Conjecture of Colliot-Thélène on zero-cycles over local fields

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k : a field,

X/k : projective smooth variety,

$$\mathrm{CH}_0(X) = \Big(\bigoplus_{\substack{x \in X \\ \text{closed point}}} \mathbb{Z} \Big) / \underset{\text{rat. equiv.}}{\sim}$$

Chow group of zero-cycles on X modulo rational equivalence.

$$A_0(X) = \mathrm{Ker}\Big(\mathrm{CH}_0(X) \xrightarrow{deg} \mathbb{Z}\Big)$$

Chow group of zero-cycles of *degree zero*

$$alb_X : A_0(X) \to Alb_{X/k}(k)$$

Albanese map

Structure of $Alb_{X/k}(k)$:

A/k : abelian variety,

$\circ$ $k = \mathbb{C} \Rightarrow A(k) = \mathbb{C}^d/\Gamma$ (Γ lattice)

$\circ$ $[k : \mathbb{Q}] < \infty \Rightarrow A(k) = \mathbb{Z}^r \oplus$ (finite)
(Mordell-Weil)

$\circ$ $[k : \mathbb{Q}_p] < \infty \Rightarrow A(k) = \mathbb{Z}_p^r \oplus$ (finite)
(Mattuck)

Known facts on $\mathrm{Ker}(alb_X)$:

$\dim(X) = 1 \Rightarrow \mathrm{Ker}(alb_X) = 0$ (Abel)

$H^0(X, \Omega^2_X) \neq 0 \;\Rightarrow \dim \mathrm{Ker}(alb_X) = \infty$
(Mumford)

Bloch-Beilinson Conjecture :

$$[k : \mathbb{Q}] < \infty \Rightarrow \mathrm{Ker}(alb_X) \text{ finite} \quad ??$$

Structure of $\mathrm{Ker}(alb_X)$ depends on
arithmetic nature of k

How about the case $[k : \mathbb{Q}_p] < \infty$?

$[k : \mathbb{Q}_p] < \infty$

$\mathcal{O}_k$: ring of integers in k

$F = \mathcal{O}_k/\mathfrak{m}_k$: residue field

$p = \mathrm{ch}(F)$: residue characteristic

Question of Colliot-Thélène :

$$(CT1) \quad \boxed{\mathrm{Ker}(alb_X) = D(X) \oplus \text{ (finite)} \quad ??}$$

$D(X)$: maximal divisible subgroup of $A_0(X)$.

$$(CT2) \qquad \boxed{D(X)_{tor} = 0 \quad ??}$$

$$\boxed{(CT1) \ + \ (CT2) \ \Rightarrow \ A_0(X)_{tor} \text{ finite}}$$

Affirmative results :

- ∘ surfaces (Saito-Sujatha)
- ∘ Varieties fibered over curves in quadrics (Colliot-Thélène/Skorobogatov, Parimala/Suresh)
- ∘ Varieties fibered over curves in Severi-Brauer varieties (Salberger)
- ∘ Products of curves (Raskind-Spiess)

Theorem (Colliot-Thélène)

X : smooth compactification of connected linear algebraic group over k. Then

$$A_0(X) = \text{ (finite) } \oplus \begin{pmatrix} p\text{-primary torsion} \\ \text{of finite exponent} \end{pmatrix}$$

Theorem (Colliot-Thélène/Raskind, Salberger)

$$H^0(X, \mathcal{O}_X) = 0 \Rightarrow \mathrm{CH}^2(X)_{tor} \text{ finite}$$

Theorem (Saito-Sato) Assume

$(*)\ \ \exists \mathcal{X}/\mathcal{O}_k,$ regular projective model of X

s.t. $Y := (\mathcal{X} \times_{\mathcal{O}_k} \mathcal{O}_k/\mathfrak{m}_k)_{red}$ is a SNCD on $\mathcal{X}$.

$$\Rightarrow \quad A_0(X) = D'(X) \oplus (finite)$$

$D'(X)$: maximal p'-divisible subgroup of $A_0(X)$

(p'-divisible $\Leftrightarrow$ divisible by $\forall n$ prime to p)

Remark: The conlucsion of the theorem would be implied by $(CT1)$ and Mattuck's theorem:

$$Alb_X(k) = \mathbb{Z}_p^r \oplus \text{ (finite)}$$

Corollary 1 ((∗) not used)

$A_0(X)$ is ℓ-divisible for almost all primes ℓ.

Corollary 2

Assume (∗) and X *rationally connected.*

$$\Rightarrow \quad A_0(X) = \text{(finite)} \;\oplus\; \begin{pmatrix} p\text{-primary torsion} \\ \text{of finite exponent} \end{pmatrix}$$

Note the exact sequence:

$$\mathrm{CH}_1(Y) \to \mathrm{CH}_1(\mathcal{X}) \to \mathrm{CH}_0(X) \to 0$$

$$|\mathrm{CH}_1(\mathcal{X})/n| < \infty \Rightarrow |\mathrm{CH}_0(X)/n| < \infty$$

R : henselian discrete valuation ring s.t.
$F = R/\mathfrak{m}_R$: finite or separably closed

$\mathcal{X}$: regular projective flat over R s.t.
$Y := (\mathcal{X} \times_R R/\mathfrak{m}_R)_{red}$: SNCD on $\mathcal{X}$.

$d = \dim(\mathcal{X}/R)$: relative dimension

Theorem (Saito-Sato)

$$\mathrm{CH}_1(\mathcal{X})/n \xrightarrow{\simeq} H^{2d}_{\text{ét}}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}(d))$$
$$(\forall n \text{ prime to } p)$$

Corollary Assume $R/\mathfrak{m}_R$ finite.

$$\mathrm{CH}_1(\mathcal{X}) = D(\mathcal{X}) \oplus \text{ (finite)}$$

$D(\mathcal{X})$: maximal p'-divisible subgroup of $\mathrm{CH}_1(\mathcal{X})$

Key ingredients in the proof:

∘ Formalism of cycle class map in étale homology for Sch/R

∘ Absolute purity (Gabber)

∘ Affine Lefschetz theorem (Gabber)

∘ Theorem of Deligne (Weil conjecture)

∘ Bertini's theorem for Sch/R (Jannsen-Saito)

∘ Classical results on Brauer groups of arithmetic surfaces (Grothendieck, Artin)

∘ Theorem of Merkur'ev-Suslin

Theorem For any prime $\ell \neq p$,

$$\mathrm{CH}_1(\mathcal{X})\{\ell\} \xrightarrow{\simeq} H^{2d-1}_{\text{ét}}(\mathcal{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))$$

(Analogue of Roitman's theorem and geometric class field theory)

Corollary Assume $R/\mathfrak{m}_R$ finite.

$$\mathrm{CH}_1(\mathcal{X}) = D(\mathcal{X}) \oplus \text{ (finite)}$$

$D(\mathcal{X})$ uniquely p'-divisible
$\Leftrightarrow n : D(\mathcal{X}) \simeq D(\mathcal{X})$ for any n prime to p

In particular

$$\mathrm{CH}_1(\mathcal{X})\{p'\} = \bigoplus_{\ell \neq p} \mathrm{CH}_1(\mathcal{X})\{\ell\} \text{ is finite}$$

Key ingredient of the proof:

Theorem (Hironaka, Cossart-Jannsen-Saito)
$(RES)_2$ holds.

$t \geq 1$: an integer

$(RES)_t$: For given

Z : excellent regular scheme,

B : SNCD on Z,

$W \subset Z$: reduced closed of dimension $\leq t$,

$\exists$: $Z \leftarrow Z_1 \leftarrow \cdots \leftarrow Z_m$: sequence of blowups s.t.

the centers of $Z_{i+1} \to Z_i$ are regular and normal crossing with the complete transforms B_i of B in Z_i,

the strict transform of W in Z_m is regular and normal crossing with B_m

$$\mathrm{CH}_1(Y) \to \mathrm{CH}_1(\mathcal{X}) \to \mathrm{CH}_0(X) \to 0$$

Infinite torsion in $\mathrm{CH}_0(X)$:

Fix:

$[k : \mathbb{Q}_p] < \infty$,

$d \geq 0$,

$$X = \{F(z) = 0\} \subset \mathbb{P}^3_k = \mathrm{Proj}(k[\underline{z}])$$

$$F(z) = \sum_I a_I \underline{z}^I \in k[\underline{z}] = k[z_0, z_1, z_2, z_3]$$

(homogeneous polynomial of degree d)

$$(\underline{z}^I = z_0^{i_0} z_1^{i_1} z_2^{i_2} z_3^{i_3}, \quad I = (i_0, i_1, i_2, i_3))$$

Theorem (Asakura-Saito) Assume

- ○ X is *generic*, i.e.
  $\{a_I\}_I$ algebraically independent over $\mathbb{Q}$
- ○ $\exists \mathcal{X}/\mathcal{O}_k$: projective smooth model of X
- ○ $d \geq 5$

$$\Rightarrow \mathrm{CH}_0(X)\{\ell\} \xrightarrow{\simeq} (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{r-1} \oplus \text{ (finite)}$$

$$r = \dim(CH^1(Y) \otimes \mathbb{Q})$$

$$Y = \mathcal{X} \times_{\mathcal{O}_k} \mathcal{O}_k/\mathfrak{m}_k$$

Bloch's exact sequence :

$$0 \to \mathrm{CH}^2(X,1)\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell \to NH^3_{\text{ét}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \to \mathrm{CH}_0(X)\{\ell\} \to 0$$

$$\Rightarrow \ (*) \ \ \mathrm{CH}_0(X)\{\ell\} \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^s \oplus \ \text{(finite)}$$

Localization sequence:

$$\mathrm{CH}^2(X,1) \xrightarrow{\partial} \mathrm{CH}^1(Y) \to \mathrm{CH}_1(\mathcal{X}) \to \mathrm{CH}_0(X) \to 0$$

$\Rightarrow$

$$\mathrm{CH}_0(X)\{\ell\} \to \mathrm{Coker}(\partial)\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell \to \mathrm{CH}_1(\mathcal{X})\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell \xrightarrow{\iota} \mathrm{CH}_0(X)\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell \to 0$$

Theorem of Saito and Sato $\Rightarrow$ $\mathrm{Ker}(\iota)$ finite

$\Rightarrow s \geq \dim(\mathrm{Coker}(\partial)\otimes\mathbb{Q})$ in $(*)$

Commutative diagram
(Langer-Saito in case $\ell = p$)

$$\begin{array}{ccc} \mathrm{CH}^2(X,1) & \xrightarrow{\rho_X} & H^1_g(G_k, V) \\ \Big\downarrow \partial & & \Big\downarrow \\ \mathrm{CH}^1(Y) & \xrightarrow{\alpha} & H^1(G_k, V)/H^1_f(G_k, V) \end{array}$$

$G_k = \mathrm{Gal}(\overline{k}/k)$,

$V = H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell(2))$,

$H^1_f \subset H^1_g \subset H^1(G_k, V)$: subspaces introduced by Bloch-Kato

$$\rho_X : \mathrm{CH}^2(X,1) \to H^1(G_k, V)$$

ℓ-adic regulator map

Key fact: α is injective

Put:

$$V_{\text{prim}} = H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell(2))/[H \cap X] \otimes \mathbb{Q}_\ell(1)$$

$$H \subset \mathbb{P}^3_k \text{ hyperplane}$$

$$[H \cap X] \in H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell(1))$$

$$\begin{aligned}\rho_{X,\text{prim}} : \mathrm{CH}^2(X, 1) \otimes \mathbb{Q}_\ell \to H^1(G_k, V) \\ \to H^1(G_k, V_{\text{prim}})\end{aligned}$$

Theorem

$$\begin{aligned}X \subset \mathbb{P}^3_k : \text{ generic of degree } d \geq 5 \\ \Rightarrow \rho_{X,\text{prim}} = 0\end{aligned}$$

$\mathbb{P}_{\mathbb{Q}}^{\binom{d+3}{3}} \supset M$: moduli of smooth surfaces of degree d in $\mathbb{P}^3/\mathbb{Q}$

$X_M \to M$: universal family.

$$X \text{ generic } \Rightarrow \exists t : \mathrm{Spec}(k) \to M \text{ s.t.}$$

- t : dominant, i.e. $\mathbb{Q}(M) \hookrightarrow k$,
- $X \simeq X_M \times_M \mathrm{Spec}(k)$ (base change via t)

$$CH^2(X,1) = \varinjlim_{(S,t_S)} CH^2(X_S,1)$$

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{t_S} & S \\ & t \searrow & \downarrow \text{smooth} \\ & & M \end{array}$$

$$X_S := X_M \times_M S$$

Commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^2(X_S,1) & \xrightarrow{\rho^{\text{ét}}_{X_S,\mathrm{prim}}} & H^1_{\text{ét}}(S, V^{\text{ét}}_{S,\mathrm{prim}}) \\ \downarrow & & \downarrow \\ \mathrm{CH}^2(X,1) & \xrightarrow{\rho_{X,\mathrm{prim}}} & H^1(G_k, V_{\mathrm{prim}}) \end{array}$$

$$\rho^{\text{ét}}_{X_S,\mathrm{prim}} : \mathrm{CH}^2(X_S,1) \to H^1_{\text{ét}}(S, V^{\text{ét}}_{S,\mathrm{prim}})$$

$$f_S : X_S := X_M \times_M S \to S$$

$$V^{\text{ét}}_{S,\mathrm{prim}} = \left(R^2(f_S)_* \mathbb{Q}_\ell(2)\right)_{\mathrm{prim}}$$

(smooth $\mathbb{Q}_\ell$-sheaf on $S_{\text{ét}}$)

Reduced to showing $\rho^{\text{ét}}_{X_S,\mathrm{prim}} = 0$

First key step : Kill $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Put $S_{\overline{\mathbb{Q}}} = S \times_{\mathbb{Q}} \overline{\mathbb{Q}}$.

Theorem

$H^0(S, V^{\text{ét}}_{S,\text{prim}}) = (V^{\text{ét}}_{S,\text{prim}})^{\pi_1(S_{\overline{\mathbb{Q}}})} = 0.$

Proof: monodromy argument, or Lefschetz principle + infinitesimal method in Hodge theory

Corollary

$H^1_{\text{ét}}(S, V^{\text{ét}}_{S,\text{prim}}) \hookrightarrow H^1_{\text{ét}}(S_{\overline{\mathbb{Q}}}, V^{\text{ét}}_{S,\text{prim}})$

Commutative diagram

$$\begin{array}{ccc}
\mathrm{CH}^2(X_S,1) & \xrightarrow{\rho^{\text{ét}}_{X_S,\mathrm{prim}}} & H^1_{\text{ét}}(S_{\overline{\mathbb{Q}}}, V^{\text{ét}}_{S,\mathrm{prim}}) \\
\downarrow & & \downarrow \simeq \\
\mathrm{CH}^2(X_{S_{\mathbb{C}}},1) & \xrightarrow{\rho^{\mathrm{an}}_{X_S,\mathrm{prim}}} & H^1_{\mathrm{an}}(S(\mathbb{C}), V^{\mathrm{an}}_{S,\mathrm{prim}}) \otimes \mathbb{Q}_\ell
\end{array}$$

$$f_{S_{\mathbb{C}}} : X_{S_{\mathbb{C}}} = X \times_S S_{\mathbb{C}} \to S_{\mathbb{C}} = S \times_{\mathbb{Q}} \mathbb{C}$$

$$V^{\mathrm{an}}_{S,\mathrm{prim}} = \left(R^2 (f_{S_{\mathbb{C}}})^{\mathrm{an}}_* \mathbb{Q}(2)\right)_{\mathrm{prim}}$$

(local system on $S(\mathbb{C})$)

$$\rho^{\mathrm{an}}_{X_S,\mathrm{prim}} : \mathrm{CH}^2(X_{S_{\mathbb{C}}},1) \to H^1_{\mathrm{an}}(S(\mathbb{C}), V^{\mathrm{an}}_{S,\mathrm{prim}})$$

analytic regulator map

Reduced to showing $\rho^{\mathrm{an}}_{X_S,\mathrm{prim}} = 0$.

Theory of mixed Hodge modules (M. Saito) :

- $H := H^1_{\mathrm{an}}(S(\mathbb{C}), V^{\mathrm{an}}_{S,\mathrm{prim}}) \in MHS$
- $\mathrm{Image}(\rho^{\mathrm{an}}_{X_S,\mathrm{prim}}) \subset F^2 H_{\mathbb{C}}$

(Hodge filtration on $H_{\mathbb{C}} = H \otimes \mathbb{C}$)

Reduced to showing:

Lemma

(1) $F^2 H_{\mathbb{C}} \hookrightarrow$ cohomolgy of de Rham complex:

$$F^2 H^2_{dR}(X_S/S)_{\mathrm{prim}} \otimes \mathcal{O}_{S_{\mathbb{C}}} \xrightarrow{\nabla} F^1 H^2_{dR}(X_S/S)_{\mathrm{prim}} \otimes \Omega^1_{S_{\mathbb{C}}/\mathbb{C}} \xrightarrow{\nabla} H^2_{dR}(X_S/S)_{\mathrm{prim}} \otimes \Omega^2_{S_{\mathbb{C}}/\mathbb{C}}.$$

(2) The complex is exact.

Ingredients of proof :

(1)

∘ Degeneration of Hodge spectral sequence with coefficients in local systems

∘ Deligne's GAGA for vector bundles with integrable connections with regular singularities

(2)

Infinitesimal method in Hodge theory
($d = \deg(X) \geq 5$ is used)

Open Problem :

How about the case $d = 4$ ($K3$ surfaces) ?

Implication to Bloch-Kato conjecture

X/k : projective smooth over k

$[k:\mathbb{Q}] < \infty$ or $[k:\mathbb{Q}_p] < \infty$.

$$\rho_X^{r,q} : \mathrm{CH}^r(X,q) \otimes \mathbb{Q}_\ell \to H^1(G_k, V)$$

ℓ-adic regulator map

$$V = H_{\text{ét}}^{2r-q-1}(X_{\overline{k}}, \mathbb{Q}_\ell(r))$$

p-adic Hodge theory $\Rightarrow$

$$\mathrm{Image}(\rho_X^{r,q}) \subset H_g^1(G_k, V)$$

(Faltings, Hyodo-Kato, Tsuji, Yamashita)

Conjecture of Bloch-Kato
(part of Tamagawa number conjecture):

$$[k:\mathbb{Q}] < \infty \Rightarrow \mathrm{Image}(\rho_X^{r,q}) = H_g^1(G_k, V)$$

Recall:

$$H^1(G_k, V) = \mathrm{Ext}^1_{R_\ell(G_k)}(\mathbb{Q}_\ell, V)$$

$R_\ell(G_k)$: category of finite dimensional $\mathbb{Q}_\ell$-spaces with continuous G_k-actions

$$[0 \to V \to E \to \mathbb{Q}_\ell \to 0] \in H^1_g(G_k, V)$$
$$\Leftrightarrow E : \text{de Rham representation of } G_k$$

$$[0 \to V \to E \to \mathbb{Q}_\ell \to 0] \in H^1_f(G_k, V)$$
$$\Leftrightarrow E : \text{crystalline representation of } G_k$$

Heuristically BK conjecture implies :

E : de Rham $\Rightarrow$ realization of mixed motive

Corollary

Variant of BK conjecture over p-adic field is FALSE.