.9) and $\{C_\lambda\}_{\lambda\in I}$ be the set of prime components of C. Fix a Cartier divisor

$$D = \sum_{\lambda\in I} m_\lambda C_\lambda \quad \text{with } m_\lambda \geq 1. \tag{9.1}$$

The first key lemma is a relation among three symbols (see Lemma 6.5).

Lemma 9.1 (three term relation). *Let x be a regular closed point of C. Let $F, Z_1, Z_2 \in \mathrm{Div}(X,C)^+$ be such that $F \Cap C$ at x, and $F \Cap Z_i$ and $Z_i \Cap C$ at x for $i = 1, 2$. Let (π, f) be a system of regular parameters such that locally at x,*

$$F = \mathrm{div}_X(f),\ Z_1 = \mathrm{div}_X(\pi - v_1 f),\ Z_2 = \mathrm{div}_X(\pi - v_2 f),\ C = \mathrm{div}_X(\pi),$$

where $v_1, v_2 \in \mathcal{O}_{X,x}^{\times}$. For $\alpha \in \mathcal{O}_{X,x}(-D)$, we have

$$\{1 + (v_1 - v_2)\alpha\}_{F,x} + \{1 + v_1\alpha\}_{Z_1,x} - \{1 + v_2\alpha\}_{Z_2,x} \in F^{(D+C)}W(X,C).$$

The second key lemma refines Lemma 7.12 to the case over a perfect field.

Lemma 9.2 (increasing order). *Let Z_a with $a \in \mathcal{P}_{D,e}(F)$ be as Definition 7.11 and take $x \in Z_a \cap C$. Let $\lambda \in I$ be such that $x \in C_\lambda$. Assume*

$$(*) \qquad H^1(X, \mathcal{O}_X(-2D - C + (e-1)F)) = H^1(C, \mathcal{O}_C(-2D + F)) = 0,$$

Assume $p \neq 2$ and $e \geq m_\lambda(p^n - 1)$ for a given integer $n > 0$. Then we have

$$\{1 + f^{e+1}\mathcal{O}_{Z_a, F\cap C}\}_{Z_a,x} \subset \widehat{F}^{(D+C)}W(X,C) + p^n \widehat{F}^{(1)}W(U). \tag{9.2}$$

Lemma 7.12 follows from Lemma 9.2 and Corollary 5.2 which implies $p^n \widehat{F}^{(1)}W(U) \subset \widehat{F}^{(D+C)}W(X,C)$ for some $n > 0$ if k is finite.

The last key lemma concerns moving elements of $W(U)$ to symbols on curves transversal to C. It refines Lemma 4.3 to the case over a perfect field. Take any dense open subset $V \subset X$ containing the generic points of C. Recall Definition 1.4.

Lemma 9.3 (moving). *Assume $p \neq 2$. For any integers $n, N > 0$, we have*

$$\widehat{F}^{(D)}W(X,C) \subset F^{(D)}_{\Cap V}W(X,C) \ +\ \widehat{F}^{(D+N\cdot C)}W(X,C) + p^n \widehat{F}^{(1)}W(U).$$

Lemma 4.3 follows from Lemma 9.3 and Corollary 5.2 as above.

In §10 through §14 we prove the following Lemmas and implications using Lemma 1.15.

In §10 we prove Lemma 10.1 and prove Lemma 9.1 using Lemma 10.1.

Lemma 9.2 is a direct consequence of Lemma 11.1 consisting of parts (1) and (2).

In §11 we prove Lemma 11.3 and state Lemma 11.6 consisting of parts (1) and (2). Using Lemma 11.3, we prove an implication Lemma 11.6(1) (resp. (2)) ⇒ Lemma 11.1(1) (resp. (2)). We also prove Lemma 11.6(1). This completes the proof of Lemma 11.1(1) and reduces Lemma 9.2 to Lemma 11.6(2).

In §12 we state Lemma 12.1 consisting of parts (1) and (2) and prove an implication Lemma 11.1(1) (resp. (2)) ⇒ Lemma 12.1(1) (resp. (2)). This completes the proof of Lemma 12.1(1) using Lemma 11.1(1) proved in §11.

In §13 we prove an implication Lemma 12.1(1) ⇒ Lemma 13.1. Lemma 11.6(2) is a direct consequence of Lemma 13.1. Thus Lemma 11.1(2) and Lemma 12.1(2) are proved by the implications shown in §11 and §12.

Finally, in §13 we prove Lemma 9.3 using Lemma 12.1(2) and Lemma 10.1.

10. Proof of Key Lemma I

Let the notation be as in §9. In this section we will prove Lemma 9.1.

Lemma 10.1. *Take a regular closed point $x \in C$ and $Z_1, Z_2 \in \mathrm{Div}(X,C)^+$ such that $x \in Z_1 \cap Z_2$ and that $Z_i \Cap C$ at x for $i = 1, 2$ (cf. Definition 1.1). Assume $(Z_1, Z_2)_x \geq e + 1$ for an integer $e \geq 1$. Then*

$$\{1 + \alpha\}_{Z_1,x} - \{1 + \alpha\}_{Z_2,x} \in F^{(D+eC)}W(X,C) \quad \textit{for } \alpha \in \mathcal{O}_{X,x}(-D).$$

The following lemma will be used several times in this paper.

Lemma 10.2. *Let X be a noetherian scheme, E be an effective Cartier divisor on X, and A be an effective Cartier divisor on E. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module such that*

$$H^1(X, \mathcal{F} \otimes \mathcal{O}_X(-E)) = H^1(E, \mathcal{F}|_E \otimes \mathcal{O}_E(-A)) = 0.$$

Then the restriction map $r_A : H^0(X, \mathcal{F}) \to H^0(A, \mathcal{F}|_A)$ is surjective.

Proof. The map r_A factors as

$$H^0(X, \mathcal{F}) \to H^0(E, \mathcal{F}|_E) \to H^0(A, \mathcal{F}|_A).$$

The first (resp. second) map is surjective due to the first (resp. second) vanishing of the cohomology group. This proves the lemma. □

Now we start the proof of Lemma 10.1. For a later purpose we assume only $e \geq 0$ (not necessarily $e \geq 1$). For an integer $d > 0$, let $\mathcal{L}(d) = |dH|$ be as Definition 7.2. Take d sufficiently large so that we can choose F in $\mathcal{L}(d)$ satisfying the conditions:

($\flat$1) $F \pitchfork C$, $x \in F$, $F \cap C \cap (Z_1 \cup Z_2 - x) = \varnothing$, and $F \pitchfork Z_i$ at x for $i = 1, 2$.

($\flat$2) Let $E = (e+1)C$ and $A \subset E$ be the part of $(e+1)F|_E$ supported at x. Then

$$H^1(X, \mathcal{O}_X(-D - 2E + (e+1)F)) = H^1(E, \mathcal{O}_X(-D - E + (e+1)F) \otimes \mathcal{O}_E(-A)) = 0.$$

Then take d' sufficiently large (compared with d chosen above) so that we can choose integral hypersurface section H' in $\mathcal{L}(d')$ satisfying

($\clubsuit$1) $(F \cap C) - x \subset H'$ and $x \notin H'$.

($\clubsuit$2) $H' \pitchfork C$ at $(F \cap C) - x$ and $H' \pitchfork F$ at $(F \cap C) - x$.

For $i = 1, 2$ we put

$$F_i = Z_i + H' \ \in \mathrm{Div}(X, C)^+. \tag{10.1}$$

Choose $\pi, \pi_D, f \in \mathcal{O}_{X, F\cap C}$ such that locally at $F \cap C$,

$$C = \mathrm{div}_X(\pi), \quad D = \mathrm{div}_X(\pi_D), \quad F = \mathrm{div}_X(f)$$

so that (π, f) is a system of regular parameters in $\mathcal{O}_{X, F\cap C}$. Let $\mathfrak{m}_{F\cap C} = (f, \pi)$ denote the radical of $\mathcal{O}_{X, F\cap C}$.

Claim 10.3. *There exist $u_1, u_2 \in \mathcal{O}^\times_{X, F\cap C}$ such that $F_i = \mathrm{div}_X(\pi - u_i f)$ locally at $F \cap C$ and that*

$$u_1 - u_2 \in (\pi - u_2 f, f^e) \subset \mathfrak{m}_{F\cap C}, \ \ u_1 - u_2 \in \mathfrak{m}_y \quad \textit{for } y \in (F \cap C) - x, \tag{10.2}$$

where $\mathfrak{m}_y$ is the maximal ideal of $\mathcal{O}_{X,y}$ (the second condition of (10.2) *is contained in the first except the case $e = 0$).*

Proof. Let $g_i \in \mathcal{O}_{X, F\cap C}$ be a local equation of F_i at $F \cap C$. By ($\clubsuit$1) we have $F \cap C \subset F_i$ so that $g_i \in \mathfrak{m}_{F\cap C}$ and we can write $g_i = a_i \pi + b_i f$ for some $a_i, b_i \in \mathcal{O}_{X, F\cap C}$. By ($\flat$1) and ($\clubsuit$2), $F_i \pitchfork F$ and $F_i \pitchfork C$ at any $y \in F \cap C$. It implies $a_i, b_i \in \mathcal{O}^\times_{X, F\cap C}$ proving the first part of the claim. By (10.1) we have

$$\mathcal{O}_{X, F\cap C}/(\pi - u_2 f) = \mathcal{O}_{F_2, F\cap C} = \mathcal{O}_{Z_2, x} \times \mathcal{O}_{H', F\cap C - x}. \tag{10.3}$$

By ($\flat$1), locally at $F \cap C - x$. we have $H' = F_1 = F_2$ so that

$$(\pi - u_1 f) - (\pi - u_2 f) = (u_1 - u_2) f \equiv 0 \in \mathcal{O}_{H', F\cap C - x}.$$

By ($\clubsuit$2), f is not a zero divisor in $\mathcal{O}_{H', F\cap C - x}$ so we get

$$(u_1 - u_2)|_{H'} = 0 \in \mathcal{O}_{H', F\cap C - x}. \tag{10.4}$$

This implies the second condition of (10.2). By the assumption of Lemma 10.1

$$(Z_1, Z_2)_x = \mathrm{length}_{\mathcal{O}_{Z_2,x}}\big(\mathcal{O}_{Z_2,x}/((u_1 - u_2) f)\big) \geq e + 1,$$

which implies $(u_1-u_2)|_{Z_2} \in (f^e)\mathcal{O}_{Z_2,x}$ noting that f generates the maximal ideal of $\mathcal{O}_{Z_2,x}$ since $Z_2 \pitchfork F$ at x. In view of (10.3) and (10.4), we get

$$(u_1-u_2)|_{F_2} \in (f^e)\mathcal{O}_{F_2,F\cap C},$$

which proves the first condition of (10.2). □

Claim 10.4. *Let the notation be as Claim 10.3. Take any $\vartheta \in \mathcal{O}_{X,x}$. For $e\geq 1$*

$$\{1+u_1^{e+1}\vartheta\pi_D\}_{Z_1,x} - \{1+u_2^{e+1}\vartheta\pi_D\}_{Z_2,x} \in F^{(D+eC)}W(X,C).$$

For $e=0$ we have

$$\{1+(u_1-u_2)\vartheta\pi_D\}_{F,x} + \{1+u_1\vartheta\pi_D\}_{Z_1,x} - \{1+u_2\vartheta\pi_D\}_{Z_2,x} \in F^{(D+C)}W(X,C).$$

By Claim 10.3, we have

$$(u_1-u_2)_{|Z_i} \in (f^e)\mathcal{O}_{Z_i,x} = \mathcal{O}_{Z_i,x}(-eC) \quad \text{for } i=1,2.$$

Hence the first assertion of Claim 10.4 implies Lemma 10.1 noting $u_1,u_2 \in \mathcal{O}_{X,x}^\times$.

We now prove Claim 10.4. Putting

$$\mathcal{F} = \mathcal{O}_X(-D-(e+1)C+(e+1)F),$$

we have an isomorphism

$$\mu : \mathcal{F}\otimes \frac{\mathcal{O}_{X,x}}{(\pi^{e+1},f^{e+1})} \simeq \mathcal{O}_{X,x}/(\pi^{e+1},f^{e+1})$$
$$c\cdot\pi_D(\frac{\pi}{f})^{e+1} \to c \bmod (\pi^{e+1},f^{e+1}) \quad (c\in\mathcal{O}_{X,x}).$$

By the assumption ($\flat$2) Lemma 10.2 implies that the natural map

$$\iota : H^0(X,\mathcal{F}) \to \mathcal{F}\otimes\mathcal{O}_{X,x}/(\pi^{e+1},f^{e+1})$$

is surjective so that we can find

$$a \in H^0(X,\mathcal{O}_X(-D-(e+1)C+(e+1)F)) \tag{10.5}$$

such that

$$\mu(\iota(a)) \equiv \vartheta \ \in \ \mathcal{O}_{X,x}/(\pi^{e+1},f^{e+1}).$$

It implies

$$a=\gamma\pi_D(\frac{\pi}{f})^{e+1} \quad \text{with } \gamma\in\mathcal{O}_{X,F\cap C} \text{ such that } \gamma-\vartheta\in(\pi^{e+1},f^{e+1})\mathcal{O}_{X,x} \tag{10.6}$$

Put

$$Z=\mathrm{div}_X(1+a)+(e+1)F. \tag{10.7}$$

By Lemma 1.15(1) $Z\in \mathrm{Div}(X,C)^+$ and $Z\cap C = F\cap C$. (10.6) implies

$$Z = \mathrm{div}_X(f^{e+1}+\pi_D\pi^{e+1}\cdot\gamma) \ \text{ locally at } F\cap C. \tag{10.8}$$

Put $b=\dfrac{\pi-u_1f}{\pi-u_2f} \in k(X)^\times$. By Claim 10.3 we have

$$\mathrm{div}_X(b) = F_1-F_2+G = Z_1-Z_2+G,$$

where G is a divisor with $G\cap F\cap C=\varnothing$. Noting $Z\cap C=F\cap C$, we may apply Lemma 1.15(3) to $1+a$ and b to get

$$\{1+a\}_{Z_1} - \{1+a\}_{Z_2} - \{b\}_Z + (e+1)\{b\}_F \ \in F^{(D+(e+1)C)}W(X,C). \tag{10.9}$$

Since $F\cap C\cap(Z_1\cup Z_2 - x)=\varnothing$ by ($\flat$1), we get for $i=1,2$

$$a|_{Z_i} \in \mathcal{O}_{Z_i,y}(-D-(e+1)C)) \quad \text{for } y\in(Z_i\cap C)-x.$$

Noting $\pi = u_i f$ in $\mathcal{O}_{Z_i,x}$ (cf. Claim 10.3) and that $Z_i \pitchfork F$ and $Z_i \pitchfork C$ at x, (10.6) implies

$$a|_{Z_i} = u_i^{e+1}\gamma\pi_D \equiv u_i^{e+1}\vartheta\pi_D \in \frac{\mathcal{O}_{Z_i,x}(-D)}{\mathcal{O}_{Z_i,x}(-D-(e+1)C)} \quad (i=1,2).$$

Hence (10.9) implies

$$\{1+u_1^{e+1}\vartheta\pi_D\}_{Z_1,x} - \{1+u_2^{e+1}\vartheta\pi_D\}_{Z_2,x} - \{b\}_Z + (e+1)\{b\}_F \in F^{(D+(e+1)C)}W(X,C).$$

Hence Claim 10.4 follows from the following claims:

(10.10) $$\{b\}_Z - (e+1)\{b\}_F \in F^{(D+eC)}W(X,C) \quad \text{for } e \geq 1,$$

(10.11) $$\{b\}_Z - \{b\}_F - \{1+(u_1-u_2)\vartheta\pi_D\}_{F,x} \in F^{(D+C)}W(X,C) \quad \text{for } e=0,$$

Claim 10.5. *There exists $b' \in \mathcal{O}^\times_{X,F\cap C}$ such that $b'|_Z = b|_Z \in \mathcal{O}_{Z,F\cap C}$ and that*

(10.12) $$(b'/b)|_F \in 1 + \mathcal{O}_{F,F\cap C}(-D-eC) \quad \textit{for } e \geq 1,$$

and that for $e=0$,

(10.13) $$(b'/b)|_F \in 1 + \mathcal{O}_{F,y}(-D-C) \quad \textit{for } y \in (F\cap C) - x,$$

(10.14) $$(b'/b)|_F \equiv 1 + (u_1-u_2)\vartheta\pi_D \in \frac{\mathcal{O}_{F,x}(-D)}{\mathcal{O}_{F,x}(-D-C)}.$$

First we finish the proof of Claim 10.4 (and hence that of Lemma 10.1) assuming the above claim. Since $b' \in \mathcal{O}^\times_{X,F\cap C}$, we may apply Lemma 1.15(4) to $1+a$ and b' to get

$$\{b'\}_Z - (e+1)\{b'\}_F \in F^{(D+(e+1)C)}W(X,C).$$

We have $\{b'\}_Z = \{b\}_Z$ since $b'|_Z = b|_Z \in \mathcal{O}_{Z,F\cap C}$. If $e \geq 1$, (10.12) implies

$$\{b'\}_F - \{b\}_F \in F^{(D+eC)}W(X,C),$$

which implies (10.10). If $e=0$, (10.13) implies $\{b'\}_{F,y} - \{b\}_{F,y} \in F^{(D+C)}W(X,C)$ for $y \in (F\cap C) - x$, and (10.14) implies

$$\{b'\}_{F,x} - \{b\}_{F,x} - \{1+(u_1-u_2)\vartheta\pi_D\}_{F,x} \in F^{(D+C)}W(X,C),$$

which implies (10.11). This completes the proof of Claim 10.4.

Proof of Claim 10.5: For $e \geq 1$ Claim 10.3 implies

(10.15) $$u_2 - u_1 = \eta(\pi - u_2 f) + \beta f^e \quad \text{with } \eta, \beta \in \mathcal{O}_{X,F\cap C}.$$

For $e=0$ we put $\beta = u_2 - u_1$ and $\eta = 0$. Then

$$b = \frac{\pi - u_1 f}{\pi - u_2 f} = 1 + \frac{(u_2-u_1)f}{\pi - u_2 f} = 1 + \eta f + \frac{\beta f^{e+1}}{\pi - u_2 f}.$$

(10.8) implies $f^{e+1}|_Z = -\gamma\pi_D\pi^{e+1} \in \mathcal{O}_{Z,F\cap C}$ so that

$$\left(\frac{\beta f^{e+1}}{\pi - u_2 f}\right)|_Z = \frac{-\beta\gamma\pi_D\pi^{e+1}\delta}{\pi^{e+1} - u_2^{e+1}f^{e+1}} = \theta|_Z \in \mathcal{O}_{Z,F\cap C},$$

where

$$\delta = \frac{\pi^{e+1} - u_2^{e+1}f^{e+1}}{\pi - u_2 f}, \ \theta = \frac{-\beta\gamma\pi_D\delta}{1+\gamma u_2^{e+1}\pi_D} \in \mathcal{O}_{X,F\cap C}.$$

Thus, taking $b' = 1 + \eta f + \theta \in \mathcal{O}_{X,F\cap C}$, we get $b|_Z = b'|_Z \in \mathcal{O}_{Z,F\cap C}$. On the other hand, we have $b|_F = 1$ so that $(b'/b)|_F = 1 + \theta \in \mathcal{O}_{F,F\cap C}$. We have $\delta|_F = \pi^e$ so that $\theta|_F \in \pi_D\pi^e\mathcal{O}_{F,F\cap C}$, which proves (10.12). Assume $e=0$. We have $\delta|_F = 1$. Hence

(10.16) $$1 + \theta|_F = 1 + \frac{(u_1-u_2)\gamma\pi_D}{1+\gamma u_2\pi_D} \in \mathcal{O}_{F,F\cap C}.$$

By (10.6) and Claim 10.3, we have

$$\gamma|_F - \vartheta|_F \in \pi\mathcal{O}_{F,x}, \quad \text{and} \quad (u_1 - u_2)|_F \in \pi\mathcal{O}_{F,y} \quad \text{for } y \in (F \cap C) - x.$$

Hence (10.16) implies (10.13) and (10.14), and Claim 10.5 is proved.

Finally we prove Lemma 9.1. In case $(Z_1, Z_2)_x \geq 2$, we have $v_1 - v_2 \in \mathfrak{m}_x$ so that $(v_1 - v_2)|_F \in \mathcal{O}_{F,x}(-C)$ and $(v_1 - v_2)|_{Z_i} \in \mathcal{O}_{Z_i,x}(-C)$ for $i = 1, 2$. Hence Lemma 9.1 follows from Lemma 10.1. We assume $Z_1 \Cap Z_2$ at x. By Lemma 10.1, we may replace F by any curve which is regular at x and tangent to F at x. Hence we may take F as in the beginning of the proof of Lemma 10.1. Then take H' and F_i for $i = 1, 2$ as before. Then Claim 10.3 and (♣1) imply $Z_i = \mathrm{div}_X(\pi - u_i f)$ locally at x for $i = 1, 2$. Hence we get $u_i - v_i \in \mathfrak{m}_x$ so that $(u_i - v_i)|_F \in \mathcal{O}_{F,x}(-C)$. Thus Lemma 9.1 follows from the second assertion of Claim 10.4.

11. Proof of Key Lemma II

Let (X, C) be in $\mathcal{C}$ (see Definition 1.9) and $\{C_\lambda\}_{\lambda \in I}$ be the set of irreducible components of C. Fix a Cartier divisor

$$D = \sum_{\lambda \in I} m_\lambda C_\lambda \quad \text{with } m_\lambda \geq 1.$$

Lemma 11.1. *Let Z_a with $a \in \mathcal{P}_{D,e}(F)$ be as Definition 7.11 and take $x \in Z_a \cap C$ and $\lambda \in I$ such that $x \in C_\lambda$. Assume*

$$H^1(X, \mathcal{O}_X(-2D - C + (e-1)F)) = H^1(C, \mathcal{O}_C(-2D + F)) = 0,$$

(1) *Assuming $e \geq m_\lambda$, we have*

$$\{1 + f^{e+1}\mathcal{O}_{Z_a, F\cap C}\}_{Z_a,x} \subset F_{\mathcal{B}}^{(2D - C_\lambda)} W(X, C) + F^{(D+C)} W(X, C).$$

(2) *Assuming $(p, 2m_\lambda) = 1$ and $e \geq m_\lambda(p^n - 1)$ for a given integer $n > 0$, we have*

$$\{1 + f^{e+1}\mathcal{O}_{Z_a, F\cap C}\}_{Z_a,x} \subset \widehat{F}^{(D+C)} W(X, C) + p^n \widehat{F}^{(1)} W(U).$$

Lemma 9.2 is a consequence of Lemma 11.1 by the following remark.

Remark 11.2. Assuming $p \neq 2$, Lemma 11.1 implies the following:

$$\{1 + f^{e+1}\mathcal{O}_{Z_a, F\cap C}\}_{Z_a,x} \subset \widehat{F}^{(D+C)} W(X, C) + p^n \widehat{F}^{(1)} W(U) \tag{11.1}$$

for $e \geq m_\lambda(p^n - 1)$. Indeed, if $m_\lambda \geq 2$, then $2D - C_\lambda \geq D + C$ and Lemma 11.1(1) implies (11.1) for $e \geq m_\lambda$. If $m_\lambda = 1$, then Lemma 11.1(2) implies (11.1) for $e \geq m_\lambda(p^n - 1)$.

In this section we prove Lemma 11.1(1). The proof of Lemma 11.1(2) will be complete in §13.

Lemma 11.3. *Let the notation be as in Definition 7.11 and the paragraph after it. Fix an integer $1 \leq \epsilon \leq e - 1$. Assume*

$$H^1(X, \mathcal{O}_X(-2D - C + \epsilon F)) = H^1(C, \mathcal{O}_C(-2D + F)) = 0. \tag{11.2}$$

For $a \in \mathcal{P}_{D,e}(F)$, we have

$$\{1 + \frac{\pi_D^2}{f^\epsilon}\mathcal{O}_{Z_a, F\cap C}\}_{Z_a} \subset \{1 + \frac{\pi_D^2 \pi}{f^\epsilon}\mathcal{O}_{Z_a, F\cap C}\}_{Z_a} + \{1 + \frac{\pi_D^2}{f}\mathcal{O}_{Z_a, F\cap C}\}_{Z_a} + F^{(2D)} W(X, C).$$

Recall $\pi, \pi_D, f \in \mathcal{O}_{X, F\cap C}$ are such that locally at $F \cap C$

$$C = \mathrm{div}_X(\pi), \; D = \mathrm{div}_X(\pi_D), \; F = \mathrm{div}_X(f)$$

so that (π, f) is a system of regular parameters in $\mathcal{O}_{X, F\cap C}$.

Remark 11.4. By (7.16), $\frac{\pi_D^2}{f^{e-1}} \equiv u^{-2}f^{e+1} \in \mathcal{O}_{Z_a,F\cap C}$ so $\frac{\pi_D^2}{f^\epsilon} \in f^{e+1}\mathcal{O}_{Z_a,F\cap C}$.

Proof. We fix $a \in \mathcal{P}_{D,e}(F)$ and write $Z = Z_a$. Take $1 + \frac{\pi_D^2}{f^\epsilon}\alpha$ with $\alpha \in \mathcal{O}_{Z,F\cap C}$. By (11.2), letting $A = (\epsilon - 1)F|_C \subset C$, Lemma 10.2 implies that the natural map

$$H^0(X, \mathcal{O}_X(-2D + \epsilon F)) \to H^0(A, \mathcal{O}_X(-2D + \epsilon F)|_A)$$

is surjective. The target group is isomorphic to

$$\mathcal{O}_X(-2D + \epsilon F) \otimes \mathcal{O}_{X,F\cap C}/(f^{\epsilon-1}, \pi) \simeq \mathcal{O}_X(-2D + \epsilon F) \otimes \mathcal{O}_{Z,F\cap C}/(f^{\epsilon-1}, \pi),$$

where the isomorphism holds since $\mathcal{O}_{X,F\cap C}(-Z) \subset (f^e, \pi) \subset (f^{\epsilon-1}, \pi)$ by (7.16) and the assumption $\epsilon \le e - 1$. Hence there exist

$$a' \in H^0(X, \mathcal{O}_X(-2D + \epsilon F))$$

and $\beta, \gamma \in \mathcal{O}_{Z,F\cap C}$ such that

$$a'|_Z = \frac{\pi_D^2}{f^\epsilon}\alpha + \frac{\pi_D^2}{f}\beta + \frac{\pi_D^2\pi}{f^\epsilon}\gamma \ \in \mathcal{O}_{Z,F\cap C}.$$

This implies

$$(1 + a')|_Z = (1 + \frac{\pi_D^2}{f^\epsilon}\alpha)(1 + \frac{\pi_D^2}{f}\beta')(1 + \frac{\pi_D^2\pi}{f^\epsilon}\gamma') \ \in \mathcal{O}^\times_{Z,F\cap C}, \tag{11.3}$$

where $\beta' = \beta v^{-1}$ with

$$v = 1 + \frac{\pi_D^2}{f^\epsilon}\alpha = 1 - \pi_D f^{e-\epsilon}u^{-1}\alpha \ \in \mathcal{O}^\times_{Z,F\cap C} \quad \text{(cf. Remark 11.4)}$$

and $\gamma' = \gamma v^{-1}(1 + \frac{\pi_D^2}{f}\beta')^{-1} \in \mathcal{O}_{Z,F\cap C}$. Put

$$Z' = \mathrm{div}_X(1 + a') + \epsilon F.$$

By Lemma 1.15(1), $Z' \in \mathrm{Div}(X, C)^+$ with $Z' \cap C = F \cap C$. Locally at $F \cap C$,

$$a' = c\frac{\pi_D^2}{f^\epsilon} \quad \text{and} \quad Z' = \mathrm{div}_X(f^\epsilon + \pi_D^2 \cdot c) \quad \text{with } c \in \mathcal{O}_{X,F\cap C}. \tag{11.4}$$

By (7.16), letting $b = f^e + \pi_D \cdot u \ \in k(X)^\times$, we have

$$\mathrm{div}_X(b) = Z + G \quad \text{for } G \in \mathrm{Div}(X) \ \text{ such that } |G| \cap F \cap C = \varnothing. \tag{11.5}$$

Noting $Z' \cap C = F \cap C$, we can apply Lemma 1.15(3) to $1 + a'$ and b to get

$$\{1 + a'\}_Z - \epsilon\{b\}_F + \{b\}_{Z'} \in F^{(2D)}W(X, C).$$

Hence (11.3) implies

$$\{1 + \frac{\pi_D^2}{f^\epsilon}\alpha\}_Z + \{1 + \frac{\pi_D^2}{f}\beta'\}_Z + \{1 + \frac{\pi_D^2\pi}{f^\epsilon}\gamma'\}_Z - \epsilon\{b\}_F + \{b\}_{Z'} \in F^{(2D)}W(X, C).$$

Thus the proof of Lemma 11.3 is reduced to showing

$$-\epsilon\{b\}_F + \{b\}_{Z'} \in F^{(2D)}W(X, C). \tag{11.6}$$

Claim 11.5. *There exists* $b' \in \pi_D \cdot \mathcal{O}^\times_{X,F\cap C}$ *such that*

$$b|_{F\cup Z'} = b'|_{F\cup Z'} \in \mathcal{O}_{F\cup Z',F\cap C}. \tag{11.7}$$

Proof. We have

$$b = f^e + \pi_D u = u\pi_D(1 - \pi_D f^{e-\epsilon}cu^{-1}) + f^{e-\epsilon}(f^\epsilon + \pi_D^2 c) \ \in \ \mathcal{O}_{X,F\cap C}.$$

The second term is divisible by a local equation of $F \cup Z'$ around $F \cap C$ due to (11.4) and the assumption $\epsilon \le e - 1$. Thus it suffices to take $b' = u\pi_D(1 - \pi_D f^{e-\epsilon}cu^{-1})$. □

By Lemma 1.15(4) applied to $1+a'$ and b', we have

$$-\epsilon\{b'\}_F + \{b'\}_{Z'} \in F^{(2D)}W(X,C).$$

which implies (11.6) since $\{b\}_F = \{b'\}_F$ and $\{b\}_{Z'} = \{b'\}_{Z'}$ thanks to (11.7) and the fact $Z' \cap C = F \cap C$. This completes the proof of Lemma 11.3. □

Let the assumption be as in Lemma 11.1. By Lemma 11.3 and Remark 11.4,

$$\begin{aligned}\{1+f^{e+1}\mathcal{O}_{Z_a,F\cap C}\}_{Z_a} &\subset \{1+\frac{\pi_D^2}{f^{e-1}}\mathcal{O}_{Z_a,F\cap C}\}_{Z_a}\\ &\subset \{1+\frac{\pi_D^2\pi}{f^{e-1}}\mathcal{O}_{Z_a,F\cap C}\}_{Z_a} + \{1+\frac{\pi_D^2}{f}\mathcal{O}_{Z_a,F\cap C}\}_{Z_a} + F^{(2D)}W(X,C)\\ &\subset \{1+\frac{\pi_D^2}{f}\mathcal{O}_{Z_a,F\cap C}\}_{Z_a} + F^{(D+C)}W(X,C)\end{aligned}$$

where the last inclusion holds since $\pi_D^2\pi/f^{e-1} \equiv -u^{-1}\pi_D\pi \in \mathcal{O}_{Z_a,F\cap C}$ by (7.16). Hence Lemma 11.1 follows from the following.

Lemma 11.6. *Fix a regular closed point $x \in C$ and let $\lambda \in I$ be such that $x \in C_\lambda$. Let $F \in \mathrm{Div}(X,C)^+$ be such that $x \in F$ and $F \pitchfork C$ at x. Take a local parameter f (resp. π) of F (resp. C) at x so that (π, f) is a system of regular parameters at x. Let $Z \in \mathrm{Div}(X,C)^+$ be such that locally at x,*

$$Z = \mathrm{div}_X(c\cdot f^e + \pi^m) \quad \text{with } c \in \mathcal{O}_{X,x}, \tag{11.8}$$

where $m > 0$ is an integer. Consider the object $(\tilde{X},\tilde{C})$ of $\mathcal{B}_X$, where $g : \tilde{X} \to X$ is the blowup at x and $\tilde{C} = g^*C$ (cf. Definition 1.10). Let $Z' \subset \tilde{X}$ be the proper transform of Z.

(1) *If $e \geq m$, we have*

$$\frac{\pi_D^2}{f} \in \mathcal{O}_{Z',Z'\cap\tilde{C}}(-g^*(2D - C_\lambda)).$$

In particular we have

$$\{1+\frac{\pi_D^2}{f}\mathcal{O}_{Z,x}\}_{Z,x} \in F_{\mathcal{B}}^{(2D-C_\lambda)}W(X,C).$$

(2) *Assuming $(p, 2m_\lambda) = 1$ and $e \geq m(p^n - 1)$ for an integer $n > 0$, we have*

$$\{1+\frac{\pi_D^2}{f}\mathcal{O}_{Z,x}\}_{Z,x} \in \widehat{F}^{(2D)}W(X,C) + p^n\widehat{F}^{(1)}W(U).$$

In this section we prove Lemma 11.6(1). The proof of Lemma 11.6(2) will be given in §13. Note that Lemma 11.1(1) follows from Lemma 11.6(1).

Let C'_λ be the proper transform of C_λ and $E = g^{-1}(x)$ be the exceptional divisor so that $g^*C_\lambda = C'_\lambda + E$. Let F' be the proper transform of F in $\tilde{X}$. We claim $Z' \cap F' \cap E = \varnothing$. Indeed f/π (resp. π) is a local parameter of F' (resp. E) at $F' \cap E$. By (11.8) and the assumption $e \geq m$, Z' is defined locally at $F' \cap E$ by $c(f/\pi)^e\pi^{e-m} + 1$, which implies the claim. Noting that f is a local parameter of E at any $y \in E - (F' \cap E)$, we have

$$\frac{\pi_D^2}{f} \in \mathcal{O}_{Z',Z'\cap E}(-2g^*D + E) \subset \mathcal{O}_{Z',Z'\cap E}(-g^*(2D - C_\lambda))$$

noting $-2g^*D + E = -2g^*D + g^*C_\lambda - C'_\lambda = -g^*(2D - C_\lambda) - C'_\lambda$. Lemma 11.6(1) follows from this.

12. Proof of Key Lemma III

Let (X, C) be in $\mathcal{C}$ (see Definition 1.9) and $\{C_\lambda\}_{\lambda\in I}$ be the set of irreducible components of C. Fix a Cartier divisor

$$D = \sum_{\lambda\in I} m_\lambda C_\lambda \quad \text{with } m_\lambda \geq 1.$$

The following lemma is a preliminary for the proof of Lemma 9.3 whose proof will be completed in §13. In this section we prove only Lemma 12.1(1), which is necessary for the proof of Lemma 13.1. Lemma 11.6(2) and hence Lemma 11.1(2) will be deduced from Lemma 13.1. Lemma 12.1(2) will be then deduced from Lemma 11.1(2).

Lemma 12.1. *Take a reduced $Z \in \mathrm{Div}(X, C)^+$ and $x \in Z \cap C$ and a dense open subset $V \subset X$ containing all generic points of C.*

(1) *We have (cf. Definition 1.4(4))*

$$\{1 + \mathcal{O}_{Z,x}(-D)\}_{Z,x} \subset F^{(D)}_{\Subset V} W(X, C) \ + \ F^{(2D-C)}_{\mathcal{B}} W(X, C) + F^{(D+C)} W(X, C),$$

(2) *Assume $p \neq 2$. For any integer $n > 0$, we have*

$$\{1 + \mathcal{O}_{Z,x}(-D)\}_{Z,x} \subset F^{(D)}_{\Subset V} W(X, C) \ + \ \widehat{F}^{(D+C)} W(X, C) + p^n \widehat{F}^{(1)} W(U).$$

Remark 12.2. (1) Letting (cf. (9.1))

$$C_0 = \sum_{\lambda\in J} C_\lambda \quad \text{with } \ J = \{\lambda \in I \mid m_\lambda \geq 2\}, \tag{12.1}$$

Lemma 12.1(1) implies

$$F^{(D)} W(X, C) \subset F^{(D)}_{\Subset V} W(X, C) \ + \ F^{(D+C_0)}_{\mathcal{B}} W(X, C),$$

noting $F^{(D+C)} W(X, C) \subset F^{(D+C_0)} W(X, C) \subset F^{(D+C_0)}_{\mathcal{B}} W(X, C)$.

(2) Assuming $p \neq 2$, Lemma 12.1(2) implies

$$F^{(D)} W(X, C) \subset F^{(D)}_{\Subset V} W(X, C) \ + \ \widehat{F}^{(D+C)} W(X, C) + p^n \widehat{F}^{(1)} W(U).$$

Proof of Lemma 12.1. Take an integer $d > 0$ large enough that the linear system $|dH - Z|$ on X ($H \subset X$ is a hyperplane section) is sufficiently very ample. By Bertini's theorem there are $F_0, F_\infty \in \mathcal{L}(d) = |dH|$ satisfying the following:

(♯1) $H^1(C, \mathcal{O}_C(-2D + F)) = 0$ for any $F \in \mathcal{L}(d)$ (cf. Remark 11.2),
(♯2) $F_0 = Z + G$ for $G \in \mathrm{Div}(X, C)^+$ such that $G \Subset C$, $G \cap C \subset V$, $G \cap C \cap Z = \varnothing$,
(♯3) $F_\infty \Subset C$ and $F_\infty \cap C \subset V$, and $F_\infty \cap C \cap F_0 = \varnothing$.

Let $L \in Gr(1, \mathcal{L}(d))$ be the line passing through F_0 and F_∞ and consider the pencil $\{F_t\}_{t\in L}$. By (♯3) the following conditions hold:

(♯4) $F_t \cap F_{t'} \cap C = \varnothing$ for $t \neq t' \in L$,
(♯5) there exists a finite subset $\Sigma \subset L$ such that $F_t \Subset C$ and $F_t \cap C \subset V$ for $t \in L - \Sigma$.

Choose an identification $L \simeq \mathbb{P}^1 = \mathrm{Proj}(k[T_0, T_1])$ such that $F_0 = F_t$ with $t = (1 : 0) \in \mathbb{P}^1$ and $F_\infty = F_t$ with $t = (0 : 1) \in \mathbb{P}^1$. Put

$$W := Z + F_\infty \in \mathrm{Div}(X, C)^+.$$

Take $q = p^N$ with N sufficiently large and a finite subset $S \subset \mathbb{P}^1(\mathbb{F}_q)\backslash(\{0, 1, \infty\} \cup \Sigma)$ with $s = \deg_L(S)$ large enough that the following conditions hold for $F \in \mathcal{L}(d)$:

(♠1) $H^1(X, \mathcal{O}_X(-2D - C + (q-2)F)) = 0$.
(♠2) $H^1(X, \mathcal{O}_X(-D - W + (s-1)F)) = H^1(W, \mathcal{O}_W(-2D + (s-1)F)) = 0$.

Put
$$\Theta = -\sum_{t\in\Lambda} F_t + (q-2)F_\infty, \quad \Lambda = \mathbb{P}^1(\mathbb{F}_q) - (S \cup \{0,\infty\}).$$
We have $\mathcal{O}_X(\Theta) \simeq \mathcal{O}_X((s-1)F_\infty)$ so that (♠2) implies
$$H^1(X, \mathcal{O}_X(-D+\Theta-W)) = H^1(W, \mathcal{O}_W(-D+\Theta)\otimes\mathcal{O}_W(-D)) = 0.$$
Thus Lemma 10.2 implies that the natural map
$$r_W : H^0(X, \mathcal{O}_X(-D - \sum_{t\in\Lambda} F_t + (q-2)F_\infty)) \to \frac{\mathcal{O}_{W,W\cap C}(-D+\Theta)}{\mathcal{O}_{W,W\cap C}(-2D+\Theta)}$$
is surjective. (♯2), (♯3) and (♯4) imply
$$Z \cap C \cap |\Theta| = \varnothing \quad \text{and} \quad \mathcal{O}_{F_\infty, F_\infty\cap C}(\Theta) = \mathcal{O}_{F_\infty,F_\infty\cap C}((q-2)F_\infty).$$
Hence the target of r_W is equal to
$$\frac{\mathcal{O}_{Z,Z\cap C}(-D)}{\mathcal{O}_{Z,Z\cap C}(-2D)} \oplus \frac{\mathcal{O}_{F_\infty,F_\infty\cap C}(-D+(q-2)F_\infty)}{\mathcal{O}_{F_\infty,F_\infty\cap C}(-2D+(q-2)F_\infty)}.$$
Since $F_\infty \pitchfork C$ we have an isomorphism
$$\nu : \frac{\mathcal{O}_{F_\infty,F_\infty\cap C}(-D+(q-2)F_\infty))}{\mathcal{O}_{F_\infty,F_\infty\cap C}(-D-C+(q-2)F_\infty))} \simeq \bigoplus_{x\in F_\infty\cap C} k(x).$$
Consider the composite map
$$\mu : H^0(X, \mathcal{O}_X(-D+\Theta)) \to \mathcal{O}_{F_\infty,F_\infty\cap C}(-D+(q-2)F_\infty)) \xrightarrow{\nu} \bigoplus_{x\in F_\infty\cap C} k(x).$$
By the surjectivity of r_W, for given $\alpha \in \mathcal{O}_{Z,x}(-D)$, one can find
$$a \in H^0(X, \mathcal{O}_X(-D - \sum_{t\in\Lambda} F_t + (q-2)F_\infty))$$
satisfying the following conditions:

(♣1) $a|_Z = \alpha \bmod \mathcal{O}_{Z,x}(-2D)$ and $a|_Z \in \mathcal{O}_{Z,y}(-2D)$ for all $y \in (Z\cap C) - x$,
(♣2) $\mu(a) = (1,\ldots,1)$.

(♣2) implies $a \in \mathcal{P}_{D,q-2}(F_\infty)$ and we put (cf. Definition 7.11)
$$Z_a = \mathrm{div}_X(1+a) + (q-2)F_\infty.$$
Take rational functions
$$\rho = \frac{T_0}{T_1 - T_0} \quad \text{and} \quad b = 1 - \rho^{q-1} \text{ on } L = \mathbb{P}^1.$$
We have
$$\mathrm{div}_X(b) = \sum_{t\in\mathbb{A}^1(\mathbb{F}_q)-\{0,1\}} F_t + (1-q)F_1 + F_0 \quad \text{with } F_0 = Z + G.$$
By (♯2), (♯3) and (♯4), no component of $\mathrm{div}_X(b)$ passes through $F_\infty \cap C$. Noting
$$b|_{F_\infty} = 1 \quad \text{and} \quad (1+a)|_{F_t} = 1 \quad \text{for } t \in \Lambda = \mathbb{P}^1(\mathbb{F}_q) - (S\cup\{0,\infty\}),$$
Lemma 1.15(1) implies

(12.2) $$W(X,C) \ni 0 = \{1+a\}_Z + \{1+a\}_G + \sum_{t\in S}\{1+a\}_{F_t} + \{1-\rho^{q-1}\}_{Z_a}.$$

(♣1) implies
$$\{1+a\}_{Z,x} - \{1+\alpha\}_{Z,x} \in F^{(2D)}W(X,C),$$
$$\{1+a\}_{Z,y} \in F^{(2D)}W(X,C) \quad \text{for } y \in (Z\cap C) - x.$$

By (♯2) and (♯4), $F_\infty \cap G \cap C = \varnothing$ and $F_\infty \cap F_t \cap C = \varnothing$ for $t \in S$ so that

$$a \in \mathcal{O}_{X,G\cap C}(-D) \quad \text{and} \quad a \in \mathcal{O}_{X,F_t\cap C}(-D) \quad \text{for } t \in S.$$

Finally, using (♯1) and (♣1), Lemma 11.1(1) with $e = q-2$ implies

$$\{1-\rho^{q-1}\}_{Z_a} \in F_{\mathcal{B}}^{(2D-C)}W(X,C) + F^{(D+C)}W(X,C).$$

Recalling $G \Cap C \subset V$ and $F_t \Cap C \subset V$ for $t \in S$, this proves Lemma 12.1(1) since the third term of (12.2) lies in $F_{\Cap V}^{(D)}W(X,C)$. Lemma 12.1(2) would follow from Lemma 11.1(2) (cf. Remark 11.2) whose proof will be complete in §13. □

13. Proof of Key Lemma IV

In this section we finish the proof of Lemma 11.6(2). Note that this completes the proof of Lemmas 11.1 and 12.1 and hence Lemma 9.2. Lemma 11.6(2) follows from the following lemma where we take $2D$ in Lemma 11.6 for D in Lemma 13.1. Let the notation be as in the beginning of §12. Recall

$$D = \sum_{\lambda\in I} m_\lambda C_\lambda \quad \text{with } m_\lambda \geq 1.$$

Lemma 13.1. *Fix a regular closed point $x \in C$ and an integer $n > 0$. Let $F \in \mathrm{Div}(X,C)^+$ be such that $x \in F$ and $F \Cap C$ at x and $\lambda \in I$ be such that $x \in C_\lambda$. Take a local parameter f (resp. π) of F (resp. C) at x so that (π, f) is a system of regular parameters at x. Let $Z \in \mathrm{Div}(X,C)^+$ be such that locally at x,*

$$Z = \mathrm{div}_X(u\cdot f^b + \pi^a) \quad \text{with } u \in \mathcal{O}_{X,x},\ a,b \in \mathbb{Z}_{>0}. \tag{13.1}$$

Assume

$$(p, m_\lambda) = 1 \text{ and } b \geq a(p^n - 1). \tag{13.2}$$

Then we have (cf. Definition 1.4)

$$\{1+\mathcal{O}_{Z,x}(-D+F)\}_{Z,x} \in \widehat{F}^{(D)}W(X,C) + p^n \widehat{F}^{(1)}W(U).$$

For the proof of Lemma 13.1, we need a preliminary lemma.

Lemma 13.2. *For a fixed integer $n > 0$, put*

$$C_P = \sum_{\lambda\in J_P} C_\lambda \quad \text{with} \quad J_P = \{\lambda \in I \mid \frac{m_\lambda}{p^n} \in \mathbb{Z}\}.$$

For $Z \in \mathrm{Div}(X,C)^+$ and $x \in Z \cap C$, we have

$$\{1+\mathcal{O}_{Z,x}(-D)\}_{Z,x} \subset F_{\mathcal{B}}^{(D+C_P)}W(X,C) + p^n F^{(C)}W(X,C).$$

Proof. By Remark 12.2(1) we have (noting $C_P \subset C_0$)

$$\{1+\mathcal{O}_{Z,x}(-D)\}_{Z,x} \subset F_{\mathcal{B}}^{(D+C_P)}W(X,C) + F_{\Cap}^{(D)}W(X,C).$$

Hence the lemma follows from the following fact: for integral $F \in \mathrm{Div}(X,C)^+$ such that $F \Cap C$ at $y \in F \cap C$, we have

$$1+\mathcal{O}_{F,y}(-D) \subset \big(1+\mathcal{O}_{F,y}(-D-C_P)\big)\cdot\big(1+\mathcal{O}_{F,y}(-C)\big)^{p^n}.$$

Indeed, letting $\lambda \in I$ be such that $y \in C_\lambda$, we have an isomorphism

$$\frac{1+\mathcal{O}_{F,y}(-D)}{1+\mathcal{O}_{F,y}(-D-C_P)} \simeq \frac{1+\mathcal{O}_{F,y}(-m_\lambda C_\lambda)}{1+\mathcal{O}_{F,y}(-(m_\lambda+1)C_\lambda)}$$

if $\lambda \in J_P$, and otherwise the group on the left hand side vanishes. The desired fact is checked by using $p^n | m_\lambda$ if $\lambda \in J_P$ and the perfectness of $\kappa(y)$ (use the equality $(1+a)^{p^n} = 1 + a^{p^n}$). □

Proof of Lemma 13.1 Taking a local parameter π_D of D at x, we want to show

$$\{1+\alpha\frac{\pi_D}{f}\}_{Z,x} \in \widehat{F}^{(D)}W(X,C) + p^n\widehat{F}^{(1)}W(U) \quad \text{for } \alpha \in \mathcal{O}_{X,x}. \tag{13.3}$$

For integers $m \geq 0$, we inductively define

$$(X_m, C'_m, E_m, x_m)$$

as follows. For $m=0$, $X_0 = X$, $C'_0 = C$, $E_0 = \varnothing$, $x_0 = x$. For $m=1$, let $g_1 : X_1 = Bl_x(X) \to X$ be the blowup at x, $C'_1 \subset X_1$ be the proper transform of C, $E_1 = g_1^{-1}(x)$ be the exceptional divisor, and $x_1 = C'_1 \cap E_1$. Assuming $(X_{m-1}, C'_{m-1}, E_{m-1}, x_{m-1})$ defined, let $g_m : X_m = Bl_{x_{m-1}}(X_{m-1}) \to X_{m-1}$, $C'_m \subset X_m$ be the proper transform of C'_{m-1}, $E_m = g_m^{-1}(x_{m-1})$, and $x_m = C'_m \cap E_m$. Let

$$\phi_m = g_m \circ \cdots \circ g_1 : X_m \to X,$$

be the composite map, $C_m = \phi_m^{-1}(C)_{\mathrm{red}}$, and $E_{i,m} \subset X_m$ be the proper transform of $E_i \subset X_i$ for $1 \leq i \leq m-1$. We also define $E_{0,1}$ as the proper transform of F. Note that (X_m, C_m) is an object of $\widehat{\mathcal{B}}_X$ but not in $\mathcal{B}_X$ (cf. Definition 1.10). We easily see that the following facts hold for $m \geq 1$:

- $(*1)$ $(\pi/f^m, f)$ is a system of regular parameters of X_m at x_m.
- $(*2)$ f is a local parameter of $E_m \subset X_m$ at any point $y \in E_m \backslash E_{m-1,m}$.
- $(*3)$ Let $Z_m \subset X_m$ be the proper transform of Z in (13.1). If $b \geq am$, Z_m is defined locally around $E_m \backslash E_{m-1,m}$ by $(\pi/f^m)^a + uf^{b-am}$. We have

$$b \geq am \iff Z_m \cap \phi_m^{-1}(x) \subset E_m \backslash E_{m-1,m},$$

$$b \geq am+1 \iff Z_m \cap \phi_m^{-1}(x) = x_m.$$

- $(*4)$ $\phi_m^* C = C'_m + mE_m + \sum_{1 \leq i \leq m-1} iE_{i,m}$,

By the assumption $(p, m_\lambda) = 1$, there is a (unique) integer m such that $1 \leq m \leq p^n - 1$ and $p^n | mm_\lambda - 1$. Take $y \in Z_m \cap \phi_m^{-1}(x)$. By (13.2) we have $b \geq a(p^n-1) \geq am$ and $(*2)$, $(*3)$ and $(*4)$ imply

$$\frac{\pi_D}{f} \in \mathcal{O}_{Z_m,y}(-\phi_m^* D + E_m) \quad \text{and} \quad \phi_m^* D - E_m \geq 0$$

and hence

$$\{1+\alpha\frac{\pi_D}{f}\}_{Z_m,y} \in F^{(\phi_m^* D - E_m)}W(X_m, C_m) \quad \text{for } \alpha \in \mathcal{O}_{X,x}.$$

By $(*4)$ the multiplicity of E_m in $\phi_m^* D - E_m$ is $mm_\lambda - 1$. Since $p^n | mm_\lambda - 1$, Lemma 13.2 with Remark 1.13 implies

$$\{1+\alpha\frac{\pi_D}{f}\}_{Z_m,y} \in F_{\mathcal{B}}^{(\phi_m^* D)}W(X_m, C_m) \ + \ p^n F^{(C_m)}W(X_m, C_m).$$

Noting $F^{(C_m)}W(X_m, C_m) \subset \widehat{F}^{(1)}W(U)$, this implies (13.3) and completes the proof of Lemma 13.1.

14. Proof of Key Lemma V

Let the notation be as in §9. In this section we prove Lemma 9.3.

Lemma 14.1. *Let $g : \tilde{X} \to X$ be the blowup at a closed point of C and $\tilde{C} = g^{-1}(C)_{\mathrm{red}}$. Then, for any integer $N > 0$, we have*

$$F_{\Cap}^{(g^* D)}W(\tilde{X}, \tilde{C}) \subset F^{(D)}W(X,C) \ + \ F^{(g^*(D+N\cdot C))}W(\tilde{X}, \tilde{C}). \tag{14.1}$$

Proof. Put $\tilde{D} = g^*D \in \mathrm{Div}(\tilde{X}, \tilde{C})^+$. It suffices to show

$$\{1 + \mathcal{O}_{F,y}(-\tilde{D})\}_{F,y} \subset F^{(D)}W(X, C) + F^{(g^*(D+N\cdot C))}W(\tilde{X}, \tilde{C})$$

for any reduced $F \in \mathrm{Div}(\tilde{X}, \tilde{C})^+$ such that $F \pitchfork \tilde{C}$ and for any $y \in F \cap \tilde{C}$.

Claim 14.2. *Let $x = g(y)$. We may assume $\kappa(x) = \kappa(y)$.*

Indeed, take a finite Galois extension k' of k and consider the diagram

$$\begin{array}{ccc} \tilde{X}' & \xrightarrow{\tilde{\phi}} & \tilde{X} \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[\phi]{} & X \end{array}$$

where $X' = X \otimes_k k'$ and $\tilde{X}' = \tilde{X} \otimes_k k'$. Let $C' = \phi^*C$ and $\tilde{C}' = \tilde{\phi}^*\tilde{C}$. Note that these are reduced since ϕ and $\tilde{\phi}$ are étale. Take a point $y' \in \tilde{X}'$ lying over y and let $x' = g'(y')$. Taking k' large enough, we may assume $\kappa(x') = \kappa(y')$. Set $F' = \tilde{\phi}^*F \in \mathrm{Div}(\tilde{X}', \tilde{C}')^+$. Then F' is reduced and $F' \pitchfork \tilde{C}'$. Thus we may assume

$$\{1 + \mathcal{O}_{F',y'}(-g'^*D')\}_{F',y'} \subset F^{(D')}W(X', C') + F^{g'^*(D'+N\cdot C')}W(\tilde{X}', \tilde{C}'), \tag{14.2}$$

where $D' = \phi^*D$. Note

$$g'^*(D' + N\cdot C') = g'^*\phi^*(D + N\cdot C) = \tilde{\phi}^*g^*(D + N\cdot C). \tag{14.3}$$

We have a commutative diagram

$$\begin{array}{ccc} 1 + \mathcal{O}_{F',y'}(-\tilde{\phi}^*\tilde{D}) & \xrightarrow{N_{F'/F}} & 1 + \mathcal{O}_{F,y}(-\tilde{D}) \\ \downarrow{\scriptstyle \{\ ,\ \}_{F',y'}} & & \downarrow{\scriptstyle \{\ ,\ \}_{F,y}} \\ F^{(\tilde{\phi}^*\tilde{D})}W(\tilde{X}', \tilde{C}') & \xrightarrow{N_{\tilde{\phi}}} & F^{(\tilde{D})}W(\tilde{X}, \tilde{C}) \\ \uparrow{\scriptstyle \hookrightarrow} & & \uparrow{\scriptstyle \hookrightarrow} \\ F^{(\phi^*D)}W(X', C') & \xrightarrow{N_{\phi}} & F^{(D)}W(X, C) \end{array}$$

where N_ϕ (resp. $N_{\tilde{\phi}}$) is induced by the norm map (1.2) for ϕ (resp. $\tilde{\phi}$), and $N_{F'/F}$ is the norm map induced by $F' \to F$. Since $F' \to F$ is étale, $N_{F'/F}$ is surjective after replacing $\mathcal{O}_{F',y'}$ and $\mathcal{O}_{F,y}$ by their henselizations. Noting Remark 1.13, (14.1) follows from (14.2) and (14.3) and

$$N_{\tilde{\phi}}\big(F^{\tilde{\phi}^*g^*(D+N\cdot C)}W(\tilde{X}', \tilde{C}')\big) \subset F^{(g^*(D+N\cdot C))}W(\tilde{X}, \tilde{C}).$$

This completes the proof of Claim 14.2.

Now we prove Lemma 14.1 assuming $\kappa(x) = \kappa(y)$. By Lemma 14.3 below, for any integer $m > 0$, there exists $G \in \mathrm{Div}(X, C)^+$ such that G is regular at x and $(G', F)_y \geq m + 1$ where G' is the proper transform of G in $\tilde{X}$. Lemma 10.1 implies

$$\{1 + \mathcal{O}_{F,y}(-\tilde{D})\}_{F,y} \subset \{1 + \mathcal{O}_{G',y}(-\tilde{D})\}_{G',y} + F^{(\tilde{D}+m\tilde{C})}W(\tilde{X}, \tilde{C}).$$

We have $m\tilde{C} \geq N \cdot g^*C$ for m sufficiently large so that

$$\{1 + \mathcal{O}_{F,y}(-\tilde{D})\}_{F,y} \subset \{1 + \mathcal{O}_{G',y}(-\tilde{D})\}_{G',y} + F^{(g^*(D+N\cdot C))}W(\tilde{X}, \tilde{C}).$$

Since G is regular at $x = g(y)$, $G' \to G$ is an isomorphism at y. Hence we have

$$\{1 + \mathcal{O}_{G',y}(-g^*D)\}_{G',y} = \{1 + \mathcal{O}_{G,x}(-D)\}_{G,x} \text{ in } W(U).$$

This completes the proof of Lemma 14.1. □

Lemma 14.3. *Let X be a smooth surface over k and $x \in X$ be a closed point and $g : X' \to X$ be the blowup at x with $E = g^{-1}(x)$. Let x' be a closed point of E such that $\kappa(x) = \kappa(x')$. Let $F \subset X'$ be an integral curve such that $x' \in F$ and $F \pitchfork E$ at x'. Then, for any integer $m > 0$, there exists a curve $G \subset X$ such that G is regular at x and $(G', F)_{x'} \geq m + 1$ where G' is the proper transform of G in X'.*

Proof. We can take a system (s,t) of regular parameters of $\mathcal{O}_{X,x}$ such that letting $E_0 = \mathrm{div}_X(s)$ and $E_0' \subset X'$ be the proper transform of E_0, $F \cap E_0' \cap E = \varnothing$. Then $E - (E_0' \cap E) = \mathrm{Spec}(\kappa(x)[t/s])$ and there is $c \in \kappa(x)$ such that x' is given by the ideal $(t/s - c)$. For $h \in \widehat{\mathcal{O}_{X,x}} \cong \kappa(x)[[s,t]]$ write

$$h = \sum_{i=0}^{\infty} h_i \quad \text{with } h_i = 0 \ \text{ or } \ h_i \in \kappa(x)[s,t] \ \text{ homogeneous of degree } i.$$

Then we define $in(h) = h_{i_h}$ with $i_h = \max\{i \mid h_i \neq 0\}$. By [ZS, Ch.VIII§7 Bilinear Lemma] we have the following.

Claim 14.4. *Assume $in(h) = QR$ for non-constant homogeneous $Q, R \in \kappa(x)[s,t]$ which have no common prime factor. Then there exist $q, r \in \kappa(x)[[s,t]]$ such that $h = qr$, $Q = in(q)$ and $R = in(r)$.*

Let $f \in \mathcal{O}_{X,x}$ be a local equation of $g(F) \subset X$. Putting $d = i_f$, we can write

$$in(f) = \sum_{1 \leq j \leq d} a_j t^j s^{d-j} \quad \text{with } a_j \in \kappa(x).$$

Then we have

$$E \times_{X'} F = \mathrm{Spec}\, \kappa(x)[z]/(\phi(z)) \quad \text{with } \phi(z) = \sum_{1 \leq j \leq d} a_j z^j.$$

Since $F \pitchfork E$ at x', this implies a decomposition $in(f) = (t - cs)H$ where $H \in \kappa(x)[t,s]$ is not divisible by $(t - cs)$. By Claim 14.4 there exist $g, f' \in \kappa(x)[[s,t]]$ such that $f = gf'$, $in(g) = (t - cs)$ and $in(f') = H$. Then, taking $G = \mathrm{div}_{X,x}(g')$ (cf. Remark 6.4) for $g' \in \mathcal{O}_{X,x}$ such that $g' \equiv g \mod (s,t)^{m+2}$ in $\widehat{\mathcal{O}_{X,x}}$, one easily see that G satisfies the desired property of Lemma 14.3. □

Put (cf. (9.1))

$$C_0 = \sum_{\lambda \in J} C_\lambda \quad \text{with } \ J = \{\lambda \in I \mid m_\lambda \geq 2\}. \tag{14.4}$$

Take a dense open subset $V \subset X$ containing all generic points of C.

Lemma 14.5. (1) *Let $g : \tilde{X} \to X$ be a map in $\mathcal{B}_X$ (cf. Definition 1.10). Note that $\tilde{C} = g^*(C)$ is reduced. Then*

$$F^{(g^*D)}W(\tilde{X}, \tilde{C}) \subset F^{(D)}W(X, C) \ + \ F_{\mathcal{B}}^{(D+C_0)}W(X, C). \tag{14.5}$$

(2) *For any integer $N > 0$, we have (cf. Definition 1.4(4))*

$$F_{\mathcal{B}}^{(D)}W(X, C) \subset F_{\pitchfork V}^{(D)}W(X, C) \ + \ F_{\mathcal{B}}^{(D+N\cdot C_0)}W(X, C).$$

Proof. To show (14.5), we first assume g is the blowup at a regular closed point x of C. By Remark 12.2(1) applied to $(\tilde{X}, \tilde{C})$, we have

$$F^{(g^*D)}W(\tilde{X}, \tilde{C}) \subset F_{\pitchfork}^{(g^*D)}W(\tilde{X}, \tilde{C}) \ + \ F_{\mathcal{B}}^{(g^*D+\tilde{C}_0)}W(\tilde{X}, \tilde{C}),$$

where $\tilde{C}_0$ is defined for g^*D as (14.4). Clearly $\tilde{C}_0 = g^*C_0$ so that

$$F_{\mathcal{B}}^{(g^*D+\tilde{C}_0)}W(\tilde{X}, \tilde{C}) = F_{\mathcal{B}}^{(g^*(D+C_0))}W(\tilde{X}, \tilde{C}) = F_{\mathcal{B}}^{(D+C_0)}W(X, C),$$

where the second equality holds by (1.3). Hence (14.5) follows from Lemma 14.1.

Now we prove (14.5) in general case by induction on the number of blown-up points. Decompose g as

$$g : \tilde{X} \xrightarrow{\psi} X' \xrightarrow{\phi} X \quad \text{with } D' = \phi^* D,\ \tilde{D} = \psi^* D' = g^* D,$$

where ϕ is in $\mathcal{B}_X$ and ψ is the blowup at a regular closed point of $C' = \phi^* C$. By the induction hypothesis applied for ϕ, we have

$$F^{(D')}W(X', C') \subset F^{(D)}W(X, C) \ + \ F_{\mathcal{B}}^{(D+C_0)}W(X, C), \tag{14.6}$$

By applying to ψ what we have shown, we get

$$F^{(\tilde{D})}W(\tilde{X}, \tilde{C}) \subset F^{(D')}W(X', C') \ + \ F_{\mathcal{B}}^{(D'+C_0')}W(X', C'), \tag{14.7}$$

where C_0' is defined for D' as (14.4). Noting that ϕ is in $\mathcal{B}_X$, we see $\phi^* C_0 = C_0'$ and

$$F_{\mathcal{B}}^{(D'+C_0')}W(X', C') = F_{\mathcal{B}}^{(\phi^*(D+C_0))}W(X', C') = F_{\mathcal{B}}^{(D+C_0)}W(X, C).$$

Thus (14.5) follows from (14.6) and (14.7) .

Next we show Lemma 14.5(2). By induction on N, we may assume $N = 1$. By Remark 12.2(1) we have

$$F^{(D)}W(X, C) \subset F_{\Subset V}^{(D)}W(X, C) \ + \ F_{\mathcal{B}}^{(D+C_0)}W(X, C).$$

Hence it suffices to show

$$F_{\mathcal{B}}^{(D)}W(X, C) \subset F^{(D)}W(X, C) \ + \ F_{\mathcal{B}}^{(D+C_0)}W(X, C).$$

This is a direct consequence of (14.5) and Definition 1.12. □

Lemma 14.6. (1) *Let $g : \tilde{X} \to X$ be a map in $\widehat{\mathcal{B}}_X$ (cf. Definition 1.10) and $\tilde{C} = g^{-1}(C)_{\mathrm{red}}$. Then, for any integer $N > 0$,*

$$F^{(g^* D)}W(\tilde{X}, \tilde{C}) \subset F^{(D)}W(X, C) \ + \ \widehat{F}^{(D+N\cdot C_0)}W(X, C). \tag{14.8}$$

(2) *For any integer $N > 0$, we have (cf. Definition 1.4(4))*

$$\widehat{F}^{(D)}W(X, C) \subset F_{\Subset V}^{(D)}W(X, C) \ + \ \widehat{F}^{(D+N\cdot C_0)}W(X, C).$$

where V is any open subset of X such that $V \cap C$ is dense in C.

Proof. (2) is shown by the same argument as the proof of Lemma 14.5(2) using (14.8) instead of (14.5). We show (14.8). Write $\tilde{D} = g^* D \in \mathrm{Div}(\tilde{X}, \tilde{C})^+$. First we assume g is the blowup at a closed point $x \in X$. Fix N in the lemma. By Lemma 14.5(2) applied to $(\tilde{X}, \tilde{C})$ and $\tilde{D}$, for any integer $M > 0$ we have

$$F^{(\tilde{D})}W(\tilde{X}, \tilde{C}) \subset F_{\Subset}^{(\tilde{D})}W(\tilde{X}, \tilde{C}) \ + \ F_{\mathcal{B}}^{(\tilde{D}+M\cdot\tilde{C}_0)}W(\tilde{X}, \tilde{C}), \tag{14.9}$$

where $\tilde{C}_0$ is defined for $\tilde{D}$ as (14.4). We have $|g^{-1}(C_0)| \subset |\tilde{C}_0|$ so that $M \cdot \tilde{C}_0 \geq N \cdot g^* C_0$ for M large enough. Then

$$F_{\mathcal{B}}^{(\tilde{D}+M\cdot\tilde{C}_0)}W(\tilde{X}, \tilde{C}) \subset F_{\mathcal{B}}^{(g^*(D+N\cdot C_0))}W(\tilde{X}, \tilde{C}) \subset \widehat{F}^{(D+N\cdot C_0)}W(X, C). \tag{14.10}$$

Thus (14.8) follows from (14.9), (14.10) and Lemma 14.1.

Now we prove Lemma 14.6 in general case by induction on the number of blown-up points. The reduction argument is similar to that of the proof of Lemma 14.5(1). Decompose g as

$$g : \tilde{X} \xrightarrow{\psi} X' \xrightarrow{\phi} X \quad \text{with } D' = \phi^* D,\ \tilde{D} = \psi^* D' = g^* D,$$

where ϕ is in $\widehat{\mathcal{B}}_X$ and ψ is the blowup at a closed point of $\phi^{-1}(C)$. By the induction hypothesis applied for ϕ, we have

$$F^{(D')}W(X', C') \subset F^{(D)}W(X, C) \ + \ \widehat{F}^{(D+N\cdot C_0)}W(X, C), \tag{14.11}$$

where $C' = \phi^{-1}(C)_{red}$. By applying to ψ what we have shown, we get

$$F^{(\tilde{D})}W(\tilde{X},\tilde{C}) \subset F^{(D')}W(X',C') \;+\; \widehat{F}^{(D'+M\cdot C'_0)}W(X',C') \tag{14.12}$$

for any integer $M > 0$, where C'_0 is defined for D' as (14.4). We have $|\phi^{-1}(C_0)| \subset |C'_0|$ so that $M \cdot C'_0 \geq N \cdot \phi^* C_0$ for M large enough. Hence

$$\widehat{F}^{(D'+M\cdot C'_0)}W(X',C') \subset \widehat{F}^{(\phi^*(D+N\cdot C_0))}W(X',C') = \widehat{F}^{(D+N\cdot C_0)}W(X,C).$$

Thus (14.8) follows from (14.11) and (14.12). □

Remark 14.7. If $D \geq 2C$, then $C_0 = C$ so Lemma 14.6(2) implies

$$\widehat{F}^{(D)}W(X,C) \subset F^{(D)}_{\Subset V}W(X,C) \;+\; \widehat{F}^{(D+N\cdot C)}W(X,C) \quad \text{for any } N > 0.$$

Lemma 14.8. *Let the assumption be as in Lemma 14.6(1). Assume $p \neq 2$. For any integers $N, n > 0$ we have*

$$F^{(g^*D)}W(\tilde{X},\tilde{C}) \subset F^{(D)}W(X,C) \;+\; \widehat{F}^{(D+N\cdot C)}W(X,C) + p^n \widehat{F}^{(1)}W(U). \tag{14.13}$$

Proof. Put $\tilde{D} = g^*D$. By the same induction argument as the proof of Lemma 14.6(1), we are reduced to the case where g is the blowup at a closed point $x \in C$. By Remark 12.2(2) applied to $\tilde{X}, \tilde{C}, \tilde{D}$, we have

$$F^{(\tilde{D})}W(\tilde{X},\tilde{C}) \subset F^{(\tilde{D})}_{\Subset}W(\tilde{X},\tilde{C}) \;+\; \widehat{F}^{(\tilde{D}+\tilde{C})}W(\tilde{X},\tilde{C}) + p^n \widehat{F}^{(1)}W(U).$$

Noting $\tilde{D} + \tilde{C} \geq 2\tilde{C}$, Remark 14.7 applied to $\tilde{X}, \tilde{C}, \tilde{D} + \tilde{C}$ implies

$$\widehat{F}^{(\tilde{D}+\tilde{C})}W(\tilde{X},\tilde{C}) \subset F^{(\tilde{D})}_{\Subset}W(\tilde{X},\tilde{C}) \;+\; \widehat{F}^{(\tilde{D}+M\cdot\tilde{C})}W(\tilde{X},\tilde{C}) \quad \text{for any } M > 0.$$

Taking M large enough so that $M \cdot \tilde{C} \geq N \cdot g^*C$, we have

$$\widehat{F}^{(\tilde{D}+M\cdot\tilde{C})}W(\tilde{X},\tilde{C}) \subset \widehat{F}^{(g^*(D+N\cdot C))}W(\tilde{X},\tilde{C}) = \widehat{F}^{(D+N\cdot C)}W(X,C).$$

Thus we get

$$F^{(\tilde{D})}W(\tilde{X},\tilde{C}) \subset F^{(\tilde{D})}_{\Subset}W(\tilde{X},\tilde{C}) \;+\; \widehat{F}^{(D+N\cdot C)}W(X,C) + p^n \widehat{F}^{(1)}W(U).$$

Now (14.13) follows from Lemma 14.1. □

Finally we prove Lemma 9.3. By Remark 12.2(2) we have

$$F^{(D)}W(X,C) \subset F^{(D)}_{\Subset V}W(X,C) \;+\; \widehat{F}^{(D+C)}W(X,C) + p^n \widehat{F}^{(1)}W(U).$$

Noting $D + C \geq 2C$, Remark 14.7 implies

$$\widehat{F}^{(D+C)}W(X,C) \subset F^{(D)}_{\Subset V}W(X,C) \;+\; \widehat{F}^{(D+N\cdot C)}W(X,C).$$

Thus it suffices to show

$$\widehat{F}^{(D)}W(X,C) \subset F^{(D)}W(X,C) \;+\; \widehat{F}^{(D+N\cdot C)}W(X,C) + p^n \widehat{F}^{(1)}W(U),$$

which follows from Lemma 14.8 and Lemma 1.11.

15. Appendix

In this section we give a sketch of a proof of Lemma 2.6. For a ring R over $\mathbb{F}_p$ and for an integer $q \geq 0$, put

$$S\widehat{C}K_q(R) = \varprojlim_{n} \mathrm{Ker}\big(K_q^{sym}(R[T]/(T^n)) \to K_q(R)\big),$$

where $K_q^{sym}(R[T]/(T^n))$ is the subgroup of $K_q(R[T]/(T^n))$ generated by symbols. For an integer $n \geq 0$ put

$$Fil^n S\widehat{C}K_q(R) = \mathrm{Ker}\big(S\widehat{C}K_q(R) \to K_q^{sym}(R[T]/(T^{n+1}))\big). \tag{15.1}$$

Let $T\widehat{C}K_q(R)$ be the p-typical part of $S\widehat{C}K_q(R)$ as defined in [B, Ch.II §7] (see also [Ka1, p.636]). Letting I_p be the set of positive integers not divisible by p, we have the projection to the p-typical part:

$$\tau_R = \sum_{n \in I_p} -\frac{\mu(n)}{n} V_n F_n : S\widehat{C}K_{q+1}(R) \to T\widehat{C}K_{q+1}(R) \tag{15.2}$$

(see [B, Ch.II §1 (2.1)] for F_n and V_n). Recall the Artin-Hasse exponential

$$E(T) := \prod_{n \in I_p} (1-T)^{-\mu(n)/n} \in T\widehat{C}K_1(R)$$

(Note $S\widehat{C}K_1(R) = 1 + T \cdot R[[T]]$ and $E(T) = \tau_R(1-T)$). There is an isomorphism

$$\psi : W(R) \xrightarrow{\cong} T\widehat{C}K_1(R)\ ;\ (a_0, a_1, a_2, \cdots) \mapsto \prod_{n \geq 0} E(a_n T^{p^n}).$$

It induces a pairing

$$W(R) \times K_q^{sym}(R) \to T\widehat{C}K_{q+1}(R)\ ;\ (x, y_1, \ldots, y_q) \mapsto (\psi(x), y_1, \ldots, y_q). \tag{15.3}$$

For a discrete valuation field L with residue field E, let

$$Res_{L/E} : S\widehat{C}K_{q+1}(L) \xrightarrow{Res_{L/E}} S\widehat{C}K_q(E) \tag{15.4}$$

be the residue map from [Ka1, §2.5 Lemma 2.5]. For an N-dimensional local field K as in (2.5), we define

$$Res_{K/\mathbb{F}_p} : S\widehat{C}K_{N+1}(K) \to S\widehat{C}K_1(\mathbb{F}_p) = (1 + T\mathbb{F}_p[[T]])^\times$$

as the composite $Tr_{k_0/\mathbb{F}_p} \circ Res_{k_1/k_0} \circ Res_{k_2/k_1} \circ \cdots Res_{K/k_{N-1}}$, where $Tr_{k_0/\mathbb{F}_p}$ is the trace map $S\widehat{C}K_1(k_0) \to S\widehat{C}K_1(\mathbb{F}_p)$. It induces natural maps (see (a) below)

$$Res_{K/\mathbb{F}_p} : T\widehat{C}K_{N+1}(K) \to T\widehat{C}K_1(\mathbb{F}_p) = W(\mathbb{F}_p),$$

$$Res^s_{K/\mathbb{F}_p} : T\widehat{C}K_{N+1}(K) \xrightarrow{Res_{K/\mathbb{F}_p}} W(\mathbb{F}_p) \to W_s(\mathbb{F}_p) \quad (s \in \mathbb{Z}_{\geq 1}).$$

By [Ka1, §3.1 Th.2 and the argument on p.662], the p-part of Ψ_K is induced by (2.1) and a pairing

$$\langle\ ,\ \rangle_K : W_s(K) \times K_N^M(K) \to W_s(\mathbb{F}_p) \simeq \mathbb{Z}/p^s\mathbb{Z}$$

which arises from the pairing (cf. (15.3))

$$W(K) \times K_N^M(K) \to T\widehat{C}K_{N+1}(K) \xrightarrow{Res^s_{K/\mathbb{F}_p}} W_s(\mathbb{F}_p).$$

Thus the commutativity of (2.6) follows from that of the following diagram

$$\begin{array}{ccccc} \mathrm{fil}_m W_s(K) & \times\ K_N^M(K)/V^m K_N^M(K) & \xrightarrow{\langle\ ,\ \rangle_K} & \mathbb{Z}/p^s\mathbb{Z} \\ \downarrow{\scriptstyle -F^s d} & \uparrow{\scriptstyle \rho_K^m} & & \uparrow{\scriptstyle p^{s-1}} \\ \mathfrak{m}_K^{-m}\Omega^1_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} E\ \times & \mathfrak{m}_K^{m-1}\Omega^{N-1}_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} E & \xrightarrow{\langle\ ,\ \rangle_\Omega} & \mathbb{Z}/p\mathbb{Z}\,, \end{array}$$

where $F^s d$ is the map in (2.3). The latter commutativity follows from

$$\begin{aligned} &Res^s_{K/\mathbb{F}_p}\{E(cT^{p^{s-1-i}}), 1 + ab_1 \cdots b_{N-1}, b_1, \ldots, b_{N-1}\} = -E(hT^{p^{s-1}}) \\ &\text{with } \ h = Res^\Omega_{K/\mathbb{F}_p}\big(c^{p^i-1} dc \wedge adb_1 \wedge \cdots \wedge db_{N-1}\big). \end{aligned} \tag{15.5}$$

for $a, b_1 \ldots, b_{N-1} \in \mathcal{O}_K$, $c \in K$ and $i \in \mathbb{Z}$ such that $v_K(a) \geq m-1$ and that $p^i v_K(c) \geq -(m-1)$ with $0 \leq i \leq s-1$ or $p^i v_K(c) \geq -m$ with $0 \leq i \leq \min\{s, \mathrm{ord}_p(m)\} - 1$. The formula is checked by using the explicit computation of the residue map in [Ka1, §2.2 and §2.4].

Here we give a sketch of the computation assuming:

($*$) K is the fraction field of the henselization of $F\{t,\pi\}$ at the prime ideal (π), where F is a finite field and $F\{t,\pi\}$ is the henselization of $F[t,\pi]$ at (t,π) (this is the only case we use in this paper).

In this case we have $N=2$ and $k_1=E:=F\{t\}[1/t]$ and $k_0=F$. We identify E with a subfield of K. The formula (15.5) is then reduced to the following formulas

$$(15.6)\qquad \begin{gathered} Res^s_{K/\mathbb{F}_p}\{E(cT^{p^{s-1-i}}),1+a\pi^{m-1}t,t\}=-E(f(c,a)T^{p^{s-1}}) \\ \text{with } f(c,a)=Res^{\Omega}_{K/\mathbb{F}_p}\big(c^{p^i-1}dc\wedge a\pi^{m-1}dt\big)\in F, \end{gathered}$$

$$(15.7)\qquad \begin{gathered} Res^s_{K/\mathbb{F}_p}\{E(cT^{p^{s-1-i}}),1+b\pi^{m},\pi\}=-E(g(c,b)T^{p^{s-1}}) \\ \text{with } g(c,b)=Res^{\Omega}_{K/\mathbb{F}_p}\big(c^{p^i-1}dc\wedge b\pi^{m}d\pi\big), \end{gathered}$$

for $a,b\in E$ and $c=\gamma\pi^{-e}$ with $\gamma\in E$ and $e,i\in\mathbb{Z}$ such that $ep^i\leq m-1$ with $0\leq i\leq s-1$ or $ep^i\leq m$ with $0\leq i\leq \min\{s,\mathrm{ord}_p(m)\}-1$. Noting

$$c^{p^i-1}dc=\frac{\gamma^{p^i}}{\pi^{ep^i}}(\frac{d\gamma}{\gamma}-e\frac{d\pi}{\pi})$$

and that $ep^i\leq m$ and $p|e$ if $ep^i=m$, we compute

$$(15.8)\qquad Res^{\Omega}_{K/\mathbb{F}_p}\big(c^{p^i-1}dc\wedge a\pi^{m-1}dt\big)=\begin{cases} -eRes^{\Omega}_{E/\mathbb{F}_p}(a\gamma^{p^i}dt) & \text{if } m=ep^i+1, \\ 0 & \text{otherwise,}\end{cases}$$

$$(15.9)\qquad Res^{\Omega}_{K/\mathbb{F}_p}\big(c^{p^i-1}dc\wedge b\pi^{m}d\pi\big)=\begin{cases} Res^{\Omega}_{E/\mathbb{F}_p}(b\gamma^{p^i-1}d\gamma) & \text{if } m=ep^i, \\ 0 & \text{otherwise.}\end{cases}$$

In what follows we only prove (15.7) ((15.6) is proved similarly). We will use the following facts (cf. [Ka1, §2.2 and §2.4]): Let L be a discrete valuation field with residue field E and $Res_{L/E}$ be as in (15.4).

(a) $Res_{L/E}$ preserves the filtration from (15.1) and commutes with F_n and V_n. In particular $Res_{L/E}\circ\tau_L=\tau_E\circ Res_{L/E}$ (cf. (15.2)).

(b) We have $S\widehat{C}K_{q+1}(\mathcal{O}_L)\subset\mathrm{Ker}(Res_{L/E})$.

(c) For $\alpha\in S\widehat{C}K_q(\mathcal{O}_L)$ with its image $\overline{\alpha}\in S\widehat{C}K_q(E)$, we have

$$Res_{L/E}(\{\alpha,\pi\})=\overline{\alpha}.$$

(d) Endow $S\widehat{C}K_q(L)$ with topology by the filtration (15.1). For $s\in\mathbb{Z}_{>0}$ let $F^sT\widehat{C}K_q(L)$ be the closed subgroup of $S\widehat{C}K_q(L)$ topologically generated by

$$\{E(aT^{p^n}),r_1,\dots,r_{q-1}\}\ \text{ and }\ \{E(aT^{p^n}),r_1,\dots,r_{q-2},T\}\quad\text{for } n\geq s,$$

where $a\in L$, $r_1,\dots,r_{q-1}\in L^\times$ (for the second element, see (e) below). Put

$$T\Phi_sK_q(L)=F^sT\widehat{C}K_q(L)/F^{s+1}T\widehat{C}K_q(L).$$

Then there is a well-defined map

$$\phi^{q,s}_L:\Omega^{q-1}_L\to T\Phi_sK_q(L)\ ;\ a\frac{dr_1}{r_1}\wedge\cdots\frac{dr_{q-1}}{r_{q-1}}\mapsto\{E(aT^{p^s}),r_1,\dots,r_{q-1}\}.$$

We have $Res_{L/E}\big(F^sT\widehat{C}K_q(L)\big) \subset F^sT\widehat{C}K_q(E)$ and the diagram

$$\begin{array}{ccc} \Omega_L^{q-1} & \xrightarrow{\phi_L^{q,s}} & T\Phi_s K_q(L) \\ {\scriptstyle Res^{\Omega}_{L/E}}\downarrow & & \downarrow{\scriptstyle Res_{L/E}} \\ \Omega_E^{q-2} & \xrightarrow{\phi_E^{q-1,s}} & T\Phi_s K_{q-1}(E)\ . \end{array}$$

is commutative (see [Ka1, §2 Prop.3]).

(*e*) For a ring R over $\mathbb{F}_p$ let $S(R) = \bigoplus_{q\geq 0} S_q(R)$ be the sub-graded ring of

$$\bigoplus_{q\geq 0} K_q^{sym}(R[[T]][T^{-1}])$$

generated by the image of $\bigoplus_{q\geq 0} K_q^{sym}(R[[T]])$ and $T \in K_1(R[[T]][T^{-1}])$. For $n \in \mathbb{Z}_{\geq 1}$ let $S^{(n)}(R) = \bigoplus_{q\geq 0} S_q^{(n)}(R)$ be the graded ideal of $S(R)$ generated by $1 + T^nR[[T]] \in S_1(R)$. Then there exists a natural map (see [Ka1, §2.2 Lemma 4])

$$\varprojlim_n S_q^{(1)}(R)/S_q^{(n)}(R) \to S\widehat{C}K_q(R),$$

which is an isomorphism if R is regular having a p-basis over $\mathbb{F}_p$ (see [Ka1, §2.2 Cor.1]). In particular the symbol $\{x, T\} \in S\widehat{C}K_q(R)$ with $x \in S\widehat{C}K_{q-1}(R)$ makes a sense. We have $\{\tau_R(x), T\} = \tau_R\{x, T\}$ by [B, Ch.II §6 Prop.(1.1)(ii)].

(*f*) For an N-dimensional local field K, [Ka1, (3) on p.679] and (a) imply

$$\tau_{\mathbb{F}_p} Res_{K/\mathbb{F}_p}\big(\{S\widehat{C}K_N(K), T\}\big) = 0.$$

Lemma 15.1. *Let K be as in $(*)$. Take integers $d, j, e, m \geq 0$ such that $j|d$ and $je \leq m$. Fix $\gamma, b \in E$ and put $\xi = 1 - \gamma\pi^{-e}T^{d/j}$ and define $\theta_k \in 1 + T\mathcal{O}_K[[T]]$ for $0 \leq k \leq j$ inductively by*

$$\theta_0 = 1 + b\pi^m,\ \theta_k = 1 + (-1)^k(\theta_0\theta_1\cdots\theta_{k-1})^{-1}b\pi^{m-ke}\gamma^kT^{kd/j}.$$

Then we have

$$\{\xi, 1 + b\pi^m, \pi\} \equiv (-1)^j\{(-1)^{1+e}\gamma^{-1}\xi, \theta_j, \pi\} \mod \mathrm{Ker}(\tau_{\mathbb{F}_p} Res_{K/\mathbb{F}_p}) \subset S\widehat{C}K_3(K).$$

Proof. For $A, B \in K[[T]]$ with $1 + B \neq 0$ we have a formula

$$\{1+TA, 1+B\} = -\{-B(1+TA), 1+(1+B)^{-1}TAB\} \text{ in } K_2(K[[T]][T^{-1}]). \tag{15.10}$$

From this and (e) we deduce

$$\{\xi, 1 + b\pi^m, \pi\} = -\{-b\pi^m\xi, \theta_1, \pi\} = -\{(-1)^{m+1}b\xi, \theta_1, \pi\} \text{ in } S\widehat{C}K_3(K).$$

If $m - e > 0$, (b) and (c) imply $\{(-1)^{m+1}b, \theta_1, \pi\} \in \mathrm{Ker}(Res_{K/\mathbb{F}_p})$ so that

$$\{\xi, 1 + b\pi^m, \pi\} \equiv -\{\xi, \theta_1, \pi\} \mod \mathrm{Ker}(Res_{K/\mathbb{F}_p}).$$

Repeat the same argument by noting $m - ke > 0$ for $k < j$ and the facts

$$\begin{aligned} \{a, \theta_k, \pi\} &\in \mathrm{Ker}(Res_{K/\mathbb{F}_p}) \text{ for } k < j \text{ and } a \in \mathcal{O}_K[[T]]^\times, \\ \{T, \theta_k, \pi\} &\in \mathrm{Ker}(\tau_{\mathbb{F}_p} Res_{K/\mathbb{F}_p}) \text{ for } k \leq j, \end{aligned}$$

which follows from (b), (c) and (f), one obtains

$$\begin{aligned}\{\xi, 1+b\pi^m, \pi\} &\equiv (-1)^{j-1}\{\xi, \theta_{j-1}, \pi\} \mod \mathrm{Ker}(\tau_{\mathbb{F}_p} Res_{K/\mathbb{F}_p}) \\ &= (-1)^j \{(1-\theta_{j-1})\xi, \theta_j, \pi\} \\ &\equiv (-1)^j\{(-1)^j b\gamma^{j-1}\pi^{m-(j-1)e}\xi, \theta_j, \pi\} \mod \mathrm{Ker}(\tau_{\mathbb{F}_p} Res_{K/\mathbb{F}_p}) \\ &= (-1)^j\{(-1)^j(1-\theta_j)^{-1} b\gamma^{j-1}\pi^{m-(j-1)e}\xi, \theta_j, \pi\} \\ &\equiv (-1)^j\{-\pi^e\gamma^{-1}\xi, \theta_j, \pi\} \mod \mathrm{Ker}(\tau_{\mathbb{F}_p} Res_{K/\mathbb{F}_p}) \\ &= (-1)^j\{(-1)^{1+e}\gamma^{-1}\xi, \theta_j, \pi\}.\end{aligned}$$

This completes the proof of Lemma 15.1. □

Now we prove (15.7). We take $d = p^{s-1}$ and $j = p^i$ in Lemma 15.1. By the lemma and (a), putting $\epsilon = (-1)^{1+e}$, we are reduced to showing

$$\tau_{\mathbb{F}_p} Res^s_{K/\mathbb{F}_p}\{\epsilon\gamma^{-1}\xi, \theta_j, \pi\} = E(hT^{p^{s-1}}), \tag{15.11}$$

$$h = \begin{cases} Res^{\Omega}_{E/\mathbb{F}_p}(b\gamma^{p^i}\dfrac{d\gamma}{\gamma}) & \text{if } m = ep^i, \\ 0 & \text{if } m > ep^i \end{cases} \tag{15.12}$$

(note $(-1)^j = -1$ by the assumption $p \neq 2$). we have

$$\{\xi, \theta_j, \pi\} \in \{S\widehat{C}K_2(K), T\} + Fil^{p^{s-1}} S\widehat{C}K_3(K) \quad (\text{cf. } (15.1)).$$

Indeed, writing $A = -\gamma\pi^{-e}$ and $B = (-1)^j(\theta_0\theta_1\cdots\theta_{j-1})^{-1}b\pi^{m-ke}\gamma^j$,

$$\{\xi, \theta_j\} = \{1+AT^{d/j}, 1+BT^d\} = -\{-BT^d(1+AT^{d/j}), 1+(1+BT^d)^{-1}ABT^{d+d/j}\}$$

by (15.10). In view of (f) we get

$$\tau_{\mathbb{F}_p} Res_{K/\mathbb{F}_p}\{\xi, \theta_j, \pi\} \;\in\; \tau_{\mathbb{F}_p}\big(Fil^{p^{s-1}} S\widehat{C}K_1(\mathbb{F}_p)\big) = \mathrm{Ker}\big(W(\mathbb{F}_p) \to W_s(\mathbb{F}_p)\big),$$

where the equality follows from [B, Ch.I §1 Remark 4.3 (ii)]. Recalling

$$\theta_j = 1 - (\theta_0\theta_1\cdots\theta_{j-1})^{-1} b\pi^{m-je}\gamma^{p^i}T^{p^{s-1}},$$

(a) and (c) imply (recall $j = p^i$)

$$Res_{K/E}\{\epsilon\gamma^{-1}, \theta_j, \pi\} = \begin{cases} \{1 - b\gamma^{p^i}T^{p^{s-1}}, \epsilon\gamma\}, & \text{if } m = ep^i, \\ 0. & \text{otherwise.} \end{cases}$$

Hence (15.11) follows from

$$\tau_{\mathbb{F}_p} Res^s_{E/\mathbb{F}_p}\{1 - b\gamma^{p^i}T^{p^{s-1}}, \epsilon\gamma\} = Res^s_{E/\mathbb{F}_p}\{E(b\gamma^{p^i}T^{p^{s-1}}), \epsilon\gamma\} = E(hT^{p^{s-1}}),$$

where h is from (15.12). The first (resp. second) equality follows from (a) (resp. (d)). This completes the proof.

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