Motives with modulus

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(AMS Summer Institute 2015)

Review of Voevodsky's theory

k: perfect field

$\mathbf{Sm}$: category of smooth schemes$/k$

$\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$: $\otimes$-triangulated category

with covariant functor

$\mathbf{Sm} \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} : X \to M(X)$

Fundamental formula:

$\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(M(Y), M(X)[i]) = H^i_{\mathrm{Nis}}(Y, C_*(X))$

$C_*(X)$: Suslin complex of X

Remark

Formula + Moving lemmas imply

comparison with Suslin homology,

Bloch's higher Chow groups

Construction of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$

Cor: **additive $\otimes$-category**

Same objects as Sm

$\mathrm{Cor}(X,Y) =$ **free abelian group gen'd by**

$Z \subset X \times Y$ **integral finite surjective**$/X$

$\mathrm{Sm}(X,Y) \hookrightarrow \mathrm{Cor}(X,Y)$ **by graphs**

$\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} =$ **pseudo-abelian envelope of**

$$K^b(\mathbf{Cor})/\langle \mathbf{MV} + \mathbf{HI} \rangle$$

MV $= \langle\ [U \times_X V] \to [U] \oplus [V] \to [X]\ \rangle$

(U,V) **elementary Nisnevich cover of** X

HI $= \langle\ [X \times \mathbf{A}^1] \to [X]\ \rangle$

$\mathrm{Cor} \to \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} : X \to M(X) = [X]$

Suslin complex (cubical version)

$\square = \mathbf{P}^1 - \{1\}$ **with faces** $0, \infty \in \mathbf{P}^1$

$\tilde{C}_n(X)(S) = \mathbf{Cor}(S \times \square^n, X)$ **(**$S \in \mathbf{Sm}$**)**

$\tilde{C}_n(X)$ **sheaf on** $\mathrm{Sm}_{\text{ét}}$

$n \to \tilde{C}_n(X)$ **cubical object of sheaves**

Definition

$$C_*(X) = \tilde{C}_*(X)/\tilde{C}_*(X)_{deg}$$

complex of sheaves on $\mathrm{Sm}_{\text{ét}}$

$H_i^S(X) := H_i(C_*(X)(k))$ **Suslin homology**

Definition **Category** $\mathbf{MCor}$

Objects: *modulus pair* $\mathscr{X} = (\overline{X}, X_\infty)$

(i) $\overline{X}$ **integral separated** k**-scheme**

(ii) X_∞ **effective Cartier divisor on** $\overline{X}$

(iii) $\mathscr{X}^o := \overline{X} - X_\infty \in \mathbf{Sm}$

$\mathrm{MCor}(\mathscr{X}, \mathscr{Y}) \subset \mathrm{Cor}(\mathscr{X}^o, \mathscr{Y}^o)$ **subgroup**

generated by $Z \subset \mathscr{X}^o \times \mathscr{Y}^o$ **such that**

(i) closure $\overline{Z} \subset \overline{X} \times \overline{Y}$ **proper/**$\overline{X}$

(ii) $p_Z^*(X_\infty \times \overline{Y}) \geq p_Z^*(\overline{X} \times Y_\infty)$

$\overline{Z}^N$ **normalization of** $\overline{Z}$

$p_Z : \overline{Z}^N \to \overline{X} \times \overline{Y}$ **projection**

MCor **is an additive** $\otimes$**-category:**

$$(\overline{X}, X_\infty) \times (\overline{Y}, Y_\infty) = (\overline{X} \times \overline{Y}, X_\infty \times \overline{Y} + \overline{X} \times Y_\infty)$$

Suslin complex with modulus

$$\overline{\square} = (\mathbf{P}^1, 1) \in \mathbf{MCor}$$

$$\tilde{C}_n(\mathscr{X})(\mathscr{Y}) = \mathbf{MCor}(\mathscr{Y} \times \overline{\square}^n, \mathscr{X})$$

$$\mathscr{X}, \mathscr{Y} \in \mathbf{MCor}$$

Definition

$$C_*(\mathscr{X}) = \tilde{C}_*(\mathscr{X}) / \tilde{C}_*(\mathscr{X})_{deg}$$

complex of presheaves on MCor

$$\mathscr{Y} = (\overline{Y}, Y_\infty) \in \mathbf{MCor}$$

$C_*(\mathscr{X})_{\mathscr{Y}}$ **complex of sheaves on** $\overline{Y}_{\text{ét}}$

$$(U \xrightarrow{f} \overline{Y}) \to C_*(\mathscr{X})(\mathscr{Y}_U)$$

$$\mathscr{Y}_U = (U, f^{-1}(Y_\infty))$$

$$H_i^S(\mathscr{X}) := H_i(C_*(\mathscr{X})(\mathrm{Spec}\, k, \emptyset))$$

Suslin homology with modulus

Hope **Construct a diagram**

$$\begin{array}{ccc} \mathbf{MCor} & \xrightarrow{M} & \mathbf{DR}^{\mathsf{eff}}_{\mathsf{gm}} \\ \downarrow \omega & & \downarrow \omega^{\mathsf{eff}}_{\mathsf{gm}} \\ \mathbf{Cor} & \xrightarrow{M} & \mathbf{DM}^{\mathsf{eff}}_{\mathsf{gm}} \end{array}$$

$$\omega(\mathscr{X}) = \overline{X} - X_\infty$$
$$\mathscr{X} = (\overline{X}, X_\infty)$$

$\mathbf{DR}^{\mathsf{eff}}_{\mathsf{gm}}$: **$\otimes$-triangulated category**

with covariant functor

$$\mathbf{MCor} \to \mathbf{DR}^{\mathsf{eff}}_{\mathsf{gm}} : \mathscr{X} \to M(\mathscr{X})$$

<u>Fundamental formula</u> :

$$\mathbf{DR}^{\mathsf{eff}}_{\mathsf{gm}}(M(\mathscr{Y}), M(\mathscr{X})[i]) = H^i_{\mathsf{Nis}}(\overline{Y}, C_*(\mathscr{X})_{\mathscr{Y}})$$

for $\mathscr{X} = (\overline{X}, X_\infty)$ **<u>proper</u> and** $\mathscr{Y} = (\overline{Y}, Y_\infty)$

Main result **Possible replacing**

$C_*(\mathscr{X})$ **by** $RC_*(\mathscr{X})$ **derived version**

Motivations

$\mathrm{DM}_{\mathsf{gm}}^{\mathsf{eff}}$ **based on** $\mathbb{A}^1$**-invariance**

$\exists$ **invariants not captured by** $\mathrm{DM}_{\mathsf{gm}}^{\mathsf{eff}}$

k , Ω_k^i , $\mathrm{Pic}(\overline{X}, X_\infty)$, $H_i^S(\mathscr{X})$

additive higher Chow groups
(Bloch-Esnault/Park)

higher Chow groups with modulus
(Binda-Kerz-S)

Wild ramification is not $\mathbb{A}^1$-invariant

Theorem **(Kerz-S)**

k **finite,** $\mathrm{ch}(k) \neq 2$

$\mathscr{X} = (\overline{X}, X_\infty) \in \mathbf{MCor}$

$$H_0^S(\mathscr{X}) \xrightarrow{\sim} \pi_1^{ab}(\mathscr{X})$$

$\pi_1^{ab}(\mathscr{X})$ **classifies coverings of** $\mathscr{X}^o$

with ramification bounded by X_∞

Definition

$\mathbf{DR}^{\mathsf{eff}}_{\mathsf{gm}}$ = **pseudo-abelian envelope of**

$$K^b(\mathbf{MCor})/\langle \mathbf{MV} + \mathbf{CI} \rangle$$

MV $= \langle\ [\mathscr{X}_{U \times_{\overline{X}} V}] \to [\mathscr{X}_U] \oplus [\mathscr{X}_V] \to [\mathscr{X}]\ \rangle$

$\mathscr{X} = (\overline{X}, X_\infty) \in \mathbf{MCor}$

(U, V) **elementary Nisnevich cover of** $\overline{X}$

$\mathscr{X}_U = (U, f^{-1}(X_\infty))$ **for étale** $f : U \to \overline{X}$

CI $= \langle\ [\mathscr{X} \times \overline{\square}] \to [\mathscr{X}]\ \rangle \qquad \overline{\square} = (\mathbf{P}^1, 1)$

$M : \mathbf{MCor} \to \mathbf{DR}^{\mathsf{eff}}_{\mathsf{gm}} : \mathscr{X} \to M(\mathscr{X}) = [\mathscr{X}]$

$\omega : \mathbf{DR}^{\mathsf{eff}}_{\mathsf{gm}} \to \mathbf{DM}^{\mathsf{eff}}_{\mathsf{gm}} \quad M(\mathscr{X}) \mapsto M(\overline{X} - X_\infty)$

$\otimes$**-triangulated functor**

Conjecture $\mathscr{X}$ **proper**

$$\mathrm{DR}^{\mathrm{eff}}_{\mathrm{gm}}(M(\mathscr{Y}), M(\mathscr{X})[i]) = H^i_{\mathrm{Nis}}(\overline{Y}, C_*(\mathscr{X})_{\mathscr{Y}})$$

Sheaf theory on $\mathbf{MCor}$

$\mathbf{MPST}$: **category of additive presheaves of abelian groups on** $\mathbf{MCor}$

Want to define $\mathbf{MNST} \subset \mathbf{MPST}$ **full subcategory of "Ninevich sheaves"**

Review of sheaf theory on $\mathbf{Cor}$

$\mathbf{PST}$: **category of additive presheaves of abelian groups on** $\mathbf{Cor}$

$\mathbf{NST} := \{F \in \mathbf{PST} \mid \gamma^* F$ **is sheaf on** $\mathbf{Sm}_{\mathrm{Nis}}\}$

$\gamma : \mathbf{Sm} \to \mathbf{Cor}$ **graph functor**

MSm and sheaves on MCor

Category MSm

Same objects as MCor

$\mathrm{MSm}(\mathscr{Y}, \mathscr{X}) =$

$\{f : \overline{Y} \to \overline{X} \mid k$-**morphisms** $f^*(X_\infty) \leq Y_\infty\}$

$\mathscr{X} = (\overline{X}, X_\infty)$, $\mathscr{Y} = (\overline{Y}, Y_\infty)$

$f \in \mathrm{MSm}(\mathscr{Y}, \mathscr{X})$ **Nisnevich cover if**

(i) $f : \overline{Y} \to \overline{X}$ **is Nisnevich cover**

(ii) $Y_\infty = f^*(X_\infty)$

$\Rightarrow$ $\mathrm{MSm}_{\mathsf{Nis}}$ **(big site),** $\mathscr{X}_{\mathsf{Nis}}$ **(small site)**

Definition $\mathrm{MNST} :=$

$\{F \in \mathrm{MPST} \mid \gamma^* F$ **is sheaf on** $\mathrm{MSm}_{\mathsf{Nis}}\}$

$\gamma : \mathrm{MSm} \to \mathrm{MCor}$ **graph functor**

Theorem

(1) $\mathbf{MNST} \hookrightarrow \mathbf{MPST}$ **has exact left adjoint**

$\Rightarrow$ $\mathbf{MNST}$ **is Grothendieck abelian**

(2) $\mathrm{Ext}^i_{\mathbf{MNST}}(\mathbb{Z}_{\mathrm{tr}}(\mathscr{X}), F) = H^i_{\mathrm{Nis}}(\overline{X}, F_{\mathscr{X}})$

$\mathscr{X} = (\overline{X}, X_\infty) \in \mathbf{MCor}$ **and** $F \in \mathbf{MNST}$

$\mathbb{Z}_{\mathrm{tr}}(\mathscr{X}) \in \mathbf{MPST}$ **represented by** $\mathscr{X} \in \mathbf{MCor}$

$F_{\mathscr{X}}$ **sheaf on** $\overline{X}_{\mathrm{Nis}}$ **defined by**

$(U \xrightarrow{f} \overline{X}) \to F(\mathscr{X}_U)$ $\mathscr{X}_U = (U, f^{-1}(X_\infty))$

(3) $\mathscr{U} \to \mathscr{X}$ **Nisnevich cover in** $\mathbf{MSm}$

$\cdots \to \mathbb{Z}_{\mathrm{tr}}(\mathscr{U} \times_{\mathscr{X}} \mathscr{U}) \to \mathbb{Z}_{\mathrm{tr}}(\mathscr{U}) \to \mathbb{Z}_{\mathrm{tr}}(\mathscr{X}) \to 0$

is exact in $\mathbf{MNST}$

Key ingredient of proof

Variant of Raynaud-Gruson platification

Theorem $\overline{X}, Y$ **integral separated**$/k$**,**

$X \subset \overline{X}$ **normal open dense**

$Z \subset X \times Y$ **integral finite surjective**$/X$

$\overline{Z} \subset \overline{X} \times Y$ **proper** $/\overline{X}$

$\Rightarrow \exists \overline{X}' \to \overline{X}$ **proper birational, isom**$/X$ **s.t.**

$\overline{Z}' \subset \overline{X}' \times Y$ **finite** $/\overline{X}'$

Definition $\overline{\square} = (\mathbf{P}^1, 1)$

$$\mathbf{DR}^{\text{eff}} = D(\mathbf{MNST})/\langle \mathbb{Z}_{\text{tr}}(\mathscr{X} \times \overline{\square}) \to \mathbb{Z}_{\text{tr}}(\mathscr{X})\rangle$$

Proposition 1 **Functor**

$K^b(\mathbf{MCor}) \to D(\mathbf{MNST})$ $[\mathscr{X}] \mapsto \mathbb{Z}_{\text{tr}}(\mathscr{X})$

induces fully faithful functor

$$\mathbf{DR}^{\text{eff}}_{\text{gm}} \to \mathbf{DR}^{\text{eff}}$$

$K^b(\mathbf{MCor}) \to D(\mathbf{MPST})$ **fully faithful**

$D(\mathbf{MPST})/I_{\text{Nis}} \xrightarrow{\sim} D(\mathbf{MNST})$

$I_{\text{Nis}} = \{$**Nis locally acyclic complexes**$\}$

Key point: I_{Nis} **generated by**

$\mathbb{Z}_{\text{tr}}(\mathscr{U} \times_{\mathscr{X}} \mathscr{V}) \to \mathbb{Z}_{\text{tr}}(\mathscr{U}) \oplus \mathbb{Z}_{\text{tr}}(\mathscr{V}) \to \mathbb{Z}_{\text{tr}}(\mathscr{X})$

$(\mathscr{U}, \mathscr{V})/\mathscr{X}$ **elementary Nisnevich cover**

Proposition 2 **The localization**

$$\pi : D(\mathbf{MNST}) \to \mathbf{DR}^{\mathrm{eff}}$$

has fully faithful right adjoint j **s.t.**

(i) Image$(j) = \{\overline{\square}$**-local complexes** $K\}$

$H^i_{\mathrm{Nis}}(\mathscr{X}, K) \xrightarrow{\sim} H^i_{\mathrm{Nis}}(\mathscr{X} \times \overline{\square}, K)\ \forall \mathscr{X} \in \mathbf{MCor}$

(ii) $j\pi = RC_* : D(\mathbf{MNST}) \to D(\mathbf{MNST})$

$RC_*(K) = \underline{\mathrm{Hom}}_{D(\mathbf{MNST})}(\mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{\bullet})_{deg}, K)$

$\mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{\bullet})_{deg}$ **degenerate part of**

cocubical object in MNST: $n \to \mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{n})$

Upshot $\mathscr{X}, \mathscr{Y} \in \mathbf{MCor}$, $\mathscr{Y} = (\overline{Y}, Y_\infty)$

$\mathbf{DR}^{\mathrm{eff}}_{\mathrm{gm}}(M(\mathscr{Y}), M(\mathscr{X})[i]) = H^i_{\mathrm{Nis}}(\overline{Y}, RC_*(\mathscr{X})_{\mathscr{Y}})$

$RC_*(\mathscr{X}) = \underline{\mathrm{Hom}}_{D(\mathbf{MNST})}(\mathbb{Z}_{\mathrm{tr}}(\overline{\square}^{\bullet})_{deg}, \mathbb{Z}_{\mathrm{tr}}(\mathscr{X}))$

Remark ∃natural isomorphism

$$C_*(\mathscr{X}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{D(\mathbf{MPST})}(\mathbb{Z}_{\mathrm{tr}}(\overline{\square}^\bullet)_{deg}, \mathbb{Z}_{\mathrm{tr}}(\mathscr{X}))$$

$\Rightarrow$ **natural map** $C_*(\mathscr{X}) \to RC_*(\mathscr{X})$

$$RC_*(\mathscr{X}) = \underline{\mathrm{Hom}}_{D(\mathbf{MNST})}(\mathbb{Z}_{\mathrm{tr}}(\overline{\square}^\bullet)_{deg}, \mathbb{Z}_{\mathrm{tr}}(\mathscr{X}))$$

"derived Suslin complex of $\mathscr{X}$"

Remark **Proposition 2 follows from Voevodsky's abstract homotopy theory using** *interval structure on* $\overline{\square}$ **coming from "multiplication map"**

$$\mu : \overline{\square} \times \overline{\square} \to \overline{\square}$$

interval structure on $\overline{\square} = (\mathbf{P}^1, 1)$

$\overline{\square} = (\mathbf{P}^1, 1) \cong \mathscr{P} = (\mathbf{P}^1, \infty)$ given by

$$\mathbf{P}^1 \to \mathbf{P}^1; t \mapsto t/(t-1)$$

$\mathscr{P}^o := \mathbf{P}^1 - \infty = \mathbf{A}^1$

$\Gamma \subset \mathbf{A}^1 \times \mathbf{A}^1 \times \mathbf{A}^1$ graph of multiplication

$$\mu : \mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$$

Lemma $\Gamma \in \mathrm{MCor}(\mathscr{P} \times \mathscr{P}, \mathscr{P})$

Proof: $\overline{\Gamma} \subset \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ closure of Γ

Modulus condition for Γ checked by

$$\overline{\Gamma} \cong Bl_{(0\times\infty,\infty\times 0)}(\mathbf{P}^1 \times \mathbf{P}^1)$$

Remark $\overline{\Gamma} \notin \mathrm{MSm}(\mathscr{P} \times \mathscr{P}, \mathscr{P})$

Essential to use MCor

$\mathscr{X}, \mathscr{Y} \in \mathbf{MCor}$, $\mathscr{Y} = (\overline{Y}, Y_\infty)$

$$\mathbf{DR}^{\mathsf{eff}}_{\mathsf{gm}}(M(\mathscr{Y}), M(\mathscr{X})[i]) = H^i_{\mathsf{Nis}}(\overline{Y}, RC_*(\mathscr{X})_{\mathscr{Y}})$$

$$RC_*(\mathscr{X}) = \underline{\mathsf{Hom}}_{D(\mathbf{MNST})}(\mathbb{Z}_{\mathsf{tr}}(\overline{\square}^\bullet)_{deg}, \mathbb{Z}_{\mathsf{tr}}(\mathscr{X}))$$

Conjecture

$C_*(\mathscr{X}) \xrightarrow{\sim} RC_*(\mathscr{X})$ **if** $\mathscr{X}$ **proper**

Conjecture follows from the following

Conjecture $F \in \mathbf{MPST}$

(i) F **is** $\overline{\square}$**-invariant** **(**$F(\mathscr{X}) \xrightarrow{\sim} F(\mathscr{X} \times \overline{\square})$**)**

(ii) F **has** M*-reciprocity*

$\Rightarrow$ F_{Nis} **is** $\overline{\square}$**-local**

($H^i_{\mathsf{Nis}}(\mathscr{X}, F) \xrightarrow{\sim} H^i_{\mathsf{Nis}}(\mathscr{X} \times \overline{\square}, F)$**)**

Compare with Voevodsky's theorem:

$F \in \mathbf{PST}$ $\mathbf{A}^1$**-invariant** $\Rightarrow$ F_{Nis} $\mathbf{A}^1$**-local**

$F \in \mathbf{MPST}$ **has** M*-reciprocity* **if**

$$\varinjlim_{\mathscr{Y} \in \mathbf{Comp}(\mathscr{X})} F(\mathscr{Y}) \xrightarrow{\sim} F(\mathscr{X}) \text{ for } \forall \mathscr{X} \in \mathbf{MCor}$$

$\mathbf{Comp}(\mathscr{X}) = \{j : \mathscr{X} \hookrightarrow \mathscr{Y} \text{ compactifications}\}$

Proposition $\mathscr{X}$ **proper**

$h_i(C_*(\mathscr{X})) \in \mathbf{MPST}$ **has** M**-reciprocity**

Expected Theorem $F \in \mathbf{MPST}$

(i) F **is** $\overline{\square}$**-invariant**

(ii) F **has** M*-reciprocity*

$\Rightarrow$ F_{Nis} **is** $\overline{\square}$**-invariant**

The conclusion is the case $i = 0$ **of**

$$H^i_{\mathsf{Nis}}(\mathscr{X}, K) \xrightarrow{\sim} H^i_{\mathsf{Nis}}(\mathscr{X} \times \overline{\square}, K) \ \forall \mathscr{X} \in \mathbf{MCor}$$

Reciprocity sheaves

A reciprocity PST **is** $F \in \mathrm{PST}$ **satisfying**

reciprocity condition **inspired by**

Rosenlicht-Serre's modulus condition

(Definition skipped)

$\mathrm{Rec} = \{$**Reciprocity** $\mathrm{PST}\} \subset \mathrm{PST}$

- **abelian subcategory closed under subobjects and quotients**
- $\mathrm{HI} := \{\mathrm{A}^1$**-invariant** $\mathrm{PST}\} \subset \mathrm{Rec}$
- $\{$**Commutative algebraic groups**$\} \subset \mathrm{Rec}$

  (e.g. $\mathrm{G}_a \in \mathrm{Rec}$**)**
- **Some of Voevodsky's basic results on** HI **can be extended to** Rec

Theorem

$$F \in \mathbf{Rec} \Rightarrow F_{\mathsf{Nis}} \in \mathbf{Rec}$$

M-reciprocity and Reciprocity

$\omega_! : \mathbf{MPST} \to \mathbf{PST}$ **exact left adjoint of**

$\omega^* : \mathbf{PST} \to \mathbf{MPST}$ $\omega^* F(\mathscr{X}) = F(\mathscr{X}^o)$

Theorem $F \in \mathbf{MPST}$

(i) F **is** $\overline{\square}$**-invariant**

(ii) F **has** M**-reciprocity**

$\Rightarrow \omega_! F \in \mathbf{PST}$ **has** *reciprocity*