ERRATUM: LEFSCHETZ THEOREM FOR ABELIAN FUNDAMENTAL GROUP WITH MODULUS

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The proof of Lemma 3.6 in [KS] is based on Claim 3.7 whose proof is not correct. In fact instead of (3-4) in the proof of Claim 3.7 one would need the vanishing of the analogous first cohomology group.

Below we give a corrected (and simplified) proof of Lemma 3.6. We would like to thank V. Srinivas for pointing out the mistake to us. We would also like to thank A. Krishna, R. Gupta and the referee for helpful comments on a preliminary version of this erratum.

Let X, D, U be as in the beginning of §3 but we don't assume X projective. Recall

$$(\mathbb{Z}/p^n\mathbb{Z})_{X|D} = \mathrm{Cone}\big(\mathrm{fil}^{\log}_{D/p} W_n\mathcal{O}_X \xrightarrow{1-F} \mathrm{fil}^{\log}_{D} W_n\mathcal{O}_X\big)[-1].$$

We have a distinguished triangle in $D^b(X)$:

$$(\mathbb{Z}/p^n\mathbb{Z})_{X|D} \to \mathrm{fil}^{\log}_{D/p} W_n\mathcal{O}_X \xrightarrow{1-F} \mathrm{fil}^{\log}_{D} W_n\mathcal{O}_X \xrightarrow{+} . \tag{1}$$

Lemma 3.6. There is a canonical isomorphism

$$\phi_{X|D} : H^1(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}) \xrightarrow{\sim} \mathrm{fil}^{\log}_D H^1(U)[p^n].$$

Proof. In the (correct) first part of the proof of Lemma 3.6 in [KS], we have constructed a canonical map $\phi_{X|D}$. In what follows we prove that it is an isomorphism. For simplicity of notation we omit the sheaf $(\mathbb{Z}/p^n\mathbb{Z})_{X|D}$ from our notation of cohomology and we just write $H^i(X, D)$ for $H^1(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D})$. For any open covering $X = X_1 \cup X_2$ Mayer-Vietoris gives in degree one the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & H^1(X,D) & \longrightarrow & H^1(X_1,D_1) \oplus H^1(X_2,D_2) & \longrightarrow & H^1(X_1\cap X_2, D_1\cap D_2) \\
& \downarrow{\scriptstyle \phi_{X|D}} & & \downarrow{\scriptstyle \phi_{X_1|D_1}\oplus\phi_{X_2|D_2}} & & \downarrow{\scriptstyle \phi_{X_1\cap X_2|D_1\cap D_2}} \\
0 \longrightarrow & \mathrm{fil}^{\log}_D H^1(U)[p^n] & \longrightarrow & \mathrm{fil}^{\log}_{D_1} H^1(U_1)[p^n] \oplus \mathrm{fil}^{\log}_{D_2} H^1(U_2)[p^n] & \longrightarrow & \mathrm{fil}^{\log}_{D_1\cap D_2} H^1(U_1\cap U_2)[p^n]
\end{array}$$

where $U_i = U \cap X_i$, $D_i = D \cap X_i$ for $i \in \{1, 2\}$. For the zero on the upper left side we use that for any non-empty open $V \subset X$ we have $H^0(V, D \cap V) = \mathbb{Z}/p^n\mathbb{Z}$. By a standard reduction based on this diagram we can assume that X is affine. For an affine X the cohomology $H^2(X, D) = 0$ by vanishing of coherent cohomology on affine schemes. So using Lemma 3.5 we can assume without loss of generality that $n = 1$.

Then, from (1) we get the following Artin-Schreier isomorphism

$$H^1(X, D) = \mathrm{Coker}(H^0(X, \mathcal{O}_X(D/p)) \xrightarrow{1-F} H^0(X, \mathcal{O}(D))), \tag{2}$$

which implies in particular that

$$\mathrm{colim}_D H^1(X, D) = H^1(U, \mathbb{Z}/p\mathbb{Z}). \tag{3}$$

Let D' be as in the proof of Claim 3.7. Let $\mathcal{W} = \mathcal{O}_{C_\lambda}(D)/\mathcal{O}_{C_\lambda}(D/p)^p$ if $p|m_\lambda$ and $\mathcal{W} = \mathcal{O}_{C_\lambda}(D)$ otherwise, and let $\mathcal{W}_\lambda$ be the stalk of $\mathcal{W}$ at the generic point λ of C_λ.

We get a commutative diagram

$$\begin{CD}
0 @>>> H^1(X,D') @>>> H^1(X,D) @>>> H^0(C_\lambda,\mathcal{W}) \\
@. @VV{\phi_{X|D'}}V @VV{\phi_{X|D}}V @VVV \\
0 @>>> \mathrm{fil}^{\log}_{D'}H^1(U)[p] @>>> \mathrm{fil}^{\log}_{D}H^1(U)[p] @>>> \mathcal{W}_\lambda
\end{CD}$$

The right vertical arrow is the obvious injective map. The upper row is exact by (2) and the commutative diagram in the proof of Claim 3.7 (for the zero on the upper left side we use the injectivity of $F:\mathcal{L}\to\mathcal{O}_{C_\lambda}(D)$ in the diagram). The exactness of the lower row follows from the claim below.

Now $\phi_{X|D}$ is injective as the composition of the injective maps

$$H^1(X,D)\to \mathrm{colim}_D H^1(X,D)\xrightarrow{\sim} H^1(U,\mathbb{Z}/p\mathbb{Z})\xrightarrow{\sim} H^1(U)[p].$$

From the commutative diagram we also deduce that

$$H^1(X,D')=\mathrm{fil}^{\log}_{D'}H^1(U)[p]\cap H^1(X,D)\subset H^1(U,\mathbb{Z}/p\mathbb{Z}). \tag{4}$$

That every element of $\mathrm{fil}^{\log}_{D}H^1(U)[p]$ is contained in the image of $\phi_{X|D}$ follows from (3) and (4) by descending induction on the multiplicities of D. □

Claim. *Let* $D=\sum_{\lambda\in I}m_\lambda C_\lambda$ *be as in* [KS, (2.5)] *and put*

$$\mathrm{fil}^{\log}_{D}\Gamma(U,\mathcal{O})=\{a\in\Gamma(U,\mathcal{O})|\ v_{K_\lambda}(a)\geq -m_\lambda\ \ \textit{for all } \lambda\in I\}.$$

Then we have

$$\mathrm{fil}^{\log}_{D}H^1(U)\cap H^1(U)[p]=\mathrm{fil}^{\log}_{D}H^1(U,\mathbb{Z}/p\mathbb{Z}):=\mathrm{Image}\big(\mathrm{fil}^{\log}_{D}\Gamma(U,\mathcal{O})\to H^1(U,\mathbb{Z}/p\mathbb{Z})\big).$$

Proof. By [KS, Def. 2.7], it suffices to show for each λ that

$$\mathrm{fil}^{\log}_{m_\lambda}H^1(K_\lambda)\cap H^1(K_\lambda)[p]=\mathrm{fil}^{\log}_{m_\lambda}H^1(K_\lambda,\mathbb{Z}/p\mathbb{Z}).$$

This follows from the commutative diagram

$$\begin{array}{ccc}
\mathrm{fil}^{\log}_{m_\lambda}H^1(K_\lambda,\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\mathrm{rsw}_p} & \mathrm{gr}^{\log}_{m_\lambda}\Omega^1_{K_\lambda} \\
\downarrow & \nearrow_{\mathrm{rsw}} & \\
\mathrm{fil}^{\log}_{m_\lambda}H^1(K_\lambda) & &
\end{array}$$

where $\mathrm{gr}^{\log}_{m_\lambda}\Omega^1_{K_\lambda}=\mathrm{fil}^{\log}_{m_\lambda}\Omega^1_{K_\lambda}/\mathrm{fil}^{\log}_{m_\lambda-1}\Omega^1_{K_\lambda}$ is the logarithmic version of [KS, (2.2)] (see [Ma, 2.6]) and rsw and rsw_p are Kato's refined Swan conductor [Kato] induced by the map

$$\mathrm{fil}^{\log}_{m_\lambda}W_s(K_\lambda)\to\mathrm{fil}^{\log}_{m_\lambda}\Omega^1_{K_\lambda}\ ;\ (a_{s-1},\ldots,a_0)\to -\sum_{i=0}^{s-1}a_i^{p^i-1}da_i, \tag{5}$$

and the kernels of rsw_p and rsw are $\mathrm{fil}^{\log}_{m_\lambda-1}H^1(K_\lambda,\mathbb{Z}/p\mathbb{Z})$ and $\mathrm{fil}^{\log}_{m_\lambda-1}H^1(K_\lambda)$ respectively: see [Ma, Rem. 3.2.12] for the last assertion for rsw. The assertion for rsw_p is directly checked by using (5) and the fact that $\mathrm{fil}^{\log}_{m_\lambda}H^1(K_\lambda,\mathbb{Z}/p\mathbb{Z})$ is the image of

$$\{a\in W_1(K_\lambda)=K_\lambda|\ v_{K_\lambda}(a)\geq -m_\lambda\}$$

under the map (2.1) in [KS] with $s=1$.

□

References

[KS] Kerz, M., Saito, S. *Lefschetz theorem for abelian fundamental group with modulus*, Algebra Number Theory **8** (2014), no. 3, 689–701.

[Kato] Kato, K. *Swan conductors for characters of degree oone in the imperfect residue field case*, Contemporary Math. **83** (1989) 101–1331.

[Ma] S. Matsuda, *On the Swan conductor in positive characteristic*, Amer. J. Math., **119** (1997), 705–739.

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