MAXIMAL COMPONENTS OF NOETHER-LEFSCHETZ LOCUS FOR BEILINSON-HODGE CYCLES

MASANORI ASAKURA AND SHUJI SAITO

Abstract. We discuss the Noether-Lefschetz locus for Beilinson's Hodge cycles on the complement of two or three hyperplane sections in a smooth projective surface in $\mathbb{P}^3$. The main theorem gives an explicit description of maximal components of the Noether-Lefschetz locus.

Introduction

In the moduli space M of smooth surfaces of degree d in $\mathbb{P}^3$ over $\mathbb{C}$, the locus of those surfaces that possess curves which are not complete intersections of the given surface with another surface is called the Noether-Lefschetz locus and denoted by M_{NL}. One can show that M_{NL} is the union of a countable number of closed algebraic subsets of M. The classical theorem of Noether-Lefschetz affirms that every component of M_{NL} has positive codimension in M when $d \geq 4$. Note that the theorem is false if $d = 3$ since a smooth cubic surface has the Picard number 7. Since the infinitesimal method in Hodge theory was introduced in [CGGH] as a powerful tool to study M_{NL}, fascinating results have been obtained concerning irreducible components of M_{NL}. First we recall the following.

Theorem 0.1 ([G1])**.** *For every irreducible component T of M_{NL},* $\operatorname{codim}(T) \geq d-3$.

The basic idea of the proof of the result is to translate the problem in the language of the infinitesimal variation of Hodge structures on a family of hypersurfaces in a projective space. Then, by the Poincaré residue representation of the cohomology of a hypersurface, the result follows from the duality theorem for the Jacobian ring associated to a hypersurface. We note that the inequality is the best possible since the family of surfaces of degree $d \geq 3$ containing a line has codimension exactly $d-3$. M.Green and C.Voisin proved the following striking theorem.

Theorem 0.2 ([G2], [V])**.** *If $d \geq 5$, the only irreducible component of M_{NL} having codimension $d-3$ is the family of surfaces of degree d containing a line.*

In this paper we study an analog of the above problem in the context of Beilinson's Hodge conjecture. For a quasi-projective smooth variety U over $\mathbb{C}$, the space of Beilinson-Hodge cycles is defined to be

$$F^qH^q(U,\mathbb{Q}(q)) := H^q(U,\mathbb{Q}(q)) \cap F^qH^q_{\mathrm{dR}}(U/\mathbb{C})$$

where $H^q(U,\mathbb{Q}(q))$ is the singular cohomology with coefficient $\mathbb{Q}(q) = (2\pi\sqrt{-1})^q\mathbb{Q}$ and F^* denotes the Hodge filtration of the mixed Hodge structure on $H^q(U,\mathbb{C}) \simeq H^q_{\mathrm{dR}}(U/\mathbb{C})$ defined by Deligne [D]. Beilinson's conjecture claims the surjectivity of the regulator map (cf. [Bl] and [Sch])

$$\mathrm{reg}^q_U \;:\; CH^q(U,q) \otimes \mathbb{Q} \longrightarrow F^qH^q(U,\mathbb{Q}(q)),$$

where $CH^q(U,q)$ is Bloch's higher Chow group [Bl]. Taking a smooth compactification $U \subset X$ with $Z = X - U$, a simple normal crossing divisor on X, we have the following formula for the value of reg^q_U on decomposable elements in $CH^q(U,q)$;

$$\mathrm{reg}^q_U(\{g_1,\ldots,g_q\}) = \mathrm{dlog} g_1 \wedge \cdots \wedge \mathrm{dlog} g_q \in H^0(X,\Omega^q_X(\log Z)) = F^qH^q_{\mathrm{dR}}(U/\mathbb{C}),$$

where $\{g_1,\dots,g_q\}\in CH^q(U,q)$ is the product of $g_j\in CH^1(U,1)=\Gamma(U,\mathcal{O}^*_{Zar})$. Beilinson's conjecture is an analog of the Hodge conjecture which claims the surjectivity of cycle class maps from Chow group to space of Hodge cycles on projective smooth varieties. The conjecture is known to hold for $q=1$ (cf. [J], Th.5.1.3) but open in general for $q\geq 2$.

The main subject to study in this paper is the *Noether-Lefschetz locus for Beilinson-Hodge cycles* on the complement of the union of a normal crossing divisor in a hypersurface in the projective space. Let $n\geq 1$ be an integer and let $X,Y_1,\dots,Y_s\subset\mathbb{P}^{n+1}$ be smooth hypersurfaces intersecting transversally and put

$$Z=\bigcup_{1\leq j\leq s} Z_j \text{ with } Z_j=X\cap Y_j \quad\text{and}\quad U=X-Z. \tag{0.1}$$

Let $H^n(U,\mathbb{Q}(n))_{\mathrm{triv}}$ be the image of the natural restriction map

$$H^n(\mathbb{P}^{n+1}-\bigcup_{1\leq j\leq s} Y_j,\mathbb{Q}(n))\longrightarrow H^n(U,\mathbb{Q}(n)).$$

In case $n\geq 2$ one can show ([AS2], Lem.3.3(2))

$$H^n(U,\mathbb{Q}(n))_{\mathrm{triv}}=\mathrm{reg}^n_U(CH^n(U,n)_{\mathrm{dec}}), \tag{0.2}$$

where $CH^n(U,n)_{\mathrm{dec}}\subset CH^n(U,n)\otimes\mathbb{Q}$ is the so-called decomposable part, the subspace generated by the image of the product map

$$\overbrace{CH^1(U,1)\otimes\cdots\otimes CH^1(U,1)}^{n\text{ times}}\longrightarrow CH^n(U,n).$$

It implies that

$$H^n(U,\mathbb{Q}(n))_{\mathrm{triv}}\subset\mathrm{Im}(\mathrm{reg}^n_U)\subset F^nH^n(U,\mathbb{Q}(n)).$$

We define

$$F^nH^n(U,\mathbb{Q}(n))_{\mathrm{prim}}:=F^nH^n(U,\mathbb{Q}(n))/H^n(U,\mathbb{Q}(n))_{\mathrm{triv}}$$

called the space of *primitive Beilinson-Hodge cycles.* Here we note the following:

Fact 0.3. $F^nH^n(U,\mathbb{Q}(n))=0$ for $s<n$.

This follows from the following facts:

(i) The graded pieces of the weight filtration on $H^n(U,\mathbb{Q}(n))$ are subquotients of pure Hodge structures $H^{n-q}(Z^{(q)},\mathbb{Q}(n-q))$ for $0\leq q\leq s$, where $Z^{(0)}=X$ and

$$Z^{(q)}=\bigcup_{1\leq j_1<\cdots<j_q\leq s} Z_{j_1}\cap\cdots\cap Z_{j_q}\quad\text{ for } q\geq 1.$$

(ii) The Hodge symmetry implies $F^pH^p(W,\mathbb{Q}(p))=0$ for $p\geq 1$ when W is a smooth projective variety W over $\mathbb{C}$.

Now fix integers $d\geq 1$ and $e_j\geq 1$ with $1\leq j\leq s$. Let M be the moduli space of sets of hypersurfaces $(X,Y_1,\dots,Y_s)$ of degree $(d,e_1,\dots,e_s)$ in $\mathbb{P}^{n+1}$ which intersect transversally. Let $(\mathcal{X},\mathcal{Y}_1,\dots,\mathcal{Y}_s)$ be the universal family over M and put

$$\mathcal{Z}=\mathcal{X}\cap(\bigcup_{1\leq j\leq s}\mathcal{Y}_j),\quad \mathcal{U}=\mathcal{X}-\mathcal{Z}.$$

For $t\in M$ let $U_t\subset X_t\supset Z_t$ be the fibers of $\mathcal{U}\subset\mathcal{X}\supset\mathcal{Z}$.

Definition 0.4. The Noether-Lefschetz locus for Beilinson-Hodge cycles on $\mathcal{U}/M$ is

$$M_{NL}=\{t\in M|\ F^nH^n(U_t,\mathbb{Q}(n))_{\mathrm{prim}}\neq 0\}$$

The analogy with the classical Noether-Lefschetz locus is explained as follows. Take $n = 2$. Instead of the map

$$H^2(\mathbb{P}^3 - \bigcup_{1\le j\le s} Y_j, \mathbb{Q}(2)) \longrightarrow F^2H^2(U, \mathbb{Q}(2))$$

we consider

$$H^2(\mathbb{P}^3, \mathbb{Q}(1)) \longrightarrow F^1H^2(X, \mathbb{Q}(1))$$

By noting that the space on the left hand side is generated by the cohomology class of a plane, and that that on the right hand side is identified with $\mathrm{Pic}(X) \otimes \mathbb{Q}$, the space defined in the same way as Definition 0.4 is nothing but the classical Noether-Lefschetz locus.

One can show by the same way as before that M_{NL} is the union of a countable number of closed analytic subsets. By analogy a series of problems on M_{NL} arise, the problems to show the counterparts of Theorems 0.1 and 0.2 in the new context. Note that in view of Fact 0.3 the problems are non-trivial only if $s \ge n$. In the previous work [AS2], the authors have shown the following result.

Theorem 0.5. *Assume $d \ge n + 2$. For every irreducible component T of M_{NL},*

$$\mathrm{codim}_M(T) \ge d + (n - 1)\min\{d, e_1, \dots, e_s\} - n - 1.$$

The estimate in Theorem 0.5 will turn out to be far from being optimal in general (see the main theorems 0.7 and 0.8 below). The basic strategy of the proof of Theorem 0.5 is the same as that of Theorem 0.1. A new input is the theory of generalized Jacobian rings developed in [AS1].

It seems a difficult problem to establish a counterpart of Theorem 0.2 in the context of Beilinson-Hodge cycles. In case $n = 1$ (case of plane curves in $\mathbb{P}^2$), it is shown in [AS2] Theorem 4.1 that the estimate in Theorem 0.5 is optimal: Every irreducible component of M_{NL} has codimension $\ge d - 2$ and a finite number of components of codimension $d - 2$ are explicitly given. It can be also shown that those are the only ones which have the maximal dimension when $s \le 2$ and $e_j = 1$ for $1 \le j \le s$. It is an open question whether it is still the case in general.

In this paper we study the problem in case $n = 2$ (case of surfaces in $\mathbb{P}^3$). By Fact 0.3 the problem is non-trivial only if $s \ge 2$. Our main results concern the special case:

$$s = 2 \text{ or } 3 \text{ and } e_j = 1 \text{ for } 1 \le j \le s, \tag{0.3}$$

but yet it reveals a new phenomenon surprising to us: There are infinitely many components of maximal dimension in M_{NL}. We expect that the method can be extended to treat the case where n is arbitrary and $s = n$ or $n + 1$ and $e_j = 1$ for $1 \le j \le s$. It is an open problem to extend our results to more general cases.

Let $P = \mathbb{C}[z_0, z_1, z_2, z_3]$ be the homogeneous coordinate ring of $\mathbb{P}^3$ and $P^l \subset P$ the subspace of homogeneous polynomials of degree $l \ge 0$. For the rest of the paper we fix an integer $d > 0$ and let $\overline{M} \subset P^d - \{0\}/\mathbb{C}^*$ be the moduli space of the smooth surfaces in $\mathbb{P}^3$. Let $M \subset \overline{M}$ (resp. $M' \subset \overline{M}$) denote the Zariski open subset of those surfaces that intersect transversally with $Y := Y_1 \cup Y_2 \cup Y_3$ (resp. $Y' := Y_1 \cup Y_2$) where $Y_j = \{z_j = 0\} \subset \mathbb{P}^3$. By definition M is a dense open subset of M'. For $t \in \overline{M}$ let X_t be the corresponding surface in $\mathbb{P}^3$ and put

$$U_t = X_t - (X_t \cap Y) \quad \text{and} \quad U'_t = X_t - (X_t \cap Y').$$

We define

$$M_{NL} := \{t \in M \mid F^2H^2(U_t, \mathbb{Q}(2))_{\mathrm{prim}} \neq 0\},$$
$$M'_{NL} := \{t \in M' \mid F^2H^2(U'_t, \mathbb{Q}(2))_{\mathrm{prim}} \neq 0\}.$$

In order to describe the irreducible components of M_{NL} and M'_{NL} of maximal dimension, we need introduce some notations.

Definition 0.6.

(1) For $\underline{c} = [c_\nu]_{1\le\nu\le d} = [c_1 : \cdots : c_d] \in \mathbb{P}^{d-1}(\mathbb{C})$, we define $T(\underline{c}) \subset M'$ as the closed subset consisting of such surfaces X that there exists $c \in \mathbb{C}^*$ and a plane $H \subset \mathbb{P}^3$ defined by a linear form $w \in P^1 - \{0\}$ for which

$$H \cap X \ = H \cap \bigcup_{1\le\nu\le d} \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3 |\ cz_1 - c_\nu z_2 = 0\}.$$

By the definition a surface in $T(\underline{c})$ is defined by an equation:

$$F = wA + \prod_{1\le\nu\le d} (cz_1 - c_\nu z_2) \text{ for some } w \in P^1,\ A \in P^{d-1},\ c \in \mathbb{C}^*.$$

(2) We fix the following datum $\big((p,q), \underline{c}, \sigma\big)$, where

(i) $p,\ q \ge 0$ are integers such that $(p,q) = 1$ and that $(p+q)$ is a divisor of d (in particular if $pq = 0$ then $(p,q) = (1,0)$ or $(0,1)$),

(ii) $\underline{c} = [c_\nu]_{1\le\nu\le r} = [c_1 : \cdots : c_r] \in \mathbb{P}^{r-1}(\mathbb{C})$ where $r := \dfrac{d}{p+q}$,

(iii) $\sigma \in \mathfrak{S}_3$, the permutation group on $(1,2,3)$.

We define $T^\sigma_{(p,q)}(\underline{c}) \subset M$ as the closed subset consisting of such surfaces X that there exists $c \in \mathbb{C}^*$ and a plane $H \subset \mathbb{P}^3$ defined by a linear form $w \in P^1 - \{0\}$ for which

$$H \cap X = H \cap \bigcup_{1\le\nu\le r} \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3 |\ cz^{p+q}_{\sigma(1)} - c_\nu z^p_{\sigma(2)} z^q_{\sigma(3)} = 0\}.$$

By the definition a surface in $T^\sigma_{(p,q)}(\underline{c})$ is defined by an equation:

$$F = wA + \prod_{1\le\nu\le r} (cz^{p+q}_{\sigma(1)} - c_\nu z^p_{\sigma(2)} z^q_{\sigma(3)}) \text{ for some } w \in P^1,\ A \in P^{d-1},\ c \in \mathbb{C}^*.$$

We will see the following facts (cf. §1):

(1) $T(\underline{c})$ (resp. $T^\sigma_{(p,q)}(\underline{c})$) is smooth irreducible of codimension $\binom{d+2}{2} - 5$ in M' (resp. M).

(2) $T(\underline{c}) \subset M'_{NL}$ (resp. $T^\sigma_{(p,q)}(\underline{c}) \subset M_{NL}$) if and only if there are roots of unity ζ_ν such that $\underline{c} = [\zeta_\nu]_{1\le\nu\le d}$ (resp. $\underline{c} = [\zeta_\nu]_{1\le\nu\le r}$). Under the condition the two form

$$\mathrm{dlog}\frac{z_2}{z_1} \wedge \mathrm{dlog}\frac{w}{z_1} \quad (\text{resp. } \mathrm{dlog}\frac{z^p_{\sigma(2)} z^q_{\sigma(3)}}{z^{p+q}_{\sigma(1)}} \wedge \mathrm{dlog}\frac{w}{z_{\sigma(1)}}\ )$$

gives rise to a primitive Beilinson-Hodge cycle, where $w \in P^1$ is as in Definition 0.6.

The main theorem of this paper is the following.

Theorem 0.7. *Assume $d \ge 4$. Let T be an irreducible component of M'_{NL}.*

(1) $\mathrm{codim}(T) \ge \binom{d+2}{2} - 5$.

(2) $\mathrm{codim}(T) = \binom{d+2}{2} - 5$ *if and only if T coincides with $T(\underline{c})$ for some $\underline{c} = [\zeta_\nu]_{1\le\nu\le d}$ where ζ_ν are roots of unity.*

(3) *Let $\underline{c}$ be as in (2). If X is a general member of $T(\underline{c})$, reg^2_U is surjective so that Beilinson's Hodge conjecture holds for $U' = X - (X \cap Y')$.*

Theorem 0.8. *Assume $d \ge 4$. Let T be an irreducible component of M_{NL}.*

(1) $\mathrm{codim}(T) \ge \binom{d+2}{2} - 5$.

(2) $\mathrm{codim}(T) = \binom{d+2}{2} - 5$ *if and only if* T *coincides with* $T^{\sigma}_{(p,q)}(\underline{c})$ *for some* $\Big((p,q), \underline{c} = [\zeta_\nu]_{1\le\nu\le r}, \sigma\Big)$ *where* ζ_ν *are roots of unity.*

(3) *Let* $\Big((p,q), \underline{c}, \sigma\Big)$ *be as in (2). If* X *is a general member of* $T^{\sigma}_{(p,q)}(\underline{c})$, reg^2_U *is surjective so that Beilinson's Hodge conjecture holds for* $U = X - (X\cap Y)$.

The paper is organized as follows. In §1 we study the subsets of the moduli space of surfaces in $\mathbb{P}^3$ defined in Definition 0.6. In §2 and §3 our problem is reduced to algebraic problems on polynomial rings via theories of infinitesimal variation of Hodge structure and Jacobian rings. The basic idea is the same as that in [G2] and [V]. A key to solving the last problem is Theorem 3.5 due to Otwinowska ([Ot], Th.2) concerning the Hilbert function of graded algebras of dimension 0. In §2 the main theorem 0.8 will be deduced from Theorem 0.8 bis, and 0.7 from 0.8. The proof of Theorem 0.8 bis occupies §3, §4 and §5.

In the appendix we will discuss an implication of Theorem 0.8 on the injectivity of the regulator map:

$$\rho_X \;:\; CH^2(X,1)\otimes\mathbb{Q} \longrightarrow H^3_D(X,\mathbb{Q}(2)),$$

where X be a member of M and $CH^2(X,1)$ is Bloch's higher Chow and $H^3_D(X,\mathbb{Q}(2))$ is the Deligne cohomology of X.

The authors are grateful to Professor A. Otwinowska for stimulating discussions and her teaching us on her work [Ot]. They thank the referee for helpful comments to improve the presentation of the paper.

1. Components of M_{NL}

Let the notation be as in Theorem 0.7 and 0.8. In this section we fix $0\in M$ and let X be the corresponding surface in $\mathbb{P}^3$. Write

$$U = X - Z,\ Z = \bigcup_{1\le j\le 3} Z_j, \quad \text{and} \quad U' = X - Z',\ Z' = \bigcup_{1\le j\le 2} Z_j,$$

where $Z_j = X\cap Y_j$. We put

$$\alpha_U := \{\frac{z_2}{z_1}, \frac{z_3}{z_1}\} \in CH^2(U,2), \tag{1.1}$$

$$\omega_U := \mathrm{dlog}\frac{z_2}{z_1}\wedge \mathrm{dlog}\frac{z_3}{z_1} \in H^0(X,\Omega^2_X(\log Z)), \tag{1.2}$$

where z_j/z_i is viewed as an element of $CH^1(U,1) = \Gamma(U,\mathcal{O}^*_{U_{Zar}})$. Note that

$$\omega_U = \mathrm{reg}^2_U(\alpha_U),$$

under the natural inclusion $F^2H^2_{\mathrm{dR}}(U/\mathbb{C}) = H^0(X,\Omega^2_X(\log Z)) \hookrightarrow H^2_{\mathrm{dR}}(U/\mathbb{C}) \simeq H^2(U,\mathbb{C})$.

Lemma 1.1.

(1) $CH^2(U',2)_{\mathrm{dec}}$ *is generated by* $\{c, z_2/z_1\}$ *with* $c\in\mathbb{C}^*$.

(2) $CH^2(U,2)_{\mathrm{dec}}$ *is generated by* α_U *and* $\{c, z_j/z_i\}$ *with* $c\in\mathbb{C}^*$ *and* $1\le i,j\le 3$.

(3) $H^2(U',\mathbb{Q}(2))_{\mathrm{triv}} = 0$ *and* $H^2(U,\mathbb{Q}(2))_{\mathrm{triv}} = \mathbb{Q}\cdot\omega_U$.

Proof. The first assertion follows from [AS2] Lem.3.3.(1). The second assertion follows from (0.2). □ □

Lemma 1.2. $T(\underline{c})$ *(resp.* $T^{\sigma}_{(p,q)}(\underline{c})$*) in Definition 0.6 is smooth irreducible of codimension* $\binom{d+2}{2} - 5$ *in* M' *(resp.* M*).*

Proof. We show the assertion only for $T(\underline{c})$. The assertion for $T^{\sigma}_{(p,q)}(\underline{c})$ is shown by the same way. By the definition a member X of $T(\underline{c})$ is defined by an equation

$$F = wA + \prod_{1\le\nu\le d}(cz_1 - c_\nu z_2) \quad \text{for some } w\in P^1,\ A\in P^{d-1},\ c\in\mathbb{C}^*,$$

and the map

$$T(\underline{c}) \to \mathbb{C}^* \times \Big((P^1-\{0\})\times(P^{d-1}-\{0\})\Big)/\mathbb{C}^*\ ;\ X\to\Big(c,(w,A)\Big),$$

is an open immersion, where $\lambda\in\mathbb{C}^*$ acts on $(P^1-\{0\})\times(P^{d-1}-\{0\})$ via $(w,A)\to(\lambda w,\lambda^{-1}A)$. The desired assertion follows easily from this. □

Assume $0\in T^{\sigma}_{(p,q)}(\underline{c})$ so that X is defined by an equation:

$$F = wA + \prod_{1\le\nu\le r}(cz^{p+q}_{\sigma(1)} - c_\nu z^p_{\sigma(2)}z^q_{\sigma(3)}) \quad (c\neq 0). \tag{1.3}$$

We note that $w\notin\sum_{1\le j\le 3}\mathbb{C}\cdot z_j$ by the assumption that X intersects Y transversally. We put

$$\xi_U := \mathrm{dlog}\frac{z^p_{\sigma(2)}z^q_{\sigma(3)}}{z^{p+q}_{\sigma(1)}}\wedge\mathrm{dlog}\frac{w}{z_{\sigma(1)}} \in H^0(X,\Omega^2_X(\log Z)), \tag{1.4}$$

and

$$\Sigma(U) := \mathbb{C}\cdot\omega_U\oplus\mathbb{C}\cdot\xi_U \subset H^0(X,\Omega^2_X(\log Z)) = F^2H^2(U,\mathbb{C}). \tag{1.5}$$

We note that ξ_U is apparently holomorphic only on $U-W$ with $W=U\cap\{w=0\}$ while it is easy to see that its residue along any irreducible component of W is zero. Rewriting (1.3) as

$$wA + \prod_{1\le\nu\le r}(cz^{p+q}_{\sigma(1)} - c_\nu z^p_{\sigma(2)}z^q_{\sigma(3)}) = wA + \prod_{\mu\in I}(cz^{p+q}_{\sigma(1)} - c_\mu z^p_{\sigma(2)}z^q_{\sigma(3)})^{e_\mu},\ (e_\mu\ge 1) \tag{1.6}$$

where $c_\mu\neq c_{\mu'}$ if $\mu\neq\mu'\in I$, W is the disjoint sum of the following smooth irreducible components for $\mu\in I$;

$$W_\mu = U\cap\{w = cz^{p+q}_{\sigma(1)} - c_\mu z^p_{\sigma(2)}z^q_{\sigma(3)} = 0\}.$$

Lemma 1.3. $\mathbb{Q}\cdot\omega_U\subset\Sigma(U)\cap H^2(U,\mathbb{Q}(2))\subset\mathbb{Q}\cdot\omega_U\oplus\mathbb{Q}\cdot\xi_U$.

Proof. The first inclusion is easy. We show the second. Without loss of generality we may assume that $\sigma\in\mathfrak{S}_3$ is the identity. Let $\phi = a\omega_U + b\xi_U\in H^0(X,\Omega^2_X(\log Z))$ with $a,b\in\mathbb{C}$ and assume $\phi\in H^2(U,\mathbb{Q}(2))$. Define

$$Z_{ij} = X\cap\{z_i = z_j = 0\}\quad(1\le i\neq j\le 3),$$
$$Z_i = X\cap\{z_i=0\},\ V_i = Z_i\cap(\bigcup_{1\le j\neq i\le 3}Z_j).$$

For $1\le i\neq j\le 3$ we consider the composite map of the successive residue maps

$$\delta_{ij}: H^0(X,\Omega^2_X(\log Z)) \xrightarrow{\mathrm{Res}_{Z_i}} H^0(Z_i,\Omega^1_{Z_i}(\log V_i)) \xrightarrow{\mathrm{Res}_{Z_{ij}}} H^0(Z_{ij},\mathcal{O}_{Z_{ij}})\cong\Psi_{ij}\otimes\mathbb{C},$$

where $\Psi_{ij} = H^0(Z_{ij},\mathbb{Q}) = \bigoplus_{x\in Z_{ij}}\mathbb{Q}$. The assumption implies

$$(*)\qquad \delta_{ij}(\phi)\in\Psi_{ij}\ \text{for any } i,\ j.$$

We have $\delta_{ji}(\phi) = -\delta_{ij}(\phi)$ and an easy residue calculation shows

$$\delta_{ij}(\omega_U) = -\underline{u}_{ij}\quad\text{for } (i,j)=(1,2),\ (2,3),\ (3,1),$$
$$\delta_{12}(\xi_U) = -p(\underline{u}_{12} - d\cdot\underline{v}_{12}),\ \delta_{31}(\xi_U) = q(\underline{u}_{31}-d\cdot\underline{v}_{31}),\ \delta_{23}(\xi_U)=0,$$

where $\underline{u}_{ij} \in \Psi_{ij}$ has component 1 at any $x \in Z_{ij}$ and $\underline{v}_{ij} \in \Psi_{ij}$ has component 1 at $x \in Z_{ij} \cap \{w = 0\}$ and 0 elsewhere. Now it is easy to see that $(*)$ implies $a, b \in \mathbb{Q}$. $\square$

Proposition 1.4. *The following conditions are equivalent.*

(1) $\xi_U \in \mathrm{Im}(\mathrm{reg}^2_U)$.
(2) $\xi_U \in H^2(U, \mathbb{Q}(2))$.
(3) $\Sigma(U) \cap H^2(U, \mathbb{Q}(2)) = \mathbb{Q} \cdot \omega_U \oplus \mathbb{Q} \cdot \xi_U$.
(4) $\underline{c} = [\zeta_\nu]_{1 \le \nu \le r} \in \mathbb{P}^{r-1}(\mathbb{C})$ *where ζ_ν are roots of unity.*

The following corollary is immediate.

Corollary 1.5. $T^\sigma_{(p,q)}(\underline{c}) \subset M_{NL}$ *if the equivalent conditions in Proposition 1.4 hold.*

In the rest of this section we prove Proposition 1.4. (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (2) are clear. (2) $\Longrightarrow$ (3) follows from Lemma 1.3. We shall prove (2) $\Longleftrightarrow$ (4) $\Longrightarrow$ (1).

We first prove (2) $\Longleftrightarrow$ (4). Consider

$$\beta = \{a \frac{z^p_{\sigma(2)} z^q_{\sigma(3)}}{z^{p+q}_{\sigma(1)}}, \frac{w}{z_{\sigma(1)}}\} \in CH^2(\tilde{U}, 2) \quad (\tilde{U} := U - W)$$

for $a \in \mathbb{C}^*$. We have the commutative diagram

$$\begin{array}{ccccc} CH^2(U,2) \otimes \mathbb{Q} & \xrightarrow{\mathrm{reg}^2_U} & F^2 H^2(U, \mathbb{Q}(2)) & \hookrightarrow & F^2 H^2_{\mathrm{dR}}(U/\mathbb{C}) \\ \downarrow & & \downarrow \iota_1 & & \downarrow \iota_2 \\ CH^2(\tilde{U},2) \otimes \mathbb{Q} & \xrightarrow{\mathrm{reg}^2_{\tilde{U}}} & F^2 H^2(\tilde{U}, \mathbb{Q}(2)) & \hookrightarrow & F^2 H^2_{\mathrm{dR}}(\tilde{U}/\mathbb{C}) \end{array}$$

and we have $\mathrm{reg}^2_{\tilde{U}}(\beta) = \iota_2(\xi_U)$ in $F^2 H^2_{\mathrm{dR}}(\tilde{U}/\mathbb{C})$. Since ι_2 is injective, (2) $\Longleftrightarrow$ (4) follows from the following assertion:

$(*)$ $\mathrm{reg}^2_{\tilde{U}}(\beta) \in \mathrm{Im}(\iota_1)$ if and only if (4) holds.

To show this we consider the commutative diagram

$$\begin{array}{ccccc} CH^2(U,2) \otimes \mathbb{Q} & \longrightarrow & CH^2(\tilde{U},2) \otimes \mathbb{Q} & \xrightarrow{\partial_{\mathcal{M}}} & CH^1(W,1) \otimes \mathbb{Q} \\ \mathrm{reg}^2_{D,U} \downarrow & & \downarrow \mathrm{reg}^2_{D,\tilde{U}} & & \cong \downarrow \mathrm{reg}^1_{D,W} \\ H^2_D(U, \mathbb{Q}(2)) & \longrightarrow & H^2_D(\tilde{U}, \mathbb{Q}(2)) & \xrightarrow{\partial_D} & H^1_D(W, \mathbb{Q}(1)) \end{array} \tag{1.7}$$

where $\mathrm{reg}^*_{D,*}$ denotes the regulator map to Deligne cohomology and $\partial_{\mathcal{M}}$ and ∂_D are the Poincare residue maps. We have the commutative diagram (cf. [EV])

$$\begin{array}{ccccc} 0 \to H^1(U, \mathbb{Q}(1)) \otimes \mathbb{C}/\mathbb{Q}(1) & \longrightarrow & H^2_D(U, \mathbb{Q}(2)) & \xrightarrow{\pi_U} & F^2 H^2(U, \mathbb{Q}(2)) \to 0 \\ \downarrow & & \downarrow & & \downarrow \iota_1 \\ 0 \to H^1(\tilde{U}, \mathbb{Q}(1)) \otimes \mathbb{C}/\mathbb{Q}(1) & \longrightarrow & H^2_D(\tilde{U}, \mathbb{Q}(2)) & \xrightarrow{\pi_{\tilde{U}}} & F^2 H^2(\tilde{U}, \mathbb{Q}(2)) \to 0 \\ \downarrow \partial_1 & & \downarrow \partial_D & & \downarrow \partial_2 \\ 0 \to H^0(W, \mathbb{Q}) \otimes \mathbb{C}/\mathbb{Q}(1) & \longrightarrow & H^1_D(W, \mathbb{Q}(1)) & \xrightarrow{\pi_W} & F^1 H^1(W, \mathbb{Q}(1)) \to 0 \end{array} \tag{1.8}$$

where ∂_i are the Poincare residue maps. The composite of $\mathrm{reg}^*_{D,*}$ with π_* coincides with the regulator map to singular cohomology. The horizontal sequences are exact. The vertical sequences are localization sequences and they are exact except the most right one. In view of (1.6) we have an isomorphism

$$\mathrm{Coker}(\partial_1) \cong \mathbb{C}/\mathbb{Q}(1) \otimes \Phi, \quad \Phi := \mathrm{Coker}(\mathbb{Q} \to \bigoplus_{\mu \in I} \mathbb{Q};\ 1 \to (e_\mu)_{\mu \in I}). \tag{1.9}$$

To see this we note that $H^1(\tilde{U},\mathbb{Q}(1))$ is Hodge type $(0,0)$ as $H^1(X,\mathbb{Q})=0$. Therefore we have the commutative diagram

$$\begin{array}{ccc} CH^1(\tilde{U},1)\otimes\mathbb{Q} & \xrightarrow{\phi} & \bigoplus_{\mu\in I}\mathbb{Q} \\ {\scriptstyle \mathrm{reg}^1_{\tilde{U}}}\big\downarrow & & \big\downarrow\cong \\ H^1(\tilde{U},\mathbb{Q}(1))=F^1H^1(\tilde{U},\mathbb{Q}(1)) & \longrightarrow & H^0(W,\mathbb{Q}) \end{array}$$

with the surjective $\mathrm{reg}^1_{\tilde{U}}$. ϕ is given by taking orders of functions along the components of W. One easily sees that $CH^1(\tilde{U},1)$ is generated by $\mathbb{C}^*$, w/z_1 and z_j/z_i with $1\le i,j\le 3$ and thus the isomorphism (1.9) follows. Now (1.8) gives rise to an exact sequence

$$F^2H^2(U,\mathbb{Q}(2))\xrightarrow{\iota_1}\mathrm{Ker}(\partial_2)\xrightarrow{\delta}\mathbb{C}/\mathbb{Q}(1)\otimes\Phi. \tag{1.10}$$

Now an easy calculation shows

$$\partial_{\mathcal{M}}(\beta)=((ac/c_\mu)^{e_\mu})_{\mu\in I}\in\bigoplus_{\mu\in I}CH^1(W_\mu,1).$$

Noting the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C}^*\otimes\mathbb{Q} & \longrightarrow & CH^1(W_\mu,1)\otimes\mathbb{Q} & & \\ & & \big\downarrow{\scriptstyle \log} & & \cong\big\downarrow{\scriptstyle \mathrm{reg}^1_{D,W_\mu}} & & \\ 0 & \longrightarrow & \mathbb{C}/\mathbb{Q}(1) & \longrightarrow & H^1_D(W_\mu,\mathbb{Q}(1)) & \xrightarrow{\pi_{W_\mu}} & F^1H^1(W_\mu,\mathbb{Q}(1))\to 0 \end{array}$$

it implies $\mathrm{reg}^2_{\tilde{U}}(\beta)\in\mathrm{Ker}(\partial_2)$ and that for δ in (1.10), we have

$$\delta(\mathrm{reg}^2_{\tilde{U}}(\beta))=\text{ the class of }(e_\mu\log(ac/c_\mu))_{\mu\in I}\text{ in }\Phi\otimes\mathbb{C}/\mathbb{Q}(1).$$

This shows that $\mathrm{reg}^2_{\tilde{U}}(\beta)\in\mathrm{Im}(\iota_1)$ if and only if there is $a\in\mathbb{C}^*$ such that ac/c_μ are roots of unity for all μ. This completes the proof of $(*)$ and hence we have $(2)\iff(4)$.

Finally we see $(4)\implies(1)$. In this case, β has a lift $\beta'\in CH^2(U,2)\otimes\mathbb{Q}$ due to the exact sequence (1.7). Then we have $\mathrm{reg}^2_U(\beta')=\xi_U\in F^2H^2(U,\mathbb{Q}(2))$ by the injectivity of ι_2. Hence we have $\xi_U\in\mathrm{Im}(\mathrm{reg}^2_U)$.

2. Infinitesimal interpretation

In this section we make the first step of the proof of Theorem 0.8. Let the assumption and the notation be as in §1. Take $\Delta\subset M$, a simply connected neighborhood of 0 in M. For $\lambda\in H^2(U,\mathbb{C})$ and $t\in\Delta$, let $\lambda_t\in H^2(U_t,\mathbb{C})$ be the flat translation of λ with respect to the Gauss-Manin connection

$$\nabla: H^2_{\mathcal{O}}(\mathcal{U}/M)\longrightarrow\Omega^1_M\otimes H^2_{\mathcal{O}}(\mathcal{U}/M),$$

where $H^p_{\mathcal{O}}(\mathcal{U}/M)$ is the sheaf of holomorphic sections of the local system $H^p_{\mathbb{C}}(\mathcal{U}/M):=R^pf_*\mathbb{C}$ with $f:\mathcal{U}\to M$, the natural morphism. We sometimes consider λ a section over Δ of $H^2_{\mathbb{C}}(\mathcal{U}/M)$ via $H^2(U,\mathbb{C})\cong\Gamma(\Delta,H^2_{\mathbb{C}}(\mathcal{U}/M))$. Putting

$$\Delta_\lambda=\{t\in\Delta|\ \lambda_t\in F^2H^2(U_t,\mathbb{C})\},$$

it is a closed analytic subset of Δ since it is defined by the vanishing of the image of λ under the map

$$\Gamma(\Delta,H^2_{\mathbb{C}}(\mathcal{U}/M))\longrightarrow\Gamma(\Delta,H^2_{\mathcal{O}}(\mathcal{U}/M)/F^2H^2_{\mathcal{O}}(\mathcal{U}/M))$$

where $F^qH^p_{\mathcal{O}}(\mathcal{U}/M) \subset H^p_{\mathcal{O}}(\mathcal{U}/M)$ is the Hodge subbundle. Taking Δ sufficiently small if necessary, we have by Lemma 1.1 (2)

$$M_{NL} \cap \Delta = \mathop{\cup}_{\lambda} \Delta_\lambda \tag{2.1}$$

as a set where λ runs over all $\lambda \in H^2(U, \mathbb{Q}(2))$ such that $\lambda \notin \mathbb{Q} \cdot \omega_U$. Let $T_t(M)$ be the tangent space of M at t, and $\Omega^1_{M,t} = \mathrm{Hom}(T_t(M), \mathbb{C})$ its dual space. By Griffiths transversality, the Gauss-Manin connection ∇ induces

$$\overline{\nabla} \;:\; H^q(X_t, \Omega^p_{X_t}(\log Z_t)) \longrightarrow \Omega^1_{M,t} \otimes H^{q+1}(X_t, \Omega^{p-1}_{X_t}(\log Z_t)),$$

and hence the pairing

$$\langle\;,\;\rangle \;:\; T_t(M) \otimes H^q(X_t, \Omega^p_{X_t}(\log Z_t)) \longrightarrow H^{q+1}(X_t, \Omega^{p-1}_{X_t}(\log Z_t))$$

for $p, q \geq 0$. Let $T \hookrightarrow \Delta_\lambda$ be any irreducible component with reduced structure. We denote the regular locus of T by T^{reg}. Since the section λ defines a flat section of $F^2H^2_{\mathcal{O}}(\mathcal{U}/M)$ along T^{reg} we have

$$T_t(T^{\mathrm{reg}}) \subset \{\partial \in T_t(M) | \ \langle \partial, \lambda_t \rangle = 0\} \quad (t \in T^{\mathrm{reg}}). \tag{2.2}$$

Theorem 0.8 bis. *Fix $\lambda \in H^0(X, \Omega^2_X(\log Z))$ with $\lambda \notin \mathbb{C} \cdot \omega_U$. Let T be an irreducible component of Δ_λ with reduced structure.*

(i) $\mathrm{codim}_\Delta(T) \geq \binom{d+2}{2} - 5$.

(ii) *Assume $d \geq 4$ and $\mathrm{codim}_\Delta(T) = \binom{d+2}{2} - 5$. If $\lambda \in H^2(U, \mathbb{Q}(2))$, then $T = T^\sigma_{(p,q)}(\underline{c}) \cap \Delta$ for some σ, p, q and $\underline{c} = [\zeta_\nu]_{1 \leq \nu \leq r}$ such that ζ_ν are roots of unity (cf. Proposition 1.4). Hence T is smooth (cf. Lemma 1.2).*

(iii) *Assume $0 \in T^\sigma_{(p,q)}(\underline{c})$ for some $\sigma, p, q, \underline{c}$. Then we have*

$$\Sigma(U) = \{\omega \in H^0(X, \Omega^2_X(\log Z)) | \ \langle \partial, \omega \rangle = 0 \textit{ for all } \partial \in T_0(T^\sigma_{(p,q)}(\underline{c}))\}.$$

Theorem 0.8 bis will be shown in the next three sections.

We deduce the main theorem 0.8 from the above. Theorem 0.8 (1) and (2) follow immediately from (i), (ii) and Corollary 1.5 in view of (2.1). Theorem 0.8 (3) follows from the following:

Claim 2.1. *Assume $0 \in T^\sigma_{(p,q)}(\underline{c})$. Then there exists a subset $E \subset \Delta_T := T^\sigma_{(p,q)}(\underline{c}) \cap \Delta$ which is the union of a countable number of proper closed analytic subsets of Δ_T such that*

$$F^2H^2(U_t, \mathbb{Q}(2)) \subset \Sigma(U_t) \quad \textit{ for all } t \in \Delta_T - E.$$

Moreover, under the assumption of Proposition 1.4 (4), we have

$$F^2H^2(U_t, \mathbb{Q}(2)) = \mathbb{Q} \cdot \omega_{U_t} \oplus \mathbb{Q} \cdot \xi_{U_t} \quad \textit{ for all } t \in \Delta_T - E,$$

where ω_{U_t}, $\xi_{U_t} \in H^2(U_t, \mathbb{Q}(2))$ are the flat translations of ω_U, $\xi_U \in H^2(U_t, \mathbb{Q}(2))$ respectively.

Proof. The second assertion follows from the first in view of Proposition 1.4. We show the first assertion. Write $H^2(U, \mathbb{Q}(2)) = \{\lambda_i\}_{i \in I}$ as a set and put

$$A = \{i \in I | \ \Delta_T \subset \Delta_{\lambda_i}\}, \ B = \{i \in I | \ \Delta_T \not\subset \Delta_{\lambda_i}\}, \ E = \Delta_T \cap (\mathop{\cup}_{i \in B} \Delta_{\lambda_i}).$$

Note that I is countable and $I = A \cup B$ and $A \cap B = \emptyset$. For $t \in \Delta_T - E$, we have $F^2H^2(U_t, \mathbb{Q}(2)) = \{\lambda_i(t)\}_{i \in A}$ so that $H^2(U_t, \mathbb{C}) \xrightarrow{\sim} \Gamma(\Delta_T, H^2_{\mathbb{C}}(\mathcal{U}/M))$ induces

$$F^2H^2(U_t, \mathbb{Q}(2)) \hookrightarrow \Gamma(\Delta_T, H^2_{\mathbb{C}}(\mathcal{U}/M) \cap F^2H^2_{\mathcal{O}}(\mathcal{U}/M)),$$

which further implies

$$F^2H^2(U_t, \mathbb{Q}(2)) \subset \mathrm{Ker}\Big(H^0(X_t, \Omega^2_{X_t}(\log Z_t)) \longrightarrow \Omega^1_{\Delta_T, t} \otimes H^1(X_t, \Omega^1_{X_t}(\log Z_t))\Big).$$

Theorem 0.8 bis (iii) implies that the last space is equal to $\Sigma(U_t)$ and the desired assertion follows. $\square$

Next we deduce Theorem 0.7 from Theorem 0.8. First we show the following:

Claim 2.2. $M \cap M'_{NL} \subset M_{NL}$.

Proof. Take $0 \in M$ and X be the corresponding surface in $\mathbb{P}^3$ that intersects transversally with $Y = Y_1 \cup Y_2 \cup Y_3$. Write $U' = X - (Y_1 \cap Y_2)$ and $U = X - Y$. By the residue calculation in the proof of Lemma 1.3, we see that $\omega_U \in H^2(U, \mathbb{Q}(2))$ does not lie in the image of $H^2(U', \mathbb{Q}(2)) \to H^2(U, \mathbb{Q}(2))$. By Lemma 1.1 it implies that the restriction map

$$F^2H^2(U', \mathbb{Q}(2))_{\mathrm{prim}} \to F^2H^2(U, \mathbb{Q}(2))_{\mathrm{prim}}$$

is injective. Hence, if $0 \in M'_{NL}$, then $0 \in M_{NL}$. $\square$

Let $T \subset M'_{NL}$ be an irreducible component with reduced structure. Take $0 \in T$ and let X be the corresponding surface in $\mathbb{P}^3$. By the assumption X, Y_1, Y_2 intersect transversally. Take a plane $L \subset \mathbb{P}^3$ such that X, Y_1, Y_2, L intersect transversally. By a coordinate transformation fixing z_1 and z_2, we may suppose $L = Y_3$. Then $0 \in T \cap M$ and $T \cap M$ is a dense open subset of T. By the above claim we have $T \cap M \subset M_{NL}$ and Theorem 0.8 implies

$$\mathrm{codim}_{M'}(T) = \mathrm{codim}_M(T \cap M) \geq \binom{d+2}{2} - 5.$$

This shows Theorem 0.7 (1). Moreover the equality holds only if $T \cap M = T^{\sigma}_{(p,q)}(\underline{c})$ for some $\Big((p,q), \underline{c}, \sigma\Big)$ as in Theorem 0.8 (2). In view of Claim 2.1, the last condition implies that there exist $a, b \in \mathbb{Q}$ with $(a,b) \neq (0,0)$ such that $\phi = a\omega_U + b\xi_U \in H^2(U, \mathbb{Q}(2))$ lies in the image of $H^2(U', \mathbb{Q}(2)) \to H^2(U, \mathbb{Q}(2))$. By the residue calculation as in the proof of Lemma 1.3, this implies that in the equation (1.3), the exponent of z_3 in $\prod\limits_{1 \leq \nu \leq r} (cz^{p+q}_{\sigma(1)} - c_\nu z^p_{\sigma(2)} z^q_{\sigma(3)})$ is 0. This implies $T^{\sigma}_{(p,q)}(\underline{c}) = T(\underline{c}') \cap M$ for some $\underline{c}' \in \mathbb{P}^d(\mathbb{C})$ and Theorem 0.7 (2) is proved. Now Theorem 0.7 (3) is a direct consequence of Theorem 0.8 (3).

3. Reduction to Jacobian rings

Let the assumption be as in §2. In this section we rephrase the theorems in §2 in terms of Jacobian rings and prove Theorem 0.8 bis (i) and (ii). Let $P = \mathbb{C}[z_0, z_1, z_2, z_3]$ be the homogeneous coordinate ring of $\mathbb{P}^3$. For an integer $l > 0$ let $P^l \subset P$ be the subspace of homogeneous polynomials of degree l. Let the assumption be as in §2 and fix $F \in P^d$ which defines $X \subset \mathbb{P}^3$. Put the ideal $J_F \subset P$ (Jacobian ideal) by

$$J_F = \langle \frac{\partial F}{\partial z_0}, z_1\frac{\partial F}{\partial z_1}, z_2\frac{\partial F}{\partial z_2}, z_3\frac{\partial F}{\partial z_3} \rangle.$$

The assumption that X transversally intersects Y is equivalent to the condition:

(3-1): J_F is complete intersection of degree $(d-1, d, d, d)$.

Write

$$R_F = P/J_F,\ J^l_F = J_F \cap P^l,\ R^l_F = \mathrm{Im}(P^l \to R_F) = P^l/J^l_F$$

and call R_F the Jacobian ring. We recall the following well-known theorem of Macaulay (cf. [GH], p.659).

Theorem 3.1 (Macaulay)**.** *There exists a natural isomorphism*

$$\tau_F \ : \ R_F^{4d-5} \xrightarrow{\sim} \mathbb{C}$$

and the pairing induced by multiplication

$$R_F^l \otimes R_F^{4d-5-l} \longrightarrow R_F^{4d-5} \xrightarrow{\tau_F} \mathbb{C}, \quad 0 \le l \le 4d-5$$

is perfect.

The relations between the Jacobian ring and cohomology group are described in the following way:

(3-2): We have the canonical surjective homomorphism

$$\psi \ : \ P^d \longrightarrow T_0(M) \ ; \ G \to \{F + \epsilon G = 0\} \subset \mathbb{P}^3_{\mathbb{C}[\epsilon]},$$

where $\mathbb{C}[\epsilon]$ is the ring of dual numbers. We have $\mathrm{Ker}(\psi) = \mathbb{C} \cdot F$.

(3-3): We have the isomorphisms

$$\phi : P^{d-1} \xrightarrow{\sim} H^0(X, \Omega_X^2(\log Z)), \quad \phi' : R_F^{2d-1} \xrightarrow{\sim} H^1(X, \Omega_X^1(\log Z)),$$

such that the diagram

$$\begin{array}{ccc} P^d \otimes P^{d-1} & \xrightarrow{\mu} & R_F^{2d-1} \\ {\scriptstyle \psi\otimes\phi}\downarrow & & \downarrow{\scriptstyle \phi'} \\ T_0(M) \otimes H^0(X, \Omega_X^2(\log Z)) & \xrightarrow{\langle\, ,\, \rangle} & H^1(X, \Omega_X^1(\log Z)) \end{array}$$

commutes up to non-zero scalar where μ is the multiplication. (If we replace the source of ϕ with R_F^{d-1}, then the target space is replaced with $H^0(X, \Omega_X^2(\log Z))/(\mathbb{C} \cdot \omega_U)$, cf. Lemma 3.2 (1) below).

(3-4): We have the following formula

$$\phi(G) = \mathrm{Res}_X \frac{G}{F z_1 z_2 z_3} \Omega \quad (G \in P^{d-1}),$$

where $\Omega = \sum_{i=0}^{3} (-1)^i z_i dz_0 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_3 \in H^0(\mathbb{P}^3, \Omega^3_{\mathbb{P}^3} \otimes \mathcal{O}(4))$ and

$$\mathrm{Res}_X \ : \ H^0(\mathbb{P}^3, \Omega^3_{\mathbb{P}^3}(\log X + Y)) \to H^0(X, \Omega_X^2(\log Z)) \quad (Y = \{z_1 z_2 z_3 = 0\} \subset \mathbb{P}^3)$$

is the residue map.

Lemma 3.2.

(1) *Putting* $\omega_F = \frac{\partial F}{\partial z_0}$*, we have* $\phi(\omega_F) = \omega_U$ *(see* (1.2) *for the definition of* ω_U*).*

(2) *Assume* $0 \in T^\sigma_{(p,q)}(\underline{c})$ *and that* X *is defined by an equation (cf. The proof of Lemma 1.2):*

$$F = wA + \prod_{1 \le \nu \le r} (c z_{\sigma(1)}^{p+q} - c_\nu z_{\sigma(2)}^p z_{\sigma(3)}^q). \tag{3.1}$$

Put

$$\xi_F = \frac{\partial w}{\partial z_0}(q z_{\sigma(2)} \frac{\partial A}{\partial z_{\sigma(2)}} - p z_{\sigma(3)} \frac{\partial A}{\partial z_{\sigma(3)}}) - \frac{\partial A}{\partial z_0}(q z_{\sigma(2)} \frac{\partial w}{\partial z_{\sigma(2)}} - p z_{\sigma(3)} \frac{\partial w}{\partial z_{\sigma(3)}}) \ \in P^{d-1}.$$

Then we have $\phi(\xi_F) = \xi_U$ *(see* (1.4) *for the definition of* ξ_U*).*

Proof. We only give a proof of (2) ((1) is easier). We may suppose $\sigma = 1$. It is enough to check the equation on $\{z_1 \neq 0\}$. Write $z_i' = z_i/z_1$ and put $F' = F(z_0', 1, z_2', z_3')$ etc. Then

$$\frac{\xi_F}{F z_1 z_2 z_3}\Omega|_{z_1\neq 0} = \frac{-\xi_F'}{F' z_2' z_3'} dz_0' \wedge dz_2' \wedge dz_3' = \frac{dF'}{F'} \mathrm{dlog}(z_2'^p z_3'^q) \wedge \frac{dw'}{w'}.$$

This yields $\phi(\xi_F) = \xi_U$ by **(3-4)**. $\square$

For $\lambda \in P^{d-1}$ consider the linear map

$$\lambda^* \ : \ P^{3d-4} \longrightarrow \mathbb{C}\ ;\ x \longmapsto \tau_F(\lambda x). \tag{3.2}$$

For an integer $l \geq 0$ define

$$\begin{aligned} I_\lambda &= \{x \in P^l |\ \lambda^*(xy) = 0 \text{ for all } y \in P^{3d-4-l}\} \\ &= \{x \in P^l |\ \lambda x = 0 \in R_F^{d-1+l}\} \end{aligned}$$

where the second equality follows from Theorem 3.1. Put $I_\lambda^l = I_\lambda \cap P^l$. Then I_λ^d is the inverse image under ψ of $\{\partial \in T_0(M)|\ \langle \partial, \lambda\rangle = 0\}$ so that (2.2) implies:

- **(3-5):** $\psi^{-1}(T_0(T^{\mathrm{reg}})) \subset I_\lambda^d$ for any irreducible component $T \subset \Delta_\lambda$ with reduced structure such that $0 \in T^{\mathrm{reg}}$.

Proposition 3.3. *Assume $\lambda \notin J_F^{d-1} = \mathbb{C}\cdot\omega_F$. Then $\dim(P^d/I_\lambda^d) \geq \binom{d+2}{2} - 5$ and the equality holds if and only if I_λ^d is complete intersection of degree $(1, d-1, d, d)$.*

Let T be as in Theorem 0.8 bis and assume $0 \in T^{\mathrm{reg}}$. **(3-5)** implies

$$\mathrm{codim}_\Delta(T) \geq \mathrm{codim}_{T_0(\Delta)}(T_0(T^{\mathrm{reg}})) \geq \dim(P^d/I_\lambda^d),$$

and hence Theorem 0.8 bis (i) follows from Proposition 3.3. Theorem 0.8 bis (ii) will be proved in the next section and Theorem 0.8 bis (iii) follows from the following:

Proposition 3.4. *Let the assumption be as in Lemma 3.2 (2).*

1. $\psi^{-1}(T_0(T_{(p,q)}^\sigma(\underline{c}))) = wP^{d-1} + J_F^d$.
2. $I_\lambda^d = wP^{d-1} + J_F^d$ *if* $\lambda = a\omega_F + b\xi_F$ *with* $b \neq 0$.
3. $\mathbb{C}\cdot\omega_F \oplus \mathbb{C}\cdot\xi_F = \{y \in P^{d-1}|\ yx = 0 \in R_F^{2d-1}$ *for all* $x \in wP^{d-1} + J_F^d\}$.

In the rest of this section we prove Propositions 3.3 and 3.4. We need the following theorem due to Otwinowska (it is shown by the same method as the proof of [Ot], Th.2).

Theorem 3.5 (Otwinowska)**.** *Let $I \subset P$ be a homogeneous ideal satisfying the conditions:*

1. *There exist an integer $N > 0$ and a non-zero linear map $\mu : P^N \to \mathbb{C}$ such that $I^l = \{x \in P^l|\ \mu(xy) = 0$ for all $y \in P^{N-l}\}$.*
2. *I contains a homogeneous ideal J which is complete intersection of degree (e_0, e_1, e_2, e_3) with $e_0 \leq e_1 \leq e_2 \leq e_3$.*
3. *There is an integer b such that $e_0 \leq b \leq e_1 - 1$ and $N + 3 = e_2 + e_3 + b$.*

For $l \geq 1$ we have

$$\dim(P^l/I^l) \geq \dim(P^l/\langle z_0, z_1^b, z_2^{e_2}, z_3^{e_3}\rangle \cap P^l).$$

Moreover, if $l \leq N - b$, the equality holds if and only if I is complete intersection of degree $(1, b, e_2, e_3)$.

We first prove Proposition 3.3. Since $\lambda \notin J_F^{d-1}$ the map λ^* is not zero. By definition (I_λ, λ^*) satisfies the condition (1) with $N = 3d-4$. I_λ contains the Jacobian ideal J_F which is complete intersection of degree $(d-1, d, d, d)$. We apply Theorem 3.5 for $(e_0, e_1, e_2, e_3, b) = (d-1, d, d, d, d-1)$, $I = I_\lambda$ and $J = J_F$. Then the assertion is straightforward by noting

$$\dim(P^d/\langle z_0, z_1^{d-1}, z_2^d, z_3^d\rangle \cap P^d) = \binom{d+2}{2} - 5.$$

Next we show Proposition 3.4. Let PGL_4 be the group of projective transformations on $\mathbb{P}^3$ and let $G \subset \mathrm{PGL}_4$ be the subgroup of such $g \in \mathrm{PGL}_4$ that $g(Y_j) = Y_j$ for all $j = 1, 2, 3$. It is evident that G naturally acts on M and $T^\sigma_{(p,q)}(\underline{c}) \subset M$ is stable under the action. Let $T^\sigma_{(p,q)}(\underline{c})_{(w,c)} \subset T^\sigma_{(p,q)}(\underline{c})$ be the closed subset of those surfaces defined by equations of the form

$$wB + \prod_{1\le\nu\le r} (cz^{p+q}_{\sigma(1)} - c_\nu z^p_{\sigma(2)} z^q_{\sigma(3)}) \quad \text{for some } B \in P^{d-1}.$$

It is easy to see that the natural map $G \times T^\sigma_{(p,q)}(\underline{c})_{(w,c)} \to T^\sigma_{(p,q)}(\underline{c})$ is smooth and surjective and that $\psi^{-1}(T_0(T^\sigma_{(p,q)}(\underline{c})_{(w,c)})) = wP^{d-1}$. The map

$$T_0(M) \xrightarrow{\psi^{-1}} P^d/\mathbb{C}\cdot F \xrightarrow{\pi} R^d_F$$

identifies R^d_F with the quotient of $T_0(M)$ by the infinitesimal action of the tangent space at the identity of G. It implies

$$\psi^{-1}(T_0(T^\sigma_{(p,q)}(\underline{c}))) = \pi^{-1}\pi\psi^{-1}(T_0(T^\sigma_{(p,q)}(\underline{c}))_{(w,c)})) = wP^{d-1} + J^d_F.$$

This completes the proof of Proposition 3.4 (1).

Let $\lambda = a\omega_F + b\xi_F$ with $b \neq 0$. Then $\lambda \notin J_F^{d-1}$ so that we have $\binom{d+2}{2} - 5 \le \dim(P^d/I^d_\lambda)$ by Proposition 3.3. An easy calculation shows $\lambda w \in J^d_F$ so that $I^d_\lambda \supset wP^{d-1} + J^d_F$. We have

$$\binom{d+2}{2} - 5 \le \dim(P^d/I^d_\lambda) \le \dim(P^d/wP^{d-1} + J^d_F)$$

$$\underset{(*)}{=} \operatorname{codim}_{T_0(M)}(T_0(T^\sigma_{(p,q)}(\underline{c}))) \le \operatorname{codim}_M(T^\sigma_{(p,q)}(\underline{c})) \underset{(**)}{\le} \binom{d+2}{2} - 5,$$

where $(*)$ from Proposition 3.4 (1), and $(**)$ from Lemma 1.2. Thus the above inequalities are all equalities so that $I^d_\lambda = wP^{d-1} + J^d_F$. This completes the proof of Proposition 3.4 (2).

Finally we show Proposition 3.4 (3). Obviously

$$\begin{aligned}\mathbb{C}\cdot\omega_F \oplus \mathbb{C}\cdot\xi_F &\subset \{y \in P^{d-1} |\ yx = 0 \in R^{2d-1}_F \text{ for all } x \in wP^{d-1} + J^d_F\} \\ &\subset \{y \in P^{d-1} |\ yw \in J^d_F\} \\ &\xrightarrow{\cong} wP^{d-1} \cap J^d_F\end{aligned}$$

where the last isomorphism is given by multiplication by w. Hence it suffices to show $\dim wP^{d-1} \cap J^d_F \le 2$. Note that w is in complete intersection with $\frac{\partial F}{\partial z_0}$, and

$$\dim\left(wP^{d-1} \cap \langle z_1\frac{\partial F}{\partial z_1}, z_2\frac{\partial F}{\partial z_2}, z_3\frac{\partial F}{\partial z_3}\rangle^d\right) \le 1$$

as wP^{d-1} has base point locus of codimension one and $\langle z_1\frac{\partial F}{\partial z_1}, z_2\frac{\partial F}{\partial z_2}, z_3\frac{\partial F}{\partial z_3}\rangle^d$ has base point locus of codimension three. Therefore some linearly independent $h_1,\ h_2 \in \langle z_1\frac{\partial F}{\partial z_1}, z_2\frac{\partial F}{\partial z_2}, z_3\frac{\partial F}{\partial z_3}\rangle^d$ defines a complete intersection ideal $\langle w, \frac{\partial F}{\partial z_0}, h_1, h_2\rangle$ of degree $(1, d-1, d, d)$. This implies $\dim wP^{d-1} \cap$

$\langle \frac{\partial F}{\partial z_0}, h_1, h_2\rangle^d = 1$ and hence we have $\dim wP^{d-1} \cap J_F^d \leq 2$. This proves the desired assertion and the proof of Proposition 3.4 is complete.

4. Proof of Theorem 0.8 bis (ii)

In this and next sections we prove Theorem 0.8 bis (ii) to complete the proof of Theorem 0.8. Let the assumption be as in Theorem 0.8 bis (ii). Without loss of generality we assume $t = 0 \in T^{\mathrm{reg}}$. Choose equations $F_t \in P^d$ that define $X_t \subset \mathbb{P}^3$ and move holomorphically for $t \in \Delta$. Let R_{F_t} be the corresponding Jacobian ring. For $t \in T^{\mathrm{reg}}$ let $I_{\lambda_t} \subset P$ be defined in the same manner as I_λ with λ replaced by $\lambda_t \in H^0(X_t, \Omega^2_{X_t}(\log Z_t))$, the flat translation of λ. For any $t \in T^{\mathrm{reg}}$ we have

$$\mathrm{codim}_\Delta(T) \geq \mathrm{codim}_{T_t(\Delta)}(T_t(T^{\mathrm{reg}})) \geq \dim(P^d/I^d_{\lambda_t}) \geq \binom{d+2}{2} - 5,$$

where $T_t(*)$ denotes the tangent space at t. The second inequality follows from **(3-5)** and the last from Proposition 3.3. Hence the assumption implies that the above inequalities are all equalities. Therefore $\psi^{-1}(T_t(T^{\mathrm{reg}})) = I^d_{\lambda_t}$. Proposition 3.3 then implies that I_{λ_t} is complete intersection of degree $(1, d-1, d, d)$ so that $I^1_{\lambda_t} = \mathbb{C}\cdot w_t$ for some $w_t \in P^1$ determined up to non-zero scalar. We easily see

$$\dim(J^d_{F_t}) = 7, \quad \dim(I^d_{\lambda_t}/w_tP^{d-1}) = \dim(P^1/\mathbb{C}\cdot w_t) + 1 + 1 = 5. \tag{4.1}$$

Lemma 4.1. *There exists $t \in T^{\mathrm{reg}}$ such that $w_t \notin \sum_{i=1}^3 \mathbb{C}\cdot z_i$.*

We will prove Lemma 4.1 in the next section. Admitting Lemma 4.1, we finish the proof of Theorem 0.8 bis (ii). Let

$$T^o = \{t \in T^{\mathrm{reg}} |\ w_t \notin \sum_{i=1}^3 \mathbb{C}\cdot z_i\}$$

which is a non-empty open subset of T by the above lemma. Without loss of generality we assume $t = 0 \in T^o$. Since $I^d_{\lambda_t} \supset w_tP^{d-1} + J^d_{F_t}$, (4.1) implies

$$\dim(w_tP^{d-1} \cap J^d_{F_t}) \geq 7 - 5 = 2. \tag{4.2}$$

Put $E_{w_t} = \mathbb{C}^3 \oplus (P^1/\mathbb{C}\cdot w_t)$. (4.2) implies that there is $\Gamma_t = [\gamma_{1,t} : \gamma_{2,t} : \gamma_{3,t} : L_t] \in \check{\mathbb{P}}(E_{w_t}) = \mathbb{P}^5$ such that

$$\sum_{i=1}^3 \gamma_{i,t} z_i \frac{\partial F_t}{\partial z_i} + L_t \frac{\partial F_t}{\partial z_0} \in w_tP^{d-1} \tag{4.3}$$

for each $t \in T^{\mathrm{reg}}$. We may choose $\Gamma_t \in \check{\mathbb{P}}(E_w)$ such that it moves holomorphically for $t \in T^o$. Put $w = w_0$ and $\Gamma = \Gamma_0 = [\gamma_1 : \gamma_2 : \gamma_3 : L]$. We now consider the morphisms

$$h : T^o \longrightarrow \check{\mathbb{P}}(P^1) = \mathbb{P}^3; \quad t \longmapsto [w_t] := \mathbb{C}\cdot w_t \tag{4.4}$$

and

$$s : h^{-1}(w) \to \check{\mathbb{P}}(E_w) = \mathbb{P}^5; \quad t \mapsto \Gamma_t. \tag{4.5}$$

We put

$$T^o_{w,\Gamma} = s^{-1}(\Gamma)^{\mathrm{reg}}.$$

By the construction we have

$$\mathrm{codim}_{T_t(T^{\mathrm{reg}})}(T_t(T^o_{w,\Gamma})) \leq 3 + 5 = 8 \text{ for } t \in T^o_{w,\Gamma}. \tag{4.6}$$

Due to **(3-2)**, (4.3) implies that for $t \in T^o_{w,\Gamma}$,

$$\psi^{-1}(T_t(T^o_{w,\Gamma})) \subset \{G \in I^d_\lambda | \sum_{i=1}^{3} \gamma_i z_i \frac{\partial G}{\partial z_i} + L\frac{\partial G}{\partial z_0} \in wP^{d-1}\}. \tag{4.7}$$

Write $wV = wP^{d-1} \cap \psi^{-1}(T_t(T^o_{w,\Gamma}))$ for a subspace $V \subset P^{d-1}$. Noting $\psi^{-1}(T_t(T^{\mathrm{reg}})) = I^d_{\lambda_t} \supset wP^{d-1}$ for $t \in T^o_{w,\Gamma}$, (4.6) implies $\mathrm{codim}_{P^{d-1}}(V) \leq 8$. (4.7) implies

$$\sum_{i=1}^{3} \gamma_i z_i \frac{\partial(wG)}{\partial z_i} + L\frac{\partial(wG)}{\partial z_0} \in wP^{d-1} \quad \text{for all } G \in V.$$

If $\sum_{i=1}^3 \gamma_i z_i \frac{\partial w}{\partial z_i} + L\frac{\partial w}{\partial z_0} \notin \mathbb{C}\cdot w$, it implies $G \in wP^{d-2}$ for all $G \in V$. Since $\mathrm{codim}_{P^{d-1}}(wP^{d-2}) = \binom{d+1}{2}$, this is a contradiction if $\binom{d+1}{2} > 8$ which holds when $d \geq 4$. Thus we get the condition:

$$\sum_{i=1}^{3} \gamma_i z_i \frac{\partial w}{\partial z_i} + L\frac{\partial w}{\partial z_0} \in \mathbb{C}\cdot w. \tag{4.8}$$

We claim that (4.7) and (4.8) imply that for each $t \in T^o_{w,\Gamma}$

$$F_t = wB_t + \prod_{1\leq\nu\leq r} (c_t z^{p+q}_{\sigma(1)} - c_{\nu,t} z^p_{\sigma(2)} z^q_{\sigma(3)}) \tag{4.9}$$

for some B_t, σ and $(c_t, c_{\nu,t})$. Since $w \notin \sum_{i=1}^3 \mathbb{C}z_i$, we may suppose $w = z_0$ by transforming by an element of G (cf. the proof of Proposition 3.3). (4.8) then reads $L \in \mathbb{C}\cdot z_0 = \mathbb{C}\cdot w$. It implies that $\gamma_1, \gamma_2, \gamma_3$ are not all zero. (4.3) now reads

$$\sum_{i=1}^{3} \gamma_i z_i \frac{\partial F_t}{\partial z_i} \in z_0 P^{d-1} \quad (t \in T^o_{z_0,\Gamma}).$$

Writing $F_t = z_0 B_t + C_t$ with C_t, a homogeneous polynomial of degree d in $\mathbb{C}[z_1, z_2, z_3]$, the above condition is equivalent to

$$\sum_{i=1}^{3} \gamma_i z_i \frac{\partial C_t}{\partial z_i} = 0.$$

Write

$$C_t = \sum_{\underline{\alpha}=(\alpha_1,\alpha_2,\alpha_3)} c_{\underline{\alpha}} z^{\underline{\alpha}}, \quad (z^{\underline{\alpha}} = z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3},\ c_{\underline{\alpha}} \in \mathbb{C})$$

and take $\underline{\alpha}$ with $c_{\underline{\alpha}} \neq 0$. The above condition implies that $\underline{\alpha}$ is an integral point lying on the sectional line ℓ in $x_1x_2x_3$-space defined by

$$\ell \ :\ \sum_{i=1}^{3} x_i - d = \sum_{i=1}^{3} \gamma_i x_i = 0,\ x_i \geq 0\ (i = 1, 2, 3).$$

Furthermore the condition **(3-1)** implies that C is divisible by neither of z_1, z_2, z_3. Letting $\pi_i : x_i = 0$ be a plane, it implies that ℓ and π_i intersect at an integral point for all $i = 1, 2, 3$. This implies that ℓ passes through one of the points $(d, 0, 0)$, $(0, d, 0)$, $(0, 0, d)$. Assuming that ℓ passes through the first point, we get $\gamma_1 = 0$ and hence $\alpha_2 : \alpha_3 = -\gamma_3 : \gamma_2 = p : q$ for some coprime non-negative integer p, q. Writing $\alpha_2 = pj$, $\alpha_3 = qj$ with $j \in \mathbb{Z}$, we get $\alpha_1 = d - (p + q)j$ since $\sum_{i=1}^3 \alpha_i = d$. The condition that ℓ and π_1 intersect at an integral point implies that $r := d/(p + q)$ is an integer and hence $\alpha_1 = (p + q)(r - j)$. Thus we can write

$$C_t = \sum_{j=0}^{r} b_{j,t} (z_1^{p+q})^{r-j} (z_2^p z_3^q)^j = \prod_{1\leq\nu\leq r} (c_t z_1^{p+q} - c_{\nu,t} z_2^p z_3^q) \quad \text{for some } b_j, c_{\nu,t}, c_t \in \mathbb{C}.$$

Hence we have (4.9).

Apply the same argument to any $t \in T^o$ and let $T^o_{w_t,\Gamma_t}$ be defined in the same way as $T^o_{w,\Gamma}$. Varying $t \in T^o$, $T^o_{w_t,\Gamma_t}$ sweep a non-empty open set $T^{oo} \subset T^o$ and we get

$$F_t = w_t B_t + \prod_{1\le\nu\le r} (c_t z^{p+q}_{\sigma(1)} - c_{\nu,t} z^p_{\sigma(2)} z^q_{\sigma(3)}) \quad (t \in T^{oo}). \tag{4.10}$$

Since w_t moves holomorphically, B_t, c_t and $c_{\mu,t}$ also move holomorphically for $t \in T^{oo}$. Since $I^d_{\lambda_t} \supset w_t P^{d-1} + J^d_{F_t}$, we have

$$\lambda_t \in \{y \in P^{d-1} | yx = 0 \text{ for all } x \in w_t P^{d-1} + J^d_{F_t}\} = \mathbb{C}\cdot\omega_{F_t} \oplus \mathbb{C}\cdot\xi_{F_t} := \Sigma(U_t)$$

where the last equality follows from Proposition 3.4 (3). Since $\lambda_t \notin \mathbb{C}\cdot\omega_{F_t}$ by the assumption $\lambda \notin \mathbb{C}\cdot\omega_F$, we get $\mathbb{C}\cdot\omega_{F_t} \oplus \mathbb{C}\cdot\lambda_t = \Sigma(U_t)$. If $\lambda_t \in H^2(U_t, \mathbb{Q}(2))$ then $\dim \Sigma(U_t) \cap H^2(U_t, \mathbb{Q}(2)) = 2$. By Lemma 1.3 it implies the condition Proposition 1.4 (3), hence (4). Thus there is a holomorphic function a_t such that $a_t c_t / c_{\mu,t}$ are roots of unity and hence $\underline{c}_t = [c_{1,t} : \cdots : c_{r,t}] \in \mathbb{P}^{r-1}(\mathbb{C})$ is constant. Therefore $T^{oo} \subset T^\sigma_{(p,q)}(\underline{c}) \cap \Delta$ and hence $T \subset T^\sigma_{(p,q)}(\underline{c}) \cap \Delta$ by taking the closure in Δ. Finally, comparing the codimensions in Δ, we conclude that the last inclusion is the equality and the proof of Proposition 3.4 is complete.

5. Proof of Lemma 4.1

In this section we prove Lemma 4.1. Assume:

$$w_t \in \sum_{i=1}^{3} \mathbb{C}\cdot z_i \quad \text{for all } t \in T^{\rm reg}.$$

We may write

$$w_t = \sum_{i=1}^{3} a_i(t) z_i \quad \text{and} \quad w = w_0 = \sum_{i=1}^{3} a_i z_i,$$

where $a_i(t)$ is a holomorphic function on $T^{\rm reg}$ with $a_i = a_i(0)$. Since $I^d_{\lambda_t} \supset w_t P^{d-1} + J^d_{F_t}$ and (4.1), we have $\dim(w_t P^{d-1} \cap J^d_{F_t}) \ge 7 - 5 = 2$ and hence there is $\Gamma_t = [\gamma_{1,t} : \gamma_{2,t} : \gamma_{3,t} : L_t] \in \check{\mathbb{P}}(E_{w_t})$ satisfies (4.3). Put $F = F_0$ and $\Gamma = \Gamma_0 = [\gamma_1 : \gamma_2 : \gamma_3 : L]$. Then (4.3) reads

$$\sum_{i=1}^{3} \gamma_i z_i \frac{\partial F}{\partial z_i} + L \frac{\partial F}{\partial z_0} \in w P^{d-1}. \tag{5.1}$$

Similarly to (4.8) we get

$$\sum_{i=1}^{3} \gamma_i a_i z_i \in \mathbb{C}\cdot \sum_{i=1}^{3} a_i z_i,$$

and hence

$$\gamma_1 a_1 : \gamma_2 a_2 : \gamma_3 a_3 = a_1 : a_2 : a_3. \tag{5.2}$$

If $L \notin \sum_{i=1}^{3} \mathbb{C}\cdot z_i$, (5.1) implies $\frac{\partial F}{\partial z_0} = 0$ at $[1:0:0:0] \in \mathbb{P}^3$, which contradicts **(3-1)**. Hence we have

$$L \in \sum_{i=1}^{3} \mathbb{C}\cdot z_i. \tag{5.3}$$

The proof is now divided into some cases. First we suppose that we are in:

Case (1): There exists $t \in T^{\rm reg}$ such that $a_i(t) \neq 0$ for all $i = 1, 2, 3$.

Without loss of generality we may suppose that $t = 0$ satisfies the above condition. (5.2) implies $\gamma_1 = \gamma_2 = \gamma_3$. If $\gamma_1 = \gamma_2 = \gamma_3 = 0$, then $L \notin \mathbb{C} \cdot w$ and (5.1) implies $\frac{\partial F}{\partial z_0} \in wP^{d-1}$ so that $\frac{\partial F}{\partial z_0} = 0$ at $[1 : 0 : 0 : 0]$, which contradicts **(3-1)**. Thus we may assume $\gamma_i = 1$ for all $i = 1, 2, 3$. By noting $d \cdot F = \sum_{i=0}^{3} z_i \frac{\partial F}{\partial z_i}$, (5.1) now reads:

$$d \cdot F + (L - z_0)\frac{\partial F}{\partial z_0} \in wP^{d-1}. \tag{5.4}$$

Claim 5.1. *$L \notin \mathbb{C} \cdot w$.*

Proof. Assume $L \in \mathbb{C} \cdot w$. (5.4) implies $dF - z_0\frac{\partial F}{\partial z_0} \in wP^{d-1}$. By the assumption $a_1 \neq 0$ we can write

$$F = wA + z_0B + C \text{ with } B \in \mathbb{C}[z_0, z_2, z_3] \cap P^{d-1},\ C \in \mathbb{C}[z_2, z_3] \cap P^d.$$

Then $\frac{\partial F}{\partial z_0} = w\frac{\partial A}{\partial z_0} + z_0\frac{\partial B}{\partial z_0} + B$ and hence $d(z_0B + C) - z_0(z_0\frac{\partial B}{\partial z_0} + B) = 0$ by noting $wP^{d-1} \cap \mathbb{C}[z_0, z_2, z_3] = 0$. It implies $C = 0$ and $(d-1)B = z_0\frac{\partial B}{\partial z_0}$. From the last equation we immediately deduce $B = cz_0^{d-1}$ with some $c \in \mathbb{C}$. Hence $F = wA + cz_0^d$, which is singular on $\{w = A = z_0 = 0\}$. It contradicts **(3-1)** and completes the proof of Claim 5.1. □

Now choose $u \in \sum_{i=1}^{3} \mathbb{C}z_i$ such that w, L, u are linearly independent and write

$$F = wA + \sum_{\nu=0}^{d} L^\nu B_\nu, \text{ with } B_\nu \in \mathbb{C}[z_0, u] \cap P^{d-\nu}.$$

In view of (5.3), (5.4) implies

$$d(\sum_{\nu=0}^{d} L^\nu B_\nu) + (L - z_0)\sum_{\nu=0}^{d} L^\nu\frac{\partial B_\nu}{\partial z_0} = \sum_{\nu=0}^{d} L^\nu(dB_\nu - z_0\frac{\partial B_\nu}{\partial z_0} + \frac{\partial B_{\nu-1}}{\partial z_0}) \in wP^{d-1}$$

where $B_{-1} = 0$ by convention. Hence we get $dB_\nu - z_0\frac{\partial B_\nu}{\partial z_0} + \frac{\partial B_{\nu-1}}{\partial z_0} = 0$ for all $\nu = 0, 1, \ldots, d$. We easily solve the equations to get $B_\nu = c(-1)^\nu\binom{d}{\nu}z_0^{d-\nu}$ for some $c \in \mathbb{C}$ independent of ν. Hence

$$F = wA + c\sum_{\nu=0}^{d}(-1)^\nu L^\nu\binom{d}{\nu}z_0^{d-\nu} = wA + c(z_0 - L)^d.$$

The equation is singular on $\{w = A = z_0 - L = 0\} \subset \mathbb{P}^3$, which contradicts **(3-1)**. This completes the proof in Case (1).

By Case (1) we may suppose $T \subset \cup_{1 \le i \le 3}\{t \in \Delta |\ a_i(t) = 0\}$. Since T is irreducible, we may suppose $a_3(t) = 0$ for all $t \in T$. Now we assume that we are in:

Case (2): There exists $t \in T^{\mathrm{reg}}$ such that $a_1(t)a_2(t) \neq 0$.

Without loss of generality we may suppose that $t = 0$ satisfies the above condition. Thus $a_3 = 0$ and $a_1a_2 \neq 0$. By replacing a smaller neighborhood of 0, we may further assume $a_1(t)a_2(t) \neq 0$ for $t \in T^{\mathrm{reg}}$. (5.2) implies $\gamma_1 = \gamma_2$. Assuming $\gamma_3 \neq 0$, (5.1) implies $z_3\frac{\partial F}{\partial z_3} = 0$ on $\{z_1 = z_2 = \frac{\partial F}{\partial z_0} = 0\}$, which contradicts **(3-1)**. Thus $\gamma_3 = 0$. If $\gamma_1 = \gamma_2 = 0$, the same argument as in the beginning of Case (1) induces a contradiction. Thus we may assume $\gamma_1 = \gamma_2 = 1$. Hence (5.1) now reads:

$$\sum_{i=1}^{2} z_i\frac{\partial F}{\partial z_i} + L\frac{\partial F}{\partial z_0} \in wP^{d-1}. \tag{5.5}$$

Claim 5.2. *$L \in \sum_{i=1}^{2} \mathbb{C} \cdot z_i$ and $L \notin \mathbb{C} \cdot w$.*

Proof. Assume $L \notin \sum_{i=1}^{2} \mathbb{C} \cdot z_i$. By (5.3) we may suppose $L = z_3 + l_1 z_1 + l_2 z_2$. Then (5.5) implies $\frac{\partial F}{\partial z_0} \in \langle z_1, z_2 \rangle$, which contradicts **(3-1)**. The proof of the second assertion is similar to that of Claim 5.1 and omitted. □

Noting $\mathbb{C}[z_0, z_1, z_2, z_3] = \mathbb{C}[z_0, w, L, z_3]$, we may write

$$F = wA + \sum_{\mu=0}^{d} z_3^{\mu} G_\mu \text{ with } A \in P^{d-1},\ G_\mu \in \mathbb{C}[z_0, L] \cap P^{d-\mu}.$$

Noting $\sum_{i=1}^{2} z_i \frac{\partial w}{\partial z_i} = w$, (5.5) implies

$$\sum_{\mu=0}^{d} z_3^{\mu} (\sum_{i=1}^{2} z_i \frac{\partial G_\mu}{\partial z_i} + L \frac{\partial G_\mu}{\partial z_0}) \in wP^{d-1}.$$

Noting $(d-\mu)G_\mu = \sum_{i=0}^{3} z_i \frac{\partial G_\mu}{\partial z_i}$ and $\frac{\partial G_\mu}{\partial z_3} = 0$, we get

$$0 = \sum_{i=1}^{2} z_i \frac{\partial G_\mu}{\partial z_i} + L \frac{\partial G_\mu}{\partial z_0} = (d-\mu) G_\mu + (L - z_0) \frac{\partial G_\mu}{\partial z_0} \quad \text{for all } \mu = 0, 1, \ldots, d.$$

We solve the last equation in the same manner as Case (1) to get $G_\mu = b_\mu (L - z_0)^{d-\mu}$ with $b_\mu \in \mathbb{C}$ and hence

$$F = wA + \sum_{\mu=0}^{d} b_\mu z_3^{\mu} (L - z_0)^{d-\mu}. \tag{5.6}$$

Claim 5.3. *Put* $\eta_F = A + \sum_{i=1}^{2} z_i \frac{\partial A}{\partial z_i} + L \frac{\partial A}{\partial z_0}$.

(1) $\phi(\eta_F) = \frac{z_3}{w} d(\frac{z_0 - L}{z_3}) \wedge \mathrm{dlog} \frac{z_1}{z_2}$.

(2) $\mathbb{C} \cdot \omega_F \oplus \mathbb{C} \cdot \eta_F = \{ y \in P^{d-1} |\ yx = 0 \in R_F^{2d-1}$ *for all* $x \in wP^{d-1} + J_F^d \}$ *(cf. Lemma 3.2).*

Proof. Noting $w\eta_F = z_1 \frac{\partial F}{\partial z_1} + z_2 \frac{\partial F}{\partial z_2} + L \frac{\partial F}{\partial z_0}$, Claim 5.3 (1) is proven by the same argument as the proof of Lemma 3.2. Claim 5.3 (2) is proven by the same argument as the proof of Proposition 3.4 (3). We omit the details. □

By Claim 5.3, $\lambda_t \in H^0(X_t, \Omega^2_{X_t}(\log Z_t))$, the flat translation of λ for $t \in T^{\mathrm{reg}}$, is written as

$$\lambda_t = f_1(t)\eta(t) + f_2(t)\omega(t) \quad (t \in T^{\mathrm{reg}}).$$

Here $f_1(t)$ and $f_2(t)$ are holomorphic functions on T^{reg} and

$$\omega(t) = \mathrm{dlog} \frac{z_2}{z_1} \wedge \mathrm{dlog} \frac{z_3}{z_1}, \quad \eta(t) = \frac{z_3}{w_t} d(\frac{z_0 - L_t}{z_3}) \wedge \mathrm{dlog} \frac{z_1}{z_2},$$

where w_t is as in the beginning of this section and

$$F_t = w_t A_t + \sum_{\mu=0}^{d} b_{\mu,t} z_3^{\mu} (L_t - z_0)^{d-\mu}, \quad L_t = l_1(t) z_1 + l_2(t) z_2$$

is the equation defining X_t such as (5.6), which varies holomorphically with $t \in T^{\mathrm{reg}}$. Recalling $Y = \cup_{1 \le j \le 3} Y_j$ with $Y_j = \{z_j = 0\} \subset \mathbb{P}^3$, write

$$Z_t = X_t \cap Y \supset Z_{3t} = X_t \cap Y_3 \supset V_t = Z_{3t} \cap (Y_1 \cup Y_2) \supset S_t = Z_{3t} \cap Y_2.$$

We consider the composite of the residue maps

$$\theta_t \ :\ H^0(X_t, \Omega^2_{X_t}(\log Z_t)) = H^0(X_t, \Omega^2_{X_t}(\log Z_t)) \xrightarrow{\mathrm{Res}_{Z_{3t}}} H^0(Z_{3t}, \Omega^1_{Z_{3t}}(\log V_t)) \xrightarrow{\mathrm{Res}_{S_t}} \mathbb{C}^{S_t} \xrightarrow{\sim} \mathbb{C}^d,$$

where the last isomorphism is obtained by choosing $\epsilon_t : \{1, 2, \ldots, d\} \xrightarrow{\sim} S_t$, an isomorphism of local systems of sets over Δ. Since λ_t is flat, we get the condition:

$$\theta_t(\lambda_t) \in \mathbb{C}^d \text{ is constant for } t \in T^{\mathrm{reg}}. \tag{5.7}$$

We shall show that the condition (5.7) induces a contradiction, which completes the proof of Lemma 4.1 in Case (2). S_t consists of distinct d-points $P_{\nu,t}$ which do not lie on Y_1. Write

$$P_{\nu,t} = [s_\nu(t) : 1 : 0 : 0] \in \mathbb{P}^3(\mathbb{C}).$$

A direct residue calculation shows

$$\theta_t(\omega(t)) = (1, \ldots, 1), \quad \theta_t(\eta(t)) = (\frac{l_1(t) - s_\nu(t)}{a_1(t)})_{1\le\nu\le d}$$

and hence

$$\theta_t(\lambda_t) = (p(t)s_\nu(t) + q(t))_{1\le\nu\le d} \text{ with } p(t) = -\frac{f_1(t)}{a_1(t)},\ q(t) = f_1(t)\frac{l_1(t)}{a_1(t)} + f_2(t).$$

Since $\lambda_t \notin \mathbb{C}\cdot\omega_t$, we have $p(t) \neq 0$. Therefore (5.7) implies that for all $1 \le \nu \le d$ we have

$$s_\nu(t) = C_\nu p(t)^{-1} - q(t)p(t)^{-1} \quad (t \in T^{\mathrm{reg}}) \tag{5.8}$$

where C_ν are constant. Letting

$$\Sigma = \{(s_1, \ldots, s_d)|\ s_\nu \in \mathbb{C},\ s_\nu \neq s_\mu \text{ for } 1 \le \nu \neq \mu \le d\},$$

we define a holomorphic map

$$\pi : \Delta \longrightarrow \Sigma; \quad t \longmapsto (s_\nu(t))_{1\le\nu\le d}.$$

Then (5.8) implies $\dim(\pi_*(T_0(T^{\mathrm{reg}}))) \le 2$. Therefore we get a contradiction if we show the following.

Claim 5.4. $\dim(\pi_*(T_0(T^{\mathrm{reg}}))) \ge d$.

Proof. Let $Q = \mathbb{C}[z_0, z_1]$ and $Q^l = P^l \cap Q$ for an integer l. Write $\overline{G} = G \bmod \langle z_2, z_3\rangle \in Q$ for $G \in P$. Consider the morphism

$$\rho\ :\ \Sigma \to N := \overset{\vee}{\mathbb{P}}(Q^d);\ \underline{s} = (s_\nu)_{1\le\nu\le d} \to [F_{\underline{s}}] \text{ with } F_{\underline{s}} = \prod_{1\le\nu\le d}(z_0 - s_\nu z_1).$$

It is finite etale and induces an isomorphism on the tangent spaces. Hence it suffices to show Claim 5.4 by replacing π with $\tilde{\pi} := \rho\circ\pi$. We then have $\tilde{\pi}(t) = [\overline{F}_t]$ and the commutative diagram

$$\begin{array}{ccc} P^d & \xrightarrow{\bmod\langle z_2,z_3\rangle} & Q^d \\ \psi\big\downarrow & & \big\downarrow\psi' \\ T_0(\Delta) & \xrightarrow{\tilde{\pi}_*} & T_{\tilde{\pi}(0)}(N) \end{array}$$

where ψ' is defined in the same way as ψ in **(3-2)** and $\mathrm{Ker}(\psi') = \mathbb{C}\cdot\overline{F}$. We have shown that $\psi^{-1}(T_0(T^{\mathrm{reg}})) = I^d_{\underline{\lambda}} \supset wP^{d-1} + J^d_{\underline{F}}$. Hence $\tilde{\pi}_*(T_0(T^{\mathrm{reg}})) \supset \psi'(z_1 Q^{d-1} + \mathbb{C}\cdot\overline{F})$. Noting $F \notin \langle z_1, z_2, z_3\rangle$ so that $\overline{F} \notin z_1 Q^{d-1}$, this implies

$$\dim(\tilde{\pi}_*(T_0(T^{\mathrm{reg}})) \ge \dim z_1 Q^{d-1} = d.$$

This completes the proof of Claim 5.4. □

By Case (2) we may assume now that we are in:

Case (3): $a_2(t) = a_3(t) = 0$ for all $t \in T^{\mathrm{reg}}$.

In this case we may assume $w = z_1$. We have

$$I_\lambda \supset I := \langle z_1 \rangle + J_F = \langle z_1, \frac{\partial F}{\partial z_0}, z_2\frac{\partial F}{\partial z_2}, z_3\frac{\partial F}{\partial z_3}\rangle$$

so that I is complete intersection of degree $(1, d-1, d, d)$. Hence $I = I_\lambda$ and $I_\lambda^d = z_1 P^{d-1} + J_F^d$. As before we can show the following.

Claim 5.5. *Put* $\kappa_F = \frac{\partial F}{\partial z_1}$.

(1) $\phi(\kappa_F) = \frac{z_0}{z_1}\mathrm{dlog}\frac{z_2}{z_0} \wedge \mathrm{dlog}\frac{z_3}{z_0}$.

(2) $\mathbb{C}\cdot\omega_F \oplus \mathbb{C}\cdot\kappa_F = \{y \in P^{d-1} |\ yx = 0 \in R_F^{2d-1}$ *for all* $x \in z_1P^{d-1} + J_F^d\}$.

As before Claim 5.5 implies

$$\lambda_t = f_1(t)\kappa(t) + f_2(t)\omega(t) \text{ for } t \in T^{\mathrm{reg}},$$

where $f_1(t)$, $f_2(t)$ and $\omega(t)$ are as before and

$$\kappa(t) = \frac{z_0}{z_1}\mathrm{dlog}\frac{z_2}{z_0} \wedge \mathrm{dlog}\frac{z_3}{z_0} \in H^0(X_t, \Omega^2_{X_t}(\log Z_t)).$$

An easy residue calculation shows $\theta_t(\lambda_t) = (f_1(t)s_\nu(t) + f_2(t))_{1\le\nu\le d}$ and the same argument as Case (2) induces a contradiction. This completes the proof of Lemma 4.1.

6. Appendix: Injectivity of regulator map

In this section we discuss an implication of Theorem 0.8 on the injectivity of the regulator map. Let X be a member of M. We are interested in the regulator map to Deligne cohomology

$$\rho_X \ :\ CH^2(X,1)\otimes\mathbb{Q} \longrightarrow H^3_D(X, \mathbb{Q}(2)),$$

where $CH^2(X,1)$ is Bloch's higher Chow group defined to be the cohomology of the complex

$$K_2(\mathbb{C}(X)) \xrightarrow{\partial_{\mathrm{tame}}} \bigoplus_{C\subset X} \mathbb{C}(C)^* \xrightarrow{\partial_{\mathrm{div}}} \bigoplus_{x\in X}\mathbb{Z},$$

where the sum on the middle term ranges over all irreducible curves on X and that on the right hand side over all closed points of X. The map ∂_{tame} is the so-called tame symbol and ∂_{div} is the sum of divisors of rational functions on curves. We have the localization exact sequence

$$CH^2(U,2) \longrightarrow CH^1(Z,1) \longrightarrow CH^2(X,1),$$

where

$$CH^1(Z,1) = \mathrm{Ker}(\bigoplus_{1\le i\le 3}\mathbb{C}(Z_i)^* \xrightarrow{\partial_{\mathrm{div}}} \bigoplus_{x\in Z}\mathbb{Z}) \quad \text{with } Z_i = X\cap Y_i.$$

By [AS2] Th.7.1 we get the following.

Theorem 6.1. *For* $t \in M - M_{NL}$, ρ_{X_t} *is injective on the subspace*

$$\Sigma_t := \mathrm{Im}(CH^1(Z_t,1) \longrightarrow CH^2(X_t,1))\otimes\mathbb{Q} \subset CH^2(X_t,1)\otimes\mathbb{Q}.$$

In this section we show there exists $t \in M - M_{NL}$ such that $\Sigma_t \neq 0$ so that Theorem 6.1 has a non-trivial implication on the injectivity of ρ_{X_t}. For this we need introduce some special locus in the moduli space M.

Definition 6.2. Let $T_{12} \subset M$ be the locus of those X defined by an equation

$$F = wA + z_1z_2B + c_1z_1^d + c_2z_2^d \text{ for some } w\in P^1,\ A\in P^{d-1},\ B\in P^{d-2},\ c_1,c_2\in\mathbb{C}^*.$$

We define T_{23} (resp. T_{31}) similarly by replacing (z_1, z_2) by (z_2, z_3) (resp. (z_3, z_1)).

We note that $T^{\sigma}_{(p,q)}(\underline{c}) \subset T_{12}$ with σ, the identity, and $p=1, q=0$. For X in T_{12} defined by such an equation as above we consider the following element

$$c_{12}(X) = ((\frac{z_2}{w})_{|Z_1}, (\frac{w}{z_1})_{|Z_2}, 1) \in \mathbb{C}(Z_1)^* \oplus \mathbb{C}(Z_2)^* \oplus \mathbb{C}(Z_3)^*.$$

It is easy to check $c_{12}(X) \in CH^1(Z,1)$. For X in T_{23} (resp. T_{31}) we define an element $c_{23}(X)$ (resp. $c_{31}(X)$) in $CH^1(Z,1)$ by the same say. Let $[c_{ij}(X)] \in CH^2(X,1)$ be the image of $c_{ij}(X) \in CH^1(Z,1)$ for $(i,j) = (1,2)$ or $(2,3)$ or $(3,1)$.

Theorem 6.3. (1) *If $d \geq 4$, $T_{12} \not\subset M_{NL}$ and $\rho_{X_t}([c_{12}(X_t)]) \neq 0$ for all $t \in T_{12} - M_{NL}$.*

(2) *If $d \geq 6$, $T_{12} \cap T_{23} \not\subset M_{NL}$ and $\rho_{X_t}([c_{12}(X_t)])$, $\rho_{X_t}([c_{23}(X_t)])$ are linearly independent for all $t \in (T_{12} \cap T_{23}) - M_{NL}$.*

(3) *If $d \geq 10$, $T_{12} \cap T_{23} \cap T_{31} \not\subset M_{NL}$ and $\rho_{X_t}([c_{12}(X_t)])$, $\rho_{X_t}([c_{23}(X_t)])$, $\rho_{X_t}([c_{23}(X_t)])$ are linearly independent for all $t \in (T_{12} \cap T_{23} \cap T_{31}) - M_{NL}$.*

Proof. Fix $0 \in M$ and let X be the corresponding surface in $\mathbb{P}^3$. By Lemma 1.1, if $0 \in M - M_{NL}$, we have

$$F^2H^2(U, \mathbb{Q}(2)) = \mathbb{Q} \cdot \mathrm{reg}^2_U(\alpha_U) \text{ with } \alpha_U = \{\frac{z_2}{z_1}, \frac{z_3}{z_1}\} \in CH^2(U,2).$$

By [AS2], Th.7.1, it implies that the kernel of the composite map

$$CH^1(Z,1) \otimes \mathbb{Q} \longrightarrow CH^2(X,1) \otimes \mathbb{Q} \xrightarrow{\rho_X} H^3_D(X, \mathbb{Q}(2))$$

is generated by

$$\partial_U(\alpha_U) = \delta := ((\frac{z_3}{z_2})_{|Z_1}, (\frac{z_1}{z_3})_{|Z_2}, (\frac{z_2}{z_1})_{|Z_2}) \in CH^1(Z,1),$$

where $\partial_U : CH^2(U,2) \to CH^1(Z,1)$.

Claim 6.4. *Write $\Lambda = \bigoplus_{1 \leq j \leq 3} \mathbb{C}(Z_j)^*$.*

(1) *Assume $0 \in T_{12}$ and that X is defined by an equation as Definition 6.2:*

$$F = wA + z_1z_2B + c_1z_1^d + c_2z_2^d.$$

Then the following elements are linearly independent in $\Lambda \otimes \mathbb{Q}$;

$$\delta,\ ((\frac{z_2}{w})_{|Z_1}, (\frac{w}{z_1})_{|Z_2}, 1).$$

(2) *Assume $0 \in T_{12} \cap T_{23}$ and that X is defined by an equation as Definition 6.2:*

$$\begin{aligned} F &= wA + z_1z_2B + c_1z_1^d + c_2z_2^d \\ &= vA' + z_2z_3B' + c'_2z_2^d + c'_3z_3^d \end{aligned}$$

Then the following elements are linearly independent in $\Lambda \otimes \mathbb{Q}$;

$$\delta,\ ((\frac{z_2}{w})_{|Z_1}, (\frac{w}{z_1})_{|Z_2}, 1),\ (1, (\frac{z_3}{v})_{|Z_2}, (\frac{v}{z_2})_{|Z_3}).$$

(3) *Assume $0 \in T_{12} \cap T_{23}$ and that X is defined by an equation as Definition 6.2:*

$$\begin{aligned} F &= wA + z_1z_2B + c_1z_1^d + c_2z_2^d \\ &= vA' + z_2z_3B' + c'_2z_2^d + c'_3z_3^d \\ &= uA'' + z_3z_1B'' + c''_3z_3^d + c''_1z_1^d \end{aligned}$$

Then the following elements are linearly independent in $\Lambda \otimes \mathbb{Q}$;

$$\delta,\ ((\frac{z_2}{w})_{|Z_1}, (\frac{w}{z_1})_{|Z_2}, 1),\ (1, (\frac{z_3}{v})_{|Z_2}, (\frac{v}{z_2})_{|Z_3}),\ ((\frac{u}{z_3})_{|Z_1}, 1, (\frac{z_1}{u})_{|Z_3}).$$

Proof. We only show Claim 6.4 (3). The others are easier and left to the readers. Assume the contrary. Then there are integers e, l, m, n not all zero such that

$$(\frac{z_2}{w})^l(\frac{u}{z_3})^n(\frac{z_3}{z_2})^e \equiv 1 \bmod z_1,$$
$$(\frac{w}{z_1})^l(\frac{z_3}{v})^m(\frac{z_1}{z_3})^e \equiv 1 \bmod z_2,$$
$$(\frac{v}{z_2})^m(\frac{z_1}{u})^n(\frac{z_2}{z_1})^e \equiv 1 \bmod z_3.$$

We note $u, v, w \notin \sum_{1\leq j\leq 3} \mathbb{C}\cdot z_j$ since otherwise it would contradict **(3-1)**. Hence the condition implies $l = m = n = e$ and u, v, w coincides up to non-zero constant. Thus we get

$$F \equiv z_1z_2B + c_1z_1^d + c_2z_2^d \equiv z_2z_3B' + c_2'z_2^d + c_3'z_3^d \equiv z_3z_1B'' + c_3''z_3^d + c_1''z_1^d \bmod w,$$

which is absurd. This completes the proof of Claim 6.4. □

By Claim 6.4, the proof of Theorem 6.3 is complete if we show that $T_{12} \not\subset M_{NL}$ (resp. $T_{12}\cap T_{23} \not\subset M_{NL}$, resp. $T_{12}\cap T_{23}\cap T_{31} \not\subset M_{NL}$) if $d \geq 4$ (resp. $d \geq 6$, resp. $d \geq 10$). Indeed we have

$$\mathrm{codim}_M(T_{12}) = \binom{d+3}{3} - \left(\binom{d+2}{3} + \binom{d}{2} + 2\right) = 2d-1.$$

One note that $T_{12}\cap T_{23}\cap T_{31} \neq \emptyset$ since the Fermat surface $z_0^d + z_1^d + z_2^d + z_3^d = 0$ belongs to it. Hence, for any irreducible component T of $T_{12}\cap T_{23}$ (resp. $T_{12}\cap T_{23}\cap T_{31}$), $\mathrm{codim}_M(T) \leq 2(2d-1)$ (resp. $\mathrm{codim}_M(T) \leq 3(2d-1)$). By Theorem 0.8 (1) it suffices to check $\binom{d+2}{2} - 5$ is greater than $2d-1$ (resp. $2(2d-1)$, resp. $3(2d-1)$) if $d \geq 4$ (resp. $d \geq 6$, resp. $d \geq 10$). This completes the proof of Theorem 6.3. □

References

[AS1] M. Asakura and S. Saito, *Generalized Jacobian rings for open complete intersections*, Math. Nachr. **279** (2006) 5–37.

[AS2] M. Asakura and S. Saito, *Noether-Lefschetz locus for Beilinson-Hodge cycles I*, Math. Z., **252** (2006), 251–273

[Bl] S. Bloch, *Algebraic cycles and higher K-theory*, Advances in Math., **61** (1986), 267–304

[CGGH] J. Carlson, M. Green, P. Griffiths, J. Harris, *Infinitesimal variations of Hodge structure (I)*, Composit. Math., **50** (1983), 109–205

[D] P. Deligne, *Théorie de Hodge II*, Publ. Math. IHES, **40** (1972), 5–57

[EV] H. Esnault and E. Viehweg, *Deligne-Beilinson cohomology*, in: Beilinson's Conjectures on Special Values of L-functions (M. Rapoport, N. Schappacher and P. Schneider, ed), Perspectives in Math., **4**, Academic Press

[G1] M. Green, *A new proof of the explicit Noether-Lefschetz theorem*, J. Differential Geometry, **27** (1988), 155-159

[G2] M. Green, *Components of maximal dimension in the Noether-Lefschetz locus*, J. Differential Geometry, **29** (1989), 295-302

[GH] P. Griffiths and J. Harris, Principles of algebraic geometry, John Wiley and Sons, NY (1978)

[J] U.Jannsen, *Deligne homology, Hodge-D-conjecture, and Motives*, in: Beilinson's Conjectures on Special Values of L-functions (M. Rapoport, N. Schappacher and P. Schneider, ed), Perspectives in Math., **4**, Academic Press

[Ot] A. Otwinowska, *Composantes de dimension maximale d'un analogue du lieu de Noether-Lefschetz*, Compositio Math. 131 (2002), no. 1, 31–50.

[Sch] P. Schneider, *Introduction to the Beilinson conjectures*, in: Beilinson's Conjectures on Special Values of L-functions (M. Rapoport, N. Schappacher and P. Schneider, ed), Perspectives in Math., **4**, Academic Press

[V] C. Voisin, *Une précision concernant le théorem de Noether*, Math. Ann., **280** (1988), 605-611