

# Surfaces over a $p$ -adic field with infinite torsion in the Chow group of 0-cycles

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## Abstract

We give an example of a projective smooth surface  $X$  over a  $p$ -adic field  $K$  such that for any prime  $\ell$  different from  $p$ , the  $\ell$ -primary torsion subgroup of  $\mathrm{CH}_0(X)$ , the Chow group of 0-cycles on  $X$ , is infinite. A key step in the proof is disproving a variant of the Bloch-Kato conjecture which characterizes the image of an  $\ell$ -adic regulator map from a higher Chow group to a continuous étale cohomology of  $X$  by using  $p$ -adic Hodge theory. With the aid of the theory of mixed Hodge modules, we reduce the problem to showing the exactness of the de Rham complex associated to a variation of Hodge structure, which is proved by the infinitesimal method in Hodge theory. Another key ingredient is the injectivity result on the cycle class map for Chow group of 1-cycles on a proper smooth model of  $X$  over the ring of integers in  $K$ , due to K. Sato and the second author.

## 1 Introduction

Let  $X$  be a smooth projective variety over a base field  $K$  and let  $\mathrm{CH}^m(X)$  be the Chow group of algebraic cycles of codimension  $m$  on  $X$  modulo rational equivalence. In case  $K$  is a number field, there is a folklore conjecture that  $\mathrm{CH}^m(X)$  is finitely generated, which in particular implies that its torsion part  $\mathrm{CH}^m(X)_{\mathrm{tor}}$  is finite. The finiteness question has been intensively studied by many authors, particularly for the case  $m = 2$  and  $m = \dim(X)$  (see nice surveys [27] and [8]).

When  $K$  is a  $p$ -adic field (namely the completion of a number field at a finite place), Rosenschon and Srinivas [29] have constructed the first example where  $\mathrm{CH}^m(X)_{\mathrm{tor}}$  is infinite. They prove that there exists a smooth projective fourfold  $X$  over a  $p$ -adic field such that the  $\ell$ -torsion subgroup  $\mathrm{CH}_1(X)[\ell]$  (see Notation) of  $\mathrm{CH}_1(X)$ , the Chow group of 1-cycles on  $X$ , is infinite for each  $\ell \in \{5, 7, 11, 13, 17\}$ .

A purpose of this paper is to give an example of a projective smooth surface  $X$  over a  $p$ -adic field such that for any prime  $\ell$  different from  $p$ , the  $\ell$ -primary torsion subgroup  $\mathrm{CH}_0(X)\{\ell\}$  (see Notation) of  $\mathrm{CH}_0(X)$ , the Chow group of 0-cycles on  $X$ , is infinite. Here we note that for  $X$  as above,  $\mathrm{CH}_0(X)\{\ell\}$  is known to always be of finite cotype over  $\mathbb{Z}_\ell$  (namely the direct sum of a finite group and a finite number of copies of  $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ . The fact follows from Bloch's exact sequence (2.3)). Thus our example presents infinite phenomena of different nature from the example in [29]. Another noteworthy point is that the phenomena discovered by our example happens rather *generically*.

To make it more precise, we prepare a notion of 'generic surfaces' in  $\mathbb{P}^3$ . Let

$$M \subset \mathbb{P}(H^0(\mathbb{P}_{\mathbb{Q}}^3, \mathcal{O}_{\mathbb{P}}(d))) \cong \mathbb{P}_{\mathbb{Q}}^{(d+3)(d+2)(d+1)/6-1}$$

be the moduli space over  $\mathbb{Q}$  of the nonsingular surfaces in  $\mathbb{P}_{\mathbb{Q}}^3$  (the subscription ' $\mathbb{Q}$ ' indicates the base field), and let

$$f : \mathcal{X} \longrightarrow M$$

be the universal family over  $M$ . For  $X \subset \mathbb{P}_K^3$ , a nonsingular surface of degree  $d$  defined over a field  $K$  of characteristic zero, there is a morphism  $t : \mathrm{Spec}K \rightarrow M$  such that  $X \cong \mathcal{X} \times_M \mathrm{Spec}K$ . We call  $X$  *generic* if  $t$  is dominant (i.e.  $t$  factors through the generic point of  $M$ ). In other words,  $X$  is generic if it is defined by an equation

$$F = \sum_I a_I z^I \quad (a_I \in K)$$

( $[z_0 : z_1 : z_2 : z_3]$  is the homogeneous coordinate of  $\mathbb{P}^3$ ,  $I = (i_0, \dots, i_3)$  are multi-indices and  $z^I = z_0^{i_0} \cdots z_3^{i_3}$ ) satisfying the following condition:

- (\*)  $a_I \neq 0$  for  $\forall I$  and  $\{a_I/a_{I_0}\}_{I \neq I_0}$  are algebraically independent over  $\mathbb{Q}$  where  $I_0 = (1, 0, 0, 0)$ .

The main theorem is:

**Theorem 1.1** *Let  $K$  be a finite extension of  $\mathbb{Q}_p$  and  $X \subset \mathbb{P}_K^3$  a nonsingular surface of degree  $d \geq 5$ . Suppose that  $X$  is generic and has a projective smooth model  $X_{\mathcal{O}_K} \subset \mathbb{P}_{\mathcal{O}_K}^3$  over the ring  $\mathcal{O}_K$  of integers in  $K$ . Let  $r$  be the Picard number (i.e. the rank of the Néron-Severi group) of the smooth special fiber of  $X_{\mathcal{O}_K}$ . Then we have*

$$\mathrm{CH}_0(X)\{\ell\} \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus r-1} \oplus (\text{finite group})$$

for  $\ell \neq p$ .

One can construct a surface with infinite torsion in the Chow group of 0-cycles in the following way. Let  $k$  be the residue field of  $K$ . Let  $Y$  be a smooth surface of degree  $d \geq 5$  in  $\mathbb{P}_k^3$  defined by an equation  $\sum_I c_I z^I$  ( $c_I \in k$ ) such that the Picard number  $r \geq 2$ . Note that

there exist such surfaces for each  $d$ . (For example if  $(p, d) = 1$ , one may choose a Fermat type surface defined by  $z_0^d - z_1^d + z_2^d - z_3^d$ . Then the intersection of  $Y$  with the hyperplane  $H \subset \mathbb{P}_k^3$  defined by  $z_0 - z_1$  is not irreducible so that  $r \geq 2$ .) Take any liftings  $\tilde{c}_I \in O_K$  and choose  $a_I \in O_K$  with  $\text{ord}(a_I) > 0$  for each indices  $I$  such that  $\{a_I\}_I$  are algebraically independent over  $\mathbb{Q}(\tilde{c}_I)$ , the subfield of  $K$  generated over  $\mathbb{Q}$  by  $\tilde{c}_I$  for all  $I$ . Let  $X \subset \mathbb{P}_K^3$  be the surface defined by the equation  $\sum_I \tilde{c}_I z^I + \sum_I a_I z^I$ . Then it is clear that  $X$  is generic and has a smooth projective model over  $O_K$  whose the special fiber is  $Y$ . Since  $Y$  has the Picard number  $r \geq 2$ ,  $\text{CH}_0(X)$  has an infinite torsion subgroup by Theorem 1.1. It is proved in [28] that if the special fiber satisfies the Tate conjecture for divisors, the geometric Picard number is congruent to  $d$  modulo 2. Thus if  $d$  is even,  $\text{CH}_0(X)$  has an infinite torsion subgroup after a suitable unramified base change. Theorem 1.1 may be compared with the finiteness results [9] and [28] on  $\text{CH}_0(X)_{\text{tor}}$  for a surface  $X$  over a  $p$ -adic field under the assumption that  $H^2(X, \mathcal{O}_X) = 0$  or, more generally, that the rank of the Néron-Severi group does not change by reduction. For a nonsingular surface  $X \subset \mathbb{P}_K^3$  of degree  $d \geq 1$ , the last condition is satisfied if  $d \leq 3$ . Hence Theorem 1.1 leaves us an interesting open question whether there is an example of a nonsingular surface  $X \subset \mathbb{P}_K^3$  of degree 4 for which  $\text{CH}_0(X)\{\ell\}$  is infinite.

A distinguished role is played in the proof of Theorem 1.1 by the  $\ell$ -adic regulator map

$$\rho_X : \text{CH}^2(X, 1) \otimes \mathbb{Q}_\ell \longrightarrow H_{\text{cont}}^1(\text{Spec}(K), H^2(X_{\bar{K}}, \mathbb{Q}_\ell(2))) \quad (X_{\bar{K}} = X \times_K \bar{K})$$

from higher Chow group to continuous étale cohomology ([21]), where  $\bar{K}$  is an algebraic closure of  $K$  and  $\ell$  is a prime different from  $\text{ch}(K)$ . It is known that the image of  $\rho_X$  is contained in the subspace

$$H_g^1(\text{Spec}(K), V) \subset H_{\text{cont}}^1(\text{Spec}(K), V) \quad (V = H^2(X_{\bar{K}}, \mathbb{Q}_\ell(2)))$$

introduced by Bloch and Kato [6]. In case  $\ell \neq p$  this is obvious since  $H_g^1 = H^1$  by definition. For  $\ell = p$  this is a consequence of a fundamental result in  $p$ -adic Hodge theory, which affirms that every representation of  $G_K = \text{Gal}(\bar{K}/K)$  arising from the cohomology of a variety over  $K$  is a de Rham representation (see the discussion after [6], (3.7.4)).

When  $K$  is a number field or a  $p$ -adic field, it is proved in [31] that  $\text{CH}^2(X)\{\ell\}$  is finite in case the image of  $\rho_X$  coincides with  $H_g^1(\text{Spec}(K), V)$ . Bloch and Kato conjecture that it should be always the case if  $K$  is a number field.

The first key step in the proof of Theorem 1.1 is to disprove the variant of the Bloch-Kato conjecture for a generic surface  $X \subset \mathbb{P}_K^3$  over a  $p$ -adic field  $K$  (see Theorem 3.6). In terms of Galois representations of  $G_K = \text{Gal}(\bar{K}/K)$ , our result implies the existence of a 1-extension of  $\mathbb{Q}_\ell$ -vector spaces with continuous  $G_K$ -action:

$$(\#) \quad 0 \rightarrow H^2(X_{\bar{K}}, \mathbb{Q}_\ell(2)) \rightarrow E \rightarrow \mathbb{Q}_\ell \rightarrow 0,$$

such that  $E$  is a de Rham representation of  $G_K$  but that there is no 1-extension of motives over  $K$ :

$$0 \rightarrow h^2(X)(2) \rightarrow M \rightarrow h(\text{Spec}(K)) \rightarrow 0$$

which gives rise to  $(\#)$  under the realization functor. The rough idea of the proof of the first key result is to relate the  $\ell$ -adic regulator map to an analytic regulator map by using the comparison theorem for étale and analytic cohomology and then to show

that the analytic regulator map is the zero map. With the aid of the theory of mixed Hodge modules ([33]), this is reduced to showing the exactness of the de Rham complex associated to a variation of Hodge structure, which is proved by the infinitesimal method in Hodge theory. This is done in §3 after in §2 we review some basic facts on the cycle class map for higher Chow groups.

Another key ingredient is the injectivity result on the cycle class map for the Chow group of 1-cycles on a proper smooth model of  $X$  over the ring  $O_K$  of integers in  $K$  due to Sato and the second author [30]. It plays an essential role in deducing the main theorem 1.1 from the first key result, which is done in §4.

Finally, in §5 Appendix, we will apply our method to produce an example of a curve  $C$  over a  $p$ -adic field such that  $SK_1(C)_{\text{tor}}$  is infinite.

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*Notations.* For an abelian group  $M$ , we denote by  $M[n]$  (resp.  $M/n$ ) the kernel (resp. cokernel) of multiplication by  $n$ . For a prime number  $\ell$  we put

$$M\{\ell\} := \bigcup_n M[\ell^n], \quad M_{\text{tor}} := \bigoplus_{\ell} M\{\ell\}.$$

For a nonsingular variety  $X$  over a field  $\text{CH}^j(X, i)$  denotes Bloch's higher Chow groups. We write  $\text{CH}^j(X) := \text{CH}^j(X, 0)$  for the (usual) Chow groups.

## 2 Review of the cycle class map and $\ell$ -adic regulator

In this section  $X$  denotes a smooth variety over a field  $K$  and  $n$  denotes a positive integer prime to  $\text{ch}(K)$ .

**2.1:** By [18] we have the cycle class map

$$c_{\text{ét}}^{i,j} : \text{CH}^i(X, j, \mathbb{Z}/n\mathbb{Z}) \rightarrow H_{\text{ét}}^{2i-j}(X, \mathbb{Z}/n\mathbb{Z}(i)),$$

where the right hand side is the étale cohomology of  $X$  with coefficients  $\mu_n^{\otimes i}$ , Tate twist of the sheaf of  $n$ -th roots of unity. The left hand side is Bloch's higher Chow group with finite coefficient which fits into the exact sequence

$$0 \rightarrow \text{CH}^i(X, j)/n \rightarrow \text{CH}^i(X, j, \mathbb{Z}/n\mathbb{Z}) \rightarrow \text{CH}^i(X, j-1)[n] \rightarrow 0. \quad (2.1)$$

In this paper we are only concerned with the map

$$c_{\text{ét}} = c_{\text{ét}}^{2,1} : \text{CH}^2(X, 1, \mathbb{Z}/n\mathbb{Z}) \rightarrow H_{\text{ét}}^3(X, \mathbb{Z}/n\mathbb{Z}(2)). \quad (2.2)$$

By [7] it is injective and its image is equal to

$$NH_{\text{ét}}^3(X, \mathbb{Z}/n\mathbb{Z}(2)) = \text{Ker}(H_{\text{ét}}^3(X, \mathbb{Z}/n\mathbb{Z}(2)) \rightarrow H_{\text{ét}}^3(\text{Spec}(K(X)), \mathbb{Z}/n\mathbb{Z}(2))),$$

where  $K(X)$  is the function field of  $X$ . In view of (2.1) it implies an exact sequence

$$0 \longrightarrow \text{CH}^2(X, 1)/n \xrightarrow{c_{\text{ét}}} NH_{\text{ét}}^3(X, \mathbb{Z}/n\mathbb{Z}(2)) \longrightarrow \text{CH}^2(X)[n] \longrightarrow 0. \quad (2.3)$$

**2.2:** We also need the cycle map to continuous étale cohomology group (cf. [21]):

$$c_{\text{cont}} : \text{CH}^2(X, 1) \longrightarrow H_{\text{cont}}^3(X, \mathbb{Z}_\ell(2)),$$

where  $\ell$  is a prime different from  $\text{ch}(K)$ . Note that in case  $K$  is a  $p$ -adic field we have

$$H_{\text{cont}}^3(X, \mathbb{Z}_\ell(2)) = \varprojlim_n H_{\text{ét}}^3(X, \mathbb{Z}/\ell^n \mathbb{Z}(2))$$

and  $c_{\text{cont}}$  is induced by  $c_{\text{ét}}$  by passing to the limit. We have the Hochschild-Serre spectral sequence

$$E_2^{i,j} = H_{\text{cont}}^i(\text{Spec}(K), H^j(X_{\bar{K}}, \mathbb{Z}_\ell(2))) \Rightarrow H_{\text{cont}}^{i+j}(X, \mathbb{Z}_\ell(2)). \quad (2.4)$$

If  $K$  is finitely generated over the prime subfield and  $X$  is proper smooth over  $K$ , the Weil conjecture proved by Deligne implies that  $H^0(\text{Spec}(K), H^3(X_{\bar{K}}, \mathbb{Q}_\ell(2))) = 0$ . The same conclusion holds if  $K$  is a  $p$ -adic field and  $X$  is proper smooth having good reduction over  $K$  (In case  $\ell \neq p$  this follows from the proper smooth base change theorem for étale cohomology. In case  $\ell = p$  one uses comparison theorems between  $p$ -adic étale and crystalline cohomology and the Weil conjecture for crystalline cohomology). Thus we get under these assumptions the following map

$$\rho_X : \text{CH}^2(X, 1) \longrightarrow H_{\text{cont}}^1(\text{Spec}(K), H^2(X_{\bar{K}}, \mathbb{Q}_\ell(2))) \quad (2.5)$$

as the composite of  $c_{\text{cont}} \otimes \mathbb{Q}_\ell$  and an edge homomorphism

$$H_{\text{cont}}^3(X, \mathbb{Q}_\ell(2)) \rightarrow H_{\text{cont}}^1(\text{Spec}(K), H^2(X_{\bar{K}}, \mathbb{Q}_\ell(2))).$$

**2.3:** For later use, we need an alternative definition of cycle class maps. For an integer  $i \geq 1$  we denote by  $\mathcal{K}_i$  the sheaf on  $X_{\text{Zar}}$ , the Zariski site on  $X$ , associated to the presheaf  $U \mapsto K_i(U)$ . By [23], 2.5, we have canonical isomorphisms

$$\text{CH}^2(X, 1) \simeq H_{\text{Zar}}^1(X, \mathcal{K}_2), \quad \text{CH}^2(X, 1, \mathbb{Z}/n\mathbb{Z}) \simeq H_{\text{Zar}}^1(X, \mathcal{K}_2/n). \quad (2.6)$$

Let  $\epsilon^{\text{ét}} : X_{\text{ét}} \rightarrow X_{\text{Zar}}$  be the natural map of sites and put

$$\mathcal{H}_{\text{ét}}^i(\mathbb{Z}/n\mathbb{Z}(r)) = R^i \epsilon_*^{\text{ét}} \mu_n^{\otimes r}.$$

The universal Chern classes in the cohomology groups of the simplicial classifying space for  $\text{GL}_n$  ( $n \geq 1$ ) give rise to higher Chern class maps on algebraic  $K$ -theory (cf. [17], [35]). It gives rise to a map of sheaves:

$$\mathcal{K}_i/n \longrightarrow \mathcal{H}_{\text{ét}}^i(\mathbb{Z}/n\mathbb{Z}(i)). \quad (2.7)$$

By [26] it is an isomorphism for  $i = 2$  and induces an isomorphism

$$H_{\text{Zar}}^1(X, \mathcal{K}_2/n) \xrightarrow{\cong} H_{\text{Zar}}^1(X, \mathcal{H}_{\text{ét}}^2(\mathbb{Z}/n\mathbb{Z}(2))). \quad (2.8)$$

By the spectral sequence

$$E_2^{pq} = H_{\text{Zar}}^p(X, \mathcal{H}_{\text{ét}}^q(\mathbb{Z}/n\mathbb{Z}(2))) \Longrightarrow H_{\text{ét}}^{p+q}(X, \mathbb{Z}/n\mathbb{Z}(2)).$$

together with the fact  $H_{\text{Zar}}^p(X, \mathcal{H}_{\text{ét}}^q(\mathbb{Z}/n\mathbb{Z}(2))) = 0$  for  $p > q$  shown by Bloch-Ogus [7], we get an injective map

$$H_{\text{Zar}}^1(X, \mathcal{H}_{\text{ét}}^2(\mathbb{Z}/n\mathbb{Z}(2))) \longrightarrow H_{\text{ét}}^3(X, \mathbb{Z}/n\mathbb{Z}(2))$$

Again by the Bloch-Ogus theory the image of the above map coincides with the coniveau filtration  $NH_{\text{ét}}^3(X, \mathbb{Z}/n\mathbb{Z}(2))$ . Combined with (2.6) and (2.8) we thus get the map

$$c_{\text{ét}} : \text{CH}^2(X, 1, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H_{\text{Zar}}^1(X, \mathcal{K}_2/n) \xrightarrow{\cong} NH_{\text{ét}}^3(X, \mathbb{Z}/n\mathbb{Z}(2)) \xrightarrow{\subset} H_{\text{ét}}^3(X, \mathbb{Z}/n\mathbb{Z}(2)).$$

It is known that this agrees with the map (2.2) (see [10] Proposition 1).

**2.4:** Now we work over the base field  $K = \mathbb{C}$ . Let  $X_{\text{an}}$  be the site on the underlying analytic space  $X(\mathbb{C})$  endowed with the ordinary topology. Let  $\epsilon^{\text{an}} : X_{\text{an}} \rightarrow X_{\text{Zar}}$  be the natural map of sites and put

$$H_{\text{an}}^i(\mathbb{Z}(r)) = R^i \epsilon_*^{\text{an}} \mathbb{Z}(r) \quad (\mathbb{Z}(r) = (2\pi\sqrt{-1})^r \mathbb{Z}).$$

Higher Chern class map then gives a map of sheaves

$$\mathcal{K}_i \longrightarrow \mathcal{H}_{\text{an}}^i(\mathbb{Z}(i)). \quad (2.9)$$

By the same argument as before, it induces a map

$$c_{\text{an}} : \text{CH}^2(X, 1) \xrightarrow{\cong} H_{\text{Zar}}^1(X, \mathcal{K}_2) \longrightarrow H_{\text{an}}^3(X(\mathbb{C}), \mathbb{Z}(2)) \quad (2.10)$$

**Lemma 2.1** *The image of  $c_{\text{an}}$  is contained in  $F^2 H_{\text{an}}^3(X(\mathbb{C}), \mathbb{C})$ , the Hodge filtration defined in [14]. In particular if  $X$  is complete, the image is torsion.*

*Proof.* Let  $\mathcal{H}_D^r(\mathbb{Z}(i))$  be the sheaf on  $X_{\text{Zar}}$  associated to a presheaf

$$U \mapsto H_D^r(U, \mathbb{Z}(i))$$

where  $H_D^\bullet$  denotes Deligne-Beilinson cohomology (cf. [15], 2.9). Higher Chern class maps to Deligne-Beilinson cohomology give rise to the map  $K_2 \rightarrow \mathcal{H}_D^2(\mathbb{Z}(2))$  and  $c_{\text{an}}$  factors as in the following commutative diagram

$$\begin{array}{ccccc} H_{\text{Zar}}^1(X, \mathcal{K}_2) & \longrightarrow & H_{\text{Zar}}^1(X, \mathcal{H}_D^2(\mathbb{Z}(2))) & \longrightarrow & H_{\text{Zar}}^1(X, \mathcal{H}_{\text{an}}^2(\mathbb{Z}(2))) \\ & & a \downarrow & & \downarrow \\ & & H_D^3(X, \mathbb{Z}(2)) & \xrightarrow{b} & H_{\text{an}}^3(X(\mathbb{C}), \mathbb{Z}(2)). \end{array}$$

Here the map  $a$  is induced from the spectral sequence

$$E_2^{pq} = H_{\text{Zar}}^p(X, \mathcal{H}_D^q(\mathbb{Z}(2))) \implies H_D^{p+q}(X, \mathbb{Z}(2))$$

in view of the fact  $H_{\text{Zar}}^p(X, \mathcal{H}_D^1(\mathbb{Z}(2))) = 0$  for  $\forall p > 0$  since  $\mathcal{H}_D^1(\mathbb{Z}(2)) \cong \mathbb{C}/\mathbb{Z}(2)$  (constant sheaf). Since the image of  $b$  is contained in  $F^2 H_{\text{an}}^3(X(\mathbb{C}), \mathbb{C})$  (see [15], 2.10), so is the image of  $c_{\text{an}}$ . Q.E.D.

**Lemma 2.2** *We have the following commutative diagram*

$$\begin{array}{ccc} \text{CH}^2(X, 1) & \xrightarrow{c_{\text{an}}} & H_{\text{an}}^3(X(\mathbb{C}), \mathbb{Z}(2)) \\ \downarrow & & \downarrow \\ \text{CH}^2(X, 1, \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{c_{\text{ét}}} & H_{\text{ét}}^3(X, \mathbb{Z}/n\mathbb{Z}(2)) \end{array} \quad (2.11)$$

Here the right vertical map is the composite

$$H_{\text{an}}^3(X(\mathbb{C}), \mathbb{Z}(2)) \rightarrow H_{\text{an}}^3(X(\mathbb{C}), \mathbb{Z}(2) \otimes \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\cong} H_{\text{ét}}^3(X, \mathbb{Z}/n\mathbb{Z}(2))$$

and the isomorphism comes from the comparison isomorphism between étale cohomology and ordinary cohomology (SGA4 $\frac{1}{2}$ , Arcata, 3.5) together with the isomorphism

$$\mathbb{Z}(1) \otimes \mathbb{Z}/n\mathbb{Z} \simeq (\epsilon^{\text{an}})^* \mu_n$$

given by the exponential map.

*Proof.* This follows from the compatibility of (2.7) and (2.9), namely the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{K}_i & \longrightarrow & \mathcal{H}_{\text{an}}^i(\mathbb{Z}(i)) \\ \downarrow & & \downarrow \\ \mathcal{K}_i/n & \longrightarrow & \mathcal{H}_{\text{ét}}^i(\mathbb{Z}/n\mathbb{Z}(i)), \end{array}$$

and it follows from the compatibility of the universal Chern classes ([17] and [35]). Q.E.D.

### 3 Counterexample to the Bloch-Kato conjecture over $p$ -adic field

In this section  $K$  denotes a  $p$ -adic field and let  $X$  be a proper smooth surface over  $K$ . We fix a prime  $\ell$  (possibly  $\ell = p$ ) and consider the map (2.5)

$$\rho_X : \text{CH}^2(X, 1) \longrightarrow H_{\text{cont}}^1(\text{Spec}(K), V) \quad (V = H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_{\ell}(2))). \quad (3.1)$$

Define the primitive part  $\tilde{V}$  of  $V$  by:

$$\tilde{V} := H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_{\ell}(2))/V_0, \quad V_0 = [H_X] \otimes \mathbb{Q}_{\ell}(1), \quad (3.2)$$

where  $[H_X] \in H_{\text{cont}}^2(X_{\bar{K}}, \mathbb{Q}_{\ell}(1))$  is the cohomology class of a hyperplane section. Noting

$$\tilde{V} \simeq \text{Ker}(H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_{\ell}(2)) \xrightarrow{\cup[H_X]} H_{\text{ét}}^4(X_{\bar{K}}, \mathbb{Q}_{\ell}(3))),$$

we get a decomposition as  $G_K$ -modules:

$$V = \tilde{V} \oplus V_0. \quad (3.3)$$

Let

$$\tilde{\rho} : \text{CH}^2(X, 1) \longrightarrow H_{\text{cont}}^1(\text{Spec}(K), \tilde{V})$$

be the induced map.

**Theorem 3.1** *Let  $X \subset \mathbb{P}_K^3$  be a generic smooth surface of degree  $d \geq 5$ . Then  $\tilde{\rho}$  is the zero map for arbitrary  $\ell$ .*

**Remark 3.2** (1) *This is an analogue of [38] 1.6 (where she worked on Deligne-Beilinson cohomology).*

(2) *Bloch-Kato [6] considers regulator maps such as (3.1) for a smooth projective variety over a number field and conjectures that its image coincides with  $H_g^1$ . We will see later (see Theorem 3.6) that the variant of the conjecture over a  $p$ -adic field is false in general.*

(3) *The construction of a counterexample mentioned in (2) hinges on the assumption that the surface  $X \subset \mathbb{P}_K^3$  is generic. One may still ask whether the image of  $l$ -adic regulator map coincides  $H_g^1$  for a proper smooth variety  $X$  over a  $p$ -adic field when  $X$  is defined over a number field.*

*Proof.* Let  $f : \mathcal{X} \rightarrow M$  be as in the introduction and let  $t : \text{Spec}(K) \rightarrow M$  be a dominant morphism such that  $X \simeq \mathcal{X} \times_M \text{Spec}(K)$ . For a morphism  $S \rightarrow M$  of smooth schemes over  $\mathbb{Q}$  let  $f_S : X_S = \mathcal{X} \times_M S \rightarrow S$  be the base change of  $f$ . The same construction of (2.5) give rise to the regulator map

$$\rho_S : \text{CH}^2(X_S, 1) \rightarrow H_{\text{cont}}^1(S, V_S),$$

where  $V_S = R^2(f_S)_* \mathbb{Q}_l(2)$  is a smooth  $\mathbb{Q}_l$ -sheaf on  $S$ . Define the primitive part of  $V_S$ :

$$\tilde{V}_S = R^2(f_S)_* \mathbb{Q}_l(2) / [H] \otimes \mathbb{Q}_l(1),$$

where  $[H] \in H^0(S, R^2(f_S)_* \mathbb{Q}_l(1))$  is the class of a hyperplane section. Let

$$\tilde{\rho}_S : \text{CH}^2(X_S, 1) \rightarrow H_{\text{cont}}^1(S, \tilde{V}_S),$$

be the induced map. Note

$$\text{CH}^2(X, 1) = \varinjlim_S \text{CH}^2(X_S, 1),$$

where  $S \rightarrow M$  ranges over the smooth morphisms which factor  $t : \text{Spec}(K) \rightarrow M$ . Note also that we have the commutative diagram for such  $S$ :

$$\begin{array}{ccc} \text{CH}^2(X_S, 1) & \xrightarrow{\tilde{\rho}_S} & H_{\text{cont}}^1(S, \tilde{V}_S) \\ \downarrow & & \downarrow \\ \text{CH}^2(X, 1) & \xrightarrow{\tilde{\rho}} & H_{\text{cont}}^1(\text{Spec}(K), \tilde{V}). \end{array}$$

Thus it suffices to show

$$H_{\text{cont}}^1(S, \tilde{V}_S) = 0.$$

Without loss of generality we suppose  $S$  is an affine smooth variety over a finite extension  $L$  of  $\mathbb{Q}$ .

**Claim 3.3** *Assume  $d \geq 4$ . The natural map*

$$H_{\text{cont}}^1(S, \tilde{V}_S) \longrightarrow H_{\text{ét}}^1(S_{\overline{\mathbb{Q}}}, \tilde{V}_S) \quad (S_{\overline{\mathbb{Q}}} := S \times_L \text{Spec}(\overline{\mathbb{Q}}))$$

*is injective.*

Indeed, by the Hochschild-Serre spectral sequence, it is enough to see  $H_{\text{ét}}^0(S_{\overline{\mathbb{Q}}}, \tilde{V}_S) = 0$ , which follows from [3], Th.6.1. (2).

By SGA4 $\frac{1}{2}$ , Arcata, Cor.(3.3) and (3.5.1) we have

$$H_{\text{ét}}^1(S_{\overline{\mathbb{Q}}}, \tilde{V}_S) \cong H_{\text{ét}}^1(S_{\mathbb{C}}, \tilde{V}_S) \simeq H_{\text{an}}^1(S(\mathbb{C}), \tilde{V}_S^{\text{an}}) \otimes \mathbb{Q}_l, \quad (S_{\mathbb{C}} := S \times_L \text{Spec}(\mathbb{C}))$$

where  $\tilde{V}_S^{\text{an}}$  is the primitive part of  $V_S^{\text{an}} = R^2(f_S^{\text{an}})_* \mathbb{Q}(2)$  with  $f_S^{\text{an}} : (X_{S_{\mathbb{C}}})_{\text{an}} \rightarrow (S_{\mathbb{C}})_{\text{an}}$ , the natural map of sites. By definition  $\tilde{V}_S^{\text{an}}$  is a local system on  $S(\mathbb{C})$  whose fiber over  $s \in S(\mathbb{C})$  is the primitive part of  $H_{\text{an}}^2(X_s(\mathbb{C}), \mathbb{Q}(2))$  for  $X_s$ , the fiber of  $X_S \rightarrow S$  over  $s$ . Due to Lemma 2.2, it suffices to show the triviality of the image of the map

$$\tilde{\rho}_S^{\text{an}} : \text{CH}^2(X_{S_{\mathbb{C}}}, 1) \longrightarrow H_{\text{an}}^1(S(\mathbb{C}), \tilde{V}_S^{\text{an}})$$



which is induced from

$$c_{\text{an}} : \text{CH}^2(X_{S_{\mathbb{C}}}, 1) \longrightarrow H_{\text{an}}^3(X_S(\mathbb{C}), \mathbb{Q}(2))$$

by using the natural map

$$H_{\text{an}}^3(X_S(\mathbb{C}), \mathbb{Q}(2)) \rightarrow H_{\text{an}}^1(S(\mathbb{C}), V_S^{\text{an}})$$

arising from the Leray spectral sequence for  $f_S^{\text{an}} : (X_{S_{\mathbb{C}}})_{\text{an}} \rightarrow (S_{\mathbb{C}})_{\text{an}}$  and the vanishing  $R^3(f_S^{\text{an}})_* \mathbb{Q}(2) = 0$ .

**Claim 3.4** *The image of  $\tilde{\rho}_S^{\text{an}}$  is contained in the Hodge filtration  $F^2 H_{\text{an}}^1(S(\mathbb{C}), \tilde{V}_S^{\text{an}} \otimes \mathbb{C})$  defined by the theory of Hodge modules [33] §4.*

This follows from the functoriality of Hodge filtrations and Lemma 2.1.

It is quite complicated to describe the Hodge filtration on  $H_{\text{an}}^1(S(\mathbb{C}), \tilde{V}_S^{\text{an}} \otimes \mathbb{C})$  precisely. However, all that we need is the following property:

**Claim 3.5** *For integers  $m, p \geq 0$  there is a natural injective map*

$$F^p H_{\text{an}}^m(S(\mathbb{C}), \tilde{V}_S^{\text{an}} \otimes \mathbb{C}) \rightarrow H_{\text{Zar}}^m(S_{\mathbb{C}}, G^p \text{DR}(\tilde{V}_S^{\text{an}}))$$

where  $G^p \text{DR}(\tilde{V}_S^{\text{an}})$  is the complex of Zariski sheaves on  $S_{\mathbb{C}}$ :

$$\begin{aligned} F^p H_{\text{dR}}^2(X_S/S)_{\text{prim}} \otimes \mathcal{O}_{S_{\mathbb{C}}} &\rightarrow F^{p-1} H_{\text{dR}}^2(X_S/S)_{\text{prim}} \otimes \Omega_{S_{\mathbb{C}}/\mathbb{C}}^1 \rightarrow \cdots \\ \cdots &\rightarrow F^{p-r} H_{\text{dR}}^2(X_S/S)_{\text{prim}} \otimes \Omega_{S_{\mathbb{C}}/\mathbb{C}}^r \rightarrow F^{p-r} H_{\text{dR}}^2(X_S/S)_{\text{prim}} \otimes \Omega_{S_{\mathbb{C}}/\mathbb{C}}^{r+1} \rightarrow \cdots \end{aligned}$$

Here  $H_{\text{dR}}^\bullet(X_S/S)$  denotes the de Rham cohomology of  $X_S/S$ , and  $H_{\text{dR}}^\bullet(X_S/S)_{\text{prim}}$  is its primitive part defined by the same way as before, and the maps are induced from the Gauss-Manin connection thanks to Griffiths transversality.

This follows from [1] Lemma 4.2. We note that its proof hinges on the theory of mixed Hodge modules. Key points are Deligne's comparison theorem for algebraic and analytic cohomology of a vector bundle with integrable connection with regular singularities (cf. [12], §6) and degeneration of the Hodge spectral sequence for cohomology with coefficients (cf. [33], (4.1.3)).

By the above claims we are reduced to showing the exactness at the middle term of the following complex:

$$F^2 H_{\text{dR}}^2(X_S/S)_{\text{prim}} \otimes \mathcal{O}_{S_{\mathbb{C}}} \rightarrow F^1 H_{\text{dR}}^2(X_S/S)_{\text{prim}} \otimes \Omega_{S_{\mathbb{C}}/\mathbb{C}}^1 \rightarrow H_{\text{dR}}^2(X_S/S)_{\text{prim}} \otimes \Omega_{S_{\mathbb{C}}/\mathbb{C}}^2 \quad (3.4)$$

This is proved by the infinitesimal method in Hodge theory. We sketch the proof. Let  $f : X_S \rightarrow S$  be the natural morphism. The assertion follows from the exactness at the middle term of the complex:

$$f_* \Omega_{X_S/S}^2 \otimes \mathcal{O}_{S_{\mathbb{C}}} \longrightarrow (R^1 f_* \Omega_{X_S/S}^1)_{\text{prim}} \otimes \Omega_{S_{\mathbb{C}}/\mathbb{C}}^1 \longrightarrow R^2 f_* \mathcal{O}_{X_S} \otimes \Omega_{S_{\mathbb{C}}/\mathbb{C}}^2 \quad (3.5)$$

and the injectivity of the complex:

$$f_*\Omega_{X_S/S}^2 \otimes \Omega_{S_C/\mathbb{C}}^1 \longrightarrow (R^1 f_*\Omega_{X_S/S}^1)_{\text{prim}} \otimes \Omega_{S_C/\mathbb{C}}^2. \quad (3.6)$$

These complexes are induced by the complex (3.4) by Griffiths transversality. In case  $S = M \subset \mathbb{P}(H^0(\mathbb{P}_{\mathbb{Q}}^3, \mathcal{O}_{\mathbb{P}}(d)))$ , these assertions are proved as follows: Let  $P = \mathbb{C}[z_0, z_1, z_2, z_3]$  and  $P^n \subset P$  be the subspace of the homogeneous polynomials of degree  $n$ . Take a point  $x \in M(\mathbb{C})$  and choose  $F \in P^d$  which defines the surface corresponding to  $x$ . Let  $R = \mathbb{C}[z_0, z_1, z_2, z_3]/(\frac{\partial F}{\partial z_0}, \dots, \frac{\partial F}{\partial z_3})$  be the Jacobian ring and  $R^n \subset R$  be the image of  $P^n$  in  $R$ . Then the fibers over  $x$  of (3.5) and (3.6) are identified with the following Koszul complexes:

$$R^{d-4} \longrightarrow R^{2d-4} \otimes (R^d)^* \longrightarrow R^{3d-4} \otimes \overset{2}{\wedge}(R^d)^*, \quad (3.7)$$

$$R^{d-4} \otimes (R^d)^* \longrightarrow R^{2d-4} \otimes \overset{2}{\wedge}(R^d)^* \quad (3.8)$$

where  $(R^d)^*$  denotes the dual space of  $R$  and the maps are induced from the multiplication  $R \otimes R \rightarrow R$ . Then the Donagi symmetrizer lemma (cf. [19], p.76) implies that (3.7) is exact at the middle term if  $d \geq 5$  and (3.8) is injective if  $d \geq 3$ , which proves the desired assertion in case  $S = M$ . The assertion in case  $S$  is dominant over  $M$  is reduced to the case  $S = M$  by an easy argument (cf. [4] §9). This completes the proof of Theorem 3.1. Q.E.D.

Let  $\mathcal{O}_K \subset K$  be the ring of integers and  $k$  be the residue field. In order to construct an example where the image of the regulator map

$$\rho_X : \text{CH}^2(X, 1) \xrightarrow{\rho_X} H_{\text{cont}}^1(\text{Spec}(K), V) \quad (V = H_{\text{ét}}^2(X_{\overline{K}}, \mathbb{Q}_{\ell}(2)))$$

is not equal to  $H_g^1(\text{Spec}(K), V)$ , we now take a proper smooth surface  $X$  having good reduction over  $K$  so that  $X$  has a proper smooth model  $X_{\mathcal{O}_K}$  over  $\text{Spec}(\mathcal{O}_K)$ . We denote the special fiber by  $Y$ . By [24] (see the diagram below 5.7 on p.341), there is a commutative diagram

$$\begin{array}{ccc} \text{CH}^2(X, 1) & \xrightarrow{\tilde{\rho}} & H_g^1(\text{Spec}(K), V) \\ \downarrow \partial & & \downarrow \\ \text{CH}^1(Y) & \xrightarrow{\alpha} & H_{\text{cont}}^1(\text{Spec}(K), V)/H_f^1(\text{Spec}(K), V) \end{array} \quad (3.9)$$

where  $H_f^1 \subset H_g^1 \subset H_{\text{cont}}^1$  are the subspaces introduced by Bloch-Kato [6] and  $\partial$  is a boundary map in localization sequence for higher Chow groups.

**Theorem 3.6** *Let  $X \subset \mathbb{P}_K^3$  be a generic smooth surface of degree  $d \geq 5$ . Assume that  $X$  has a projective smooth model  $X_{\mathcal{O}_K} \subset \mathbb{P}_{\mathcal{O}_K}^3$  over  $\mathcal{O}_K$  and let  $Y \subset \mathbb{P}_k^3$  be its special fiber.*

- (1) *The image of  $\partial \otimes \mathbb{Q}$  is contained in the subspace of  $\text{CH}^1(Y) \otimes \mathbb{Q}$  generated by the class  $[H_Y]$  of a hyperplane section of  $Y$ .*
- (2) *Let  $r$  be the Picard number of  $Y$ . Then*

$$\dim_{\mathbb{Q}_{\ell}}(H_g^1(\text{Spec}(K), V)/\text{Image}(\rho_X)) \geq r - 1.$$

*Proof.* Recall  $V = \widetilde{V} \oplus V_0$ , a decomposition as  $G_K$ -modules (cf. (3.3)). Let  $W \subset CH^2(X, 1)$  be the image of  $\mathbb{Z} \cdot [H_X] \otimes K^\times$  under the product map  $CH^1(X) \otimes K^\times \rightarrow CH^2(X, 1)$ . Then it is easy to see  $\rho_X$  induces an isomorphism

$$W \otimes \mathbb{Q}_\ell \simeq H_g^1(\mathrm{Spec}(K), V_0) = H_{\mathrm{cont}}^1(\mathrm{Spec}(K), V_0)$$

and that  $\partial(W) = \mathbb{Z} \cdot [H_Y] \subset CH^1(Y)$ . Hence (1) follows from Theorem 3.1 together with injectivity of  $\alpha$  in (3.9) proved by [24], Lemma 5-7.

As for (2) we first note that  $\dim_{\mathbb{Q}_\ell}(H_{\mathrm{cont}}^1(\mathrm{Spec}(K), V_0)/H_f^1(\mathrm{Spec}(K), V_0)) = 1$  (see [6], 3.9). Moreover the same argument (except using the Tate conjecture) in the last part of §5 of [24] shows

$$\dim_{\mathbb{Q}_\ell}(CH^1(Y) \otimes \mathbb{Q}_\ell) \leq \dim_{\mathbb{Q}_\ell}(H_g^1(\mathrm{Spec}(K), V)/H_f^1(\mathrm{Spec}(K), V)).$$

Hence (2) follows from (1). Q.E.D.

**Remark 3.7** *Let the assumption be as in Corollary 3.6. Then we have*

$$\dim_{\mathbb{Q}_\ell}(H_g^1(\mathrm{Spec}(K), V)/\mathrm{Image}(\rho_X)) \geq \begin{cases} r-1 & \ell \neq p \\ r-1 + (h^{0,2} + h^{1,1} - 1)[K : \mathbb{Q}_p] & \ell = p \end{cases}$$

where  $h^{p,q} := \dim_K H^q(X, \Omega_{X/K}^p)$  denotes the Hodge number. Moreover the equality holds if and only if the Tate conjecture for divisors on  $Y$  holds. This follows from Theorem 3.1 and the computation of  $\dim_{\mathbb{Q}_\ell} H_g^1(\mathrm{Spec}(K), V)$  using [6] 3.8 and 3.8.4. The details are omitted.

## 4 Proof of Theorem 1.1

Let  $K$  be a  $p$ -adic field and  $\mathcal{O}_K \subset K$  the ring of integers and  $k$  the residue field. Let us consider schemes

$$\begin{array}{ccccc} X & \xrightarrow{j} & X_{\mathcal{O}_K} & \xleftarrow{i} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_K) & \longleftarrow & \mathrm{Spec}(k) \end{array} \quad (4.1)$$

where all vertical arrows are projective and smooth of relative dimension 2 and the diagrams are Cartesian. We have a boundary map in localization sequence for higher Chow groups with finite coefficients

$$\partial : CH^2(X, 1, \mathbb{Z}/n\mathbb{Z}) \rightarrow CH^1(Y)/n.$$

For a prime number  $\ell$ , it induces

$$\partial_\ell : CH^2(X, 1, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \rightarrow CH^1(Y) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell,$$

where  $CH^2(X, 1, \mathbb{Q}_\ell/\mathbb{Z}_\ell) := \varinjlim_n CH^2(X, 1, \mathbb{Z}/\ell^n\mathbb{Z})$ .

**Theorem 4.1** For  $\ell \neq p := \text{ch}(k)$ ,  $\partial_\ell$  is surjective and has finite kernel. Hence we have

$$\text{CH}^2(X, 1, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus r} + (\text{finite group})$$

where  $r$  is the rank of  $\text{CH}^1(Y)$ .

Theorem 1.1 is an immediate consequence of Theorems 3.6 (1), 4.1, and the exact sequence (2.1):

$$0 \rightarrow \text{CH}^2(X, 1) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \rightarrow \text{CH}^2(X, 1, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \rightarrow \text{CH}^2(X)\{\ell\} \rightarrow 0.$$

*Proof.* Write  $\Lambda = \mathbb{Q}_\ell/\mathbb{Z}_\ell$ . We have a commutative diagram:

$$\begin{array}{ccccccc} \text{CH}^2(X, 1, \Lambda) & \xrightarrow{\partial} & \text{CH}^1(Y) \otimes \Lambda & \xrightarrow{i_*} & \text{CH}^2(X_{\mathcal{O}_K}) \otimes \Lambda & \xrightarrow{j^*} & \text{CH}^2(X) \otimes \Lambda \\ \downarrow c_1 & & \downarrow c_2 & & \downarrow c_3 & & \downarrow c_4 \\ H_{\text{ét}}^3(X, \Lambda(2)) & \xrightarrow{\partial_{\text{ét}}} & H_{\text{ét}}^2(Y, \Lambda(1)) & \xrightarrow{i_*} & H_{\text{ét}}^4(X_{\mathcal{O}_K}, \Lambda(2)) & \xrightarrow{j_{\text{ét}}^*} & H_{\text{ét}}^4(X, \Lambda(2)) \end{array}$$

Here the upper exact sequence arises from localization theory for higher Chow groups with finite coefficient ([25], Theorem 1.7) and the lower from localization theory for étale cohomology together with absolute purity [16]. The vertical maps are étale cycle class maps. By (2.3)  $c_1$  is injective. Noting  $\text{CH}^1(Y) = H^1(Y, \mathbb{G}_m)$ ,  $c_2$  is injective by the Kummer theory. It is shown in [30] that  $c_3$  is an isomorphism. Hence the diagram reduces the proof of Theorem 4.1 to showing that  $\text{Ker}(\partial_{\text{ét}})$  and  $\text{Ker}(j_{\text{ét}}^*)$  are finite. This is an easy consequence of the proper base change theorem for étale cohomology and the Weil conjecture ([13]). For the former we use also an exact sequence

$$H_{\text{ét}}^3(X_{\mathcal{O}_K}, \Lambda(2)) \rightarrow H_{\text{ét}}^3(X, \Lambda(2)) \xrightarrow{\partial_{\text{ét}}} H_{\text{ét}}^2(Y, \Lambda(1)).$$

Q.E.D.

## 5 Appendix: $SK_1$ of curves over $p$ -adic fields

Let  $C$  be a proper smooth curve over a field  $K$  and consider  $\text{CH}^2(C, 1)$ . By [23], 2.5, we have an isomorphism

$$\text{CH}^2(C, 1) \simeq H_{\text{Zar}}^1(C, \mathcal{K}_2) \simeq SK_1(C).$$

By definition

$$SK_1(C) = \text{Coker}(K_2(K(C)) \xrightarrow{\delta} \bigoplus_{x \in C_0} K(x)^\times),$$

where  $K(C)$  is the function field of  $C$ ,  $C_0$  is the set of the closed points of  $C$ , and  $K(x)$  is the residue field of  $x \in C_0$ , and  $\delta$  is given by the tame symbols. The norm maps  $K(x)^\times \rightarrow K^\times$  for  $x \in C_0$  induce

$$N_{C/K} : SK_1(C) \rightarrow K^\times.$$

We write  $V(C) = \text{Ker}(N_{C/K})$ .

When  $K$  is a  $p$ -adic field, it is known by class field theory for curves over local field ([34]), that  $V(C)$  is a direct sum of its maximal divisible subgroup and a finite group. An interesting question is whether the divisible subgroup is uniquely divisible, or equivalently whether  $SK_1(C)_{\text{tor}}$  is finite. In case the genus  $g(C) = 1$  affirmative results have been obtained in [32] and [2]. The purpose of this section is to show that the method in the previous sections gives rise to an example of a curve  $C$  of  $g(C) \geq 2$  such that  $SK_1(C)_{\text{tor}}$  is infinite.

Let  $C$  be as in the beginning of this section and let  $n$  be a positive integer prime to  $\text{ch}(K)$ . We have the cycle class map

$$c_{\text{ét}} : \text{CH}^2(C, 2, \mathbb{Z}/n\mathbb{Z}) \rightarrow H_{\text{ét}}^2(C, \mathbb{Z}/n\mathbb{Z}(2)). \quad (5.1)$$

The main result of [26] implies that the above map is an isomorphism. In view of the exact sequence (cf. (2.1)):

$$0 \rightarrow \text{CH}^2(C, 2)/n \rightarrow \text{CH}^2(C, 2, \mathbb{Z}/n\mathbb{Z}) \rightarrow SK_1(C)[n] \rightarrow 0,$$

we get the exact sequence ([37] 23.4):

$$0 \rightarrow \text{CH}^2(C, 2)/n \rightarrow H_{\text{ét}}^2(C, \mathbb{Z}/n\mathbb{Z}(2)) \rightarrow SK_1(C)[n] \rightarrow 0. \quad (5.2)$$

We will use also cycle class map to continuous étale cohomology:

$$c_{\text{cont}} : \text{CH}^2(C, 2) \otimes \mathbb{Q}_\ell \rightarrow H_{\text{cont}}^2(C, \mathbb{Q}_\ell(2))$$

where  $\ell$  is any prime number different from  $\text{ch}(K)$ . When  $K$  is a  $p$ -adic field, one easily shows

$$H_{\text{cont}}^2(C, \mathbb{Q}_\ell(2)) \simeq H_{\text{cont}}^1(\text{Spec}(K), H_{\text{ét}}^1(C_{\overline{K}}, \mathbb{Q}_\ell(2))) \quad (5.3)$$

by using the Hochschild-Serre spectral sequence (2.4). Hence we get the map

$$\rho_C : \text{CH}^2(C, 2) \otimes \mathbb{Q}_\ell \rightarrow H_{\text{cont}}^1(\text{Spec}(K), H_{\text{ét}}^1(C_{\overline{K}}, \mathbb{Q}_\ell(2))). \quad (5.4)$$

Note that  $\rho_C$  is trivial if  $C$  has good reduction and  $\ell \neq p$ , since the group on the right hand side is trivial. The last fact is a consequence of the proper smooth base change theorem for étale cohomology and the weight argument.

Let  $M_g$  be the moduli space of tri-canonically embedded projective nonsingular curves of genus  $g \geq 2$  over the base field  $\mathbb{Q}$  (cf. [11]), and let  $f : \mathcal{C} \rightarrow M_g$  be the universal family.

**Definition 5.1** *Let  $C$  be a proper smooth curve over a field  $K$  of characteristic zero. We say  $C$  is generic if there is a dominant morphism  $\text{Spec}(K) \rightarrow M_g$  such that  $C \cong \mathcal{C} \times_{M_g} \text{Spec}(K)$ .*

**Theorem 5.2** *Let  $K$  be a  $p$ -adic field and let  $C$  be a generic curve of genus  $g \geq 2$  over  $K$ . Then  $\rho_C$  is the zero map for all  $\ell$ . We have an isomorphism*

$$SK_1(C)_{\text{tor}} \cong H_{\text{ét}}^2(C, \mathbb{Q}/\mathbb{Z}(2)) \left( := \varinjlim_n H_{\text{ét}}^2(C, \mathbb{Z}/n\mathbb{Z}(2)) \right).$$

**Remark 5.3** *Theorem 5.2 is comparable with the main result of [20] where they worked on Deligne-Beilinson cohomology.*

*Proof.* The second assertion follows easily from the first in view of (5.2). The first assertion is shown by the same method as the proof of Theorem 3.1, noting the following fact (cf. [20] §3): Let  $S \rightarrow M_g$  be a dominant smooth morphism and put  $f : C_S := \mathcal{C} \times_{M_g} S \rightarrow S$ . Then the map

$$f_*\Omega_{C_S/S}^1 \longrightarrow R^1 f_*\mathcal{O}_{C_S} \otimes \Omega_{S/\mathbb{Q}}^1$$

induced from the Gauss-Manin connection is injective. Q.E.D.

**Corollary 5.4** *Let  $C$  be as in Theorem 5.2. Assume the Jacobian variety  $J(C)$  has semistable reduction over  $K$ . Let  $\mathcal{J}$  be the Neron model of  $J$  with  $\mathcal{J}_s$ , its special fiber. Let  $r$  be the dimension of the maximal split torus in  $\mathcal{J}_s$ . For a prime  $\ell$ , we have*

$$SK_1(C)\{\ell\} \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{r_\ell} \oplus (\text{finite group}),$$

where  $r_\ell = r$  for  $\ell \neq p$  and  $r_p = r + 2g[K : \mathbb{Q}_p]$ .

For example  $SK_1(C)\{\ell\}$  is infinite for any  $\ell$  if  $C$  is a Mumford curve (i.e. a proper smooth curve with semistable reduction over  $K$  such that the irreducible components are isomorphic to  $\mathbb{P}_k^1$  and that they intersect each other at  $k$ -rational points, where  $k$  is the residue field of  $K$ ), which is generic in the sense of Definition 5.1.

Theorem 5.4 follows from Theorem 5.2 and the following:

**Lemma 5.5** *Let  $C$  be proper smooth curve over a  $p$ -adic field  $K$ . Assume  $J(C)$  has semistable reduction over  $K$  and let  $r_\ell$  be as above. Then*

$$\dim_{\mathbb{Q}_\ell} H_{\text{cont}}^2(C, \mathbb{Q}_\ell(2)) = \dim_{\mathbb{Q}_\ell} H_{\text{cont}}^1(\text{Spec}(K), V) = r_\ell. \quad (V = H_{\text{ét}}^1(C_{\overline{K}}, \mathbb{Q}_\ell(2))).$$

*Proof.* The first equality follows from (5.3). By [22], Th. 5 and Cor. 7 (p. 354–355) we have  $H_{\text{cont}}^0(\text{Spec}(K), V) = 0$  and  $\dim_{\mathbb{Q}_\ell} H_{\text{cont}}^2(\text{Spec}(K), V) = r$ . Lemma 5.5 now follows from computation of Euler-Poincaré characteristic given in [36], II 5.7. Q.E.D.

**Remark 5.6** *Using [6], 3.8.4 together with a well-known fact on the structure as  $\text{Gal}(\overline{K}/K)$ -module of the Tate module of an abelian variety over  $K$  (cf. [39], Exposé IX), one can show that  $H_{\text{cont}}^1(\text{Spec}(K), V) = H_g^1(\text{Spec}(K), V)$ . Hence, if  $C$  is a generic curve of genus  $\geq 2$ , then the map  $\rho_C$  (5.4) does not surject on to  $H_g^1$  if  $r_\ell \geq 1$ . This gives another counterexample to a variant of the Bloch-Kato conjecture for  $p$ -adic fields.*

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