Proceedings of the International Congress of Mathematicians
Hyderabad, India, 2010

Cohomological Hasse principle and motivic cohomology of arithmetic schemes

Shuji SAITO

Abstract. In 1985 Kazuya Kato formulated a fascinating framework of conjectures which generalize the Hasse principle for the Brauer group of a global field to the so-called cohomological Hasse principle for an arithmetic scheme X. He defined an invariant $KH_a(X)$ ($a \geq 0$), called the Kato homology of X, that reflects the arithmetic nature of X. As a generalization of the classical Hasse principle, Kato conjectured the vanishing of $KH_a(X) = 0$ for $a > 0$, when X is a proper smooth variety over a finite field, or a regular scheme proper and flat over the ring of integers in a number field or in a local field. The conjecture turns out to play a significant rôle in arithmetic geometry. We will explain recent progress on the conjecture and its implications on finiteness of motivic cohomology, special values of zeta functions, a generalization of higher dimensional class field theory, and a geometric application to quotient singularities.

Mathematics Subject Classification (2000). Primary 14C25; Secondary 19E15.

Keywords. Hasse principle, motivic cohomology, zeta function, higher class field theory

Contents

Introduction

A fundamental fact in number theory is the Hasse principle for the Brauer group of a global field K, which is a global-local principle for a central simple algebra A over K:

$$A \simeq M_n(K) \text{ if and only if } A \otimes_K K_x \simeq M_n(K_x) \text{ for all places } x \text{ of } K,$$

where $M_n(*)$ is the matrix algebra and K_x is the completion of K at x. In 1985 Kazuya Kato [K] formulated a fascinating framework of conjectures which generalizes this fact to higher dimensional *arithmetic schemes,* namely schemes of finite type over a finite field or the ring of integers in a number field or a local field. For an integer $n > 0$ and for an arithmetic scheme X, he defined a collection of $\mathbb{Z}/n\mathbb{Z}$-modules

$$KH_a(X, \mathbb{Z}/n\mathbb{Z}) \quad (a \geq 0)$$

which we call the Kato homology of X. The Hasse principle for the Brauer group of a global field K is equivalent to the vanishing $KH_1(X, \mathbb{Z}/n\mathbb{Z}) = 0$ for all $n > 0$, where $X = \mathrm{Spec}(\mathcal{O}_K)$ with the ring $\mathcal{O}_K$ of integers in K. As a generalization of this fact, he proposed the following conjecture called the cohomological Hasse principle.

Conjecture 0.1. *Let X be either a proper smooth variety over a finite field, or a regular scheme proper flat over the ring of integers in a number field or in a local field. Then*

$$KH_a(X, \mathbb{Z}/n\mathbb{Z}) = 0 \quad \textit{for } a > 0.$$

There is work on the conjecture by Kato [K], Colliot-Thélène [CT] and Jannsen-Saito [JS1], where the vanishing $KH_a(X, \mathbb{Z}/n\mathbb{Z}) = 0$ for small degree a is shown. The first aim of this article is to report on the recent progress on the conjecture, the work of U. Jannsen, M. Kerz and the author, which proves the vanishing in all degrees under suitable conditions. The second aim is to give applications of these results. It turns out that the cohomological Hasse principle plays a significant rôle in arithmetic geometry, in particular in the study of motivic cohomology of arithmetic schemes.

Motivic cohomology is an important object to study in arithmetic geometry. It includes the ideal class group and the unit group of a number field, and the Chow groups of algebraic varieties. It is closely related to zeta-functions of algebraic varieties over a finite field or an algebraic number field. One of the important open problems is the conjecture that motivic cohomology of regular arithmetic schemes is finitely generated, a generalization of the known finiteness results on the ideal class group and the unit group of a number field (Minkowski and Dirichlet), and the group of the rational points on an abelian variety over a number field (Mordell-Weil). There have been only few results on the conjecture except the cases stated above and the one-dimensional case (Quillen). In [JS2] it was found that the Kato homology $KH_a(X, \mathbb{Z}/n\mathbb{Z})$ fills a gap between motivic cohomology with finite coefficient and étale cohomology of X. Thus, thanks to known finiteness results on étale cohomology, the cohomological Hasse principle implies new finiteness results on motivic cohomology.

We will also give other implications. One is a result on special values of the zeta function $\zeta(X,s)$ of a smooth projective variety over a finite field, which expresses

$$\zeta(X,0)^* := \lim_{s\to 0} \zeta(X,s)\cdot(1-q^{-s})$$

by the cardinalities of the torsion subgroups of motivic cohomology groups of X. It may be viewed as a geometric analogue of the analytic class number formula for the Dedekind zeta function of a number field.

Another application is a generalization of the higher dimensional class field theory by Schmidt-Spiess [ScSp] which describes the abelian fundamental group of a smooth variety over a finite field by using its Suslin homology of degree 0. Suslin homology is an algebraic analogue of singular homology for topological spaces and is compared to motivic homology defined by Voevodsky. We generalize the work of Schmidt-Spiess to its higher-degree variant and establish an isomorphism between Suslin homology of higher degree and the dual of étale cohomology.

Finally we give an application to a geometric problem on singularities. A consequence is the vanishing of weight homology groups of the exceptional divisors of desingularizations of quotient singularities.

The paper is organized as follows.

In §1 we give a brief review on motivic cohomology. There are mainly two ways of definition. The first one is due to Voevodsky [V1] who constructed the triangulated category of motives and defined motivic (co)homology as the space of maps in this category. We will not go into details of this construction but we explain another (more concrete) definition of motivic (co)homology given by Bloch's higher Chow group and Suslin's homology.

In §2 we state the finiteness conjecture of motivic cohomology and recall some known results on the conjecture. As a tool to approach the conjecture, we introduce the cycle class map from motivic cohomology to étale cohomology constructed by Bloch [B1] and Geisser-Levine [GL] and K.Sato [Sat2].

In §3 we state the Kato conjectures on the cohomological Hasse principle together with a lemma which affirms that the Kato homology controls the kernel and cokernel of the cycle class map introduced in §2.

In §4 we recall all known results on the Kato conjectures and give a very rough sketch of the proof of the most recent result due to Kerz-Saito [KeS], [Sa3].

In §5 we state some new results on the finiteness conjecture of motivic cohomology as an application of the result of Kerz-Saito.

In §6 we give its application to special values of the zeta function of a smooth projective variety over a finite field.

In §7 we give as another application a higher-degree variant of the higher dimensional class field theory of Schmidt-Spiess [ScSp].

In §8 we explain a geometric application to quotient singularities.

The author is grateful to Prof. J.-L. Colliot-Thélène and Prof. T. Geisser for their helpful comments on the first version of this paper.

1. Motivic cohomology

The purpose of this section is to give a quick review on motivic cohomology. We start with the class number formula for an algebraic number field K:

$$\lim_{s\to 0} \zeta_K(s)\cdot s^{-\rho_0} = -\frac{|Cl(K)|\cdot R_K}{|(\mathcal{O}_K^\times)_{\rm tors}|} \tag{1.1}$$

where $\zeta_K(s)$ is the Dedekind zeta function of K, ρ_0 is the rank of the unit group $\mathcal{O}_K^\times$ of the ring $\mathcal{O}_K$ of integers, $(\mathcal{O}_K^\times)_{\rm tors}$ is the torsion part of $\mathcal{O}_K^\times$ (namely the group of the roots of unity in K), $Cl(K)$ is the ideal class group of $\mathcal{O}_K$, and R_K is Dirichlet's regulator.

The philosophical question arises whether one could view the above formula as an arithmetic index theorem:

$$\boxed{\text{index (analytic invariant)}} = \boxed{\text{characteristic class (e.g. Euler characteristic)}}$$

An answer to the question is given by motivic cohomology

$$H^i_M(X,\mathbb{Z}(r))$$

which is defined for a scheme X (satisfying a reasonable condition) and for integers i and r. Indeed, in case $X=\mathrm{Spec}(\mathcal{O}_K)$ with $\mathcal{O}_K$ as above, we have

$$Cl(K)=H^2_M(X,\mathbb{Z}(1)),\quad \mathcal{O}_K^\times = H^1_M(X,\mathbb{Z}(1)).$$

Motivic cohomology theory may be considered universal cohomology theory in view of the existence of regulator maps to other cohomology theories, defined according to the context where X lives:

$$H^i_M(X,\mathbb{Z}(r)) \rightarrow \begin{array}{l} H^i_B(X,\mathbb{Z}(r)) \text{ (Betti cohomology)} \\ H^i_D(X,\mathbb{Z}(r)) \text{ (Deligne cohomology)} \\ H^i_{\text{ét}}(X,\mathbb{Z}_\ell(r)) \text{ (étale cohomology)} \\ H^i_{crys}(X/W(k)) \text{ (crystalline cohomology)} \\ \cdots\cdots\cdots \end{array}$$

Dirichlet's regulator map that defines R_K in (1.1) can be viewed as a special case of the regulator map to Deligne cohomology.

Another important property of motivic cohomology is its relation to algebraic K-theory via the spectral sequence for smooth X

$$E_2^{p,q} = H^p_M(X,\mathbb{Z}(-\frac{q}{2})) \Rightarrow K_{-p-q}(X) \tag{1.2}$$

which is an algebraic analogue of the Atiyah-Hirzebruch spectral sequence for topological K-theory (see [Gra2] and [Le]).

Here we introduce two kinds of constructions of motivic cohomology. The first one is due to Voevodsky [V1] who constructed $DM(k)$, the triangulated category of motives over a field k. It is a tensor category equipped with a functor

$$M : Sm/k \to DM(k)\ ;\ X \to M(X)$$

where Sm/k is the category of smooth schemes over the field k. Motivic cohomology and homology of $X \in Sm/k$ are then defined as the space of maps in $DM(k)$:

$$H^i_M(X,\mathbb{Z}(r)) = \mathrm{Hom}_{DM(k)}(M(X),\mathbb{Z}(r)[i]),$$

$$H_i^M(X,\mathbb{Z}(r)) = \mathrm{Hom}_{DM(k)}(\mathbb{Z}(r)[i],M(X))$$

respectively, where $\mathbb{Z}(1)$ is a distinguished object in $DM(k)$ called the Tate object. It is invertible for the tensor structure and $\mathbb{Z}(r)$ for $r \in \mathbb{Z}$ is the r-th tensor power of $\mathbb{Z}(1)$. We do not go into details on $DM(k)$.

Another (more concrete) definition of motivic (co)homology is given by

$\mathrm{CH}^r(X,q)$, Bloch's higher Chow group ([B2], [Le])

$H_i^S(X,\mathbb{Z})$, Suslin homology ([SV1], [Sc])

defined for a scheme X of finite type over a field or a Dedekind domain. We note that $\mathrm{CH}^r(X,q)$ for $q = 0$ is the Chow group of algebraic cycles on X of codimension r modulo rational equivalence. We have the following comparison result ([V4], [MVW], Lecture 19):

Theorem 1.1. *For a smooth scheme X over a field, we have natural isomorphisms*

$$H^i_M(X,\mathbb{Z}(r)) \simeq \mathrm{CH}^r(X,2r-i), \quad H_i^M(X,\mathbb{Z}(0)) \simeq H_i^S(X,\mathbb{Z}).$$

Before going to a brief review of the definition of Bloch's higher Chow group and Suslin homology, we first recall the singular homology of a topological space X:

$$H_q(X,\mathbb{Z}) := H_q(s(X,\bullet))$$

where $s(X,\bullet)$ is the singular chain complex:

$$\cdots \to s(X,q) \xrightarrow{\partial} s(X,q-1) \xrightarrow{\partial} \cdots \to s(X,0),$$

$$s(X,q) = \bigoplus_{\Gamma} \mathbb{Z}[\Gamma], \quad \Gamma \text{ ranges over all continuous maps } \Delta^q_{top} \to X.$$

Here

$$\Delta^q_{top} = \Big\{(x_0,x_1,\ldots,x_q) \in \mathbb{R}^{q+1} \ \Big|\ \sum_{0\le i\le q} x_i = 1,\ x_i \ge 0\Big\}$$

is the standard simplex and the boundary map ∂ is the alternating sum of the restrictions to the faces of codimension 1 in Δ^q_{top}.

The definition of Bloch's higher Chow group and Suslin homology is an algebraic analogue of the above construction. Here we assume that X is of finite type over a field k while it is possilbe to treat more general cases (cf. [Le] and [Sc]). The standard simplex is replaced by its algebraic analogue

$$\Delta^q = \mathrm{Spec}\big(k[t_0,\ldots,t_q]/(\sum_{i=0}^{q} t_i - 1)\big)\ ,$$

whose faces are $\Delta^s = \{t_{i_1} = \cdots = t_{i_{q-s}} = 0\} \subset \Delta^q$. We have two kinds of analogues of $s(X,q)$ given by the spaces of algebraic cycles on $X \times \Delta^q$:

$$z^r(X,q) = \bigoplus_{\Gamma\subset X\times\Delta^q} \mathbb{Z}[\Gamma], \quad c_0(X,q) = \bigoplus_{\Xi\subset X\times\Delta^q} \mathbb{Z}[\Xi]$$

where Γ (resp. Ξ) ranges over all integral closed subschemes of $X \times \Delta^q$, which have codimension r and intersect properly all faces $\Delta^s \subset \Delta^q$ (resp. which are finite surjective over Δ^q). One may be tempted to take Γ and Ξ as maps of schemes $f : \Delta^q \to X$ but this does not give a correct answer (such f give rise to algebraic cycles on $X \times \Delta^q$ by taking its graphs but there are not sufficiently many maps of schemes).

These groups fit into the so-called *cycle complexes* (graded homologically)

$$z^r(X,\bullet)\ :\ \cdots \to z^r(X,q) \xrightarrow{\partial} z^r(X,q-1) \xrightarrow{\partial} \cdots \xrightarrow{\partial} z^r(X,0),$$

$$c_0(X,\bullet)\ :\ \cdots \to c_0(X,q) \xrightarrow{\partial} c_0(X,q-1) \xrightarrow{\partial} \cdots \xrightarrow{\partial} c_0(X,0).$$

Bloch's higher Chow group and Suslin homology are defined as the homology groups of these complexes:

$$\mathrm{CH}^r(X,q) := H_q\big(z^r(X,\bullet)\big),$$

$$H_q^S(X,\mathbb{Z}) := H_q\big(c_0(X,\bullet)\big).$$

One may also consider the versions with finite coefficients:

$$\mathrm{CH}^r(X,q;\mathbb{Z}/n\mathbb{Z}) := H_q\big(z^r(X,\bullet)\otimes\mathbb{Z}/n\mathbb{Z}\big),$$

$$H_q^S(X,\mathbb{Z}/n\mathbb{Z}) := H_q\big(c_0(X,\bullet)\otimes\mathbb{Z}/n\mathbb{Z}\big).$$

In what follows, for a regular scheme of finite type over a perfect field or a Dedekind domain, we denote (cf. Theorem 1.1)

$$\begin{aligned} &H^i_M(X,\mathbb{Z}(r)) = \mathrm{CH}^r(X,2r-i),\\ &H^i_M(X,\mathbb{Z}/n\mathbb{Z}(r)) = \mathrm{CH}^r(X,2r-i;\mathbb{Z}/n\mathbb{Z}). \end{aligned} \tag{1.3}$$

We have an exact sequence

$$0 \to H^i_M(X,\mathbb{Z}(r))/n \to H^i_M(X,\mathbb{Z}/n\mathbb{Z}(r)) \to H^{i+1}_M(X,\mathbb{Z}(r))[n] \to 0 \tag{1.4}$$

where $M[n] = \mathrm{Ker}(M \xrightarrow{n} M)$ for an abelian group M.

2. Finiteness conjecture on motivic cohomology

A fundamental question in arithmetic geometry is the following.

Conjecture 2.1. *For a regular scheme X of finite type over $\mathbb{F}_p$ or $\mathbb{Z}$, $H^q_M(X, \mathbb{Z}(r))$ is finitely generated.*

In view of the spectral sequence (1.2), the conjecture would imply that the algebraic K-groups $K_i(X)$ of X are finitely generated, which is the so-called Bass conjecture. The above conjecture is a basis of the conjectures on special values of zeta functions of arithmetic varieties due to Beilinson and Bloch-Kato.

Remark 2.2. For a (not necessarily regular) scheme X of finite type over $\mathbb{F}_p$ or $\mathbb{Z}$, $\mathrm{CH}^r(X, q)$ is conjectured to be finitely generated. Indeed this follows from Conjecture 2.1 by the localization sequence for higher Chow groups.

In §5 we will present new finiteness results on motivic cohomology. Very little had been known about the conjecture except the following results. Let X be a regular scheme of finite type over $\mathbb{F}_p$ or $\mathbb{Z}$.

Theorem 2.3. *$H^q_M(X, \mathbb{Z}(1))$ is finitely generated for all integers q.*

In fact, we have

$$H^q_M(X, \mathbb{Z}(1)) = \mathrm{CH}^1(X, 2-q) = \begin{cases} \mathrm{Pic}(X) & q = 2 \\ \Gamma(X, \mathcal{O}_X^\times) & q = 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore the above theorem is a consequence of the finiteness results on the ideal class group and the unit group for the ring of integers in a number field (Minkowski and Dirichlet), and the Mordell-Weil theorem on the rational points of an abelian variety over a number field.

Theorem 2.4. *If $\dim(X) = 1$, $H^q_M(X, \mathbb{Z}(r))$ is finitely generated up to torsion.*

This follows from the fact that $K_i(X)$ is finitely generated, due to Quillen [Q] (see also [Gra1]), together with a result on the degeneracy of the spectral sequence (1.2) up to torsion ([Le], Theorem 11.7). As for the torsion part of $H^q_M(X, \mathbb{Z}(r))$, one can show that it is finite assuming the Bloch-Kato conjecture stated later in this section (see Theorem 2.7 and [Le], Theorems 14.3 and 14.5).

Theorem 2.5. *$H^{2d}_M(X, \mathbb{Z}(d))$ is finitely generated where $d = \dim(X)$.*

Note that $H^{2d}_M(X, \mathbb{Z}(d))$ coincides with the Chow group $\mathrm{CH}_0(X)$ of zero cycles on X modulo rational equivalence. Theorem 2.5 is a consequence of higher unramified class field theory due to Bloch[B1] and Kato-Saito[KS1]:

Theorem 2.6. *Let X be a regular scheme proper over $\mathbb{F}_p$ or $\mathbb{Z}$. Assume $X(\mathbb{R}) = \emptyset$ for simplicity. Then the higher reciprocity map*

$$\rho_X \ : \ \mathrm{CH}_0(X) \to \pi_1^{ab}(X)$$

is an isomorphism if X is flat over $\mathbb{Z}$, and injective with dense image otherwise.

Here $\pi_1^{ab}(X)$ is the abelian fundamental group of X and the definition of ρ_X will be given in §8. Theorem 2.5 follows from Theorem 2.6 and the finiteness result of $\pi_1^{ab}(X)$ due to Katz-Lang.

A way to approach Conjecture 2.1 is to use the cycle class map. Let X be a regular scheme of finite type over a perfect field or the ring $\mathcal{O}_k$ of integers in a number field or in a local field. Under a technical condition (which is necessary only in the case X is flat over $\mathcal{O}_k$ and n is not invertible in $\mathcal{O}_k$), there is a cycle class map

$$\rho_X : H^i_M(X, \mathbb{Z}/n\mathbb{Z}(r)) \to H^i_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(r)) \tag{2.1}$$

from the motivic cohomology with finite coefficient to the étale cohomology with suitable coefficient (explained below). The constructions of the cycle class map are due to Bloch [B1] and Geisser-Levine [GL] and K.Sato [Sat2]. The target group of the cycle class map varies according to the context: In case n is invertible on X,

$$H^i_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(r)) = H^i_{\text{ét}}(X, \mu_n^{\otimes r}),$$

where μ_n is the étale sheaf of the n-th roots of unity. In case X is smooth over a perfect field k and $n = mp^\nu$ with $p = \text{ch}(k)$ and $(p, m) = 1$,

$$H^i_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(r)) = H^i_{\text{ét}}(X, \mathbb{Z}/m\mathbb{Z}(r)) \oplus H^{i-r}(X, W_\nu\Omega^r_{X,log}), \tag{2.2}$$

where $W_\nu\Omega^r_{X,log}$ is the logarithmic part of the de Rham-Witt sheaf $W_\nu\Omega^r_X$ ([Il1], I 5.7). Finally, in case X is flat over $\mathcal{O}_k$ and and n is not invertible on $\mathcal{O}_k$, $H^i_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(r))$ is the hyper cohomology of a certain object of the derived category of complexes of étale sheaves, which is defined by K. Sato [Sat1] as an étale incarnation of the motivic complex on X with finite coefficient.

We note that the target group of the cycle class map is known to be finite. Thus the injectivity of the map would imply a finiteness result for motivic cohomology of X. Indeed we have the following result due to Suslin-Voevodsky [SV2] and Geisser-Levine [GL] (see also K.Sato [Sat2]).

Theorem 2.7. *Let X be as above. Assume* $\mathbf{(BK)}^r_{X,\ell}$ *(see below) for every prime ℓ dividing n. Then the cycle class map*

$$\rho_X : H^i_M(X, \mathbb{Z}/n\mathbb{Z}(r)) \to H^i_{ét}(X, \mathbb{Z}/n\mathbb{Z}(r))$$

is an isomorphism for $i \leq r$ and injective for $i = r + 1$.

In case X is smooth over a perfect field of characteristic $p > 0$ and $n = p^r$, this is a theorem of Geisser-Levine, which is used in Sato's work for the mixed characteristic case.

Corollary 2.8. *Let X be as above and assume X is of finite type over $\mathbb{F}_p$ or $\mathbb{Z}$. Then $H^i_M(X, \mathbb{Z}/n\mathbb{Z}(r))$ is finite for $i \leq r + 1$.*

We now explain the condition $\mathbf{(BK)}^t_{X,\ell}$. For a field L and a prime ℓ and an integer $t > 0$, we have the Galois symbol map

$$h^t_{L,\ell} : K^M_t(L)/\ell \to H^t(L, \mathbb{Z}/\ell\mathbb{Z}(t))$$

where $H^*(L, \mathbb{Z}/\ell\mathbb{Z}(t)) = H^*_{\text{ét}}(\text{Spec}(L), \mathbb{Z}/\ell\mathbb{Z}(t))$ is the Galois cohomology of L and $K^M_t(L)$ denotes the Milnor K-group of L. It is conjectured that $h^t_{L,\ell}$ is surjective. The conjecture is called the Bloch-Kato conjecture in case $l \neq \text{ch}(L)$ (the case $l = \text{ch}(L)$ is known to hold due to Bloch-Gabber-Kato [BK]). The surjectivity of $h^t_{L,\ell}$ is known if $t = 1$ (the Kummer theory) or $t = 2$ (Merkurjev-Suslin [MS]) or $\ell = 2$ (Voevodsky [V1]). Recently a proof of the conjecture has been announced by Rost and Voevodsky (see [SJ] and [V2], and [HW], [V3], [We1] and [We2] for details).

For a scheme X, we introduce the condition:

$\mathbf{(BK)}^t_{X,\ell}$: $h^t_{L,\ell}$ is surjective for any field L finitely generated over a residue field of X.

3. Cohomological Hasse principle

In this section we discuss the cohomological Hasse principle which generalizes the following theorem of Hasse-Minkowski to higher dimensional arithmetic schemes. It plays an important role in the study of motivic cohomology of arithmetic schemes (see Lemma 3.6 below).

Theorem 3.1. *A quadratic form with rational coefficients*

$$a_1X_1^2 + \cdots a_nX_n^2 \quad (a_1, \dots, a_n \in \mathbb{Q})$$

has a non-trivial zero in $\mathbb{Q}$ if and only if it has in $\mathbb{R}$ and $\mathbb{Q}_p$ for every prime p.

In general, a quadratic form over a field k with $\text{ch}(k) \neq 2$:

$$X^2 - aY^2 - bZ^2 \quad (a, b \in k^\times)$$

has a non-trivial zero in k if and only if

$$h(a) \cup h(b) = 0 \in H^2(k, \mathbb{Z}/2\mathbb{Z})$$

where $h : k^\times/2 \simeq H^1(k, \mathbb{Z}/2\mathbb{Z})$ is the Kummer isomorphism, and

$$\cup : H^1(k, \mathbb{Z}/2\mathbb{Z}) \times H^1(k, \mathbb{Z}/2\mathbb{Z}) \to H^2(k, \mathbb{Z}/2\mathbb{Z})$$

is the cup product. Therefore the case $n = 3$ (which is the most crucial to the proof) of the theorem is equivalent to the injectivity of the restriction map

$$H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}) \to \bigoplus_{p \in P_{\mathbb{Q}}} H^2(\mathbb{Q}_p, \mathbb{Z}/2\mathbb{Z}) \ \oplus\ H^2(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$$

where $P_{\mathbb{Q}}$ is the set of the rational primes. Moreover we have the residue isomorphism for $p \in P_{\mathbb{Q}}$:

$$\partial_p : H^2(\mathbb{Q}_p, \mathbb{Z}/2\mathbb{Z}) \simeq H^1(\mathbb{F}_p, \mathbb{Z}/2\mathbb{Z}), \tag{3.1}$$

and Theorem 3.1 is equivalent to the injectivity of the residue map:

$$H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\partial} \bigoplus_{p \in P_{\mathbb{Q}}} H^1(\mathbb{F}_p, \mathbb{Z}/2\mathbb{Z}) \oplus H^2(\mathbb{R}, \mathbb{Z}/2\mathbb{Z}). \tag{3.2}$$

This fact has been extended to the following.

Theorem 3.2. *(Brauer-Hasse-Noether and Witt) Let X be either* $\mathrm{Spec}(\mathcal{O}_K)$ *with $\mathcal{O}_K$ the ring of integers in a number field or in a local field, or a proper smooth curve over a finite field. Let K be the function field of X. For simplicity, in case K is a number field, we assume that n is odd or that $X(\mathbb{R}) = \emptyset$ (namely K is totally imaginary). Then the residue map*

$$H^2(K, \mathbb{Z}/n\mathbb{Z}(1)) \xrightarrow{\partial} \bigoplus_{x \in X_{(0)}} H^1(\kappa(x), \mathbb{Z}/n\mathbb{Z}) \tag{3.3}$$

is injective, where $X_{(0)}$ is the set of the closed points of X, $\kappa(x)$ is the residue field of $x \in X$, and $\mathbb{Z}/n\mathbb{Z}(1)$ is defined as in (2.2).

We remark that there is a natural isomorphism

$$H^2(K, \mathbb{Z}/n\mathbb{Z}(1)) \simeq Br(K)[n],$$

where $Br(K)$ is the Brauer group of K (the set of equivalence classes of central simple algebras over K endowed with a suitable group structure). Thus Theorem 3.2 in case K is a global field, is equivalent to the Hasse principle for the Brauer group of K, namely the following global-local principle for such an algebra A:

$$A \simeq M_n(K) \Leftrightarrow A \otimes_K K_x \simeq M_n(K_x) \quad (\forall x \in X_{(0)})$$

where $M_n(*)$ is the matrix algebra and K_x is the completion of K at x.

In 1985 K.Kato [K] formulated a fascinating framework of conjectures which generalize Theorem 3.2 to higher dimensional *arithmetic schemes X,* namely a scheme of finite type over a finite field or the ring of integers in a number field or a local field. He defined a complex of abelian groups $KC_\bullet(X, \mathbb{Z}/n\mathbb{Z})$ (now called the Kato complex of X):

$$\cdots \xrightarrow{\partial} \bigoplus_{x \in X_{(a)}} H^{a+1}(x, \mathbb{Z}/n\mathbb{Z}(a)) \xrightarrow{\partial} \bigoplus_{x \in X_{(a-1)}} H^a(x, \mathbb{Z}/n\mathbb{Z}(a-1)) \xrightarrow{\partial} \cdots$$

$$\cdots \xrightarrow{\partial} \bigoplus_{x \in X_{(1)}} H^2(x, \mathbb{Z}/n\mathbb{Z}(1)) \xrightarrow{\partial} \bigoplus_{x \in X_{(0)}} H^1(x, \mathbb{Z}/n\mathbb{Z})$$

Here $H^*(x,\mathbb{Z}/n\mathbb{Z}(a))$ is the Galois cohomology of the residue fields $\kappa(x)$ of x and $\mathbb{Z}/n\mathbb{Z}(a)$ is defined as in (2.2). The term in degree a is the direct sum of the Galois cohomology group for $x \in X_{(a)}$, where

$$X_{(a)} = \{x \in X \mid \dim \overline{\{x\}} = a\},$$

the set of those points of X whose closure in X has dimension a. Note that $x \in X_{(a)}$ if and only if $\mathrm{trdeg}_{\mathbb{F}_p}\kappa(x) = a$ or $\mathrm{trdeg}_{\mathbb{Q}}\kappa(x) = a-1$.

In case X is as in Theorem 3.2, $KC_\bullet(X,\mathbb{Z}/n\mathbb{Z})$ coincides with the complex (3.3), and the assertion of Theorem 3.2 is equivalent to the vanishing of the first homology group $H_1\big(KC_\bullet(X,\mathbb{Z}/n\mathbb{Z})\big)$.

We define Kato homology of an arithmetic scheme X as

$$KH_a(X,\mathbb{Z}/n\mathbb{Z}) = H_a\big(KC_\bullet(X,\mathbb{Z}/n\mathbb{Z})\big) \quad (a \geq 0). \tag{3.4}$$

We will also use

$$KH_a(X,\mathbb{Q}/\mathbb{Z}) = \varinjlim_n KH_a(X,\mathbb{Z}/n\mathbb{Z}),$$

$$KH_a(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell) = \varinjlim_n KH_a(X,\mathbb{Z}/\ell^n\mathbb{Z}),$$

where ℓ is a prime. Kato notices that Theorem 3.2 admits the following conjectural generalization.

Conjecture 3.3. *Let X be a proper smooth variety over a finite field. Then*

$$KH_a(X,\mathbb{Z}/n\mathbb{Z}) = 0 \quad (\forall a > 0).$$

We remark that Geisser [Ge2] defined Kato homology with integral coefficient and studied an integral version of Conjecture 3.3.

Conjecture 3.4. *Let X be a regular scheme proper flat over the ring $\mathcal{O}_k$ of integers in a number field. Assume*

(∗) *either n is odd or k is totally imaginary.*

Then

$$KH_a(X,\mathbb{Z}/n\mathbb{Z}) = 0 \quad (\forall a > 0).$$

We note that the assumption (∗) may be removed by modifying $KH_a(X,\mathbb{Q}/\mathbb{Z})$ (see [JS1] Conjecture C on page 482).

Conjecture 3.5. *Let X be a regular scheme proper and flat over* $\mathrm{Spec}(\mathcal{O}_k)$ *where $\mathcal{O}_k$ is the ring of integers in a local field. Then*

$$KH_a(X,\mathbb{Z}/n\mathbb{Z}) = 0 \quad \textit{for } a \geq 0.$$

The relationship of Kato homology of an arithmetic scheme to its motivic cohomology is explained in the following lemma (see [JS2], Lemma 6.2).

Lemma 3.6. *Let X be a connected regular scheme of finite type over a finite field or the ring $\mathcal{O}_k$ of integers in a number field or of a local field with $d = \dim(X)$. For an integer $i \geq 0$, assume $\mathbf{(BK)}^t_{X,\ell}$ with $t = 2d - i + 1$ (see §2). Then the cycle class map* (2.1)

$$\rho_X : H^i_M(X, \mathbb{Z}/n\mathbb{Z}(r)) \to H^i_{ét}(X, \mathbb{Z}/n\mathbb{Z}(r))$$

is an isomorphism for $r > d := \dim(X)$, and there is an exact sequence

$$KH_{2d-i+2}(X, \mathbb{Z}/n\mathbb{Z}) \to H^i_M(X, \mathbb{Z}/n\mathbb{Z}(d)) \xrightarrow{\rho_X} H^i_{ét}(X, \mathbb{Z}/n\mathbb{Z}(d)) \to KH_{2d-i+1}(X, \mathbb{Z}/n\mathbb{Z}).$$

4. Results on Cohomological Hasse principle

In this section we state the known results on the Kato conjectures 3.3, 3.4 and 3.5. Let X be as in the conjectures. As explained, the Kato conjectures in case $\dim(X) = 1$ rephrase the classical fundamental facts on the Brauer group of a global field and a local field.

Kato [K] proved Conjectures 3.3, 3.4, and 3.5 in case $\dim(X) = 2$. He deduced it from higher class field theory for X proved in [KS2] and [Sa1]. For X of dimension 2 over a finite field, the vanishing of $KH_2(X, \mathbb{Z}/n\mathbb{Z})$ in Conjecture 3.3 had been earlier established in [CTSS] (prime-to-p-part), and by M. Gros [Gr] for the p-part.

The first result after [K] is the following:

Theorem 4.1. *(Saito [Sa2]) Let X be a smooth projective 3-fold over a finite field F. Then $KH_3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ for any prime $\ell \neq \mathrm{ch}(F)$.*

This result was immediately generalized to the following:

Theorem 4.2. *(Colliot-Thélène [CT], Suwa [Sw]) Let X be a smooth projective variety over a finite field. Then*

$$KH_a(X, \mathbb{Q}/\mathbb{Z}) = 0 \quad \text{for } 0 < a \leq 3$$

[CT] handled the prime-to-p part where $p = \mathrm{ch}(F)$, and Suwa [Sw] later adapted the technique of [CT] to handle the p-part. A tool in [Sa2] is a class field theory of surfaces over local fields, while the technique in [CT] is global and different from that in [Sa2].

The arithmetic version of the above theorem was established in the following:

Theorem 4.3. *(Jannsen-Saito [JS1]) Let X be a regular projective flat scheme over $S = \mathrm{Spec}(\mathcal{O}_k)$ where k is a number field or a local field. Fix a prime ℓ. Assume that for any closed point $v \in S$, the reduced part of $X_v = X \times_S v$ is* *a divisor with simple normal crossings on X and that X_v is reduced if $v|\ell$. For simplicity, if k is a number field, we assume that $\ell \neq 2$ or k is totally imaginary. Then*

$$KH_a(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0 \quad \textit{for } 0 < a \leq 3$$

The following theorem is a direct consequence of [J], Theorem 0.5. It reduces Conjecture 3.4 to Conjecture 3.5 for Kato homology with $\mathbb{Q}/\mathbb{Z}$-coefficient.

Theorem 4.4. *(Jannsen [J]) Let X be a regular projective flat scheme over $S = \mathrm{Spec}(\mathcal{O}_k)$ where k is a number field. For each closed point $v \in S$, let $S_v = \mathrm{Spec}(\mathcal{O}_{k_v})$ where k_v is the completion of k at v and write $X_{S_v} = X \times_S S_v$. Fix a prime ℓ and assume for simplicity that $\ell \neq 2$ or k is totally imaginary. Then we have a natural isomorphism*

$$KH_a(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \simeq \bigoplus_{v \in S_{(0)}} KH_a(X_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \quad \textit{for } a > 0.$$

In the next theorem, Conjecture 3.3 is shown assuming resolution of singularities.

Theorem 4.5. *(Jannsen [J], Jannsen-Saito [JS2]) Let X be a projective smooth variety of dimension d over a finite field. Let $t \geq 1$ be an integer. Then we have*

$$KH_a(X, \mathbb{Q}/\mathbb{Z}) = 0 \quad \textit{for } 0 < a \leq t$$

if either $t \leq 4$ or $\mathbf{(RS)}_d$, or $\mathbf{(RES)}_{t-2}$ (see below) holds.

$\mathbf{(RS)}_d$: For any X integral and proper of dimension$\leq d$ over F, there exists a proper birational morphism $\pi : X' \to X$ such that X' is smooth over F. For any U smooth of dimension$\leq d$ over F, there is an open immersion $U \hookrightarrow X$ such that X is projective smooth over F with $X - U$ a divisor with simple normal crossings on X.

$\mathbf{(RES)}_t$: For any smooth projective variety X over F, any divisor Y with simple normal crossings on X with $U = X - Y$, and any integral closed subscheme $W \subset X$ of dimension$\leq t$ such that $W \cap U$ is regular, there exists a birational proper map $\pi : X' \to X$ such that X' is projective smooth over F and $\pi^{-1}(U) \simeq U$, and that $Y' = X' - \pi^{-1}(U)$ is a divisor with simple normal crossings on X', and that the proper transform of W in X' is regular and intersects transversally with Y'.

We note that a proof of $\mathbf{(RES)_2}$ is given in [CJS] based on an idea of Hironaka, which enables us to obtain the unconditional vanishing of Kato homology in degree $a \leq 4$.

The above approach has been improved to remove the assumptions $\mathbf{(RS)}_d$ and $\mathbf{(RES)}_t$ on resolution of singularities, at least if we are restricted to the prime-to-$\mathrm{ch}(F)$ part:

Theorem 4.6. *(Kerz-Saito [KeS], [Sa3]) Let X be a proper smooth variety over a finite field F. For a prime $\ell \neq \mathrm{ch}(F)$, we have $KH_a(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ for $a > 0$.*

A key to the proof is the following refinement of de Jong's alteration theorem due to Gabber (see [Il2]).

Theorem 4.7. *(Gabber) Let F be a perfect field and X be a variety over F. Let $W \subset X$ be a proper closed subscheme. Let ℓ be a prime different from $\mathrm{ch}(F)$. Then there exists a projective morphism $\pi : X' \to X$ such that*

- *X' is smooth over F and the reduced part of $\pi^{-1}(W)$ is a divisor with simple normal crossings on X.*
- *π is generically finite of degree prime to ℓ,*

The same technique proves the following arithmetic version as well:

Theorem 4.8. *(Kerz-Saito [KeS], [Sa3]) Let X be a regular scheme, proper flat scheme over a henselian discrete valuation ring with finite residue field F. Then, for every prime $\ell \neq \mathrm{ch}(F)$, we have $KH_a(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ for $a \geq 0$.*

We remark that one can prove the above results with $\mathbb{Z}/\ell^n\mathbb{Z}$-coefficient instead of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-coefficient by using the Bloch-Kato conjecture:

Theorem 4.9. *Let X and ℓ be as in Theorem 4.6 or Theorem 4.8. Assume $\mathbf{(BK)}^t_{X,\ell}$ holds. Then we have $KH_a(X, \mathbb{Z}/\ell^n\mathbb{Z}) = 0$ for $0 < a \leq t$.*

In the rest of this section we give a very rough sketch of the proof of Theorem 4.6. We fix a finite field F and work in the category $\mathcal{C}$ of schemes separated of finite type over F. We first recall the following:

Definition 4.10. Let $\mathcal{C}_*$ be the category with the same objects as $\mathcal{C}$, but morphisms are the proper maps in $\mathcal{C}$. Let Ab be the category of abelian group. A homology theory $H = \{H_a\}_{a\in\mathbb{Z}}$ on $\mathcal{C}$ is a sequence of covariant functors:

$$H_a(-) :\ \mathcal{C}_* \to Ab$$

satisfying the following conditions:

(i) For each open immersion $j : V \hookrightarrow X$ in $\mathcal{C}$, there is a map $j^* : H_a(X) \to H_a(V)$, associated to j in a functorial way.

(ii) If $i : Y \hookrightarrow X$ is a closed immersion in X, with open complement $j : V \hookrightarrow X$, there is a long exact sequence (called localization sequence)

$$\cdots \xrightarrow{\partial} H_a(Y) \xrightarrow{i_*} H_a(X) \xrightarrow{j^*} H_a(V) \xrightarrow{\partial} H_{a-1}(Y) \longrightarrow \cdots.$$

(The maps ∂ are called the connecting morphisms.) This sequence is functorial with respect to proper maps or open immersions, in an obvious way.

It is an easy exercise to check that Kato homology (3.4)

$$KH(-,\Lambda)=\{KH_a(-,\Lambda)\}_{a\in\mathbb{Z}}\quad(\Lambda=\mathbb{Z}/n\mathbb{Z},\ \mathbb{Q}/\mathbb{Z},\ \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

provides us with a homology theory on $\mathcal{C}$. Another homology theory which we use is the étale homology theory $H^{\text{ét}}(-,\Lambda)$ on $\mathcal{C}$ given by

$$H_a^{\text{ét}}(X,\Lambda):=H^{-a}(X_{\text{ét}},Rf^!\Lambda)\quad\text{for } f:X\to\operatorname{Spec}(F)\text{ in }\mathcal{C}.$$

where $Rf^!$ is the right adjoint of $Rf_!$ defined in [SGA 4], XVIII, 3.1.4. Using a result of [JSS], we can identify $KH_a(X,\Lambda)$ with an E^2-term of the niveau spectral sequence to get the following map as an edge homomorphism

$$\epsilon_a:\ H_{a-1}^{\text{ét}}(X,\Lambda)\to KH_a(X,\Lambda)\quad\text{for each } a\geq 1\text{ and } X\in\mathcal{C}.$$

This gives rise to a natural transformation of homology theories

$$\epsilon:\ H^{\text{ét}}(-,\Lambda)[-1]\to KH(-,\Lambda).$$

We now keep our attention to the above homology theories in case $\Lambda=\mathbb{Q}_\ell/\mathbb{Z}_\ell$ with $\ell\neq\operatorname{ch}(F)$ and in this case we simply write $KH_a(X)$ and $H_a^{\text{ét}}(X)$. For each integer $d>0$ consider the following condition:

$\mathbf{KC}(d)$: For any connected $X\in\mathcal{C}$ with $\dim(X)\leq d$ which is proper and smooth over F we have $KH_a(X)=0$ for $a\geq 1$.

We prove $\mathbf{KC}(d)$ by induction on d. One of the basic ingredients in the proof is a result of Jannsen and Saito [JS2] relying on weight arguments [D] which implies the following:

Claim 4.11. *Assume* $\mathbf{KC}(d-1)$. *Let $X\in\mathcal{C}$ be connected proper smooth over F, and let Y be a divisor with simple normal crossings on X such that one of the irreducible components of Y is ample. Put $U=X-Y$. Then the composite map*

$$\delta_a:\ H_{a-1}^{ét}(U)\xrightarrow{\epsilon_a}KH_a(U)\xrightarrow{\partial}KH_{a-1}(Y)$$

is injective for $1\leq a\leq d$ and surjective for $a\geq 2$.

Now we sketch a proof of $\mathbf{KC}(d-1)\Longrightarrow\mathbf{KC}(d)$. Let $X\in\mathcal{C}$ be a connected proper smooth over F with $\dim(X)=d$. Fix an element $\alpha\in KH_a(X)$ for $a\geq 1$. We have to show $\alpha=0$. From the construction of ϵ_a, it is easy to see that there is a dense open subscheme $j:U\to X$ satisfying the condition

$$(*)\qquad j^*(\alpha)\text{ is in the image of }\epsilon_a:H_{a-1}^{\text{ét}}(U)\to KH_a(U).$$

Suppose for the moment that $Y=X-U$ is a divisor with simple normal crossings on X. Then one can use a Bertini argument to find a hypersurface section $H\hookrightarrow X$ such that $Y\cup H$ is a divisor with simple normal crossings. Replacing Y by $Y\cup H$

and U by $U - U \cap H$, the condition $(*)$ is preserved. Consider the commutative diagram

$$\begin{array}{ccccccccc}
KH_{a+1}(U) & \xrightarrow{\partial} & KH_a(Y) & \longrightarrow & KH_a(X) & \xrightarrow{j^*} & KH_a(U) & \xrightarrow{\partial} & KH_{a-1}(Y) \\
\uparrow{\scriptstyle \epsilon_{a+1}} & \nearrow{\scriptstyle \delta_{a+1}} & & & & & \uparrow{\scriptstyle \epsilon_a} & \nearrow{\scriptstyle \delta_a} & \\
H_a^{\text{ét}}(U) & & & & & & H_{a-1}^{\text{ét}}(U) & &
\end{array}$$

By the assumption $\mathbf{KC}(d-1)$, Claim 4.11 implies that the map δ_a is injective and the map δ_{a+1} is surjective. A simple diagram chase shows that $\alpha = 0$.

In the general case in which $Y \hookrightarrow X$ is not necessarily a divisor with simple normal crossings we use Theorem 4.7 to find an alteration $f : X' \to X$ of degree prime to ℓ such that $f^{-1}(Y)$ is a divisor with simple normal crossings. We then construct a pullback map

$$f^* : KH_a(X) \to KH_a(X')$$

which allows us to conduct the above argument for $f^*(\alpha) \in KH_a(X')$. This implies $f^*(\alpha) = 0$ and taking the pushforward gives $f_* f^*(\alpha) = \deg(f)\,\alpha = 0$. Since $\deg(f)$ is prime to ℓ we conclude $\alpha = 0$ and therefore we have finished the proof.

The construction of the necessary pullback map on Kato homology, especially in the arithmetic case, and its compatibility with the pullback map on etale homology are the most severe technical difficulties. This problem is solved using Rost's version of intersection theory and the method of deformation to normal cones [R].

5. Application: Finiteness of motivic cohomology

In the following sections we present some applications of the results on the cohomological Hasse principle of §4. The first apllication is on the finiteness conjecture for motivic cohomology.

Theorem 5.1. *Let X be a quasi-projective scheme over either a finite field F or a henselian discrete valuation ring with finite residue field F. Let $n > 0$ be an integer prime to* $\mathrm{ch}(F)$ *and assume* $\mathbf{(BK)}^t_{X,\ell}$ *for all primes $\ell | n$ and integers $t \geq 0$. Then* $\mathrm{CH}^r(X, q; \mathbb{Z}/n\mathbb{Z})$ *is finite for all $r \geq \dim(X)$ and $q \geq 0$.*

Proof When X is regular and projective over the base, the assertion follows from Theorem 4.9 and Lemma 3.6. The general case is reduced to the special case by using the localization sequence for $\mathrm{CH}^r(X, q; \mathbb{Z}/n\mathbb{Z})$ and Gabbers's theorem 4.7 (and its variant for schemes over a discrete valuation ring). For simplicity we only treat the case over a finite field F. We may assume $n = \ell^m$ for a prime $\ell \neq \mathrm{ch}(F)$. We proceed by induction on $\dim(X)$. First we remark that the localization sequence for higher Chow groups implies that for a dense open subscheme $U \subset X$, the finiteness of $\mathrm{CH}^r(X,q;\mathbb{Z}/n\mathbb{Z})$ for all $r \geq \dim(X)$ and q is equivalent to that of $\mathrm{CH}^r(U,q;\mathbb{Z}/n\mathbb{Z})$. Thus it suffices to show the assertion for any smooth variety U over F. If U is an open subscheme of a smooth projective variety X over F, we have already seen that the assertion holds for X and hence for U by the above remark. In general Gabbers's theorem 4.7 implies that there exist an open subscheme V of a smooth projective variety X over F, an open subscheme W of U, and a finite étale morphism $\pi : V \to W$ of degree prime to ℓ. We know that the assertion holds for V so that it holds for W by a standard norm argument. This completes the proof by the above remark. □

We note that the above theorem implies an affirmative result on the Bass conjecture. Let $K'_i(X,\mathbb{Z}/n\mathbb{Z})$ be Quillen's higher K-groups with finite coefficients constructed from the category of coherent sheaves on X (which coincide with the algebraic K-groups with finite coefficients constructed from the category of vector bundles when X is regular).

Corollary 5.2. *Under the assumption of Theorem 5.1, $K'_i(X,\mathbb{Z}/n\mathbb{Z})$ is finite for $i \geq \dim(X) - 2$.*

Proof Theorem 2.7 implies that $\mathrm{CH}^r(X,q;\mathbb{Z}/n\mathbb{Z})$ is finite for $r \leq q+1$. Hence the assertion follows from Theorem 5.1 and the Atiyah-Hirzebruch spectral sequence (see [Le] for its construction in the most general case):

$$E_2^{p,q} = \mathrm{CH}^{-q/2}(X,-p-q;\mathbb{Z}/n\mathbb{Z}) \Rightarrow K'_{-p-q}(X,\mathbb{Z}/n\mathbb{Z})$$

(note $E_2^{p,q}$ can be nonzero only if $q \leq 0$ and $p+q \leq 0$). □

6. Application: Special values of zeta functions

Let X be a smooth projective variety over a finite field F. We consider the zeta function

$$\zeta(X,s) = \prod_{x \in X_{(0)}} \frac{1}{1 - N(x)^{-s}} \quad (s \in \mathbb{C})$$

where $N(x)$ is the cardinality of the residue field $\kappa(x)$ of x. The infinite product converges absolutely in the region $\{s \in \mathbb{C} \mid \Re(s) > \dim(X)\}$ and can be continued to the whole s-plane as a meromorphic function. Indeed the fundamental results of Grothendieck and Deligne imply that

$$\zeta(X,s) = \prod_{0 \leq i \leq 2d} P_X^i(q^{-s})^{(-1)^{i+1}},$$

where $P_X^i(t) \in \mathbb{Z}[t]$, and that for every integer r

$$\zeta(X,r)^* := \lim_{s \to r} \zeta(X,s) \cdot (1 - q^{r-s})^{\rho_r}$$

is a rational number, where $\rho_r = -\mathrm{ord}_{s=r}\zeta(X,s)$. The problem is to express these values in terms of arithmetic invariants associated to X. It has been studied by Milne [Mil] (who used étale cohomology) and Lichtenbaum [Li] (who used (conjectural) étale motivic complexes) and Geisser [Ge1] (who used Weil-étale cohomology). As an application of Theorem 4.9, we get the following new result on the problem.

Theorem 6.1. *Let X be a smooth projective variety over a finite field F. Let $p = \mathrm{ch}(F)$ and $d = \dim(X)$.*

(1) For all integers j, the torsion part $H^j_M(X,\mathbb{Z}(d))_{tors}$ of $H^j_M(X,\mathbb{Z}(d))$ is finite modulo the p-primary torsion subgroup. Moreover, $H^j_M(X,\mathbb{Z}(d))_{tors}$ is finite if $d \le 4$.

(2) We have the equality up to a power of p:

$$\zeta(X,0)^* = \prod_{0\le j\le 2d} |H^j_M(X,\mathbb{Z}(d))_{tors}|^{(-1)^j} \tag{6.1}$$

The equality holds also for the p-part if $d \le 4$.

Remark 6.2. Let $X = \mathrm{Spec}(\mathcal{O}_K)$ where $\mathcal{O}_K$ is the ring of integers in a number field. The formula (6.1) should be compared with the formula

$$\lim_{s\to 0} \zeta(X,s)\cdot s^{-\rho_0} = -\frac{|H^2_M(X,\mathbb{Z}(1))_{\mathrm{tors}}|}{|H^1_M(X,\mathbb{Z}(1))_{\mathrm{tors}}|}\cdot R_K$$

which is obtained by rewriting the class number formula (1.1) using motivic cohomology. Thus (6.1) may be viewed as a geometric analogue of the class number formula. Note that the regulator R_K does not appear in (6.1) since $H^j_M(X,\mathbb{Z}(d))$ is (conjecturally) finite for $j \neq 2d$.

Proof of Theorem: For simplicity we treat only the case $\ell \neq \mathrm{ch}(F)$. Put

$$H^j_M(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) = \varinjlim_n H^j_M(X,\mathbb{Z}/\ell^n\mathbb{Z}(d)),$$

$$H^j_{\text{ét}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) = \varinjlim_n H^j_{\text{ét}}(X,\mathbb{Z}/\ell^n\mathbb{Z}(d)).$$

By Theorem 4.6 and Lemma 3.6 we have an isomorphism

$$H^j_M(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \simeq H^j_{\text{ét}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(d)). \tag{6.2}$$

By (1.4) we have an exact sequence

$$0 \to H^j_M(X,\mathbb{Z}(d))\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell \to H^j_M(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \to H^{j+1}_M(X,\mathbb{Z}(r))\{\ell\} \to 0 \tag{6.3}$$

where $M\{\ell\}$ denotes the ℓ-primary torsion part for an abelian group M. Assuming $j \neq 2d$, one can show using Deligne's proof of the Weil conjecture [D] and a theorem of Gabber [Ga] that $H^j_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))$ is finite and trivial for almost all ℓ (see [CTSS], Theorem 2). Thus (6.2) and (6.3) imply $H^j_M(X, \mathbb{Z}(d)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0$ and we get an isomorphism of finite groups

$$H^{j+1}_M(X, \mathbb{Z}(r))\{\ell\} \simeq H^j_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)). \tag{6.4}$$

This shows the first assertion (1). For the proof of (2), we use the formula

$$\zeta(X,0)^* = \frac{[H^0_{\text{ét}}(X,\mathbb{Z})_{\text{tors}}][H^2_{\text{ét}}(X,\mathbb{Z})_{\text{cotor}}][H^4_{\text{ét}}(X,\mathbb{Z})]\cdots}{[H^1_{\text{ét}}(X,\mathbb{Z})][H^3_{\text{ét}}(X,\mathbb{Z})][H^5_{\text{ét}}(X,\mathbb{Z})]\cdots} \tag{6.5}$$

due to Milne [Mil], Theorem 0.4. Here $H^0_{\text{ét}}(X,\mathbb{Z}) = \mathbb{Z}$, $H^1_{\text{ét}}(X,\mathbb{Z}) = 0$, and $H^j_{\text{ét}}(X,\mathbb{Z})$ is finite for $j \geq 3$, and the cotorsion part $H^2_{\text{ét}}(X,\mathbb{Z})_{\text{cotor}}$ of $H^2_{\text{ét}}(X,\mathbb{Z})$ is finite. By arithmetic Poincaré duality we have

$$H^{2d-i}_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \simeq \text{Hom}(H^{i+1}_{\text{ét}}(X, \mathbb{Z}_\ell), \mathbb{Q}_\ell/\mathbb{Z}_\ell),$$

where $H^{i+1}_{\text{ét}}(X,\mathbb{Z}_\ell) = \varprojlim_n H^{i+1}_{\text{ét}}(X, \mathbb{Z}/\ell^n\mathbb{Z})$, and this group is finite for $i \geq 1$. Thus the desired assertion follows from the following isomorphisms

$$H^j_{\text{ét}}(X,\mathbb{Z}_\ell) \simeq H^j_{\text{ét}}(X,\mathbb{Z})\{\ell\} \quad \text{for } j \geq 3,$$
$$H^2_{\text{ét}}(X,\mathbb{Z}_\ell) \simeq H^2_{\text{ét}}(X,\mathbb{Z})_{\text{cotor}}\{\ell\},$$

which can be easily shown by using the exact sequence of étale sheaves

$$0 \to \mathbb{Z} \xrightarrow{\ell^n} \mathbb{Z} \to \mathbb{Z}/\ell^n\mathbb{Z} \to 0.$$

□

7. Application: Higher class field theory

Another application of Theorem 4.9 is a generalization of the higher dimensional class field theory by Schmidt-Spiess [ScSp] which describes the abelian fundamental group of a smooth scheme over a finite field by using its Suslin homology of degree 0 (see (7.2) below). The generalization is its higher-degree variant and establishes an isomorphism between the Suslin homlogy of higher degree and the dual of étale cohomology (see Theorem 7.2 below).

We start with a brief review of higher dimensional class field theory. Let X be a regular scheme of finite type over $\mathbb{F}_p$ or $\mathbb{Z}$. Higher dimensional class field theory aims at describing all relations among the Frobenius elements

$$\sigma_x \in \pi_1^{ab}(X)$$

associated to closed points of X. Here $\pi_1^{ab}(X)$ is the abelian fundamental group of X, which classifies the abelian finite étale coverings of X. To be more precise, let

$X_{(0)}$ be the set of the closed points of X. For $x \in X_{(0)}$, the residue field $\kappa(x)$ of x is finite. The closed immersion $x \to X$ induces $\rho_x : \pi_1^{ab}(x) \to \pi_1^{ab}(X)$ and $\pi_1^{ab}(x)$ is the absolute Galois group of $\kappa(x)$ which is topologically generated by the q-th power map where $q = |\kappa(x)|$. The Frobenius element $\sigma_x \in \pi_1^{ab}(X)$ is defined as its image under ρ_x. This defines the map

$$\rho_X : Z_0(X) \to \pi_1^{ab}(X) \; ; \; (n_x)_{x \in X_{(0)}} \to \prod_{x \in X_{(0)}} (\sigma_x)^{n_x}$$

where $Z_0(X) = \bigoplus_{x \in X_{(0)}} \mathbb{Z}$ is the group of zero cycles on X. It was shown by Lang that the image of ρ_X is dense in $\pi_1^{ab}(X)$ and the problem is to determine its kernel.

The question was first answered by Kato-Saito [KS2], which used the higher idele class group of X defined as the cohomolgy group of the sheaf of relative Milnor K-group with respect to the Nisnevich topology. Unfortunately, the description of the kernel of ρ_X in this formulation is not direct and does not give a clear answer to the above question except in the case where X is proper over the base. It is the higher unramified class field theory stated as an (almost) isomorphism:

$$\rho_X \; : \; \mathrm{CH}_0(X) \to \pi_1^{ab}(X)$$

(see Theorem 2.6). Recall

$$\mathrm{CH}_0(X) = \mathrm{Coker}\big(\bigoplus_{y \in X_{(1)}} \kappa(y)^\times \xrightarrow{\delta} \bigoplus_{x \in X_{(0)}} \mathbb{Z} \big)$$

where $X_{(1)}$ is the set of the generic points of the integral curves on X and δ is given by taking the divisors of functions on those curves.

An essential improvement has been given by the following theorem due to Schmidt-Spiess [ScSp] and Kerz-Schmidt-Wiesend [W], [KeSc] (see also [Sz]).

Theorem 7.1. *Let X be a connected regular scheme of finite type over $\mathbb{F}_p$ or $\mathbb{Z}$. Let $n > 0$ be an integer prime to the characteristic of the function field of X. Then $\rho_X : Z_0(X) \to \pi_1^{ab}(X)$ induces an isomorphism*

$$\mathrm{Coker}\Big(\bigoplus_{y \in X_{(1)}} \kappa(y)_{\Sigma,n} \xrightarrow{\delta} \bigoplus_{x \in X_{(0)}} \mathbb{Z}/n\mathbb{Z} \Big) \simeq \pi_1^{ab}(X)/n$$

For $y \in X_{(1)}$, $\kappa(y)_{\Sigma,n}$ is a subgroup of $\kappa(y)^\times$ defined as follows. For simplicity we restrict to the case X is over $\mathbb{F}_p$. Let $C \subset X$ be the closure of y in X, $\widetilde{C}$ be its normalization, and $\overline{C}$ be the smooth compactification of $\widetilde{C}$, and put $\Sigma_y = \overline{C} - \widetilde{C}$. Then

$$\kappa(y)_{\Sigma,n} = \big\{ a \in \kappa(y)^\times \; \big| \; a \in \big(\kappa(y)_x^\times\big)^n \quad \text{for all } x \in \Sigma_y \big\}$$

where $\kappa(y)_x$ is the completion of $\kappa(y)$ at x.

In what follows we assume that X is smooth over a finite field. Schmidt-Spiess [ScSp] have established a canonical isomorphism

$$\operatorname{Coker}\Big(\bigoplus_{y\in X_{(1)}} \kappa(y)_{\Sigma,n} \xrightarrow{\delta} \bigoplus_{x\in X_{(0)}} \mathbb{Z}/n\mathbb{Z}\Big) \simeq H_0^S(X,\mathbb{Z}/n\mathbb{Z}). \tag{7.1}$$

(A similar isomorphism was shown by Schmidt [Sc] when X is flat over $\mathbb{Z}$ under a certain tameness condition). Thus Theorem 7.1 can be rephrased as a canonical isomorphism

$$H_0^S(X,\mathbb{Z}/n\mathbb{Z}) \simeq \pi_1^{ab}(X)/n. \tag{7.2}$$

As an application of Theorem 4.9, we can extend this to the following.

Theorem 7.2. ([KeS]) *Let X be a connected smooth scheme over a finite field $\mathbb{F}_q$ and let $n>0$ be an integer prime to $\mathrm{ch}(\mathbb{F}_q)$. Then there exists a canonical isomorphism for all integers $i\geq 0$*

$$H_i^S(X,\mathbb{Z}/n\mathbb{Z}) \simeq H_{ét}^{i+1}(X,\mathbb{Z}/n\mathbb{Z})^* := \operatorname{Hom}\big(H_{ét}^{i+1}(X,\mathbb{Z}/n\mathbb{Z}),\mathbb{Z}/n\mathbb{Z}\big),$$

where $H_i^S(X,\mathbb{Z}/n\mathbb{Z})$ is the Suslin homology defined in §1. In particular $H_i^S(X,\mathbb{Z}/n\mathbb{Z})$ is finite.

The case $i=0$ of Theorem 7.2 is reduced to the isomorphism (7.2) by the natural isomorphism $H^1_{\text{ét}}(X,\mathbb{Z}/n\mathbb{Z})^* \simeq \pi_1^{ab}(X)/n$.

Remark 7.3. Let X be separated of finite type over a finite field F. Assuming resolution of singularities over F, Geisser [Ge3] proved that $H_i^S(X,\mathbb{Z}/n\mathbb{Z})$ is finite for all integers i and n.

8. Application: Resolution of quotient singularities

We fix a field k and assume $\mathrm{ch}(k)=0$. Let $\mathcal{C}$ be the category $\mathcal{C}$ of separated schemes of finite type over k. Let $\mathcal{S}\subset\mathcal{C}$ be the subcategory of smooth projective schemes over k. Fix an abelian group Λ. Based on work of Gillet and Soulé [GS], Jannsen ([J], Theorem 5.9) proved the following.

Theorem 8.1. *There exists a homology theory (cf. Definition 4.10)*

$$H^W(-,\Lambda): \mathcal{C}_* \to \mathcal{A}b$$

such that for all $X\in\mathcal{S}$, we have

$$H_a^W(X,\Lambda) = \begin{cases} \Lambda^{\pi_0(X)} & a=0 \\ 0 & a\neq 0, \end{cases}$$

where $\pi_0(X)$ is the set of connected components of X. We call $H_a^W(X,\Lambda)$ the weight homology group of X with coefficient Λ.

We briefly review the construction of [GS]. To a simplicial object in $\mathcal{S}$:

$$X_\bullet \ : \quad \cdots X_2 \ \begin{matrix} \xrightarrow{\delta_0} \\ \xleftarrow{s_0} \\ \xrightarrow{\delta_1} \\ \xleftarrow{s_1} \\ \xrightarrow{\delta_2} \end{matrix} \ X_1 \ \begin{matrix} \xrightarrow{\delta_0} \\ \xleftarrow{s_0} \\ \xrightarrow{\delta_1} \end{matrix} \ X_0,$$

we associate a chain complex of abelian groups

$$W(X_\bullet, \Lambda): \ \cdots \to \Lambda^{\pi_0(X_n)} \xrightarrow{\partial} \Lambda^{\pi_0(X_{n-1})} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Lambda^{\pi_0(X_0)},$$

where $\partial : \Lambda^{\pi_0(X_n)} \to \Lambda^{\pi_0(X_{n-1})}$ is defined as $\partial = \sum_{a=0}^{n} (-1)^a \partial_a$ with

$$\partial_a : \Lambda^{\pi_0(X_n)} \to \Lambda^{\pi_0(X_{n-1})} \ ; \ (x_i)_{i \in \pi_0(X_n)} \to \Big(\sum_{\delta_a(i)=j} x_i \Big)_{j \in \pi_0(X_{n-1})} .$$

For $X \in \mathcal{C}$ choose an open immersion $j : X \to \overline{X}$ with $\overline{X} \in \mathcal{C}$ proper over k and let $i : Y = \overline{X} - X \to \overline{X}$ be the closed immersion for the complement. By [GS] 1.4, one can find a diagram

$$\begin{array}{ccc} Y_\bullet & \xrightarrow{i_\bullet} & \overline{X}_\bullet \\ \downarrow \pi_Y & & \downarrow \pi_X \\ Y & \xrightarrow{i} & \overline{X} \end{array} \tag{8.1}$$

where $Y_\bullet$ and $\overline{X}_\bullet$ are simplicial objects in $\mathcal{S}$ and π_X and π_Y are hyperenvelopes. To this diagram one associates a complex

$$Cone\big(W(Y_\bullet, \Lambda) \xrightarrow{i_{\bullet *}} W(\overline{X}_\bullet, \Lambda)\big).$$

By [GS], 1.4, the image $W(X, \Lambda)$ of the above complex in the homotopy category of chain complexes of abelian groups depends only on X and not on a choice of the diagram (8.1). The homology theory in Theorem 8.1 is defined as

$$H_a^W(X, \Lambda) := H_a(W(X, \Lambda)) \quad \text{ for } X \in \mathcal{C}.$$

For example, if X is a divisor with simple normal crossings on a smooth projective variety over k, $H_a^W(X, \Lambda)$ is a homology group of the complex:

$$\cdots \to \Lambda^{\pi_0(X^{(a)})} \xrightarrow{\partial} \Lambda^{\pi_0(X^{(a-1)})} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Lambda^{\pi_0(X^{(1)})}.$$

where $X_1, \ldots, X_N$ are the irreducible components of X and

$$X^{(a)} = \coprod_{1 \le i_i < \cdots < i_a \le N} X_{i_1, \ldots, i_a} \quad (X_{i_1, \ldots, i_a} = X_{i_1} \cap \cdots \cap X_{i_a}) \ ,$$

and the differentials ∂ are obvious ones.

Theorem 8.2. *([KeS2]) Let X be a quasi-projective smooth variety over k with action of a finite group G. Let X/G be the geometric quotient ([Mu], Ch.II §7, SGA 1 V §1).*

(1) Assume that X is projective. Then $H^W_a(X/G,\mathbb{Z}) = 0$ for all $a > 0$.

(2) Assume that the singular locus Z of X/G is proper over k. Let $\pi : Y \to X/G$ be a proper birational morphism such that Y is smooth over k and π is an isomorphism over outside Z. Let E be the reduced part of $\pi^{-1}(Z)$. Then $H^W_a(E,\mathbb{Z}) \simeq H^W_a(Z,\mathbb{Z})$ for all a. In particular, if Z is regular, then $H^W_a(E,\mathbb{Z}) = 0$ for all $a > 0$.

Here we explain an idea of the proof of Theorem 8.2(1). The second assertion (2) is an easy consequence of (1). Since $H^W_a(Y,\mathbb{Z})$ for $Y \in \mathcal{C}$ is finitely generated, it suffices to show the assertion for the weight homology group with coefficient $\Lambda = \mathbb{Q}/\mathbb{Z}$. Without loss of generality, we assume that k is finitely generated over $\mathbb{Q}$. Then the basic idea of the proof is to introduce an arithmetic invariant $KH_a(Y)$ for $Y \in \mathcal{C}$ which is defined without referring to desingularizations (or hyperenvelopes) and to show the following facts:

(∗1) $H^W_a(Y,\mathbb{Q}/\mathbb{Z}) \simeq KH_a(Y)$ for all $Y \in \mathcal{C}$.

(∗2) $KH_a(X/G) = 0$ for $a \neq 0$, where X/G is as in Theorem 8.2(1).

To define such an invariant we consider

$$H_a(X) := \varinjlim_n \operatorname{Hom}\big(H^a_c(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z}),\mathbb{Q}/\mathbb{Z}\big) \quad \text{for } X \in Ob(\mathcal{C}). \tag{8.2}$$

where $H^a_c(X_{\text{ét}},\mathbb{Z}/n\mathbb{Z})$ is the étale cohomology with compact support (cf. [JS2], Example 2.5). It provides a homology theory on $\mathcal{C}$ and gives rise to the niveau spectral sequence:

$$E^1_{p,q}(X) = \bigoplus_{x\in X_{(p)}} H_{p+q}(x) \;\Rightarrow\; H_{p+q}(X) \quad \text{with } H_a(x) = \varinjlim_{V\subseteq\overline{\{x\}}} H_a(V). \tag{8.3}$$

Here the limit is over all non-empty open subschemes $V \subseteq \overline{\{x\}}$. The affine Lefschetz theorem implies $E^1_{p,q}(X) = 0$ for $q < 0$ and the desired arithmetic invariant $KH_a(X)$ is defined as $E^2_{a,0}(X)$, an E^2 term of the spectral sequence. By the same techniques as the proof of Theorems 4.5 and 4.6, one can prove the following.

Theorem 8.3. *For $X \in \mathcal{S}$, we have*

$$KH_a(X) = \begin{cases} (\mathbb{Q}/\mathbb{Z})^{\pi_0(X)} & a = 0 \\ 0 & a \neq 0, \end{cases}$$

The assertion (∗1) follows from Theorem 8.3 and a result of Jannsen [J], Theorem 5.13. We note that the proof of Theorem 8.3 uses the weight argument ([D]) and requires the assumption that k is finitely generated. In order to show the assertion (∗2), we apply the same argument to the equivariant version of (8.2). We fix a finite group G and let $\mathcal{C}_G$ be the category of quasi-projective schemes over k with a G-action. We consider

$$H_a^G(X) := \varinjlim_n \operatorname{Hom}\big(H_c^a(G; X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}), \mathbb{Q}/\mathbb{Z}\big) \quad \text{for } X \in Ob(\mathcal{C}). \tag{8.4}$$

Here

$$H_c^a(G; X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}) := \mathbb{R}\Gamma(G, \mathbb{R}\Gamma(\overline{X}_{\text{ét}}, j_!\mathbb{Z}/n\mathbb{Z}))),$$

is the equivariant étale cohomology with compact support, where $j : X \hookrightarrow \overline{X}$ is any equivariant compactification of X, and $R\Gamma(G, -)$ is the derived functor of taking G-invariants. This provides a homology theory on $\mathcal{C}_G$ and the equivariant version $KH_a^G(X)$ of $KH_a(X)$ is defined as an E^2-term of the associated niveau spectral sequence. Then (∗2) follows from the following.

Theorem 8.4. *([KeS2]) Let $X \in \mathcal{C}_G$ be smooth over k.*

(1) We have a natural isomorphism $KH_a^G(X) \simeq KH_a(X/G)$ for all a.

(2) If X is projective, we have

$$KH_a^G(X) = \begin{cases} (\mathbb{Q}/\mathbb{Z})^{\pi_0(X/G)} & a = 0 \\ 0 & a \neq 0, \end{cases}$$

References

[B1] S. Bloch, *Higher Algebraic K-theory and class field theory for arithmetic surfaces*, Ann. of Math. **114** (1981), 229–265.

[B2] S. Bloch, *Algebraic cycles and higher algebraic K-theory*, Adv. Math. **61** (1986), 267–304.

[BK] S. Bloch and K. Kato, *p-adic étale cohomology*, Publ. Math. IHES **63** (1986), 107–152.

[CJS] V. Cossart, U. Jannsen and S. Saito, *Resolution of singularities for embedded surfaces*, in preparation (see www.mathematik.uni-regensburg.de/Jannsen).

[CT] J.-L. Colliot-Thélène, *On the reciprocity sequence in the higher class field theory of function fields*, Algebraic K-Theory and Algebraic Topology (Lake Louise, AB, 1991), (J.F. Jardine and V.P. Snaith, ed), 35–55, Kluwer Academic Publishers, 1993.

[CTSS] J.-L. Colliot-Thélène, J.-J. Sansuc and C. Soulé, *Torsion dans le groupe de Chow de codimension deux*, Duke Math. J. **50** (1983), 763–801.

[D] P. Deligne, *La conjecture de Weil II*, Publ. Math. IHES **52** (1981), 313–428.

[Ga] O. Gabber, *Sur la torsion dans la cohomologie l-adique d'une variété*, C. R. Acad. Sc. Paris Sér. I Math. **297** (1983), 179–182.

[Ge1] T. Geisser, *Weil-étale cohomology over finite fields*, Math. Ann. **330** (2004), 665–692.

[Ge2] T. Geisser, *Arithmetic homology and an integral version of Kato's conjecture*, To appear in J. Reine Angew.

[Ge3] T. Geisser, *On Suslin's singular homology and cohomology*, preprint.

[GL] T. Geisser and M. Levine, *The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky*, J. Reine Angew. **530** (2001), 55–103.

[Gra1] D. Grayson, *Finite generation of K-groups of a curve over a finite field (after Quillen)*, Lecture Notes in Math. **966** (1982), Springer-Verlag, Berlin, 69–90.

[Gra2] D. Grayson, *The motivic spectral sequence*, in: Handbook of K-theory 1, eds. E. friedlander, D. Grayson, Springer (2005).

[Gr] M. Gros, *Sur la partie p-primaire du groupe de Chow de codimension deux*, Comm. Algebra **13** (1985), 2407–2420.

[GS] H. Gillet and C. Soulé, *Descent, motives and K-theory*, J. Reine Angew. **478** (1996), 127–176.

[HW] C. Weibel, *Axioms for the Norm Residue Isomorphism*, K-theory Preprint Archives, http://www.math.uiuc.edu/K-theory/0809/

[Il1] L. Illusie, *Complexe de De Rham-Witt et cohomologie cristalline*, Ann. Ec. Norm. Sup. 4 série **12** (1979), 501–661.

[Il2] L. Illusie, *On Gabber's refined uniformization*, http://www.math.u-psud.fr/∼illusie/

[J] U. Jannsen, *Hasse principles for higher dimensional fields*, http://arxiv.org/abs/0910.2803.

[JS1] U. Jannsen and S. Saito, *Kato homology of arithmetic schemes and higher class field theory*, Documenta Math. Extra Volume: Kazuya Kato's Fiftieth Birthday (2003), 479–538

[JS2] U. Jannsen and S. Saito, *Kato conjecture and motivic cohomology over finite fields*, http://arxiv.org/abs/0910.2815

[JSS] U. Jannsen, S. Saito and K. Sato, *Etale duality for constructible sheaves on arithmetic schemes*, http://arxiv.org/abs/0910.3759.

[K] K. Kato, *A Hasse principle for two dimensional global fields*, J. für die reine und angew. Math. **366** (1986), 142–183.

[KS1] K. Kato and S. Saito, *Unramified class field theory of arithmetic surfaces*, Ann. of Math. **118** (1985), 241–275.

[KS2] K. Kato and S. Saito, *Global class field theory of arithmetic schemes*, Contemporary Math. **55**(1986), 255–331.

[KeSc] M. Kerz and A. Schmidt *Covering data and higher dimensional global class field theory* J. of Number Theory **129** (2009), 2569–2599

[KeS] M. Kerz and S. Saito, *Kato conjecture and motivic cohomology for arithmetic schemes*, in preparation.

[KeS2] M. Kerz and S. Saito, *Equivariant Kato conjecture and the weight homology of quotient schemes*, in preparation.

[Le] M. Levine, *K-theory and motivic cohomology of schemes*, preprint.

[Li] S. Lichtenbaum, *Values of zeta functions at non-negative integers*, In: Number theory, Noordwijkerhout 1983, Lecture Notes in Math. —**1068**, 127–138.

[Mil] J.S. Milne, *Values of zeta functions of varieties over finite fields*, American J. of Math. **108** (1986), 297–360.

[MS] A.S. Merkurjev and A.A. Suslin, *K-cohomology of Severi-Brauer Varieties and the norm residue homomorphism*, Math. USSR Izvestiya **21** (1983), 307–340.

[Mu] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research. Bombay, Oxford University Press (1970).

[MVW] A.S. Merkurjev and A.A. Suslin, *Lecture notes on motivic cohomology*, Clay Math. Monographs **2**, Amerian Math. Society, Clay Math. Institute

[P] B. Poonen, *Bertini theorems over finite fields*, Ann. of Math. **160** (2004), 1099–1127.

[Q] D. Quillen, *Finite generation of the group K_i of rings of algebraic integers*, Lecture Notes in Math. **341** (1973), Springer-Verlag, Berlin, 195–214.

[R] M. Rost, *Chow groups with coefficients*, Doc. Math. J. **1** (1996), 319–393.

[Sa1] S. Saito, *Unramified class field theory of arithmetic schemes*, Ann. of Math. **121** (1985), 251–281.

[Sa2] S. Saito *Cohomological Hasse principle for a threefold over a finite field*, in: Algebraic K-theory and Algebraic Topology, NATO ASI Series, **407** (1994), 229–241, Kluwer Academic Publishers

[Sa3] S. Saito, *Recent progress on the Kato conjecture*, http://www.lcv.ne.jp/∼smaki/en/preprints/index.html.

[Sat1] K. Sato, *p-adic étale Tate twists and duality of arithmetic schemes* (with an appendix by Hagihara, K.). Ann. Sci. Éc. Norm. Sup. (4) **40** (2007), 519–588

[Sat2] K. Sato, *Higher cycle class maps for p-adic étale Tate twists and the image of p-adic regulators*, in preparation

[SJ] A. Suslin and S. Joukhovitski, *Norm Varieties* J. Pure Appl. Alg. **206** (2006), 245–276.

[Sc] A. Schmidt, *Singular homology of arithmetic schemes* Algebra Number Theory **1**(2) (2007), 183–222.

[ScSp] A. Schmidt and M. Spiess, *Singular homology and class field thoery of varieties over finite fields* J. Reine Angew. Math. **527** (2000), 13–37.

[SV1] A. Suslin and V. Voevodsky, *Singular homology of abstract algebraic varieties* Invent. Math. **123** (1996), 61–94.

[SV2] A. Suslin and V. Voevodsky, *Bloch-Kato conjecture and motivic cohomology with finite coefficients*, in: Cycles, Transfer, and Motivic Homology Theories, Annals of Math. Studies **143**, Princeton University Press, 2000.

[Sw] N. Suwa, *A note on Gersten's conjecture for logarithmic Hodge-Witt sheaves*, K-theory **9** (1995), 245–271.

[Sz] T. Szamuely, *Corps de classes des schémas arithmétiques* http://www.renyi.hu/szamuely/publ.html.

[V1] V. Voevodsky, *Triangulated categories of motives over a field*, Cycles, Tansfers, and Motivic Homology Theories, Annals of Math. Studies, Princeton University Press

[V2] V. Voevodsky, *On motivic cohomology with $\mathbb{Z}/l$-coefficients*, K-theory Preprint Archives, http://www.math.uiuc.edu/K-theory/0639/

[V3] V. Voevodsky, *Motivic Eilenberg-MacLane spaces*, K-theory Preprint Archives, http://www.math.uiuc.edu/K-theory/0864/

[V4] V. Voevodsky, *Motivic cohomology are isomorphic to higher Chow groups*, Int. Math. Res. Not. **7** (2002), 351–355.

[W] G. Wiesend, *Class field theory for arithmetic schemes*, Math. Z. **256**(4) (2007), 717–729.

[We1] C. Weibel, *Axioms for the Norm Residue Isomorphism*, K-theory Preprint Archives, http://www.math.uiuc.edu/K-theory/0809/

[We2] C. Weibel, *Patching the Norm Residue Isomorphism Theorem*, K-theory Preprint Archives, http://www.math.uiuc.edu/K-theory/0844/

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo, 153-8914 JAPAN
E-mail: sshuji@msb.biglobe.ne.jp