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Beilinson's Hodge Conjecture with Coefficients for Open Complete Intersections

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Dedicated to Professor J.P. Murre on the occasion of his 75th birthday

8.1 Introduction

Let U be a smooth algebraic variety over $\mathbb{C}$ and let U^{an} be the analytic site on $U(\mathbb{C})$, the associated analytic space. An important object to study in algebraic geometry is the regulator map from the higher Chow group ([7]) to the singular cohomology of U (cf. [18])

$$\mathrm{reg}_U^{p,q} : CH^q(U, 2q-p) \otimes \mathbb{Q} \to (2\pi\sqrt{-1})^q W_{2q} H^p(U^{\mathrm{an}}, \mathbb{Q}) \cap F^q H^p(U^{\mathrm{an}}, \mathbb{C}),$$

where F^* and W_* denote the Hodge and the weight filtrations of the mixed Hodge structure on the singular cohomology defined by Deligne [8]. For the special case $p = q$, we get

$$\mathrm{reg}_U^q : CH^q(U, q) \otimes \mathbb{Q} \to H^q(U^{\mathrm{an}}, \mathbb{Q}(q)) \cap F^q H^q(U^{\mathrm{an}}, \mathbb{C}). \quad (\mathbb{Q}(q) = (2\pi\sqrt{-1})^q \mathbb{Q})$$

Beilinson's Hodge conjecture claims the surjectivity of reg_U^q (cf. [11, Conjecture 8.5]). In [4] we studied the problem in case U is an open complete intersection, namely U is the complement in a smooth complete intersection X of a simple normal crossing divisor $Z = \cup_{j=1}^s Z_j$ on X such that $Z_j \subset X$ is a smooth hypersurface section. One of the main results affirms that reg_U^q is surjective if the degree of the defining equations of X and Z_j are sufficiently large and if U is general in an appropriate sense. Indeed, under the assumption we have shown a stronger assertion that reg_U^q is surjective even restricted on the subgroup $CH^q(U, q)_{dec}$ of decomposable elements in $CH^q(U, q)$, which is not true in general. In order to explain this, let $K_q^M(\mathcal{O}(U))$ be the Milnor K-group of the ring $\mathcal{O}(U) = \Gamma(U, \mathcal{O}_{Zar})$ (see

§8.2.3 for its definition). We have the natural map

$$\sigma_U : K_q^M(\mathcal{O}(U)) \to CH^q(U,q)$$

induced by cup product and the natural isomorphism

$$K_1^M(\mathcal{O}(U)) = \Gamma(U, \mathcal{O}_{Zar}^*) \xrightarrow{\cong} CH^1(U,1)$$

and $CH^q(U,q)_{dec}$ is defined to be its image. Note that we have the following formula for the value of reg_U^q on decomposable elements;

$$\begin{aligned}\mathrm{reg}_U^q(\{g_1,\ldots,g_q\}) &= [g_1]\cup\cdots\cup[g_q] \in H^q(U^{\mathrm{an}},\mathbb{Q}(q))\\ &= \mathrm{dlog}\, g_1\wedge\cdots\wedge\mathrm{dlog}\, g_q \in H^0(X,\Omega^q_{X/\mathbb{C}}(\log Z)) = F^qH^q(U^{\mathrm{an}},\mathbf{C})\end{aligned}$$

where $g_j \in \mathcal{O}(U)^*$ for $1 \le j \le q$ and $[g_j] \in H^1(U^{\mathrm{an}},\mathbb{Q}(1))$ is the image of g_j under the map $\mathcal{O}(U)^* \to H^1(U^{\mathrm{an}},\mathbb{Z}(1))$ induced by the exponential sequence

$$0 \to \mathbb{Z}(1) \to \mathcal{O}_{U^{\mathrm{an}}} \xrightarrow{\exp} \mathcal{O}^*_{U^{\mathrm{an}}} \to 0.$$

In what follows we are mainly concerned with the map

$$\mathrm{reg}_U^q : K_q^M(\mathcal{O}(U))\otimes\mathbb{Q} \to H^q(U^{\mathrm{an}},\mathbb{Q}(q)) \cap F^qH^q(U^{\mathrm{an}},\mathbb{C}) \tag{8.1}$$

which is the composition of the regulator map and σ_U.

Now we consider the following variant of the above problem. Assume that we are given a smooth algebraic variety S over $\mathbb{C}$ and a smooth surjective morphism $\pi : U \to S$ over $\mathbb{C}$. Let $\pi_*^{\mathrm{an}} : U^{\mathrm{an}} \to S^{\mathrm{an}}$ be the associated morphism of sites. Assume that the fibers of π are affine of dimension m. Then $R^b\pi_*^{\mathrm{an}}\mathbb{Q} = 0$ for $b > m$ and we have the natural map

$$\alpha : H^{m+q}(U^{\mathrm{an}},\mathbb{Q}(m+q)) \to H^q(S^{\mathrm{an}}, R^m\pi_*^{\mathrm{an}}\mathbb{Q}(m+q))$$

which is an edge homomorphism of the Leray spectral sequence

$$E_2^{a,b} = H^a(S^{\mathrm{an}}, R^b\pi_*^{\mathrm{an}}\mathbb{Q}(m+q)) \Rightarrow H^{a+b}(U^{\mathrm{an}},\mathbb{Q}(m+q)).$$

Note that $H^a(S^{\mathrm{an}}, R^b\pi_*^{\mathrm{an}}\mathbb{Q}(m+q))$ carries in a canonical way a mixed Hodge structure and α is a morphism of mixed Hodge structures ([17] and [1]). Let

$$\mathrm{reg}_{U/S}^{m+q} : K_{m+q}^M(\mathcal{O}(U))\otimes\mathbb{Q} \to H^q(S^{\mathrm{an}}, R^m\pi_*^{\mathrm{an}}\mathbb{Q}(m+q)) \cap F^{m+q} \tag{8.2}$$

be the composition of reg_U^{m+q} and α where $F^t \subset H^q(S^{\mathrm{an}}, R^m\pi_*^{\mathrm{an}}\mathbb{C})$ denotes the Hodge filtration. In this paper we study $\mathrm{reg}_{U/S}^{m+q}$ in case U/S is a family of open complete intersections, namely in case that the fibers of π are open complete intersections. Roughly speaking, our main results affirm that

$\mathrm{reg}^{m+q}_{U/S}$ is surjective for $q=0,1$ if $\pi : U \to S$ is the pullback of the universal family of open complete intersection of sufficiently high degree via a dominant smooth morphism from S to the moduli space. Let $d_i, e_j \geq 0$ ($1 \leq i \leq r$, $1 \leq j \leq s$) be fixed integers. Let $\mathbf{M} = \mathbf{M}(d_1, \cdots, d_r; e_1, \cdots, e_s)$ be the moduli space of the sets $(X_{1,o}, \ldots, X_{r,o}; Y_{1,o} \ldots, Y_{s,o})$ of smooth hypersurfaces in $\mathbb{P}^n$ of degree $d_1, \cdots, d_r; e_1, \cdots, e_s$ respectively which intersect transversally with each other. Let $f : S \to \mathbf{M}$ be a morphism of finite type with $S = \operatorname{Spec} R$ nonsingular affine and let $X_i \to S$ and $Y_j \to S$ be the pullback of the universal families of hypersurfaces over $\mathbf{M}$. Put

$$X = X_1 \cap \cdots \cap X_r \quad \text{and} \quad U = X \setminus \bigcup_{1 \leq j \leq s} X \cap Y_j$$

with the natural morphisms $\pi : U \to S$. Put

$$\mathbf{d} = \sum_{i=1}^{r} d_i, \quad \delta_{\min} = \min_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} \{d_i, e_j\}, \quad d_{\max} = \max_{1 \leq i \leq r} \{d_i\}.$$

Theorem 8.1.1. *(see §8.3) Assume f is dominant smooth.*

(1) Assuming $\delta_{min}(n-r-1) + \mathbf{d} \geq n+1$,

$$\mathrm{reg}^m_{U/S} : K^M_m(\mathcal{O}(U)) \otimes \mathbb{Q} \to H^0(S^{\mathrm{an}}, R^m \pi^{\mathrm{an}}_* \mathbb{Q}_U(m+1))$$

is surjective.

(ii) Assuming

$$\delta_{min}(n-r-1) + \mathbf{d} \geq n+2,\ \delta_{min}(n-r) + \mathbf{d} \geq n+1+d_{\max},\ \delta_{min} \geq 2,$$

$$\mathrm{reg}^{m+1}_{U/S} : K^M_{m+1}(\mathcal{O}(U)) \otimes \mathbb{Q} \to H^1(S^{\mathrm{an}}, R^m \pi^{\mathrm{an}}_* \mathbb{Q}_U(m+1)) \cap F^{m+1}$$

is surjective.

The method of the study is the infinitesimal method in Hodge theory and is a natural generalization of that in [3] and [4]. To explain this, we now work over an arbitrary algebraically field k of characteristic zero which will be fixed in the whole paper. Let $f : S \to \mathbf{M}$ and $\pi : U \to S$ be defined over k as above. Following Katz and Oda ([12]), we have the algebraic Gauss Manin connection on the de Rham cohomology (see §8.2.2)

$$\nabla : H^\bullet_{\mathrm{dR}}(U/S) \longrightarrow H^\bullet_{\mathrm{dR}}(U/S) \otimes_R \Omega^1_{R/k}. \tag{8.3}$$

The map ∇ is extended to $H^\bullet_{\mathrm{dR}}(U/S) \otimes_R \Omega^q_{R/k} \longrightarrow H^\bullet_{\mathrm{dR}}(U/S) \otimes_R \Omega^{q+1}_{R/k}$ by imposing the Leibniz rule

$$\nabla(e \otimes \omega) = \nabla(e) \wedge \omega + e \otimes d\omega \tag{8.4}$$

and it induces the complex

$$\mathrm{Gr}_F^{p+1} H^m_{\mathrm{dR}}(U/S)\otimes\Omega^{q-1}_{R/k} \longrightarrow \mathrm{Gr}_F^{p} H^m_{\mathrm{dR}}(U/S)\otimes\Omega^{q}_{R/k} \longrightarrow \mathrm{Gr}_F^{p-1} H^m_{\mathrm{dR}}(U/S)\otimes\Omega^{q+1}_{R/k},$$

where $m = \dim(U/S)$ and $F^\bullet$ denotes the Hodge filtration:

$$F^p H^q_{\mathrm{dR}}(U/S) = H^q_{Zar}(X, \Omega^{\geq p}_{X/S}(\log Z)) \subset H^q_{\mathrm{dR}}(U/S).$$

The cohomology at the middle term of the complex has been studied in [3] when $1 \leq p \leq m-1$.

In the study of the variant of Beilinson's Hodge conjecture, a crucial role will be played by the kernel of the following map:

$$\overline{\nabla}_q : F^m H^m_{\mathrm{dR}}(U/S) \otimes \Omega^q_{R/k} \longrightarrow \mathrm{Gr}_F^{m-1} H_{\mathrm{dR}}(U/S) \otimes \Omega^{q+1}_{R/k} \quad (q \geq 0).$$

which arises as the special case $p = m$ in the above complex. The key result is, roughly speaking, that when $f : S \to \mathbf{M}$ factors as $S \xrightarrow{g} T \xrightarrow{i} \mathbf{M}$ where g is smooth and i is a regular immersion of small codimension, then the kernel of $\overline{\nabla}_q$ is generated by the image of

$$\mathrm{dlog} : K^M_{m+q}(\mathcal{O}(U)) \longrightarrow F^m H^m_{\mathrm{dR}}(U/S) \otimes \Omega^q_{R/k}$$

(see §8.2.3 for its definition). In case $k = \mathbb{C}$ it implies the surjectivity of $\mathrm{reg}^{m+q}_{U/S}$ (8.2) for $q = 0$ and 1 by using the known surjectivity of the map (8.1) for $U = S$.

The main tool for the proof of the above key result is the theory of generalized Jacobian rings developed by the authors in [3]. It describe the Hodge cohomology groups of U and the Gauss-Manin connection $\overline{\nabla}_q$ in terms of multiplication of the rings, so that the various problems can be translated into algebraic computations in Jacobian rings. We show several computational results on Jacobian rings in §8.4 and §8.5. The basic techniques for this were developed by M.Green, C.Voisin and Nori. We note that a key to the computational results is Proposition 8.5.5, which is proved in [3] as a generalization of Nori's connectivity theorem ([14]) to open complete intersections.

Notation and Conventions

For an abelian group M, we write $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$.

8.2 The Main Theorem

Throughout the paper, we work over an algebraically closed field k of characteristic zero.

8.2.1 Setup

We fix integers $n \geq 2$, $r, s \geq 1$, $n \geq r$ and $d_1, \cdots, d_r$, $e_1, \cdots, e_s \geq 1$. We put

$$\begin{aligned} \mathbf{d} &= \sum_{i=1}^{r} d_i, \quad \mathbf{e} = \sum_{j=1}^{s} e_j, \quad \delta_{\min} = \min_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} \{d_i, e_j\}, \\ d_{\max} &= \max_{1 \leq i \leq r} \{d_i\}, \quad e_{\max} = \max_{1 \leq j \leq s} \{e_j\}. \end{aligned}$$

Let $P = k[X_0, \cdots, X_n]$ be the polynomial ring over k and P^d denote the subspace of the homogeneous polynomials of degree d. Then the space $P^d - \{0\}$ parametrizes hypersurfaces in $\mathbb{P}^n$ of degree d with a chosen defining equation. Let

$$\widetilde{\mathbf{M}} = \widetilde{\mathbf{M}}(d_1, \cdots, d_r; e_1, \cdots, e_s) \subset \prod_{i=1}^{r} (P^{d_i} - \{0\}) \times \prod_{j=1}^{s} (P^{e_j} - \{0\})$$

be the Zariski open subset such that the associated divisor $X_{1,o} + \cdots + X_{r,o} + Y_{1,o} + \cdots + Y_{s,o}$ to any point $o \in \widetilde{\mathbf{M}}$ is a simple normal crossing divisor on $\mathbb{P}^n$, namely all $X_{i,o}$ and $Y_{j,o}$ are nonsingular and they intersect transversally with each other. Put $X_o = X_{1,o} \cap \cdots \cap X_{r,o}$ and $Z_{j,o} = X_o \cap Y_{j,o}$. Then X_o is a nonsingular complete intersection of dimension $n - r$, and $\sum_{j=1}^{s} Z_{j,o}$ is a simple normal crossing divisor on X_o.

Let $f : S \to \widetilde{\mathbf{M}}$ be a morphism of finite type with $S = \operatorname{Spec} R$ nonsingular affine. We write $P_R = P \otimes_k R$ and $P_R^\ell = P^\ell \otimes_k R$. Let

$$F_i \in P_R^{d_i} \ (1 \leq i \leq r) \quad \text{and} \quad G_j \in P_R^{e_j} \ (1 \leq j \leq s) \tag{8.5}$$

be the pullback of the universal polynomials over the moduli space. We denote by X, X_i, Y_j and Z_j the associated families of the complete intersections X_o, $X_{i,o}$, $Y_{j,o}$ and divisors $Z_{j,o}$ respectively. Thus we get the smooth morphisms:

$$\pi_X : X \longrightarrow S, \quad \pi_{X_i} : X_i \longrightarrow S, \quad \pi_{Y_j} : Y_j \longrightarrow S, \quad \pi_{Z_j} : Z_j \longrightarrow S. \tag{8.6}$$

We write

$$X_* = \sum_{i=1}^{r} X_i, \quad Y_* = \sum_{j=1}^{s} Y_j, \quad Z_* = \sum_{j=1}^{s} Z_j.$$

Put $U = X - Z_*$ and we get $\pi : U \to S$, a family of open complete intersections.

8.2.2 Gauss-Manin connection

For an integer $q \geq 0$ we have the *Gauss-Manin connection*

$$\nabla : H^{\bullet}_{\mathrm{dR}}(U/S) \longrightarrow H^{\bullet}_{\mathrm{dR}}(U/S) \otimes \Omega^1_{R/k}. \tag{8.7}$$

Here $H^{\bullet}_{\mathrm{dR}}(U/S)$ is the de Rham cohomology defined as

$$H^k_{\mathrm{dR}}(U/S) = H^k_{\mathrm{Zar}}(X, \Omega^{\bullet}_{X/S}(\log Z_*)) = \Gamma(S, R^k\pi_{X*}\Omega^{\bullet}_{X/S}(\log Z_*)),$$

where the second equality follows from the assumption that S is affine. It is an integrable connection and satisfies the Griffiths transversality:

$$\nabla(F^p H^{\bullet}_{\mathrm{dR}}(U/S)) \subset F^{p-1} H^{\bullet}_{\mathrm{dR}}(U/S) \otimes \Omega^1_{R/k} \tag{8.8}$$

with respect to the Hodge filtration

$$F^p H^{\bullet}_{\mathrm{dR}}(U/S) := H^{\bullet}_{\mathrm{Zar}}(X, \Omega^{\geq p}_{X/S}(\log Z_*)). \tag{8.9}$$

We are interested in $H^{n-r}_{\mathrm{dR}}(U/S)$ (Since X is a complete intersection, the cohomology in other degrees is not interesting). We denote by

$$\nabla_q : F^{n-r} H^{n-r}_{\mathrm{dR}}(U/S) \otimes \Omega^q_{R/k} \longrightarrow F^{n-r-1} H^{n-r}_{\mathrm{dR}}(U/S) \otimes \Omega^{q+1}_{R/k} \tag{8.10}$$

the map given by (8.4). Noting

$$F^p H^{n-r}_{\mathrm{dR}}(U/S)/F^{p+1} H^{n-r}_{\mathrm{dR}}(U/S) \simeq H^{n-r-p}(X, \Omega^p_{X/S}(\log Z_*)),$$

(8.8) implies that ∇_q induces

$$\overline{\nabla}_q : H^0(X, \Omega^{n-r}_{X/S}(\log Z_*)) \otimes \Omega^q_{R/k} \longrightarrow H^1(X, \Omega^{n-r-1}_{X/S}(\log Z_*)) \otimes \Omega^{q+1}_{R/k}. \tag{8.11}$$

Our main theorem gives an explicit description of $\mathrm{Ker}(\overline{\nabla}_q)$ under suitable conditions. For its statement we need more notations.

8.2.3 Milnor K-theory

We denote by $K^M_\ell(\mathcal{A})$ the Milnor K-group of a commutative ring $\mathcal{A}$ ([13, 19]). By definition, it is the quotient of $\mathcal{A}^{*\otimes \ell}$ by the subgroup generated by

$$a_1 \otimes \cdots \otimes a_\ell, \quad (a_i + a_j = 0 \text{ or } 1 \text{ for some } i \neq j).$$

The element represented by $a_1 \otimes \cdots \otimes a_\ell$ is called the Steinberg symbol, and written by $\{a_1, \cdots, a_\ell\}$. We have

$$\{a_1, \cdots, a_i, \cdots, a_j, \cdots, a_\ell\} = -\{a_1, \cdots, a_j, \cdots, a_i, \cdots, a_\ell\} \quad \text{for } i \neq j$$

following from the expansion $\{ab, -ab\} = \{a, b\} + \{b, a\} + \{a, -a\} + \{b, -b\}$.

Let $\mathcal{O}(U) = \Gamma(U_{Zar}, \mathcal{O}_U)$ be the ring of regular functions on U. We have the dlog map

$$\mathrm{dlog} : K_\ell^M(\mathcal{O}(U)) \longrightarrow H^0(\Omega^\ell_{X/k}(\log Z_*)), \quad \{h_1, \cdots, h_\ell\} \longmapsto \frac{dh_1}{h_1} \wedge \cdots \wedge \frac{dh_\ell}{h_\ell} \tag{8.12}$$

Assuming $\ell \geq n - r = \dim(X/S)$, there is the unique map

$$\upsilon_X : \Omega^\ell_{X/k}(\log Z_*) \longrightarrow \Omega^{n-r}_{X/S}(\log Z_*) \otimes \Omega^{\ell-n+r}_{S/k}. \tag{8.13}$$

such that its composition with $\Omega^{n-r}_{X/k}(\log Z_*) \otimes \Omega^{\ell-n+r}_{S/k} \to \Omega^\ell_{X/k}(\log Z_*)$ is the identity map $\Omega^{n-r}_{X/k}(\log Z_*) \otimes \Omega^{\ell-n+r}_{S/k} \to \Omega^{n-r}_{X/S}(\log Z_*) \otimes \Omega^{\ell-n+r}_{S/k}$. Let

$$\psi_\ell^M : K_\ell^M(\mathcal{O}(U)) \longrightarrow H^0(\Omega^{n-r}_{X/S}(\log Z_*)) \otimes \Omega^{\ell-n+r}_{R/k} = F^{n-r} H^{n-r}_{\mathrm{dR}}(U/S) \otimes \Omega^{\ell-n+r}_{R/k}.$$

be the composition of υ_X and dlog. Its image is contained in $\mathrm{Ker}(\nabla_{\ell-n+r})$ since it lies in the image of $H^0(\Omega^\ell_{X/k}(\log Z_*))$. Thus we get the map

$$\psi_\ell^M : K_\ell^M(\mathcal{O}(U)) \longrightarrow \mathrm{Ker}(\nabla_{\ell-n+r}) \tag{8.14}$$

We will also consider the induced maps

$$K^M_{\ell+n-r}(\mathcal{O}(U)) \otimes_{\mathbb{Z}} \Omega^{q-\ell, d=0}_{R/k} \longrightarrow \mathrm{Ker}(\nabla_q); \quad \xi \otimes \omega \mapsto \psi^M_{\ell+n-r}(\xi) \wedge \omega,$$

$$K^M_{\ell+n-r}(\mathcal{O}(U)) \otimes_{\mathbb{Z}} \Omega^{q-\ell}_{R/k} \longrightarrow \mathrm{Ker}(\overline{\nabla}_q); \quad \xi \otimes \omega \mapsto \psi^M_{\ell+n-r}(\xi) \wedge \omega,$$

where $\Omega^{\bullet, d=0}_{R/k} = \mathrm{Ker}(d : \Omega^\bullet_{R/k} \to \Omega^\bullet_{R/k})$ is the module of closed forms.

Now we construct some special elements in $K_\ell^M(\mathcal{O}(U))$. Let $\ell \geq 1$ be an integer. We define $\bigwedge^{\ell}(G_j)$ as the $\mathbb{Q}$-vector space spanned by symbols υ_J indexed by multi-indices $J = (j_0, \cdots, j_\ell)$ $(1 \leq j_k \leq s)$ with relations

$$\upsilon_{j_0 \cdots j_p \cdots j_q \cdots j_\ell} = -\upsilon_{j_0 \cdots j_q \cdots j_p \cdots j_\ell} \quad \text{for } 0 \leq p \neq q \leq \ell \tag{8.15}$$

and

$$\sum_{k=0}^{\ell+1} (-1)^k e_{j_k} \upsilon_{j_0 \cdots \widehat{j_k} \cdots j_{\ell+1}} = 0. \tag{8.16}$$

We formally put $\bigwedge^{0}(G_j) = \mathbb{Q}$. By convention, $\bigwedge^{\ell}(G_j) = 0$ if $s = 0$ or 1. We easily see

$$\dim_{\mathbb{Q}} \bigwedge^{\ell}(G_j) = \binom{s-1}{\ell} \text{ with basis } \{\upsilon_{1 j_1 \cdots j_\ell} \,;\, 2 \leq j_1 < \cdots < j_\ell \leq s\},$$

and $\bigwedge^{\ell}(G_j) = 0$ if $\ell \geq s$. Let $G_j^{e_i}/G_i^{e_j}|_X$ be the restriction on X of a rational function $G_j^{e_i}/G_i^{e_j}$ on $\mathbb{P}^n_R = \mathrm{Proj}(R[X_0, \ldots, X_n])$. Then we have a natural homomorphism

$$sym_\ell : \bigwedge^{\ell}(G_j) \longrightarrow K_\ell^M(\mathcal{O}(U))_{\mathbb{Q}}. \tag{8.17}$$

$$v_J \mapsto g_J := e_{j_0}^{-\ell+1}\left\{G_{j_1}^{e_{j_0}}/G_{j_0}^{e_{j_1}}|_X, \cdots, G_{j_\ell}^{e_{j_0}}/G_{j_0}^{e_{j_\ell}}|_X\right\} \quad (J = (j_0, \ldots, j_\ell))$$

Putting $g_j = G_j/X_0^{e_j}|_X$, a calculation shows

$$\mathrm{dlog}(g_J) = \sum_{\nu=0}^{\ell}(-1)^\nu e_{j_\nu}\frac{dg_{j_0}}{g_{j_0}} \wedge \cdots \wedge \widehat{\frac{dg_{j_\nu}}{g_{j_\nu}}} \wedge \cdots \wedge \frac{dg_{j_\ell}}{g_{j_\ell}} \quad \text{on } \{X_0 \neq 0\}. \tag{8.18}$$

The maps $\psi_\bullet^M$ and sym_ℓ induce a homomorphism

$$\Psi^q_{U/S} : \bigoplus_{\ell=0}^{q} \bigwedge^{\ell+n-r}(G_j) \otimes_{\mathbb{Q}} \Omega^{q-\ell}_{R/k} \longrightarrow \mathrm{Ker}(\overline{\nabla}_q); \quad g_I \otimes \eta \mapsto \psi_\ell^M(g_I) \bigwedge \eta. \tag{8.19}$$

The main theorem affirms that this map is an isomorphism under suitable conditions. In order to give the precise statement we need to introduce some notations.

8.2.4 Statement of the Main Theorem

Let $T_{R/k}$ be the derivation module of R over k which is the dual of $\Omega^1_{R/k}$. A derivation $\theta \in T_{R/k}$ acts on $P_R = P \otimes_k R = R[X_0, \ldots, X_n]$ by $\mathrm{id}_P \otimes \theta$. Introducing indeterminants $\mu_1, \ldots, \mu_r, \lambda_1, \ldots, \lambda_s$, we define an R-linear homomorphism

$$\Theta = \Theta_{(F_i, G_j)} : T_{R/k} \longrightarrow A_1(0), \quad \theta \mapsto \sum_{i=1}^{r}\theta(F_i)\mu_i + \sum_{j=1}^{s}\theta(G_j)\lambda_j. \tag{8.20}$$

where

$$A_1(0) = \bigoplus_{i=1}^{r} P_R^{d_i}\mu_i \bigoplus \bigoplus_{j=1}^{s} P_R^{e_j}\lambda_j \quad (P_R^\ell = P^\ell \otimes_k R) \tag{8.21}$$

We note that Θ is surjective (resp. an isomorphism) if $f : S = \mathrm{Spec}(R) \to \widetilde{M}$ is étale (resp. smooth). Put

$$W = \mathrm{Im}(\Theta) \subset A_1(0).$$

It is a finitely generated R-module.

For an ideal $I \subset P_R$ we denote by $A_1(0)/I$ the quotient of $A_1(0)$ by the submodule

$$\bigoplus_{i=1}^{r}(I \cap P_R^{d_i})\mu_i \bigoplus \bigoplus_{j=1}^{s}(I \cap P_R^{e_j})\lambda_j.$$

For a variety V over k we denote by $|V|$ the set of the closed points of V. Let $\alpha \in |\mathbb{P}_R^n|$ and $x \in S = \mathrm{Spec}(R)$ be its image with $\kappa(x)$, its residue field. Let $\mathfrak{m}_{\alpha,x} \subset P_x := P \otimes_R \kappa(x)$ be the homogeneous ideal defining α in $\mathrm{Proj}(P_x)$ and let $\mathfrak{m}_\alpha \subset P_R$ be the inverse image of $\mathfrak{m}_{\alpha,x}$. The evaluation at α induces an isomorphism (note $\kappa(x) = k$)

$$v_\alpha : A_1(0)/\mathfrak{m}_\alpha \simeq \bigoplus_{i=1}^{r} k \cdot \mu_i \bigoplus \bigoplus_{j=1}^{s} k \cdot \lambda_j. \tag{8.22}$$

We now introduce the conditions for $\Psi^q_{U/S}$ to be an isomorphism. We fix an integer $q \geq 0$. Consider the following four conditions.

(I) Both W and $A_1(0)/W$ are locally free R-modules. We put

$$c = \mathrm{rank}_R(A_1(0)/W).$$

(II) W has no base points: $W \to A_1(0)/\mathfrak{m}_\alpha$ is surjective for $\forall \alpha \in |\mathbb{P}_R^n|$.

(III)$_q$ One of the following conditions holds:

- (i) $q = 0$ and $\delta_{\min}(n-r-1) + \mathbf{d} - n - 1 \geq c$,
- (ii) $q = 1$, $\delta_{\min}(n-r-1) + \mathbf{d} - n - 1 \geq c+1$ and $\delta_{\min}(n-r) + \mathbf{d} - n - 1 - d_{\max} \geq c$,
- (iii) $\delta_{\min}(n-1) - n - 1 \geq c + q$.

(IV)$_q$ For any $x \in |S|$ and any $1 \leq j_1 < \cdots < j_{n-r} \leq s$, there exist $q+1$ points $\alpha_0, \cdots, \alpha_q \in |X \cap Y_{j_1} \cap \cdots \cap Y_{j_{n-r}}|$ lying over x such that the map

$$W \to A_1(0)/(J' + \mathfrak{m}_{\alpha_0}) \bigoplus \cdots \bigoplus A_1(0)/(J' + \mathfrak{m}_{\alpha_q})$$

is surjective. Here $J' \subset A_1(0)$ denotes the R-submodule generated by the elements

$$L \cdot \left(\sum_{i=1}^{r} \frac{\partial F_i}{\partial X_k}\mu_i + \sum_{j=1}^{s} \frac{\partial G_j}{\partial X_\nu}\lambda_j\right) \quad \text{with } 0 \leq \nu \leq n \text{ and } L \in P_R^1.$$

Remark 8.2.1. **(I)** holds if f factors as $S \xrightarrow{g} T \xrightarrow{i} \widetilde{M}$ where g is smooth and i is a regular immersion. In this case $c = \mathrm{codim}_{\widetilde{M}}(T)$.

Remark 8.2.2. In view of (8.22), **(II)** holds if $F_i\mu_i,\ G_j\lambda_j \in W$ for $\forall i, j$ and $J' \subset W$.

Remark 8.2.3. $\mathbf{(IV)}_q$ always holds if $s \le n-r+1$. Indeed we will see (cf. 8.5.3) that for any $1 \le j_1 < \cdots < j_{n-r} \le s$ and any $\alpha \in |X \cap Y_{j_1} \cap \cdots \cap Y_{j_{n-r}}|$, $A_1(0)/(J'+\mathfrak{m}_\alpha)$ is a k-vector space of dimension $s-1-(n-r)$ and $A_1(0)/(J'+\mathfrak{m}_\alpha) = 0$ if $s-1 \le n-r$.

Remark 8.2.4. $\mathbf{(IV)}_q$ holds if $W = A_1(0)$ and $\delta_{\min} \ge q$ (cf. §8.2.1). In this case the natural map

$$A_1(0) \longrightarrow \bigoplus_{i=0}^{q} A_1(0)/\mathfrak{m}_{\alpha_i} \tag{8.23}$$

is surjective for arbitrary $(q+1)$-points $\alpha_i \in |\mathbb{P}^n_R|$ $(0 \le i \le q)$ lying over a point $x \in |S|$. To see this it suffices to show that

$$P^q_x \longrightarrow \bigoplus_{i=0}^{q} P^q_x/(\mathfrak{m}_{\alpha_i,x} \cap P^q_x) \tag{8.24}$$

is surjective. Let $H_i \in P^1_x$ $(0 \le i \le q)$ be a linear form such that $H_i(\alpha_j) \ne 0$ for $j \ne i$ and $H_i(\alpha_i) = 0$. Then the images of $H'_i := H_0 \cdots \widehat{H_i} \cdots H_q \in P^q_x$ for $0 \le i \le q$ generate the right hand side of (8.24).

Main Theorem. *Fix an integer $q \ge 0$.*

i) *Assuming* $\mathbf{(IV)}_q$, $\Psi^q_{U/S}$ *is injective.*
ii) *Assuming* $\mathbf{(I)}$, $\mathbf{(II)}_q$, $\mathbf{(III)}$ *and* $\mathbf{(IV)}_q$, $\Psi^q_{U/S}$ *is an isomorphism.*

In order to clarify the technical conditions of the Main Theorem, we explain in the next section its implications on the image of the regulator map (8.2). The proof of the Main Theorem will be given in the sections following the next.

8.3 Implications of the Main Theorem

Let

$$\Omega^{\bullet,d=0}_{R/k} = \mathrm{Ker}(\Omega^{\bullet}_{R/k} \xrightarrow{d} \Omega^{\bullet+1}_{R/k})$$

be the module of closed differential forms.

Theorem 8.3.1. *Fix an integer $q \ge 0$ and assume* $\mathbf{(I)}$, $\mathbf{(II)}$, $\mathbf{(III)}_q$ *and* $\mathbf{(IV)}_{q+1}$ *in the Main Theorem. Then the map ψ^M_ℓ (cf.* (8.14)*) induces an isomorphism*

$$\bigoplus_{\ell=0}^{q} \bigwedge^{\ell+n-r} (G_j) \otimes_{\mathbb{Q}} \Omega^{q-\ell,d=0}_{R/k} \xrightarrow{\cong} \mathrm{Ker}(\nabla_q), \tag{8.25}$$

where $\nabla_q : F^{n-r}H^{n-r}_{\mathrm{dR}}(U/S) \otimes \Omega^q_{R/k} \to F^{n-r-1}H^{n-r}_{\mathrm{dR}}(U/S) \otimes \Omega^{q+1}_{R/k}$.

Proof We first note that $\mathbf{(IV)}_{q+1} \Longrightarrow \mathbf{(IV)}_q$ by definition. Consider the following commutative diagram

$$\begin{array}{ccc}
\bigoplus_{\ell=0}^{q} \bigwedge^{\ell+n-r}(G_j) \otimes_{\mathbb{Q}} \Omega^{q-\ell}_{R/k} & \xrightarrow{\Psi'} & F^{n-r}H^{n-r}_{\mathrm{dR}}(U/S) \otimes \Omega^q_{R/k} \\
{\scriptstyle \mathrm{id}\otimes d}\downarrow & & \downarrow{\scriptstyle \nabla_q} \\
\bigoplus_{\ell=0}^{q+1} \bigwedge^{\ell+n-r}(G_j) \otimes_{\mathbb{Q}} \Omega^{q+1-\ell}_{R/k} & \xrightarrow{\Psi''} & F^{n-r-1}H^{n-r}_{\mathrm{dR}}(U/S) \otimes \Omega^{q+1}_{R/k}.
\end{array}$$

Ψ' is injective and its image is $\ker \overline{\nabla}_q$ by the Main Theorem (ii). Ψ'' is injective by the Main Theorem (i). Thus the assertion follows by diagram chase. $\square$

The second implication of the Main theorem concerns the Hodge filtration on cohomology with coefficients. The Gauss-Manin connection (cf. (8.7)) gives rise to the following complex of Zariski sheaves on S

$$H^{n-r}_{\mathrm{dR}}(U/S) \xrightarrow{\nabla} H^{n-r}_{\mathrm{dR}}(U/S) \otimes \Omega^1_{S/k} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} H^{n-r}_{\mathrm{dR}}(U/S) \otimes \Omega^{\dim S}_{S/k}. \qquad (8.26)$$

which is denoted by $H^{n-r}_{\mathrm{dR}}(U/S) \otimes \Omega^{\bullet}_{S/k}$. We define the de Rham cohomology with coefficients as the hypercohomology

$$H^q_{\mathrm{dR}}(S, H^{n-r}_{\mathrm{dR}}(U/S)) = H^q_{\mathrm{Zar}}(S, H^{n-r}_{\mathrm{dR}}(U/S) \otimes \Omega^{\bullet}_{S/k}).$$

It is a finite dimensional k-vector space. It follows from the theory of mixed Hodge modules by Morihiko Saito ([17]) that $H^{\bullet}_{\mathrm{dR}}(S, H^{n-r}_{\mathrm{dR}}(U/S))$ carries in a canonical way the Hodge filtration and the weight filtration $W_{\bullet}$ denoted by

$$F^p H^{\bullet}_{\mathrm{dR}}(S, H^{n-r}_{\mathrm{dR}}(U/S)) \quad \text{and} \quad W_p H^{\bullet}_{\mathrm{dR}}(S, H^{n-r}_{\mathrm{dR}}(U/S))$$

respectively. (Arapura [1] has recently given a simpler proof of this fact.) In case $k = \mathbb{C}$ there is the comparison isomorphism between the de Rham cohomology and the Betti cohomology ([9, Thm.6.2])

$$H^q(S^{\mathrm{an}}, R^{n-r}\pi^{\mathrm{an}}_*\mathbb{C}_U) \simeq H^q_{\mathrm{dR}}(S, H^{n-r}_{\mathrm{dR}}(U/S)) \quad (\pi : U \to S) \qquad (8.27)$$

which preserves the Hodge and weight filtrations on both sides defined by M. Saito. It endows $H^{\bullet}(S^{\mathrm{an}}, R^{n-r}\pi^{\mathrm{an}}_*\mathbb{Q}_U)$ with a mixed Hodge structure..

Define the subcomplex G^i of $H^{n-r}_{\mathrm{dR}}(U/S) \otimes \Omega^{\bullet}_{S/k}$ as

$$\begin{aligned} F^i H^{n-r}_{\mathrm{dR}}(U/S) \to F^{i-1}H^{n-r}_{\mathrm{dR}}(U/S) \otimes \Omega^1_{S/k} \to \cdots \\ \cdots \to F^{i-\dim S}H^{n-r}_{\mathrm{dR}}(U/S) \otimes \Omega^{\dim S}_{S/k} \end{aligned} \qquad (8.28)$$

where $F^\bullet H^{n-r}_{\mathrm{dR}}(U/S)$ is the Hodge filtration as in (8.9). If S were proper over k, we would have

$$F^i H^\bullet_{\mathrm{dR}}(S, H^{n-r}_{\mathrm{dR}}(U/S)) = H^\bullet_{\mathrm{Zar}}(S, G^i).$$

When S is not proper, there is in general only a natural injection (cf. [2, Lemma 4.2])

$$F^i H^\bullet_{\mathrm{dR}}(S, H^{n-r}_{\mathrm{dR}}(U/S)) \hookrightarrow H^\bullet_{\mathrm{Zar}}(S, G^i) \quad (\forall i \geq 0). \tag{8.29}$$

The precise description of the Hodge filtration on the de Rham cohomology with coefficients is more complicated in general.

Theorem 8.3.2. *Fix an integer $q \geq 0$. Let $S \subset \overline{S}$ be a smooth compactification with $\partial S := \overline{S} - S$, a normal crossing divisor on $\overline{S}$. Assuming* **(I)**, **(II)**, $\mathbf{(III)}_q$ *and* $\mathbf{(IV)}_{q+1}$ *in the main theorem, we have an isomorphism*

$$\bigoplus_{\ell=0}^{q} \bigwedge^{\ell+n-r} (G_j) \otimes_{\mathbb{Q}} \Gamma(\overline{S}, \Omega^{q-\ell}_{\overline{S}/k}(\log \partial S)) \xrightarrow{\cong} F^{n-r+q} H^q_{\mathrm{dR}}(S, H^{n-r}_{\mathrm{dR}}(U/S)). \tag{8.30}$$

Proof We have the following commutative diagram

$$\begin{array}{ccc}
\bigoplus_{\ell=0}^{q} \bigwedge^{\ell+n-r} (G_j) \otimes_{\mathbb{Q}} \Gamma(\overline{S}, \Omega^{q-\ell}_{\overline{S}/k}(\log \partial S)) & \xrightarrow{\phi} & F^{n-r+q} H^q_{\mathrm{dR}}(S, H^{n-r}_{\mathrm{dR}}(U/S)) \\
{\scriptstyle e}\downarrow\cap & & {\scriptstyle a}\downarrow\cap \\
\bigoplus_{\ell=0}^{q} \bigwedge^{\ell+n-r} (G_j) \otimes_{\mathbb{Q}} \Omega^{q-\ell, d=0}_{R/k} & \xrightarrow{\cong} & H^q(S, G^{n-r+q}) \\
{\scriptstyle b}\downarrow & & \downarrow \\
\bigoplus_{\ell=0}^{q} \bigwedge^{\ell+n-r} (G_j) \otimes_{\mathbb{Q}} H^{q-\ell}_{\mathrm{dR}}(S/k) & \xrightarrow{c} & H^q_{\mathrm{dR}}(S, H^{n-r}_{\mathrm{dR}}(U/S))
\end{array} \tag{8.31}$$

where by definition

$$H^t_{\mathrm{dR}}(S/k) = H^t(\overline{S}, \Omega^\bullet_{\overline{S}/k}(\log \partial S))$$

and the map b comes from the isomorphism

$$H^t(\overline{S}, \Omega^\bullet_{\overline{S}/k}(\log \partial S)) \simeq H^t(S, \Omega^\bullet_{S/k}) \simeq \Omega^{t,d=0}_{R/k}/d\Omega^{t-1}_{R/k} \tag{8.32}$$

due to [8, II (3.1.11)] and it is surjective. The map a comes from (8.29). The map e comes from [8, II (3.2.14)]. The bijection in the middle row is the composition of the isomorphism in Theorem 8.3.1 and the isomorphism

$$\mathrm{Ker}(\nabla_\ell) \simeq H^\ell(S, G^{\ell+n-r}) \quad \text{for } \forall \ell \geq 0. \tag{8.33}$$

The map c is induced by the composition

$$\begin{aligned}\bigoplus_{\ell=0}^{q} \bigwedge^{\ell+n-r} (G_j) \otimes_{\mathbb{Q}} \Omega_{R/k}^{q-\ell,d=0} &\longrightarrow \mathrm{Ker}(\nabla_\ell) \otimes_{\mathbb{Q}} H_{\mathrm{dR}}^{q-\ell}(S/k) \\ &\longrightarrow H^\ell(S, H_{\mathrm{dR}}^{n-r}(U/S) \otimes \Omega_{S/k}^\bullet) \otimes H^{q-\ell}(S, \Omega_{S/k}^\bullet) \\ &\longrightarrow H^q(S, H_{\mathrm{dR}}^{n-r}(U/S) \otimes \Omega_{S/k}^\bullet)\end{aligned}$$

where the first map is induced by $\psi_{\ell+n-r}^M$ (8.14) and (8.32), the second by (8.33), and the last by cup product. We claim that there is a map ϕ which makes the upper square of the diagram (8.31) commute. Indeed let V be the source of c. Endowing V with the Hodge filtration defined by

$$F^p\big(\bigwedge^t (G_j) \otimes_{\mathbb{Q}} H_{\mathrm{dR}}^u(S/k)\big) = \bigwedge^t (G_j) \otimes_{\mathbb{Q}} F^{p-t} H_{\mathrm{dR}}^u(S/k)$$

with $F^{p-t}H_{\mathrm{dR}}^u(S/k) = H^u(\overline{S}, \Omega_{\overline{S}/k}^{\geq p-t}(\log \partial S))$, c respects the Hodge filtrations. Noting

$$F^{n-r+q}V = \bigoplus_{\ell=0}^{q} \bigwedge^{\ell+n-r} (G_j) \otimes_{\mathbb{Q}} H^0(\overline{S}, \Omega_{\overline{S}/k}^{q-\ell}(\log \partial S)),$$

we see that c induces ϕ as desired. The injectivity of ϕ follows from that of e. To show its surjectivity, note that $\mathrm{Im}(c)$ contains $F^{n-r+q}H_{\mathrm{dR}}^q(S, H_{\mathrm{dR}}^{n-r}(U/S))$ by the diagram. This shows $F^{n-r+q}\,\mathrm{Coker}(c) = 0$. By strictness of the Hodge filtration, we get the surjectivity of ϕ. This completes the proof of the theorem. $\square$

In what follows we assume $k = \mathbb{C}$. Take $S \subset \overline{S}$, a smooth compactification with $\partial S := \overline{S} - S$, a normal crossing divisor on $\overline{S}$. Write for $t \geq 0$

$$H_{\mathbb{Q}}^{t,0}(S) := H^t(S^{\mathrm{an}}, \mathbb{Q}(t)) \cap F^t H^t(S^{\mathrm{an}}, \mathbb{C}) = H^t(S^{\mathrm{an}}, \mathbb{Q}(t)) \cap H^0(\overline{S}, \Omega_{\overline{S}/\mathbb{C}}^t(\log \partial S))$$

Write $m = n - r = \dim(U/S)$. Let

$$\mathrm{reg}_{U/S}^{m+\ell} : K_{m+\ell}^M(\mathcal{O}(U)) \otimes \mathbb{Q} \to H^\ell(S^{\mathrm{an}}, R^m \pi_*^{\mathrm{an}} \mathbb{Q}(m+\ell)) \cap F^{m+\ell}$$

be as (8.2). It induces for $q \geq 0$

$$\begin{aligned}\lambda_q : \textstyle\bigoplus_{\ell=0}^q K_{m+\ell}^M(\mathcal{O}(U)) \otimes_{\mathbb{Q}} H_{\mathbb{Q}}^{q-\ell,0}(S) &\to \\ \to H^q(S^{\mathrm{an}}, R^m \pi_*^{\mathrm{an}} \mathbb{Q}_U(m+q)) &\cap F^{m+q}.\end{aligned} \tag{8.34}$$

Theorem 8.1.1 follows from the following corollaries in view of Remarks 8.2.1, 8.2.2, 8.2.3.

Corollary 8.3.3. *Fix an integer* $q \geq 1$ *and assume* **(I)**, **(II)**, $\mathbf{(III)}_q$ *and* $\mathbf{(IV)}_{q+1}$ *in the Main Theorem. Then the map* (8.2)

$$\mathrm{reg}^{m+q}_{U/S} : K^M_{m+q}(\mathcal{O}(U)) \otimes \mathbb{Q} \to H^q(S^{\mathrm{an}}, R^m\pi^{\mathrm{an}}_*\mathbb{Q}_U(m+q)) \cap F^{m+q}$$

is surjective for $q = 1$. *More generally* $\mathrm{reg}^{m+q}_{U/S}$ *is surjective if the regulator map for* S*:*

$$\mathrm{reg}^t_S : K^M_t(\mathcal{O}(S)) \otimes \mathbb{Q} \to H_{\mathbb{Q}}(S)^{t,0}$$

is surjective for $1 \leq \forall t \leq q$.

Proof The first assertion of 8.3.3 follows from the second in view of the fact that reg^1_S is surjective, namely $H^{1,0}_{\mathbb{Q}}(S)$ is generated by $\operatorname{dlog}\mathcal{O}(S)^*$. The second assertion is a direct consequence of the following isomorphism induced by λ_q:

$$\bigoplus_{\ell=0}^{q} \bigwedge^{\ell+m} (G_j) \otimes_{\mathbb{Q}} H_{\mathbb{Q}}(S)^{q-\ell,0} \xrightarrow{\cong} H^q(S^{\mathrm{an}}, R^m\pi^{\mathrm{an}}_*\mathbb{Q}_U(m+q)) \cap F^{m+q}. \tag{8.35}$$

which follows from Theorem 8.3.2. □

Corollary 8.3.4. *Assuming* **(I)**, **(II)**, $\mathbf{(III)}_q$ *and* $\mathbf{(IV)}_{q+1}$ *for* $q = 0$, λ_0 *induces an isomorphism*

$$\bigwedge^m (G_j) \xrightarrow{\cong} H^0(S^{\mathrm{an}}, R^m\pi^{\mathrm{an}}_*\mathbb{Q}_U(m)).$$

Proof Applying (8.35), we have an isomorphism

$$\bigwedge^m (G_j) \xrightarrow{\cong} H^0(S^{\mathrm{an}}, R^m\pi^{\mathrm{an}}_*\mathbb{Q}_U(m)) \cap F^m. \tag{8.36}$$

We need show that the right hand side is equal to $H^0(S^{\mathrm{an}}, R^m\pi^{\mathrm{an}}_*\mathbb{Q}_U)$. It suffices to show that $H^0(S^{\mathrm{an}}, R^m\pi^{\mathrm{an}}_*\mathbb{Q})$ is pure of type (m, m). We need a result from [3, Theorem (III)], which implies that the map

$$\overline{\nabla} : \mathrm{Gr}^p_F H^m_{\mathrm{dR}}(U/S) \longrightarrow \mathrm{Gr}^{p-1}_F H^m_{\mathrm{dR}}(U/S) \otimes \Omega^1_{R/\mathbb{C}}$$

is injective for all $1 \leq p \leq m-1$ under the assumption of Corollary 8.3.4. It implies

$$\mathrm{Ker}(\nabla) \cap F^1 H^m_{\mathrm{dR}}(U/S) = \mathrm{Ker}(\nabla) \cap F^m H^m_{\mathrm{dR}}(U/S),$$

where $\nabla : H^m_{\mathrm{dR}}(U/S) \to H^m_{\mathrm{dR}}(U/S) \otimes \Omega^1_{R/\mathbb{C}}$ is the algebraic Gauss-Manin connection. Noting $H^0(S^{\mathrm{an}}, R^m\pi^{\mathrm{an}}_*\mathbb{C}_U) \xrightarrow{\sim} \mathrm{Ker}(\nabla)$ under the comparison isomorphism (8.27), it implies

$$F^1 H^0(S^{\mathrm{an}}, R^m \pi_*^{\mathrm{an}} \mathbb{C}_U) = F^m H^0(S^{\mathrm{an}}, R^m \pi_*^{\mathrm{an}} \mathbb{C}_U). \tag{8.37}$$

Consider the mixed Hodge structure $H := H^0(S^{\mathrm{an}}, R^m \pi_*^{\mathrm{an}} \mathbb{Q}_U)$. By the Hodge symmetry (8.37) implies

$$H^{p,q} := Gr_F^p Gr_{\overline{F}}^q Gr_{p+q}^W H = 0 \quad \text{unless } (p,q) = (m,0), (m,m), (0,m).$$

Hence it suffices to show $H^{m,0} = 0$. Putting $V = F^m H^0(S^{\mathrm{an}}, R^m \pi_{X*}^{\mathrm{an}} \mathbb{C}_X)$ where $\pi_X : X \to S$ is as in (8.6), we have the surjection $V \to H^{m,0}$ while $V = 0$ by Theorem 8.3.2 applied to the case $s = 0$. This completes the proof. □

8.4 Theory of Generalized Jacobian Rings

We introduce the generalized Jacobian ring. It describes the Hodge cohomology groups $H^\bullet(\Omega^\bullet_{X/S}(\log Z_*))$ of open complete intersections, and enables us to identify the Gauss-Manin connection (cf. (8.11)) with the multiplication of rings. The computational results in this section will play a key role in the proof of the Main Theorem (see §8.5.2).

8.4.1 Fundamental results on generalized Jacobian ring

Recall the notations in §8.2.1. Let

$$A = P_R[\mu_1, \cdots, \mu_r, \lambda_1, \cdots, \lambda_s] = R[X_0, \ldots, X_n, \mu_1, \cdots, \mu_r, \lambda_1, \cdots, \lambda_s]$$

be the polynomial ring over P_R with indeterminants $\mu_1, \cdots, \mu_r, \lambda_1, \cdots, \lambda_s$. For $q \in \mathbb{Z}$ and $\ell \in \mathbb{Z}$, we put

$$A_q(\ell) = \bigoplus_{a_1+\cdots+a_r+b_1+\cdots+b_s=q} P_R^{m(a,b,\ell)} \cdot \mu_1^{a_1} \cdots \mu_r^{a_r} \lambda_1^{b_1} \cdots \lambda_s^{b_s}$$

with $m(a,b,\ell) = \sum_{i=1}^r a_i d_i + \sum_{j=1}^s b_j e_j + \ell$. Here a_i and b_j run over non-negative integers satisfying $a_1 + \cdots + a_r + b_1 + \cdots + b_s = q$. By convention, $A_q(\ell) = 0$ for $q < 0$. Note that the notation in 8.21 is compatible with the above definition.

The Jacobian ideal $J = J(F_1, \cdots, F_r, G_1, \cdots, G_s)$ is defined to be the ideal of A generated by

$$\sum_{i=1}^r \frac{\partial F_i}{\partial X_k} \mu_i + \sum_{j=1}^s \frac{\partial G_j}{\partial X_k} \lambda_j, \quad F_\ell, \quad G_{\ell'} \lambda_{\ell'} \quad (0 \le k \le n,\ 1 \le \ell \le r,\ 1 \le \ell' \le s).$$

The quotient ring $B = A/J$ is called the *generalized Jacobian ring* ([3]). We put

$$J_q(\ell) = A_q(\ell) \cap J \quad \text{and} \quad B_q(\ell) = A_q(\ell)/J_q(\ell).$$

We now recall some fundamental results from [3].

Theorem 8.4.1 ([3], Theorem (I)). *Suppose $n \geq r+1$. For each integer $0 \leq p \leq n-r$ there is a natural isomorphism*

$$\phi : B_p(\mathbf{d}+\mathbf{e}-n-1) \xrightarrow{\cong} H^p(X, \Omega^{n-r-p}_{X/S}(\log Z_*))$$

and the following diagram is commutative up to a scalar in $R^\times$:

$$\begin{array}{ccc} B_p(\mathbf{d}+\mathbf{e}-n-1) & \xrightarrow{\epsilon} & B_{p+1}(\mathbf{d}+\mathbf{e}-n-1) \otimes A_1(0)^* \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle 1\otimes\Theta^*} \\ H^p(X, \Omega^{n-r-p}_{X/S}(\log Z_*)) & \xrightarrow{\overline{\nabla}} & H^{p+1}(X, \Omega^{n-r-p-1}_{X/S}(\log Z_*)) \otimes \Omega^1_{R/k} \end{array} \tag{8.38}$$

where $\overline{\nabla}$ is induced by the Gauss-Manin connection:

$$\nabla : F^{n-r-p} H^{n-r}_{\mathrm{dR}}(U/S) \longrightarrow F^{n-r-p-1} H^{n-r}_{\mathrm{dR}}(U/S) \otimes \Omega^1_{R/k}$$

and ϵ is induced from the multiplication

$$B_p(\mathbf{d}+\mathbf{e}-n-1) \otimes A_1(0) \to B_{p+1}(\mathbf{d}+\mathbf{e}-n-1)$$

and Θ^* is the dual of the map (8.20)

$$\Theta : T_{R/k} \to A_1(0); \quad \theta \mapsto \sum_{i=1}^{r} \theta(F_i)\mu_i + \sum_{j=1}^{s} \theta(G_j)\lambda_j.$$

The second fundamental result is the duality theorem for generalized Jacobian rings. For an R-module we denote $M^* = \mathrm{Hom}_R(M, R)$.

Theorem 8.4.2 ([3], Theorem (II)). *There is a natural map (called the trace map)*

$$\tau : B_{n-r}(2(\mathbf{d}-n-1)+\mathbf{e}) \to R.$$

Let

$$h_p : B_p(\mathbf{d}-n-1) \to B_{n-r-p}(\mathbf{d}+\mathbf{e}-n-1)^*$$

be the map induced by the following pairing induced by the multiplication

$$B_p(\mathbf{d}-n-1) \otimes B_{n-r-p}(\mathbf{d}+\mathbf{e}-n-1) \to B_{n-r}(2(\mathbf{d}-n-1)+\mathbf{e}) \xrightarrow{\tau} R.$$

Then h_p is bijective if $1 \leq p \leq n-r$, and surjective if $p = 0$. The kernel of h_0 is a locally free R-module of rank $\binom{s-1}{n-r}$.

8.4.2 Generalized Jacobian rings à la M. Green

We review from [3, §2] a "sheaf theoretic" definition of generalized Jacobian ring. This sophisticated definition originates from M.Green ([10]). It is useful for various computations (cf. §8.4.3 and §8.5.2).

Put

$$\mathbb{E} = \mathbb{E}_0 \bigoplus \mathbb{E}_1 \quad \text{with } \mathbb{E}_0 = \bigoplus_{i=1}^{r} \mathcal{O}(d_i) \text{ and } \mathbb{E}_1 = \bigoplus_{j=1}^{s} \mathcal{O}(e_j)$$

which is a locally free sheaf on $\mathbb{P}^n = \mathbb{P}^n_R$. We consider the projective space bundle

$$\pi : \mathbb{P} := \mathbb{P}(\mathbb{E}) \longrightarrow \mathbb{P}^n.$$

Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathbb{E})}(1)$ be the tautological line bundle. We have the Euler exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \pi^*\mathbb{E}^* \otimes \mathcal{L} \longrightarrow T_{\mathbb{P}/\mathbb{P}^n} \longrightarrow 0. \tag{8.39}$$

We consider the effective divisors

$$\mathbb{Q}_i := \mathbb{P}\Big(\bigoplus_{1\le\alpha\ne i\le r} \mathcal{O}(d_\alpha) \bigoplus \mathbb{E}_1\Big) \hookrightarrow \mathbb{P}(\mathbb{E}) \quad \text{for } 1 \le i \le r,$$

$$\mathbb{P}_j := \mathbb{P}\Big(\mathbb{E}_0 \bigoplus \bigoplus_{1\le\beta\ne j\le r} \mathcal{O}(e_\beta)\Big) \hookrightarrow \mathbb{P}(\mathbb{E}) \quad \text{for } 1 \le j \le s,$$

and let

$$\mu_i \in H^0(\mathbb{P}, \mathcal{L} \otimes \pi^*\mathcal{O}(-d_i)), \quad \lambda_j \in H^0(\mathbb{P}, \mathcal{L} \otimes \pi^*\mathcal{O}(-e_j))$$

be the global sections associated to these. We put

$$\sigma = \sum_{i=1}^{r} F_i\mu_i + \sum_{j=1}^{s} G_j\lambda_j \ \in \Gamma(\mathbb{P}, \mathcal{L}).$$

Let $\Sigma_{\mathcal{L}}$ be the sheaf of differential operators on $\mathcal{L}$ of order ≤ 1, defined as:

$$\begin{aligned}\Sigma_{\mathcal{L}} = \mathcal{D}\mathit{iff}^{\,\le 1}(\mathcal{L}) &= \{P \in \mathcal{E}nd_k(\mathcal{L}) \ ; \ Pf - fP \text{ is } \mathcal{O}_{\mathbb{P}}\text{-linear for } \forall f \in \mathcal{O}_{\mathbb{P}}\} \\ &\simeq \mathcal{L} \otimes \mathcal{D}\mathit{iff}^{\,\le 1}(\mathcal{O}_{\mathbb{P}}) \otimes \mathcal{L}^*.\end{aligned}$$

(It might be helpful to mention that $\Sigma_{\mathcal{L}}$ is a prolongation bundle.) By definition it fits into an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \Sigma_{\mathcal{L}} \longrightarrow T_{\mathbb{P}} \longrightarrow 0 \tag{8.40}$$

with extension class

$$-c_1(\mathcal{L}) \in \mathrm{Ext}^1(T_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}) \simeq \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}}, \Omega^1_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}) \simeq H^1(\mathbb{P}, \Omega^1_{\mathbb{P}}).$$

Letting $U \subset \mathbb{P}^n$ be an affine subspace and $x_1, \cdots, x_n$ be its coordinates, $\Gamma(\pi^{-1}(U), \Sigma_{\mathcal{L}})$ is generated by the following sections

$$\frac{\partial}{\partial x_i},\ \lambda_i\frac{\partial}{\partial \lambda_j},\ \lambda_i\frac{\partial}{\partial \mu_j},\ \mu_i\frac{\partial}{\partial \lambda_j},\ \mu_i\frac{\partial}{\partial \mu_j},\ \mathcal{O}_{\mathbb{P}}\text{-linear maps.} \tag{8.41}$$

The section σ defines a map

$$j(\sigma) : \Sigma_{\mathcal{L}} \longrightarrow \mathcal{L}, \quad P \longmapsto P(\sigma),$$

which is surjective by the assumption that X_*+Y_* is a simple normal crossing divisor. It gives rise to the exact sequence

$$0 \longrightarrow T_{\mathbb{P}}(-\log \mathcal{Z}) \longrightarrow \Sigma_{\mathcal{L}} \xrightarrow{j(\sigma)} \mathcal{L} \longrightarrow 0, \tag{8.42}$$

where $\mathcal{Z} \subset \mathbb{P}$ is the zero divisor of σ. Put

$$\mathbb{Q}_* = \mathbb{Q}_1 + \cdots + \mathbb{Q}_r \quad \text{and} \quad \mathbb{P}_* = \mathbb{P}_1 + \cdots + \mathbb{P}_s,$$

and define $\Sigma_{\mathcal{L}}(-\log \mathbb{P}_*)$ to be the inverse image of $T_{\mathbb{P}}(-\log \mathbb{P}_*)$ via the map in (8.40). We then have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \Sigma_{\mathcal{L}}(-\log \mathbb{P}_*) \longrightarrow T_{\mathbb{P}}(-\log \mathbb{P}_*) \longrightarrow 0. \tag{8.43}$$

Moreover (8.42) gives rise to an exact sequence

$$0 \longrightarrow T_{\mathbb{P}}(-\log(\mathcal{Z} + \mathbb{P}_*)) \longrightarrow \Sigma_{\mathcal{L}}(-\log \mathbb{P}_*) \xrightarrow{j(\sigma)} \mathcal{L} \longrightarrow 0. \tag{8.44}$$

Lemma 8.4.3. *For integers k and ℓ, put $A_k(\ell)_\Sigma = H^0(\mathcal{L}^k \otimes \pi^*\mathcal{O}(\ell))$ and*

$$J_k(\ell)_\Sigma = \mathrm{Im}\left(H^0(\Sigma_{\mathcal{L}}(-\log \mathbb{P}_*) \otimes \mathcal{L}^{k-1} \otimes \pi^*\mathcal{O}(\ell)) \xrightarrow{j(\sigma)\otimes 1} H^0(\mathcal{L}^k \otimes \pi^*\mathcal{O}(\ell))\right).$$

Then we have

$$A_k(\ell) = A_k(\ell)_\Sigma, \quad J_k(\ell) = J_k(\ell)_\Sigma.$$

Proof See [3, Lem.(2-2)]. □

Thus we have obtained another definition of the generalized Jacobian ring.

8.4.3 Some computational results

We keep the notations in §8.4.2. In what follows we simply write $\Sigma = \Sigma_{\mathcal{L}}(-\log \mathbb{P}_*)$.

Lemma 8.4.4. *We have*

$$H^w(\mathbb{P}, \bigwedge^p \Sigma^* \otimes \mathcal{L}^\nu \otimes \pi^* \mathcal{O}(\ell)) = 0$$

if one of the following conditions holds:

(i) $p - \nu \le r + s - 1$ *and* $\nu \le -1$,
(ii) $p - \nu \ge n + 1$ *and* $\nu \ge -s + 1$.
(iii) $w > 0$, $\nu \ge -s + 1$, $\ell \ge 0$ *and* $(\nu, \ell) \ne (0, 0)$.

Proof See [3, Thm.(4-1)]. □

Proposition 8.4.5. *Let k be an integer.*

(i) $H^\nu(\bigwedge^k \Sigma^*) = 0$ *for any* $k \ge 0$ *and* $\nu \ne 0,\ n$.
(ii) $H^0(\bigwedge^k \Sigma^*)$ *is a locally free R-module of rank* $\binom{s-1}{k}$.
(iii) $H^n(\bigwedge^k \Sigma^*)$ *is a locally free R-module of rank* $\binom{s-1}{k-n-1}$. *(Note* $\binom{x}{\ell} = 0$ *for* $\ell < 0$ *by convention.)*

The rest of this section is devoted to the proof of Proposition 8.4.5.

Recall that there is an exact sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}}(\log \mathbb{P}_*) \longrightarrow \Sigma^* \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow 0 \tag{8.45}$$

with the extension class

$$c_1(\mathcal{L})|_{\mathbb{P}-\mathbb{P}_*} \in \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}}, \Omega^1_{\mathbb{P}}(\log \mathbb{P}_*)) = H^1(\Omega^1_{\mathbb{P}}(\log \mathbb{P}_*)).$$

It gives rise to the short exact sequence

$$0 \longrightarrow \Omega^\bullet_{\mathbb{P}}(\log \mathbb{P}_*) \longrightarrow \bigwedge^\bullet \Sigma^* \longrightarrow \Omega^{\bullet-1}_{\mathbb{P}}(\log \mathbb{P}_*) \longrightarrow 0, \tag{8.46}$$

and we have the long exact sequence

$$\begin{aligned} \cdots H^\nu(\Omega^k_{\mathbb{P}}(\log \mathbb{P}_*)) \longrightarrow H^\nu(\bigwedge^k \Sigma^*) \longrightarrow \\ H^\nu(\Omega^{k-1}_{\mathbb{P}}(\log \mathbb{P}_*)) \xrightarrow{\delta} H^{\nu+1}(\Omega^k_{\mathbb{P}}(\log \mathbb{P}_*)) \cdots, \end{aligned} \tag{8.47}$$

where δ is induced by the cup-product with $c_1(\mathcal{L})|_{\mathbb{P}-\mathbb{P}_*} \in H^1(\Omega^1_{\mathbb{P}}(\log \mathbb{P}_*))$.

The first step is to write down $H^\bullet(\Omega^\bullet_{\mathbb{P}}(\log \mathbb{P}_*))$ and δ explicitly. We prepare some notations. For an integer $k \geq 1$ we put

$$\Delta_k = \{I = (i_1, \cdots, i_k) |\ 1 \leq i_1 < \cdots < i_k \leq s\}.$$

For $I = (i_1, \cdots, i_k) \in \Delta_k$ we write $\mathbb{P}_I = \mathbb{P}_{i_1} \cap \cdots \cap \mathbb{P}_{i_1}$. For $k = 0$ we put $\Delta_0 = \{\varnothing\}$ and $\mathbb{P}_\varnothing = \mathbb{P}$ by convention. To compute $H^\nu(\Omega^k_{\mathbb{P}}(\log \mathbb{P}_*))$ we first note the isomorphisms

$$H^\nu(\Omega^k_{\mathbb{P}}(\log \mathbb{P}_*)) \simeq \mathrm{Gr}^k_F H^{\nu+k}(\mathbb{P} - \mathbb{P}_*) \simeq \mathrm{Gr}^W_{2k} H^{\nu+k}(\mathbb{P} - \mathbb{P}_*),$$

where $H^*(\mathbb{P} - \mathbb{P}_*) = H^*(\mathbb{P}, \Omega^\bullet_{\mathbb{P}/S}(\log \mathbb{P}_*))$, and

$$F^p H^*(\mathbb{P} - \mathbb{P}_*) = H^*(\mathbb{P}, \Omega^{\geq p}_{\mathbb{P}/S}(\log \mathbb{P}_*)) \subset H^*(\mathbb{P}, \Omega^\bullet_{\mathbb{P}/S}(\log \mathbb{P}_*))$$

is the Hodge filtration and $W_p H^*(\mathbb{P} - \mathbb{P}_*) \subset H^*(\mathbb{P} - \mathbb{P}_*)$ denotes the weight filtration induced by the spectral sequence

$$E_2^{pq} = \bigoplus_{I \in \Delta_q} H^p(\mathbb{P}_I) \Longrightarrow H^{p+q}(\mathbb{P} - \mathbb{P}_*) \tag{8.48}$$

where $H^*(\mathbb{P}_I) = H^*(\mathbb{P}_I, \Omega^\bullet_{\mathbb{P}_I/S})$ (cf. [8]). We note $E_2^{p,q} = 0$ unless $0 \leq q \leq s$. Since the spectral sequence (8.48) degenerates at E_3, $H^\nu(\Omega^k_{\mathbb{P}}(\log \mathbb{P}_*))$ is isomorphic to the cohomology at the middle term of the following complex

$$\bigoplus_{I_1 \in \Delta_{k-\nu+1}} H^{2\nu-2}(\mathbb{P}_{I_1}) \longrightarrow \bigoplus_{I_2 \in \Delta_{k-\nu}} H^{2\nu}(\mathbb{P}_{I_2}) \longrightarrow \bigoplus_{I_3 \in \Delta_{k-\nu-1}} H^{2\nu+2}(\mathbb{P}_{I_3}). \tag{8.49}$$

The arrows in (8.49) are described as follows. Let $I_1 = (i_1, \cdots, i_k) \in \Delta_k$ and $I_2 \in \Delta_{k-1}$. If $I_2 \not\subset I_1$, then $H^{2\nu-2}(\mathbb{P}_{I_1}) \to H^{2\nu}(\mathbb{P}_{I_2})$ is the zero map. If $I_2 = (i_1, \cdots, \widehat{i_p}, \cdots, i_k)$, then it is $(-1)^{p-1}\phi_{I_1 I_2}$ where $\phi_{I_1 I_2}$ is the Gysin map. In order to describe it in more convenient way we introduce some notations. Let

$$S_R = R[x, y]$$

be the polynomial ring and S_R^ν be the set of homogeneous polynomials of degree ν. We put

$$Q_I(x, y) = \begin{cases} \displaystyle\prod_{i=1}^r (x - d_i y) \cdot \prod_{j \notin I} (x - e_j y) & \text{if } I \neq \varnothing, \\ \displaystyle\prod_{i=1}^r (x - d_i y) \cdot \prod_{j=1}^s (x - e_j y) & \text{if } I = \varnothing, \end{cases}$$

Lemma 8.4.6. i) *There is an isomorphism of graded rings:*

$$S_R/(Q_I(x,y),y^{n+1}) \xrightarrow{\cong} H^*(\mathbb{P}_I);\ x \mapsto c_1(\mathcal{L})|_{\mathbb{P}_I},\ y \mapsto \pi^* c_1(\mathcal{O}(1))|_{\mathbb{P}_I},$$

where we recall $\pi : \mathbb{P} = \mathbb{P}(\mathbb{E}) \to \mathbb{P}^n$.

ii) *For* $I \subset I'$ *we have the commutative diagram*

$$\begin{array}{ccc} H^*(\mathbb{P}_I) & \xrightarrow{\cong} & R[x,y]/(Q_I(x,y),y^{n+1}) \\ {\scriptstyle\psi_{II'}}\downarrow & & \downarrow \\ H^*(\mathbb{P}_{I'}) & \xrightarrow{\cong} & R[x,y]/(Q_{I'}(x,y),y^{n+1}) \end{array}$$

where the left vertical map is the restriction map and the right vertical map is the natural surjection.

iii) *If* $I' = I \cup \{j\}$ *with* $j \notin I$ *we have the commutative diagram*

$$\begin{array}{ccc} H^*(\mathbb{P}_{I'}) & \xrightarrow{\cong} & R[x,y]/(Q_{I'}(x,y),y^{n+1}) \\ {\scriptstyle\phi_{II'}}\downarrow & & \downarrow{\scriptstyle x-e_jy} \\ H^*(\mathbb{P}_I) & \xrightarrow{\cong} & R[x,y]/(Q_I(x,y),y^{n+1}) \end{array}$$

where the left vertical map is the Gysin map and the right vertical map is the multiplication by $x - e_j y$.

Proof The first assertion is well-known, and the second assertion follows immediately from the first. To show the last assertion, we note that $\phi_{II'}$ is the Poincaré dual of $\psi_{II'}$ and the composite $\phi_{II'}\psi_{II'} : H^*(\mathbb{P}_I) \to H^{*+2}(\mathbb{P}_I)$ is the multiplication by the class $c_1(\mathbb{P}_{I'})|_{\mathbb{P}_I}$ of the divisor $\mathbb{P}_{I'}$ in $\mathbb{P}_I$. Hence the assertion follows by noting $c_1(\mathbb{P}_{I'})|_{\mathbb{P}_I} = c_1(\mathbb{P}_j)|_{\mathbb{P}_I} = x - e_j y$. □

For $I = (i_1, \ldots, i_\ell) \in \Delta_\ell$ write $\lambda_I = \lambda_{i_1} \wedge \cdots \wedge \lambda_{i_\ell}$ and $\lambda_I = 1$ if $I = \varnothing$. Lemma 8.4.6 provides us with an isomorphism

$$\bigoplus_{I\in\Delta_\ell} S_R^p/(Q_I(x,y),y^{n+1}) \otimes \lambda_I \xrightarrow{\cong} \bigoplus_{I\in\Delta_\ell} H^{2p}(\mathbb{P}_I) \quad \text{for each } p \geq 0.$$

Under this isomorphism, the arrows in (8.49) are identified with

$$d_\lambda : \xi \otimes \lambda_I \longmapsto \sum_{k=1}^{\ell} (-1)^{k-1}(x - e_{i_k}y)\xi \otimes \lambda_{i_1} \wedge \cdots \wedge \widehat{\lambda_{i_k}} \wedge \cdots \wedge \lambda_{i_\ell}. \tag{8.50}$$

Thus we have obtained the following result.

Lemma 8.4.7. *For any integers k and ν, $H^\nu(\Omega^k_{\mathbb{P}}(\log \mathbb{P}_*))$ is isomorphic to the cohomology at the middle term of the complex*

$$\bigoplus_{I_1\in\Delta_{k-\nu+1}} S_R^{\nu-1}/(Q_{I_1}(x,y),y^{n+1})\otimes\lambda_{I_1} \xrightarrow{d_\lambda} \bigoplus_{I_2\in\Delta_{k-\nu}} S_R^{\nu}/(Q_{I_2}(x,y),y^{n+1})\otimes\lambda_{I_2} \xrightarrow{d_\lambda} \bigoplus_{I_3\in\Delta_{k-\nu-1}} S_R^{\nu+1}/(Q_{I_3}(x,y),y^{n+1})\otimes\lambda_{I_3} \quad (8.51)$$

with d_λ defined as in (8.50) *(Note that by convention $\bigoplus_{I\in\Delta_\ell}(\cdots)=0$ unless $0\le\ell\le s$).*

In order to calculate the cohomology at the middle term of the complex (8.51), we introduce new symbols $\varepsilon_1,\cdots,\varepsilon_s$ and write $\varepsilon_I=\varepsilon_{i_1}\wedge\cdots\wedge\varepsilon_{i_\ell}$ for $I=(i_1,\dots,i_\ell)\in\Delta_\ell$. Consider the following diagram

$$\begin{tikzcd}
 0 \arrow[d] & 0 \arrow[d] \\
 \bigoplus_{I\in\Delta_\ell} S_R\otimes\varepsilon_I \arrow[r, "d_\varepsilon"] \arrow[d, "a_I"'] & \bigoplus_{I'\in\Delta_{\ell-1}} S_R\otimes\varepsilon_{I'} \arrow[d, "a_{I'}"] \\
 \bigoplus_{I\in\Delta_\ell} S_R\otimes\lambda_I \bigoplus S_R\otimes\varepsilon_I \arrow[r, "d_\lambda+d_\varepsilon"] \arrow[d, "b_I"'] & \bigoplus_{I'\in\Delta_{\ell-1}} S_R\otimes\lambda_{I'} \bigoplus S_R\otimes\varepsilon_{I'} \arrow[d, "b_{I'}"] \\
 \bigoplus_{I\in\Delta_\ell} S_R\otimes\lambda_I \arrow[r, "d_\lambda"] \arrow[d] & \bigoplus_{I'\in\Delta_{\ell-1}} S_R\otimes\lambda_{I'} \arrow[d] \\
 \bigoplus_{I\in\Delta_\ell} S_R/(Q_I,y^{n+1})\otimes\lambda_I \arrow[r, "d_\lambda"] \arrow[d] & \bigoplus_{I'\in\Delta_{\ell-1}} S_R/(Q_{I'},y^{n+1})\otimes\lambda_{I'} \arrow[d] \\
 0 & 0
\end{tikzcd} \quad (8.52)$$

where d_ε is given by:

$$d_\varepsilon:\xi\otimes\varepsilon_I\longmapsto\sum_{k=1}^{\ell}(-1)^{k-1}\xi\otimes\varepsilon_{i_1}\wedge\cdots\wedge\widehat{\varepsilon_{i_k}}\wedge\cdots\wedge\varepsilon_{i_\ell}, \quad (8.53)$$

a_I and b_I are given by:

$$a_I:\xi\otimes\varepsilon_I\mapsto\xi Q_I\otimes\lambda_I+\xi y^{n+1}\otimes\varepsilon_I \quad\text{and}\quad b_I:\xi_1\otimes\lambda_I+\xi_2\otimes\varepsilon_I\mapsto(\xi_1y^{n+1}-\xi_2Q_I)\otimes\lambda_I.$$

One can easily check that the diagram is commutative and the vertical sequences are exact. Put

$$E^\ell = \mathrm{Ker}\left(\bigoplus_{I\in\Delta_\ell} S_R^0 \otimes \varepsilon_I \xrightarrow{d_\varepsilon} \bigoplus_{I\in\Delta_{\ell-1}} S_R^0 \otimes \varepsilon_I\right). \tag{8.54}$$

Note $E^\ell = 0$ unless $0 \le \ell \le s$ by convention.

Claim 8.4.8. i) *The following sequence is exact:*

$$0 \longrightarrow \bigoplus_{I\in\Delta_s} S_R \otimes \varepsilon_I \xrightarrow{d_\varepsilon} \bigoplus_{I\in\Delta_{s-1}} S_R \otimes \varepsilon_I \xrightarrow{d_\varepsilon} \cdots \longrightarrow \bigoplus_{I=\varnothing} S_R \otimes 1 \longrightarrow 0.$$

ii) *Assume $s \ge 2$ and $e_i \ne e_j$ for some $i \ne j$. Then, for an integer $\ell \ge 0$, the following sequence is exact:*

$$\bigoplus_{I\in\Delta_\ell} S_R^0 \otimes \lambda_I \xrightarrow{d_\lambda} \bigoplus_{I\in\Delta_{\ell-1}} S_R^1 \otimes \lambda_I \xrightarrow{d_\lambda} \cdots \longrightarrow \bigoplus_{I=\varnothing} S_R^\ell \otimes 1 \longrightarrow 0.$$

iii) *Assume $e = e_1 = \cdots = e_s$. Then the cohomology at the middle term of the complex*

$$\bigoplus_{I_1\in\Delta_{\ell+1}} S_R^{p-1} \otimes \lambda_{I_1} \xrightarrow{d_\lambda} \bigoplus_{I_2\in\Delta_\ell} S_R^p \otimes \lambda_{I_2} \xrightarrow{d_\lambda} \bigoplus_{I_3\in\Delta_{\ell-1}} S_R^{p+1} \otimes \lambda_{I_3} \tag{8.55}$$

is isomorphic to $S_R^p/(x-ey) \otimes E^\ell$.

Proof (i) Easy (and well-known).
(ii) Let V_0 be a free R-module with basis $\lambda_1, \cdots, \lambda_s$. Let $c : \mathcal{O}_{\mathbb{P}^1} \otimes V_0 \to \mathcal{O}_{\mathbb{P}^1}(1)$ be the map of locally free sheaves on $\mathbb{P}^1 = \mathrm{Proj}(S_R)$, defined by $\lambda_j \mapsto x - e_j y$. This is surjective by the assumption. It gives rise to the Koszul complex

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbb{P}^1}(\ell-s) \otimes \bigwedge^s V_0 \to \mathcal{O}_{\mathbb{P}^1}(\ell-s+1) \otimes \bigwedge^{s-1} V_0 \to \cdots \\ \cdots \to \mathcal{O}_{\mathbb{P}^1}(\ell-1) \otimes V_0 \to \mathcal{O}_{\mathbb{P}^1}(\ell) \to 0. \end{aligned} \tag{8.56}$$

We decompose (8.56) into the following sequences:

$$0 \to \mathcal{O}_{\mathbb{P}^1}(\ell-s) \otimes \bigwedge^s V_0 \to \cdots \to \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \bigwedge^{\ell+1} V_0 \to V_1 \to 0. \tag{8.57}$$

$$0 \to V_1 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \otimes \bigwedge^{\ell} V_0 \to \cdots \to \mathcal{O}_{\mathbb{P}^1}(\ell-1) \otimes V_0 \to \mathcal{O}_{\mathbb{P}^1}(\ell) \to 0. \tag{8.58}$$

Noting $H^0(\mathcal{O}_{\mathbb{P}^1}(\ell)) = S_R^\ell$ and that (8.58) gives an acyclic resolution of V_1, it suffices to show $H^i(\mathbb{P}^1, V_1) = 0$ for $i \geq 1$ which is obvious for $i \geq 2$. To show $H^1(V_1) = 0$ it suffices to prove $H^0(\mathcal{O}(-2) \otimes V_1^*) = 0$ by the Serre duality. By (8.57) there is an injection $\mathcal{O}(-2) \otimes V_1^* \hookrightarrow \mathcal{O}(-1) \otimes \bigwedge^{\ell+1} V_0^*$. The assertion follows from this.

(iii) Since $e = e_1 = \cdots = e_s$, we have the following commutative diagram

$$\begin{array}{ccccc} \bigoplus_{I_1 \in \Delta_{\ell+1}} S_R^{p-1} \otimes \varepsilon_{I_1} & \xrightarrow{d_\epsilon} & \bigoplus_{I_2 \in \Delta_{\ell}} S_R^{p-1} \otimes \varepsilon_{I_2} & \xrightarrow{d_\epsilon} & \bigoplus_{I_3 \in \Delta_{\ell-1}} S_R^{p-1} \otimes \varepsilon_{I_3} \\ {\scriptstyle \iota_0}\downarrow & & {\scriptstyle \iota_1}\downarrow & & {\scriptstyle \iota_2}\downarrow \\ \bigoplus_{I_1 \in \Delta_{\ell+1}} S_R^{p-1} \otimes \lambda_{I_1} & \xrightarrow{d_\lambda} & \bigoplus_{I_2 \in \Delta_{\ell}} S_R^{p} \otimes \lambda_{I_2} & \xrightarrow{d_\lambda} & \bigoplus_{I_3 \in \Delta_{\ell-1}} S_R^{p+1} \otimes \lambda_{I_3} \end{array}$$

where $\iota_i : \xi \otimes \varepsilon_I \mapsto (x - ey)^i \xi \otimes \lambda_I$. Note that ι_0 is bijective and ι_i are injective for $i > 0$. Due to (i), the cohomology group of the complex (8.55) is isomorphic to

$$\operatorname{Ker}\left(\bigoplus_{I_2 \in \Delta_\ell} S_R^p/(x - ey) \otimes \lambda_{I_2} \xrightarrow{d_\lambda} \bigoplus_{I_3 \in \Delta_{\ell-1}} S_R^{p+1}/(x - ey)^2 \otimes \lambda_{I_3} \right). \quad (8.59)$$

The map $S_R^p/(x-ey) \to S_R^{p+1}/(x-ey)^2$, given by multiplication with $(x-ey)$, is injective. Hence (8.59) is isomorphic to

$$\operatorname{Ker}\left(\bigoplus_{I_2 \in \Delta_\ell} S_R^p/(x - ey) \otimes \varepsilon_{I_2} \xrightarrow{d_\varepsilon} \bigoplus_{I_3 \in \Delta_{\ell-1}} S_R^{p}/(x - ey) \otimes \varepsilon_{I_3} \right) = S_R^p/(x-ey) \otimes E^\ell.$$

This completes the proof. □

Combining Lemmas 8.4.7, 8.4.8 and (8.52), we get the following explicit description of $H^\nu(\Omega_{\mathbb{P}}^k(\log \mathbb{P}_*))$.

Lemma 8.4.9. *Let $E^\bullet$ be as in* (8.54), *and put*

$$\Lambda^\ell = \operatorname{Ker}\left(\bigoplus_{I \in \Delta_\ell} S_R^0 \otimes \lambda_I \xrightarrow{d_\lambda} \bigoplus_{I \in \Delta_{\ell-1}} S_R^1 \otimes \lambda_I \right).$$

i) *Assume $s \geq 2$ and $e_i \neq e_j$ for some $i \neq j$. Then we have*

$$H^\nu(\Omega_{\mathbb{P}}^k(\log \mathbb{P}_*)) \simeq \begin{cases} \Lambda^k & \text{if } \nu = 0 \\ \Lambda^{k-n-1} & \text{if } \nu = n \\ 0 & \text{otherwise} \end{cases}$$

(*Note $\Lambda^\ell = 0$ unless $0 \leq \ell \leq s$ by convention.*)

ii) *Assume* $e = e_1 = \cdots = e_s$. *Then we have for* $\forall k,\ \nu \geq 0$

$$H^\nu(\Omega^k_{\mathbb{P}}(\log \mathbb{P}_*)) \simeq S^\nu_R/(y^{n+1}, x - ey) \otimes E^{k-\nu} \simeq R[y]^\nu/(y^{n+1}) \otimes E^{k-\nu}.$$

In order to complete the proof of Proposition 8.4.5 we need the following Lemma, which gives an explicit description of the map δ in (8.47).

Lemma 8.4.10. i) *Assume* $s \geq 2$ *and* $e_i \neq e_j$ *for some* $i \neq j$. *Then* $c_1(\mathcal{L})|_{\mathbb{P}-\mathbb{P}_*} = 0$ *and* $\delta = 0$.

ii) *Assume* $e = e_1 = \cdots = e_s$. *Then*

$$\pi^* : H^1(\Omega^1_{\mathbb{P}^n}) \longrightarrow H^1(\Omega^1_{\mathbb{P}}(\log \mathbb{P}_*))$$

is injective and $c_1(\mathcal{L})|_{\mathbb{P}-\mathbb{P}_*} = \pi^* c_1(\mathcal{O}(e))$. *The map* δ *in* (8.47) *is identified with the multiplication by* $ey \otimes 1$ *under the isomorphisms in Lemma 8.4.7.*

Proof (i) Noting $\mathcal{O}(\mathbb{P}_j) \simeq \mathcal{L} \otimes \pi^*\mathcal{O}(-e_j)$ and $c_1(\mathbb{P}_j)|_{\mathbb{P}-\mathbb{P}_*} = 0$, we have

$$c_1(\mathcal{L})|_{\mathbb{P}-\mathbb{P}_*} = \pi^* c_1(\mathcal{O}(e_j)) = e_j \pi^* c_1(\mathcal{O}(1)) \quad \text{for } 1 \leq \forall j \leq s.$$

By the assumption this implies $c_1(\mathcal{L})|_{\mathbb{P}-\mathbb{P}_*} = 0$ and $\pi^* c_1(\mathcal{O}(1)) = 0$.
(ii) The first assertion follows from the existence of an isomorphism $\mathbb{P} \simeq \mathbb{P}^n \times \mathbb{P}^{r+s-1}$ such that $\mathbb{P}_j$ corresponds to $\mathbb{P}^n \times H_j$ with H_j a hyperplane. The second assertion has been already shown in (i). To show the last we first note that the cup product for $H^\nu(\Omega^k_{\mathbb{P}}(\log \mathbb{P}_*))$ is induced by the cup product

$$H^{2i}(\mathbb{P}_I) \otimes H^{2j}(\mathbb{P}_I) \to H^{2(i+j)}(\mathbb{P}_{I+J})$$

when one identifies $H^\nu(\Omega^k_{\mathbb{P}}(\log \mathbb{P}_*))$ with the cohomology at the middle term of the complex (8.49). Here $\mathbb{P}_{I+J} = \mathbb{P}_I \cap \mathbb{P}_J$ if $I \cap J = \varnothing$ and $H^{2(i+j)}(\mathbb{P}_{I+J}) = 0$ otherwise by convention. Under the isomorphisms of Lemma 8.4.6, it is identified with

$$S^i_R/(Q_I(x,y), y^{n+1}) \otimes S^j_R/(Q_J(x,y), y^{n+1}) \to S^{i+j}_R/(Q_{I+J}(x,y), y^{n+1}),$$

$$(f \otimes \lambda_I) \otimes (g \otimes \lambda_J) \mapsto fg \otimes (\lambda_I \bigwedge \lambda_J).$$

Since δ is induced by the cup product with $c_1(\mathcal{L})|_{\mathbb{P}-\mathbb{P}_*} \in H^1(\Omega^1_{\mathbb{P}}(\log \mathbb{P}_*))$, the desired assertion follows by noting $c_1(\mathcal{L})|_{\mathbb{P}-\mathbb{P}_*}$ corresponds to ey under the isomorphism $H^1(\Omega^1_{\mathbb{P}}(\log \mathbb{P}_*)) \simeq S^1_R/(Q_\varnothing(x,y), y^{n+1})$ due to Lemma 8.4.7. $\square$

Finally we can complete the proof of Proposition 8.4.5. First assume $s \geq 2$ and $e_i \neq e_j$ for some $i \neq j$. The assertion follows from (8.47), Lemma 8.4.10 (i) and Lemma 8.4.9 (i) by noting that the R-module Λ^ℓ is locally free of rank $\binom{s-2}{\ell}$ due to Claim 8.4.8 (ii). (Compare the coefficients of $(1-x)^{s-2} = (1-x)^s \cdot (1-x)^{-2} = (\sum_p (-1)^p \binom{s}{p} x^p) \cdot (\sum_q qx^q)$ to get $\binom{s-2}{\ell} = \binom{s}{\ell} - 2\binom{s}{\ell-1} + \cdots$.) Next assume $e = e_1 = \cdots = e_s$. By (8.47), Lemma 8.4.10 (ii) and Lemma 8.4.9 (ii), we have an exact sequence

$$
\begin{aligned}
R[y]^{\nu-1}/(y^{n+1}) \xrightarrow{ey\otimes 1} R[y]^{\nu}/(y^{n+1}) \otimes E^{k-\nu} \longrightarrow H^\nu(\bigwedge^k \Sigma^*) \\
\longrightarrow R[y]^{\nu}/(y^{n+1}) \otimes E^{k-\nu-1} \xrightarrow{ey\otimes 1} R[y]^{\nu+1}/(y^{n+1}) \otimes E^{k-\nu-1}.
\end{aligned}
$$

The desired assertion follows by noting that the R-module E^ℓ is locally free of rank $\binom{s-1}{\ell}$ due to Claim 8.4.8 (i). This completes the proof of Proposition 8.4.5.

8.5 Proof of the Main Theorem

In this section we prove the Main Theorem stated in §8.2.4.

8.5.1 Proof of (i)

Let

$$
\Psi^q_{U/S} : \bigoplus_{\ell=0}^{q} \bigwedge^{\ell+n-r} (G_j)_X \otimes_{\mathbb{Q}} \Omega^{q-\ell}_{R/k} \longrightarrow H^0(\Omega^{n-r}_{X/S}(\log Z_*)) \otimes \Omega^q_{R/k} \tag{8.60}
$$

be the map (8.19). We show the stronger assertion that $\Psi^q_{U/S} \otimes_R \kappa(x)$ is injective for any $x \in |S|$ assuming $\mathbf{(IV)}_q$. We fix $x \in |S|$. Without loss of generality we may assume $j_1 = 1, \cdots, j_{n-r} = n-r$ in $\mathbf{(IV)}_q$. We may work in an étale neighbourhood of x to assume that R is a strict henselian local ring with the closed point $x \in \mathrm{Spec}(R)$ and that the 0-dimensional scheme defined by $F_1 = \cdots = F_r = G_1 = \cdots = G_{n-r} = 0$ in $\mathbb{P}^n_R$ is a disjoint union of copies of $\mathrm{Spec}(R)$.

For an integer $\ell \geq 1$ let $\bigwedge^\ell (G_j)_{j\neq 1}$ be the subspace of $\bigwedge^\ell (G_j)$ generated by such $g_{j_0\cdots j_\ell}$ that $j_\nu \neq 1$ for $0 \leq \forall\nu \leq \ell$ (cf. (8.17)). We have the exact sequence

$$
0 \to \bigwedge^\ell (G_j)_{j\neq 1} \to \bigwedge^\ell (G_j) \xrightarrow{\tau} \bigwedge^{\ell-1} (G_j)_{j\neq 1} \to 0,
$$

where τ is characterized by the condition that $\tau(g_{1j_1\cdots j_\ell}) = -g_{j_1\cdots j_\ell}$ and that it annihilates $\bigwedge^{\ell}(G_j)_{j\neq 1}$. Put $Z_*^{(1)} = Z_2 + \cdots + Z_s$ where we recall that $Z_j \subset X$ is a smooth hypersurface section defined by G_j. Consider the residue map along Z_1:

$$\mathrm{Res} : \Omega^{n-r}_{X/S}(\log Z_*) \to \Omega^{n-r-1}_{Z_1/S}(\log Z_*^{(1)} \cap Z_1);\ dg_1/g_1 \wedge \omega \mapsto \omega|_{Z_1},$$

where g_1 is a local equation of Z_1. By (8.18) one sees that $\mathrm{Res}\circ\Psi^q_{U/S}$ factors through τ and we get the following commutative diagram:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\bigoplus_{\ell=0}^{q} \bigwedge^{\ell+n-r} (G_j)_{j\neq 1} \otimes_{\mathbb{Q}} \Omega^{q-\ell}_{R/k} & \longrightarrow & H^0(\Omega^{n-r}_{X/S}(\log Z_*^{(1)})) \otimes_R \Omega^q_{R/k} \\
\downarrow & & \downarrow \\
\bigoplus_{\ell=0}^{q} \bigwedge^{\ell+n-r} (G_j) \otimes_{\mathbb{Q}} \Omega^{q-\ell}_{R/k} & \xrightarrow{\Psi^q_{U/S}} & H^0(\Omega^{n-r}_{X/S}(\log Z_*)) \otimes_R \Omega^q_{R/k} \\
\downarrow{\scriptstyle \tau\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{Res}\otimes\mathrm{id}} \\
\bigoplus_{\ell=0}^{q} \bigwedge^{\ell+n-r-1} (G_j)_{j\neq 1} \otimes_{\mathbb{Q}} \Omega^{q-\ell}_{R/k} & \longrightarrow & H^0(\Omega^{n-r-1}_{Z_1/S}(\log Z_*^{(1)} \cap Z_1)) \otimes_R \Omega^q_{R/k} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

By the diagram and induction we are reduced to show the injectivity of $\Psi^q_{U/S} \otimes_R \kappa(x)$ in case $s = 1$ or $n - r = 0$. If $s = 1$, the assertion is clear because $\bigwedge^{\ell}(G_j) = 0$ by convention. We consider the case $n - r = 0$. Then

$$X = \{F_1 = \cdots = F_n = 0\} \subset \mathbb{P}^n_R \quad \text{and} \quad Y_j = \{G_j = 0\} \subset \mathbb{P}^n_R \quad (1 \le j \le s).$$

By the assumption we have $X = \coprod_{\beta\in X(R)} \mathrm{Spec}(R)$ where $X(R)$ is the set of sections of $X \to \mathrm{Spec}(R)$. The map (8.60) becomes

$$\Psi : \bigoplus_{\ell=0}^{q} \bigwedge^{\ell}(G_j) \otimes_{\mathbb{Q}} \Omega^{q-\ell}_{R/k} \longrightarrow H^0(\mathcal{O}_X) \otimes_R \Omega^q_{R/k}$$

By Nakayama's lemma, condition $\mathbf{(IV)}_q$ in the Main Theorem implies:

$(*)$ *There are $q+1$ points $\beta_0, \cdots, \beta_q \in X(R)$ such that the map*

$$W \xrightarrow{\Theta} A_1(0)/(J' + \mathfrak{m}_{\beta_0}) \oplus \cdots \oplus A_1(0)/(J' + \mathfrak{m}_{\beta_q}),$$

is surjective, where $\mathfrak{m}_\beta \subset P_R$ *denotes the homogeneous ideal defining* β *in* $\mathbb{P}^n_R = \mathrm{Proj}(P_R)$.

We note

$$H^0(\mathcal{O}_X) = \bigoplus_{\beta \in X(R)} R \cdot [\beta]$$

and put

$$H^0(\mathcal{O}_X)' = \bigoplus_{0 \le \nu \le q} R \cdot [\beta_\nu].$$

It suffices to show the injectivity of $\Psi' \otimes_R \kappa(x)$ where

$$\Psi' : \bigoplus_{\ell=0}^{q} \bigwedge^{\ell}(G_j) \otimes_{\mathbb{Q}} \Omega^{q-\ell}_{R/k} \longrightarrow H^0(\mathcal{O}_X)' \otimes_R \Omega^q_{R/k} \tag{8.61}$$

is the composite of Ψ with the projection $H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_X)'$. We have

$$\Psi'(v_{1j_1\cdots j_\ell} \otimes \eta) = \sum_{\nu=0}^{q} [\beta_\nu] \otimes \epsilon_{j_1\nu} \wedge \cdots \wedge \epsilon_{j_\ell \nu} \wedge \eta \quad (\eta \in \Omega^{q-\ell}_{R/k}),$$

$$\text{with} \quad \epsilon_{j\nu} := \mathrm{dlog}((G_j^{e_1}/G_1^{e_j})(\beta_\nu)) \in \Omega^1_{R/k}.$$

Hence the desired assertion follows from the following two lemmas.

Lemma 8.5.1. *The log forms* $\epsilon_{j\nu}$ $(2 \le j \le s,\ 0 \le \nu \le q)$ *are linearly independent in* $\Omega^1_{R/k} \otimes \kappa(x)$.

Lemma 8.5.2. *Let* Ω *be a finite dimensional vector space over a field* k *of characteristic zero. Suppose that* $\epsilon_{j\nu} \in \Omega$ $(1 \le j \le s,\ 0 \le \nu \le q)$ *are linearly independent. For given* $\eta_t \in \bigwedge^t \Omega$ *with* $0 \le t \le q$, *put for* $0 \le \nu \le q$ *(the product is wedge product)*

$$\left.\begin{aligned} \omega_\nu \;=\; & \eta_q + \textstyle\sum_{j=1}^{s} \eta_{q-1}\epsilon_{j\nu} + \sum_{1 \le j_1 < j_2 \le s} \eta_{q-2}\epsilon_{j_1\nu}\epsilon_{j_2\nu} + \cdots \\ & \cdots + \textstyle\sum_{1 \le j_1 < \cdots < j_q \le s} \eta_0 \epsilon_{j_1\nu} \cdots \epsilon_{j_q\nu} \quad \in \bigwedge^q \Omega. \end{aligned}\right\} \tag{8.62}$$

If $\omega_\nu = 0$ *for* $0 \le \forall \nu \le q$, *then* $\eta_t = 0$ *for* $0 \le \forall t \le q$.

Now we prove the above lemmas. Lemma 8.5.1 follows from the condition $(*)$ by noting the following:

Lemma 8.5.3. *For* $\beta \in X(R)$, $A_1(0)/(J' + \mathfrak{m}_\beta)$ *is a free* R*-module of rank* $s-1$, *and the dual of* $T_{R/k} \to A_1(0)/(J' + \mathfrak{m}_\beta)$ *induced by* Θ *is given by the matrix*

$$(\mathrm{dlog}\,\frac{G_2^{e_1}}{G_1^{e_2}}(\beta), \cdots, \mathrm{dlog}\,\frac{G_s^{e_1}}{G_1^{e_s}}(\beta))$$

for a suitable choice of a basis of $A_1(0)/(J'+\mathfrak{m}_\beta)$.

Proof Giving $\beta=(\beta_0:\cdots:\beta_n)$ in the homogeneous coordinate of $\mathbb{P}^n_R$,

$$\mathfrak{m}_\beta=(\beta_i X_j-\beta_j X_i)_{0\le i<j\le n}\subset P_R=R[X_0,\cdots,X_n].$$

We may assume without loss of generality that $\beta_0=1$. We write $\partial_\ell:=\partial/\partial X_\ell$ for $0\le\ell\le n$ and $L(\beta):=L(1,\beta_1,\cdots,\beta_n)\in R$ for any homogeneous polynomial L. Then we have an isomorphism

$$A_1(0)/(J'+\mathfrak{m}_\beta)\cong\big(\bigoplus_{i=1}^n R\cdot X_0^{d_i}\mu_i\bigoplus\bigoplus_{j=1}^s R\cdot X_0^{e_j}\lambda_j\big)/J'_\beta$$

$$\text{with}\quad J'_\beta=\sum_{\ell=0}^n R\cdot(\sum_{i=1}^n\partial_\ell F_i(\beta)X_0^{d_i}\mu_i+\sum_{j=1}^s\partial_\ell G_j(\beta)X_0^{e_j}\lambda_j).$$

Using the fact

$$\det\begin{pmatrix}\partial_1F_1(\beta)&\cdots&\partial_1F_n(\beta)\\ \vdots&&\vdots\\ \partial_nF_1(\beta)&\cdots&\partial_nF_n(\beta)\end{pmatrix}\in R^*,$$

we get

$$A_1(0)/(J'+\mathfrak{m}_\beta)\cong\big(\bigoplus_{j=1}^s R\cdot X_0^{e_j}\lambda_j\big)/R\cdot(\sum_{j=1}^s e_jG_j(\beta)X_0^{e_j}\lambda_j),\tag{8.63}$$

which is a free R-module of rank $(s-1)$. To prove the second assertion we first show for $\theta\in T_{R/k}$

$$\Theta(\theta)\equiv\theta(G_1(\beta))\cdot X_0^{e_1}\lambda_1+\cdots+\theta(G_s(\beta))\cdot X_0^{e_s}\lambda_s$$
$$\mod J'+(X_\ell-\beta_\ell X_0)_{1\le\ell\le n}.\tag{8.64}$$

In fact

$$\begin{aligned}
\sum_{j=1}^{s}\theta(G_j(\beta))\cdot X_0^{e_j}\lambda_j &= \sum_{j=1}^{s}(\theta G_j)(\beta)\cdot X_0^{e_j}\lambda_j + \sum_{j=1}^{s}\sum_{\ell=0}^{n}\frac{\partial G_j}{\partial X_\ell}(\beta)\cdot\theta(\beta_\ell)\cdot X_0^{e_j}\lambda_j \\
&\equiv \sum_{j=1}^{s}(\theta G_j)(\beta)\cdot X_0^{e_j}\lambda_j - \sum_{i=1}^{n}\sum_{\ell=0}^{n}\frac{\partial F_i}{\partial X_\ell}(\beta)\cdot\theta(\beta_\ell)\cdot X_0^{d_i}\mu_i \mod J' \\
&\underset{(**)}{=} \sum_{j=1}^{s}(\theta G_j)(\beta)\cdot X_0^{e_j}\lambda_j + \sum_{i=1}^{n}(\theta F_i)(\beta)\cdot X_0^{d_i}\mu_i \\
&\equiv \sum_{j=1}^{s}(\theta G_j)\lambda_j + \sum_{i=1}^{n}(\theta F_i)\mu_i \mod (X_\ell - \beta_l X_0)_{1\le\ell\le n} \\
&= \Theta(\theta).
\end{aligned}$$

Here $(**)$ follows from $0 = \theta(F_i(\beta)) = (\theta F_i)(\beta) + \sum_{\ell=0}^{n}(\partial_\ell F_i)(\beta)\cdot\theta(\beta_\ell)$ (note $F_i(\beta) = 0$ since $\beta \in X(R)$). Let $\{(X_0^{e_j}\lambda_j)^*\}_{1\le j\le s}$ be the dual basis of $\bigoplus_{j=1}^{s} R\cdot X_0^{e_j}\lambda_j$. Then

$$\frac{1}{e_jG_j(\beta)}(X_0^{e_j}\lambda_j)^* - \frac{1}{e_1G_1(\beta)}(X_0^{e_1}\lambda_1)^* \quad (2\le j\le s)$$

is a basis of the dual module of the right hand side of (8.63). By (8.64), we see that the dual of $T_{R/k} \to A_1(0)/(J' + \mathfrak{m}_\beta)$ is given by

$$\frac{1}{e_jG_j(\beta)}(X_0^{e_j}\lambda_j)^* - \frac{1}{e_1G_1(\beta)}(X_0^{e_1}\lambda_1)^* \longmapsto \frac{1}{e_1e_j}\,\mathrm{dlog}\,\frac{G_j^{e_1}}{G_1^{e_j}}(\beta).$$

This completes the proof of Lemma 8.5.1. □

Finally we prove Lemma 8.5.2. We prove the assertion by induction on $q \ge 0$. If $q = 0$, it is clear. Let ϵ_{il}^* $(1\le i\le s,\ 0\le \ell\le q)$ be a linear form on Ω such that $\epsilon_{il}^*(\epsilon_{i'\ell'}) = 0$ if $(i,\ell) \ne (i',\ell')$ and $\epsilon_{il}^*(\epsilon_{il}) = 1$. For $1 \le \nu \ne \ell \le q$ we have

$$\begin{aligned}
(\omega_\nu, \epsilon_{il}^*) &= (\eta_q, \epsilon_{il}^*) + \sum_{j=1}^{s}(\eta_{q-1}, \epsilon_{il}^*)\epsilon_{j\nu} + \cdots + \sum_{1\le j_1<\cdots<j_{q-1}\le s}(\eta_1, \epsilon_{il}^*)\epsilon_{j_1\nu}\cdots\epsilon_{j_{q-1}\nu} \\
&= 0 \in \bigwedge^{q-1}\Omega.
\end{aligned}$$

By induction this implies $(\eta_t, \epsilon_{il}^*) = 0$ for $0 \le \forall t, \forall \ell \le q$ and $1 \le \forall i \le s$. Then

$$0 = (\omega_\nu, \epsilon_{j_1\nu}^*\cdots\epsilon_{j_{q-t}\nu}^*) = \eta_t.$$

This completes the proof of Lemma 8.5.2.

8.5.2 Proof of (ii) : Case $T_{R/k} \simeq W$

Since we have proved the injectivity of $\Psi^q_{U/S} \otimes_R \kappa(x)$ for $\forall x \in |S|$, it suffices to show that the kernel of $\overline{\nabla}_q$ is a locally free R-module of the same rank as the source of $\Psi^q_{U/S}$. More precisely we want to show the following.

Lemma 8.5.4. *Assuming* **(I)**, **(II)** *and* $\mathbf{(III)}_q$ *in the Main Theorem,* $\mathrm{Ker}(\overline{\nabla}_q)$ *is a locally free* R*-module of rank*

$$\sum_{k\geq 0} \binom{s-1}{n-r+k} \cdot \mathrm{rank} \bigwedge^{q-k} T_{R/k}. \tag{8.65}$$

In this subsection we show Lemma 8.5.4 assuming $\Theta : T_{R/k} \to W$ is an isomorphism. We note that W is a locally free R-module by **(I)**. By Theorem 8.4.1, $\overline{\nabla}_q$ is identified with the map

$$B_0(\mathbf{d}+\mathbf{e}-n-1) \otimes \bigwedge^{q} W^* \longrightarrow B_1(\mathbf{d}+\mathbf{e}-n-1) \otimes \bigwedge^{q+1} W^*$$

induced by the multiplication $B_q(\ell) \otimes W \to B_{q+1}(\ell)$. By the duality theorem (Theorem 8.4.2), the dual of the map fits into the commutative diagram

$$\begin{array}{ccc} B_1(\mathbf{d}+\mathbf{e}-n-1)^* \otimes \bigwedge^{q+1} W & \longrightarrow & B_0(\mathbf{d}+\mathbf{e}-n-1)^* \otimes \bigwedge^{q} W \\ \cong \uparrow & & \uparrow \iota \\ B_{n-r-1}(\mathbf{d}-n-1) \otimes \bigwedge^{q+1} W & \xrightarrow{\Phi} & B_{n-r}(\mathbf{d}-n-1) \otimes \bigwedge^{q} W. \end{array} \tag{8.66}$$

The diagram induces an exact sequence

$$0 \to \mathrm{Coker}(\Phi) \to \big(\mathrm{Ker}(\overline{\nabla}_q)\big)^* \to \mathrm{Coker}(\iota) \to 0.$$

Due to Theorem 8.4.2 ι is injective and its cokernel is a locally free R-module of rank $\binom{s-1}{n-r} \cdot \mathrm{rank} \bigwedge^{q} W$. Therefore it suffices to show that the cokernel of the map Φ is a locally free R-module of rank

$$\sum_{k\geq 1} \binom{s-1}{n-r+k} \cdot \mathrm{rank} \bigwedge^{q-k} W. \tag{8.67}$$

In order to show this we recall the notations in §8.4.2 and §8.4.3. For integers k, h and ℓ we put

$$\mathbf{M}_{k,h}(\ell) = \bigwedge^{n+r+s-h} \Sigma^* \otimes \mathcal{L}^{r+k-h} \otimes \pi^* \mathcal{O}(\ell - \mathbf{d} + n + 1)$$

and

$$C_{k,h}(\ell) = H^0(\mathbb{P}, \mathbf{M}_{k,h}(\ell)).$$

Due to Lemma 8.4.3, there is an exact sequence

$$C_{k,1}(\ell) \longrightarrow C_{k,0}(\ell) \longrightarrow B_k(\ell) \longrightarrow 0. \tag{8.68}$$

We now need the following two results Lemmas 8.5.5 and 8.5.6. The first result is a direct consequence of [3, Lem.(7-4)]. It is a generalization of Nori's connectivity [14] to the case of open complete intersections and is the key to the proof of the Main Theorem.

Proposition 8.5.5. *Putting $C_{k,h} = C_{k,h}(\mathbf{d} - n - 1)$, the Koszul complex*

$$\begin{aligned} C_{n-r+h-1,h} \otimes \bigwedge^{q-h+1} W \quad &\to \quad C_{n-r+h,h} \otimes \bigwedge^{q-h} W \\ &\to \quad C_{n-r+h+1,h} \otimes \bigwedge^{q-h-1} W \to C_{n-r+h+2,h} \otimes \bigwedge^{q-h-2} W \end{aligned}$$

is exact for $\forall h \geq 0$ assuming **(II)** *and* $\mathbf{(III)}_q$ *in the Main Theorem.*

Putting $\mathbf{M}_{k,h} = \mathbf{M}_{k,h}(\mathbf{d} - n - 1)$, (8.44) induces the following exact sequence (cf. [3, Lem.(5-1)])

$$0 \to \mathbf{M}_{n-r+k,n+r+s} \to \mathbf{M}_{n-r+k,n+r+s-1} \to \cdots \to \mathbf{M}_{n-r+k,0} \to 0. \tag{8.69}$$

Lemma 8.5.6. *Let $k \geq 1$ be an integer. Then the complex*

$$0 \to C_{n-r+k,n+r+s} \to C_{n-r+k,n+r+s-1} \to \cdots \to C_{n-r+k,0} \to 0$$

induced by (8.69) *is exact except at the term $C_{n-r+k,k-1}$, and the cohomology group at this term is isomorphic to $H^n(\bigwedge^{r+s-k} \Sigma^*)$.*

Before proving Lemma 8.5.6, we complete the proof of Lemma 8.5.4 assuming $T_{R/k} \simeq W$. We write $B_\bullet = B_\bullet(\mathbf{d} - n - 1)$. Consider the following commutative diagram:

$$\begin{tikzcd}
 & C_{n-r-1,1}\otimes \overset{q+1}{\bigwedge} W \arrow[r] \arrow[d] & C_{n-r-1,0}\otimes \overset{q+1}{\bigwedge} W \arrow[r, "j_1"] \arrow[d] & B_{n-r-1}\otimes \overset{q+1}{\bigwedge} W \arrow[d, "\Phi"] \\
 & C_{n-r,1}\otimes \overset{q}{\bigwedge} W \arrow[r] \arrow[d] & \boxed{C_{n-r,0}\otimes \overset{q}{\bigwedge} W} \arrow[r, "j_2"] \arrow[d] & B_{n-r}\otimes \overset{q}{\bigwedge} W \\
C_{n-r+1,2}\otimes \overset{q-1}{\bigwedge} W \arrow[r] \arrow[d] & \boxed{C_{n-r+1,1}\otimes \overset{q-1}{\bigwedge} W} \arrow[r] \arrow[d] & \boxed{C_{n-r+1,0}\otimes \overset{q-1}{\bigwedge} W} \arrow[d] & \\
\boxed{C_{n-r+2,2}\otimes \overset{q-2}{\bigwedge} W} \arrow[r] \arrow[d] & \boxed{C_{n-r+2,1}\otimes \overset{q-2}{\bigwedge} W} \arrow[r] \arrow[d] & \vdots & \\
\boxed{C_{n-r+3,2}\otimes \overset{q-3}{\bigwedge} W} \arrow[r] \arrow[d] & \vdots & & \\
\vdots & & &
\end{tikzcd}$$

Starting from the third row, each horizontal sequence is exact except at the term $C_{n-r+k,k-1}\otimes \overset{q-k}{\bigwedge} W$ $(k \geq 1)$ by Lemma 8.5.6. The horizontal sequences in the first and second row are exact, and the maps j_1 and j_2 are surjective (cf. (8.68)). The vertical sequences are exact at the boxed terms by Proposition 8.5.5. A diagram chase now shows that there is a finite decreasing filtration $U^\bullet$ on $\operatorname{Coker}\Phi$ such that $U^0 \operatorname{Coker}\Phi = \operatorname{Coker}\Phi$ and that

$$\operatorname{Gr}_U^{k-1} \operatorname{Coker}\Phi \simeq H^n(\overset{r+s-k}{\bigwedge} \Sigma^*) \otimes \overset{q-k}{\bigwedge} W \quad (k \geq 1).$$

This shows that the cokernel of Φ is a locally free R-module. Moreover, by Proposition 8.4.5 (iii), we have

$$\operatorname{rank}(\operatorname{Coker}\Phi) = \sum_{k\geq 1} \binom{s-1}{r+s-k-n-1} \cdot \operatorname{rank} \overset{q-k}{\bigwedge} W, \tag{8.70}$$

which is equal to (8.67). (Note $\binom{x}{\ell} = 0$ for $\ell < 0$ by convention.) This completes the proof of Lemma 8.5.4 assuming $T_{R/k} \simeq W$.

Now we prove Lemma 8.5.6. Noting $\mathbf{M}_{n-r+k,n+k} = \overset{r+s-k}{\bigwedge} \Sigma^*$, we decompose (8.69) into the following exact sequences:

$$0 \to \mathbf{M}_{n-r+k,n+r+s} \to \cdots \to \mathbf{M}_{n-r+k,n+k+1} \to N_1 \to 0, \tag{8.71}$$

$$0 \longrightarrow N_1 \longrightarrow \bigwedge^{r+s-k} \Sigma^* \longrightarrow N_2 \longrightarrow 0, \tag{8.72}$$

$$0 \to N_2 \to \mathbf{M}_{n-r+k,n+k-1} \to \cdots \to \mathbf{M}_{n-r+k,0} \to 0. \tag{8.73}$$

By Lemma 8.4.4 i), $H^w(\mathbf{M}_{n-r+k,h}) = 0$ for $\forall w \geq 0$ and $\forall h \geq n+k+1$. Hence $H^w(N_1) = 0$ and $H^w(\bigwedge^{r+s-k} \Sigma^*) = H^w(N_2)$ for $\forall w \geq 0$. On the other hand, since $H^w(\bigwedge^{\bullet}\Sigma^* \otimes \mathcal{L}^\nu) = 0$ if $\nu > 0$ and $w > 0$ by Lemma 8.4.4 (iii), we have $H^w(\mathbf{M}_{n-r+k,h}) = 0$ for $\forall w > 0$ and $0 \leq \forall h \leq n+k-1$. This means that (8.73) is a flabby resolution of N_2. Therefore, for $0 \leq h \leq n+k-2$, the cohomology group at $C_{n-r+k,h}$ is isomorphic to $H^{n+k-1-h}(\bigwedge^{r+s-k} \Sigma^*)$. Now the assertion follows from Proposition 8.4.5 (i).

8.5.3 Proof of (ii) : General Case

It remains to show 8.5.4 in case $\Theta : T_{R/k} \to W$ is not necessarily an isomorphism. Setting $I = \mathrm{Ker}(T_{R/k} \to W)$, we get the exact sequence:

$$0 \longrightarrow I \longrightarrow T_{R/k} \longrightarrow W \longrightarrow 0. \tag{8.74}$$

Since W is a locally free R-module, so is I. By the argument in §8.5.2 it suffices to show that the cokernel of the map

$$B_{n-r-1}(\mathbf{d}-n-1) \otimes \bigwedge^{q+1} T_{R/k} \longrightarrow B_{n-r}(\mathbf{d}-n-1) \otimes \bigwedge^{q} T_{R/k} \tag{8.75}$$

is a locally free R-module of rank

$$\sum_{k\geq 1} \binom{s-1}{n-r+k} \cdot \mathrm{rank} \bigwedge^{q-k} T_{R/k}. \tag{8.76}$$

(8.74) gives rise to a filtration $U^\bullet$ on $\bigwedge^{q} T_{R/k}$ such that $U^i/U^{i+1} = (\bigwedge^{q-i} W) \otimes (\bigwedge^{i} I)$. Since I annihilates $B_\bullet(\mathbf{d}-n-1)$, the map (8.75) admits a filtration whose graded quotients for $0 \leq i \leq q$ are given by

$$B_{n-r-1}(\mathbf{d}-n-1) \otimes \bigwedge^{q+1-i} W \otimes \bigwedge^{i} I \longrightarrow B_{n-r}(\mathbf{d}-n-1) \otimes \bigwedge^{q-i} W \otimes \bigwedge^{i} I. \tag{8.77}$$

By what we have shown in §8.5.2 the cokernel of the map (8.75) is a locally free R-module of rank

$$\sum_{i=0}^{q} \left(\sum_{k\geq 1} \binom{s-1}{n-r+k} \cdot \mathrm{rank}(\bigwedge^{q-i-k} W) \right) \cdot \mathrm{rank}(\bigwedge^{i} I). \tag{8.78}$$

It is easy to see that the numbers (8.76) and (8.78) are equal (left to the reader).

This completes the proof of the Main Theorem.

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